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COMPLETE SPECTRAL THEORY FOR MATRICES OVER A FIELD WHOSE GRAPH IS A STAR

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Abstract: For matrices over a field $\mathbb{F}$, whose graph is a star, any characteristic polynomial may occur if $|\mathbb{F}|$ is large enough. Depending upon the diagonal entries, some linear factors will have to occur, but given this, the characteristic polynomial is still arbitrary. For smaller fields, a characterization of achievable polynomials is given. The geometrically multiple eigenvalues are easily identified, and, given this, the Jordan structure is completely determined. It turns out that no eigenvalue may enjoy more that one block of size greater than one, a restriction not present in all trees.

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1. Introduction

We follow standard matricial notation as in [3].

Let $\mathbb{F}$ be a field and G an undirected graph on n vertices. By

$$A = (a_{i\,j}) \in \mathfrak{F}(G)$$

we mean an n-by-n matrix with entries from $\mathbb{F}$, in which for $i \neq j$, $a_{i\,j} \neq 0$ iff $\{i,\, j\}$ is an edge of G. There is no restriction on diagonal entries other than that they be in $\mathbb{F}$. Thus, $a_{i\,j} \neq 0$ iff $a_{j\,i} \neq 0$, and A is *combinatorially symmetric.* The *star* on n vertices, S_n, is the tree with one central vertex and $n-1$ vertices pendent from it. Our interest here is in the spectral theory for matrices in $\mathfrak{F}(S_n)$, including the Jordan canonical form (JCF), understood in a natural way for arbitrary fields, and characteristic polynomials over $\mathbb{F}$. We give a complete description of what can happen.

Sometimes matrices in $\mathfrak{F}(S_n)$ are called *arrow matrices*, and there seems to be interest in them for a number of reasons (see, e.g., [7, 8]).

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Recently a geometric multiplicity theory has been developed for matrices in $\mathfrak{F}(T)$ when T is a tree [6]; this generalizes a prior multiplicity theory for real symmetric and Hermitian matrices [4, 5], though there are notable differences. This will be of use to us. Also, for the path on n vertices, P_n, the arbitrariness of characteristic polynomials, and more has been discussed for $\mathfrak{F}(P_n)$, [1]. However here Jordan structure is very simple as no eigenvalue may have geometric multiplicity more than one. The case of stars is much richer.

In the case of algebraically closed fields, because of the *additive inverse eigenvalue problem* [2], it is clear that any characteristic polynomial may occur in $\mathfrak{F}(G)$, and, in particular in $\mathfrak{F}(S_n)$. However, the IEP gives no insight into Jordan structure, nor into non-algebraically closed fields which we are able to analyze.

2. Supporting Facts

Without loss of generality we may assume

$$A = \begin{bmatrix} b_1 & a_2 & \cdots & a_n \\ 1 & b_2 & \text{\Large 0} & \\ \vdots & & \ddots & \\ & \text{\Large 0} & & \\ 1 & & & b_n \end{bmatrix} \tag{1}$$

in which $b_1, \dots, b_n, a_2, \dots, a_n \in \mathbb{F}$, because of diagonal similarity and permutation similarity. We assume $a_i \neq 0$, $i = 2, \dots, n$.

First we collect the facts we need in order to give our main results in the next section. The first of these motivates our primary result, by indicating that we cannot expect to say more. It follows from the new geometric multiplicity theory [6], as the center vertex of a star can be the only Parter vertex, or could be proven, in this case, via a straightforward matrix calculation. Let $\mathrm{m}_A(\lambda)$ denote the (algebraic) multiplicity of λ as an eigenvalue of A and $\mathrm{gm}_A(\lambda)$ the geometric multiplicity.

Theorem 2.1. *For $A \in \mathfrak{F}(S_n)$, $\mathrm{gm}_A(\alpha) = k \geq 2$ iff α appears exactly $k+1$ times among $b_2, \dots, b_n$. For any $\beta \in \sigma(A)$, if $\beta \notin \{b_2, \dots, b_n\}$ then $\mathrm{gm}_A(\beta) = 1$, and if γ appears once among $b_2, \dots, b_n$, $\gamma \notin \sigma(A)$.*

Next we prove a fact that we will need for our Jordan canonical form results, by a matrix calculation.

