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DESCENT DATA AND ABSOLUTE KAN EXTENSIONS

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ABSTRACT: The fundamental construction underlying descent theory, the lax descent category, comes with a functor that forgets the *descent data*. We prove that, in any 2-category $\mathfrak{A}$ with lax descent objects, the forgetful morphisms create all absolute Kan extensions. As a consequence of this result, we get a *monadicity theorem* which says that a right adjoint functor is monadic if and only if it is, up to the composition with an equivalence, a functor that forgets descent data. In particular, within the classical context of *descent theory*, we show that, in a fibred category, the forgetful functor between the category of internal actions of a precategory a and the category of internal actions of the underlying discrete precategory is monadic if and only if it has a left adjoint. This proves that one of the implications of the celebrated Bénabou-Roubaud theorem does not depend on the so called Beck-Chevalley condition. Namely, we show that, in a fibred category with pullbacks, whenever an effective descent morphism induces a right adjoint functor, the functor is monadic.

KEYWORDS: effective descent morphisms, internal actions, indexed categories, creation of absolute Kan extensions, Beck's monadicity theorem, Bénabou-Roubaud theorem, descent theory, monadicity theorem.
MATH. SUBJECT CLASSIFICATION (2010): 18D05, 18C15, 18A22, 18A30, 18A40, 18F99, 11R32, 13B05.

Introduction

The (lax) *descent objects* [44, 46, 33, 36], the 2-dimensional limits underlying *descent theory* [18, 19, 23, 46, 34], play an important role in 2-dimensional universal algebra [27, 6, 30, 32]. They can be seen as 2-dimensional analogues of the equalizer. While equalizers encompass equality and commutativity of diagrams in 1-dimensional category theory, the (lax) descent objects encompass 2-dimensional coherence: morphism (or 2-cell) plus coherence equations [30, 32, 33]. For this reason, results on the lax descent objects [46, 34] (or on descent theory) usually shed light to a wide range of situations [5, 9, 20, 30, 33, 34, 36].

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As shown in [23], in the classical case of the 2-category Cat of categories, *internal category theory* provides a useful perspective to introduce *descent theory* [19, 18]. The lax descent category can be seen as a generalization of the 2-functor

$$\mathsf{Mon}(\mathsf{Set})^{\mathrm{op}} \to \mathsf{Cat}, \quad m \mapsto m\text{-}\mathsf{Set}$$

in which $\mathsf{Mon}(\mathsf{Set})$ denotes the usual category of monoids (of the cartesian monoidal category Set), and m-Set is the category of sets endowed with actions of the monoid m, usually called m-sets.

Recall that every small category a (internal category in Set) has an underlying truncated simplicial set, called the underlying *precategory* [20, 21],

$$\underline{\mathsf{Cat}}(\mathrm{j}-, a) : \Delta_3