Lemma 2.2. *For* $A \in \mathfrak{F}(S_n)$*, if* α *appears exactly* k *times among* $b_2, \dots, b_n$*, then* $\operatorname{rank}(A - \alpha I) = n - k + 1$ *and* $\operatorname{rank}[(A - \alpha I)^2] \geq n - k$*.*

Proof: We may suppose that A has the form (1) and, wlog, that $\alpha = 0$ and that the k appearances are in the last k diagonal positions. Then,

$$A = \begin{bmatrix} 0 & a_2 & \cdots & a_n \\ 1 & & & \\ & & 0 & \\ \vdots & & & \\ 1 & & & \end{bmatrix} + \begin{bmatrix} b_1 & & & & & & \\ & b_2 & & & & & \\ & & \ddots & & & 0 & \\ & & & b_{n-k} & & & \\ & 0 & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

Let A_1 be the first summand and A_2 the second. Since the first $n-k+1$ rows of A are linearly independent and each of the last $k-1$ rows repeat the one above them, rank$A = n-k+1$ as claimed. Now, with $a = \sum_{i=2}^{n} a_i$

$$A^2 = (A_1 + A_2)^2 = A_1^2 + A_1 A_2 + A_2 A_1 + A_2^2$$

$$= \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a_2 & \dots & a_n \\ 0 & a_2 & \dots & a_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & a_2 & \dots & a_n \end{bmatrix} + \begin{bmatrix} 0 & b_2 a_2 & \dots & b_{n-k} a_{n-k} & 0 & \dots & 0 \\ b_1 & & & & & & \\ \vdots & & & 0 & & & \\ b_1 & & & & & & \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & b_1 a_2 & \dots & b_1 a_n \\ b_2 & & & \\ \vdots & & & \\ b_{n-k} & & 0 & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix} + \begin{bmatrix} b_1^2 & & & & & & \\ & b_2^2 & & & & & \\ & & \ddots & & & 0 & \\ & & & b_{n-k}^2 & & & \\ & 0 & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

$$
= \begin{bmatrix}
b_1^2 + a & a_2(b_1+b_2) & a_3(b_1+b_3) & & \dots & a_{n-k}(b_1+b_{n-k}) & a_{n-k+1}b_1 & \dots & a_n b_1 \\
b_1+b_2 & a_2+b_2^2 & a_3 & & \dots & a_{n-k} & a_{n-k+1} & \dots & a_n \\
b_1+b_3 & a_2 & a_3+b_3^2 & a_4 & \dots & a_{n-k} & a_{n-k+1} & \dots & a_n \\
\vdots & \vdots & & \ddots & & & \vdots & & \vdots \\
b_1+b_{n-k} & a_2 & a_3 & a_4 & \dots & a_{n-k}+b_{n-k}^2 & a_{n-k+1} & \dots & a_n \\
b_1 & a_2 & a_3 & a_4 & \dots & a_{n-k} & a_{n-k+1} & \dots & a_n \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \\
b_1 & a_2 & a_3 & a_4 & \dots & a_{n-k} & a_{n-k+1} & \dots & a_n
\end{bmatrix}
$$

Using row $n-k+1$ and then column $n-k+1$ this matrix reduces, by elementary operations, to

$$
\begin{bmatrix}
a & a_2b_2 & a_3b_3 & \dots & a_{n-k}b_{n-k} & 0 & \dots & 0 & \\
b_1 & b_2^2 & 0 & \dots & 0 & 0 & \dots & 0 & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & & \vdots & \\
b_{n-k} & 0 & 0 & \dots & b_{n-k}^2 & 0 & \dots & 0 & \\
b_1 & a_2 & a_3 & \dots & a_{n-k} & a_{n-k+1} & \dots & a_n & \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0
\end{bmatrix}
$$

Since $b_2, \dots, b_{n-k}$ are assumed nonzero, as well as $a_2, \dots, a_n$, rows $2, \dots\ n-k+1$ are linearly independent, so $\operatorname{rank}[(A-\alpha I)^2] \geq n-k$. This completes the proof. ■

Finally, we give a technical lemma that explains why GF_2 is an exception to the first result of the next section.

Lemma 2.3. *Let* $\mathbb{F}$ *be a field,* $\mathbb{F} \neq GF_2$*, and* $a \in \mathbb{F}$*. For each* $r \in \mathbb{N}$*,* $r \geq 2$*, there are nonzero elements* $a_1, a_2, \dots, a_r \in \mathbb{F}$ *such that* $a = a_1 + a_2 + \cdots a_r$*.*

Proof: If $(r-1)1 \neq a$ take $a_1 = \cdots = a_{r-1} = 1$, $a_r = a - (r-1)1$. If $(r-1)1 = a$, pick a nonzero element $k \in \mathbb{F}$, $k \neq 1$ and take $a_1 = \cdots = a_{r-2} = 1$, $a_{r-1} = k$, $a_r = a - (r-2)1 - k$. ■

We will apply this fact both when $a = 0$ and $a \neq 0$. For GF_2, when r is odd and $a = 0$, the claim is false and when r is even and $a \neq 0$, the claim is false.

3. Main Results

Now we may give our primary results. The first of these says that for the star, we may have any characteristic polynomial, with any values on the pendent vertices, subject to the algebraic limitations of Theorem 2.1, for any field, with the exception of GF_2. That exception actually occurs and is because of Lemma 2.3. Using this we determine for which fields the characteristic polynomials achievable for the star are arbitrary (as the entries corresponding to the pendent vertices vary) in Theorem 3.5. This is essentially when $|\mathbb{F}| \geq n-1$. When $|\mathbb{F} < n-1$, some field elements must occur as roots of the characteristic polynomial. The precise restrictions are characterized in this case, leaving the characteristic polynomial otherwise arbitrary (Theorem 3.10). Then, we are able to characterize the possible Jordan canonical forms (JCF's) in terms of the diagonal structure and the characteristic polynomial.

Theorem 3.1. *Let* $\mathbb{F}$ *be a field,* $\mathbb{F} \neq GF_2$*, and* $A \in \mathfrak{F}(S_n)$*, as shown in (1). Let* $\alpha_1\,, \ldots, \alpha_t$*, with multiplicities* $m_1\,, \ldots, m_t$*, respectively, be a rearrangement of the distinct items among* $b_2\,, \ldots, b_n$*. If* $p(x)$ *is any monic polynomial of degree* n *over* $\mathbb{F}$ *such that, for* $i = 1\,, \ldots, t$, $(x-\alpha_i)^{m_i-1}|p(x)$ *if* $m_i > 1$*, and* $(x-\alpha_i) \not| p(x)$ *if* $m_i = 1$*, then* $b_1\,, a_2\,, \ldots, a_n$*, may be chosen in* $\mathbb{F}$ *so that* $p_A(x) = p(x)$*.*

Proof: Let

$$g(x) = \prod_{i=1}^{t} (x-\alpha_i)^{m_i},$$

$d(x) = \gcd(p(x), g(x))$, $p(x) = p_1(x)d(x)$, $g(x) = g_1(x)d(x)$. It may happen that $g_1(x) = 1$ (see example below); if this is not the case, from the divisibility conditions of the hypothesis of the theorems follows that each root of $g_1(x)$ has multiplicity one and are some of the roots of $g(x)$. Without loss of generality, we may suppose $g_1(x) = (x-\alpha_1)\cdots(x-\alpha_s)$, $0 \leq s \leq t$ ($s = 0$ corresponding to the case $g_1(x) = 1$). As $\deg p(x) = \deg g(x) + 1$ we have also $\deg p_1(x) = \deg g_1(x) + 1$. From Euclidian division there are $b \in \mathbb{F}$, $r(x) \in \mathbb{F}[x]$ such that

$$p_1(x) = (x-b)g_1(x) - r(x)\,. \tag{2}$$

with $r(x) = 0$ or $\deg r(x) > \deg g_1(x)$; note that we have $r(x) = 0$ if and only if $g_1(x) = 1$.

For $r(x) \neq 0$ consider the partial fraction decomposition of $\frac{r(x)}{g_1(x)}$,

$$\frac{r(x)}{g_1(x)} = \sum_{i=1}^{s} \frac{A_i}{x - \alpha_i} \tag{3}$$

with $A_i \neq 0$ (otherwise α_i will not be a root of g_1), $i = 1, \cdots, s$. Now use Lemma 2.3: For each i, $1 \leq i \leq s$ choose m_i nonzero elements in $\mathbb{F}$, $a_{i_1}, \ldots, a_{i_{m_i}}$ suck that $a_{i_1} + \cdots + a_{i_{m_i}} = A_i$. For $s + 1 \leq i \leq t$, α_i is not a root of $g_1(x)$ but is a root of $g(x)$; according to the hypothesis of the theorem this is only possible if $m_i > 1$, So again using Lemma 2.3, for each i, $s+1 \leq i \leq t$, there are m_i nonzero elements in $\mathbb{F}$, $a_{i_1}, \ldots, a_{i_{m_i}}$ suck that $a_{i_1} + \cdots + a_{i_{m_i}} = 0$. Equation 3 can be written as:

$$\frac{r(x)}{g_1(x)} = \sum_{i=1}^{s} \sum_{l=1}^{m_i} \frac{a_{i_l}}{x - \alpha_i} + \sum_{i=s+1}^{t} \sum_{l=1}^{m_i} \frac{a_{i_l}}{x - \alpha_i}\,. \tag{4}$$

This equality is also valid for $r(x) = 0$, considering the first double summand as zero. Using the fact that

$$\frac{p(x)}{g(x)} = \frac{p_1(x)}{g_1(x)}$$

we get from 2 and 4:

$$p(x) = (x - b)g(x) - \sum_{i=1}^{s} \sum_{l=1}^{m_i} a_{i_l} \frac{g(x)}{x - \alpha_i} + \sum_{i=s+1}^{t} \sum_{l=1}^{m_i} a_{i_l} \frac{g(x)}{x - \alpha_i}. \tag{5}$$

Now take for A the matrix

$$A = \begin{bmatrix}
b & a_{1_1} & \cdots & a_{1_{m_1}} & \cdots & a_{t_1} & \cdots & a_{t_{m_t}} \\
1 & \alpha_1 & {\Large 0} & & & & & \\
\vdots & {\Large 0} & \ddots & & & & & \\
1 & & & \alpha_1 & & & & \\
\vdots & & & & \ddots & & & \\
1 & & & & & \alpha_t & & \\
\vdots & & & & & & \ddots & \\
1 & & & & & & & \alpha_t
\end{bmatrix} \tag{6}$$

The Laplace expansion of $\det(xI - A)$along the first row and first column gives the second summand of 4. So the characteristic polynomial of this matrix A will be the desired polynomial. ■

We present some examples of the construction described in the previous theorem.

Example 3.2. *Suppose we want to find a matrix A in the form (1) (say over the rationals)*

$$A = \begin{bmatrix} b_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

with characteristic polynomial $p(x) = x^5 - 2x^4 - x^3 + 2x^2 + 4$. The conditions of Theorem 3.1 are clearly satisfied, so we begin by dividing $p(x)$ by $g(x) = x(x-1)(x+1)(x+2) = x^4 - 2x^3 + 2x$, to obtain

$$p(x) = (x - 0)g(x) + 4$$

(so $b_1 = 0$). Now we perform the partial fraction decomposition of $\frac{4}{g(x)}$:

$$\frac{4}{x(x-1)(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1} + \frac{d}{x-2}$$

The usual calculations give us $a_2 = a = 2$, $a_3 = b = -2$, $a_4 = c = -\frac{2}{3}$, $a_5 = d = \frac{2}{3}$ and A is determined. Note that the above calculation can, in fact, be carried out over any field other than GF_2 and fields of characteristic 3. For fields of characteristic 3 we have (on the diagonal of A), $2 = -1$ and so, according to Theorem 3.1, 2 must be a root of $p(x)$, which is not the case (Note also the values obtained for a_4 and a_5).

Example 3.3. *Suppose we want to find a matrix A (over any field $\mathbb{F}$ other than GF_2) in the form*

$$A = \begin{bmatrix} b_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with characteristic polynomial $p(x) = x^2(x-1)^3$. The conditions of Theorem 3.1 are satisfied, so again we begin by dividing $p(x)$ by $g(x) = x^2(x-1)^2$, to obtain

$$p(x) = (x-0)g(x)$$

(so $b_1 = 1$). We have $r(x) = 0$ so we just have to find nonzero a_2, a_3, a_4, $a_5 \in \mathbb{F}$ such that $a_2 + a_3 = 0$, $a_4 + a_5$ (see Lemma 2.3). We may just take $a_2 = 1\, a_3 = -1$, $a_4 = 1$, $a_5 = -1$.

Remark 3.4. When $\mathbb{F}$ is GF_2 the conclusion of Theorem 3.1 is not valid. Note that the A in 1 becomes

$$A = \begin{bmatrix} b_1 & 1 & \cdots & 1 \\ 1 & b_2 & 0 & \\ \vdots & 0 & \ddots & \\ 1 & & & b_n \end{bmatrix},$$

and $b_2, \ldots, b_n \in \{0, 1\}$. They may be arranged wlog so that there are k 1's followed by l 0's, so that $k + l = n - 1$. Then by Theorem 2.1,

$$(x-1)^{k-1}x^{l-1}|p_A(x).$$

The quotient will be a monic cubic polynomial; over GF_2 there are $2^3 = 8$ such polynomials. For each k, l pair, only 2 possibilities (b=0 or b=1) occur (so that we do not have the conclusions of Theorem 3.1). However, as k, l vary (4 possibilities for the 2 parities means all 8 cubics do appear), a straightforward, but tedious, calculation gives:

$b_1 = 0$, k even, l even; quotient: $x^2(x-1)$;
$b_1 = 0$, k even, l odd; quotient: $(x-1)^3$;
$b_1 = 0$, k odd, l even; quotient: $x(x^2+x+1)$;
$b_1 = 0$, k odd, l odd; quotient: x^3+x^2+1;
$b_1 = 1$, k even, l even; quotient: $(x-1)^2x$;

$b_1 = 1$, k even, l odd; quotient: $(x-1)(x^2+x+1)$;
$b_1 = 1$, k odd, l even; quotient: x^3;
$b_1 = 1$, k odd, l odd; quotient: x^3+x+1.
So each cubic appears exactly once.

Next, we show that the characteristic polynomials that occur, over a field, are arbitrary, with certain field exceptions that we precisely determine.

Theorem 3.5. *Let n be a positive integer and $\mathbb{F}$ a field with at least $n-1$ elements. If $p(x)$ is a polynomial of degree n over $\mathbb{F}$, then there is a matrix in $\mathfrak{F}(S_n)$ with characteristic polynomial $p(x)$, except when $n = 2^k$ and $\mathbb{F} = GF_{2^k}$. In the latter case $p(x)$ cannot be $\prod_{i=1}^{n}(x-\alpha_i)$, $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, distinct, but all other polynomials do occur.*

Proof: First, identify the set

$$R = \{\alpha_1, \ldots, \alpha_r\}$$

of field elements that occur at least once as a root of $p(x)$. If $2r \leq n-1$, choose each element of R to appear exactly twice among $b_2, \ldots, b_n$, and complete this sequence with pairwise distinct elements of $\mathbb{F}$ (note the number of field elements is sufficient). Then, these α's will appear as roots of $P_A(x)$, which is otherwise arbitrary by Theorem 3.1 and A can be chosen such that $p(x) = p_A(x)$; so the proof is complete in this case .

If $2r > n-1$, pick $\left\lfloor \dfrac{n-1}{2} \right\rfloor$ elements of R and choose them to appear twice among $b_2, \ldots, b_n$, so that they will occur as roots of $p_A(x)$ which will be otherwise arbitrary by Theorem 3.1. If $n-1$ is even all b_i, $2 \leq i \leq n$ will be fixed and the proof is again complete. If $n-1$ is odd and $R \neq \mathbb{F}$ we may choose the remaining b_i, $2 \leq i \leq n$ from $\mathbb{F} \backslash R$ and again we are done. The case in which $|\mathbb{F}| = n = 2^k$ (the remaining part of the case $n-1$ odd) provides the only difficulty. That difficulty is limited to the case $R = \mathbb{F}$ and then $p(x)$ is the polynomial with each field element occurring exactly once as a root. This polynomial can not be achieved (at least one field element must occur an odd number of times among $b_2, \ldots, b_n$, but all the others can be completing the proof. ∎

Remark 3.6. There is a subtlety to the statement of Theorem 3.5 that should be mentioned. Since there are no polynomials in $\mathbb{F}[x]$ of degree k with exactly $k-1$ roots in $\mathbb{F}$, a formal exclusion of such polynomials from the statement is unnecessary.

Corollary 3.7. *For any infinite field* $\mathbb{F}$ *and any polynomial* $p(x)$ *over* $\mathbb{F}$ *of degree* n*, there is a matrix* $A \in \mathfrak{F}(S_n)$ *whose characteristic polynomial is* $p(x)$*.*

When the field is finite and sufficiently small relative to n, the characteristic polynomial of $A \in \mathfrak{F}(S_n)$ is no longer arbitrary because repeats must occur among $b_2, \dots, b_n$.

Example 3.8. *Suppose that* $\mathbb{F} = GF_5$ *and* $n = 12$*. Then among the 11 diagonal entries* $b_1, b_2, \dots, b_{12}$ *of* $A \in \mathfrak{F}(S_{12})$*, there must be at least* $6 = 11-5$ *that are duplicates of ones that appeared in a prior position. Depending upon the pattern of duplication, Theorem 3.1 guaranties that duplicate field elements will appear as roots of the characteristic polynomial at least a certain number of times. For example, if the field elements are* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ *with frequencies* $f_1 = 2, f_2 = 1, f_3 = 4, f_4 = 3, f_5 = 1$ *among* $b_1, b_2, \dots, b_{12}$ *then the polynomial*

$$p(x) = (x - \alpha_1)(x - \alpha_3)^3(x - \alpha_4)^2$$

must divide the characteristic polynomial of A *(and* α_2 *and* α_5 *will not appear as roots in this case). Notice that this always accounts for at least* $n-1-|\mathbb{F}| = 6$ *roots of of the characteristic polynomial and that there could be more.*

Regardless of the distribution of repetitions, if there are only 6, any non-repeated filed elements, will appear exactly once among the b_i*'s and thus not appear as a root of the characteristic polynomial.*

Example 3.9. *There is a more subtle restriction on the characteristic polynomial if* n *is even and* $|\mathbb{F}| + 1 < n \le 2|\mathbb{F}|$*. For example, again let* $\mathbb{F} = GF_5$ *and suppose* $n = 10$*. Now, at least 4 repetitions are guaranteed among* $b_1, b_2, \dots, b_{10}$*. Then the characteristic polynomial can not have each field element exactly once as a root. (This restriction is not implied by the one identified in the prior example.) To have each element no more than once, none can appear more than twice (according to Theorem 3.1). To have four repetitions among the* b_i*'s, we must see 4 field elements exactly twice each. Then, the fifth field element must appear exactly once, and thus cannot be a root.*

The restrictions identified in the above examples are the only ones that prevent the characteristic polynomial $p(x)$ of $A \in \mathfrak{F}(S_n)$ from being arbitrary. We formalize this in the following two-part theorem.

Theorem 3.10. *Let n be a positive integer, $\mathbb{F}$ a field with $|\mathbb{F}| < n-1$, $p(x)$ a monic polynomial of degree n over $\mathbb{F}$. If n is not both even and $\leq 2|\mathbb{F}|$, then there is a matrix $A \in \mathfrak{F}(S_n)$ with characteristic polynomial $p(x)$ if and only if $p(x) = p_1(x)p_2(x)$, with $p_1(x)$, $p_2(x) \in \mathbb{F}[x]$, $\deg p_1(x) \geq n-1-|\mathbb{F}|$ and $p_1(x)$ having all its roots in $\mathbb{F}$.*

If n is even and $n \leq 2|\mathbb{F}|$, then the above restrictions on $p(x)$ still apply, and there is exactly one additional restriction on $p(x)$: It cannot have every element of $\mathbb{F}$ exactly once as a root.

Proof: Suppose that the field elements are $\alpha_1, \ldots, \alpha_{|\mathbb{F}|}$, the number of appearances of α_i on the diagonal of $A(1)$ is t_i and that the desired multiplicity of α_i as a root of $p(x) = p_A$ (x) is m_i (possibly 0), $i = 1, \ldots, |\mathbb{F}|$. Suppose that only $m_1 > 0, \ldots, m_k > 0$. Then, by Theorem 3.1, this information is consistent if and only if:
(1) $t_i \in \{0, 2, 3, \ldots, m_i+1\}$, $1 \leq i \leq k$ and $t_i \in \{0, 1\}$, $k+1 \leq i \leq |\mathbb{F}|$.
Of course, we must also have:

(2) $\displaystyle\sum_{i=1}^{|\mathbb{F}|} t_i = n-1.$

Now for the necessity of the first restriction: $\sum m_i \geq n-1-|\mathbb{F}|$, notice that each time $t_i > 1$, we have $m_i > t_i - 1$, which is the the number of repetitions (appearances beyond 1) of α_i. Since $n-1-|\mathbb{F}|$ is the fewest possible total repetitions of all the α_i,the inequality is verified giving the first stated restriction on $p_A(x)$.

For the second restriction, if $m_1 = m_2 = \cdots = m_{|\mathbb{F}|} = 1$, then $t_i \in \{0, 2\}$, $1 \leq i \leq |\mathbb{F}|$. But then $\sum_{i=1}^{|\mathbb{F}|} t_i$ is even contradicting (2) when n is even.

For the sufficiency of the conditions, we use the desired multiplicities (meeting the restrictions) to make an assignment of the α_i's to the diagonal of $A(1)$, *ie* choice of t_i's consistent with (1) and (2), insuring that the multiplicities are possible by Theorem 3.1. For each i, $1 \leq i \leq k$, let $t_i = m_1 + 1$, $t_2 = m_2 + 1$ and so on. If $(\sum m_i) + k = n-1$ no further assignment is necessary and the proof is complete. If $(\sum m_i) + k < n-1$, then there will be unassigned field elements (that do not occur as roots of $p(x)$), and these may be assigned one at a time until $n-1$ total assignments are reached. The first restriction insures and verifies sufficiency in this case. If $(\sum m_i) + k > n-1$, we are allowed by (1) to decrease a selection of assignments (t_i's), so as to make $\sum t_i = n-1$, as long as no $t_i = 1$ with $m_i > 0$. (We may also assign additional α_i's, if any, singly.) This may always be done, except in the situation in which $m_i = 1$, $i = 1, \dots, |\mathbb{F}|$ and n is even. In this event any downward adjustment of t_i would have to be by 2 and there would be no opportunity to recover. But this situation is precluded by the second restriction of the theorem. ■

Remark 3.11. Note that the second restriction in Theorem 3.10 remains valid when $n > 2|\mathbb{F}|$ and n is even. but it is subsumed in the first restriction in that event.

Normally, the JCF is defined over an algebraically closed field, because we would like similarity to the special form by a matrix over the field. We may, of course, think of it over general fields by considering an extension field in which all the eigenvalues of the given matrix lie, or by simply thinking of it in terms of *Jordan blocks* or ranks of powers of $(A - \lambda I)$, [3]. With this in mind, we may completely determine the possible JCF's among $A \in \mathfrak{F}(S_n)$ and see how each occurs.

Theorem 3.12. *Let $\mathbb{F}$ be a field and $A \in \mathfrak{F}(S_n)$. Then for any eigenvalue $\lambda \in \sigma(A)$ at most one Jordan block associated with λ is of size greater than 1. Moreover, the number of blocks associated with λ is greater than 1 only when λ appears at least three times among $b_2, \dots, b_n$; in this event the number of blocks is one less than the number of such appearances. In particular the number of blocks is 1 if $\lambda \notin \mathbb{F}$, or λ apeears exactly twice or not at all.*

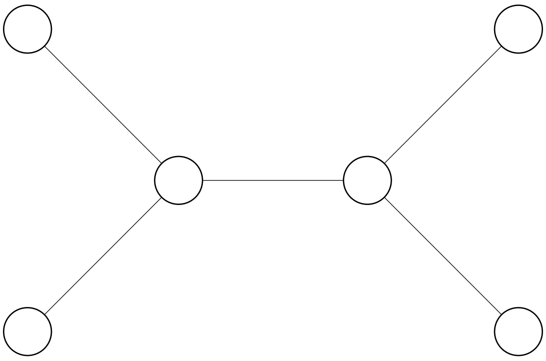
Proof: According to Lemma 2.2 squaring of A can increase the rank deficiency by at most one. This means that (if 0 is an eigenvalue), all Jordan blocks associated with 0 are 1-by-1 or just one block has size greater than 1. Application of this observation, which is independent of the field, to $(A - \lambda I)$ show that the same is true for any eigenvalue λ. (Note that, as two different eigenvalues may have geometric multiplicity less than algebraic multiplicity, we may have a block of size more than one for different eigenvalues). Since the number blocks associated with λ is its geometric multiplicity, then the second statement follows from Theorem 2.1. If $\lambda \notin \mathbb{F}$, it cannot appear among $b_2, \dots, b_n$ so that it may have only one block. The last two observations are similarly clear. ■

Now it is clear how to determine the JCF of A in any particular instance, The characteristic polynomial of A determines the eigenvalues of A and, thus, which numbers have at least one block, whether they are in $\mathbb{F}$ or not. The $b_2, \dots, b_n$ determine which eigenvalues have two or more blocks (three or more appearances). They can only be in the field. Then the algebraic multiplicity is determined by the characteristic polynomial and all of this will go into one block, except for the 1-by-1 blocks required by the geometric multiplicity. Note that this is rather different from a general matrix.

4. Other trees

Now, for paths and stars, the characteristic polynomial is arbitrary over $\mathbb{R}$, [1] . For other trees this is not yet know, but we suspect that the characteristic polynomial remains arbitrary. This is all in the not-necessarily symmetric case.

In the case of paths and stars, for each eigenvalue, the JCF can have at most one block of size more than 1. This does not remain true for other trees. For example, let T be the tree



and let

$$A = \begin{bmatrix} 0 & z & a & -a & 0 & 0 \\ 1 & y & 0 & 0 & c & -c \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}(T)\,.$$

Then $\operatorname{rank}(A) = 4$, $0 \in \sigma(A)$ has geometric multiplicity 2 and

$$A^2 = \begin{bmatrix} z & zy & 0 & 0 & zc & -zc \\ y & z+y^2 & a & -a & yc & -yc \\ 0 & z & a & -a & 0 & 0 \\ 0 & z & a & -a & 0 & 0 \\ 1 & y & 0 & 0 & c & -c \\ 1 & y & 0 & 0 & c & -c \end{bmatrix}.$$

Since A^2 row reduces to

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & a & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & y & 0 & 0 & c & -c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$\operatorname{rank}(A^2) = 2$. Since $\operatorname{rank}(A^3) = 2$ as well, this means that A has two 2-by-2 Jordan blocks associated with the eigenvalue 0.

An additional interesting question is what about a tree determines the number of Jordan blocks of size greater than 1 associated with a given multiple eigenvalue.

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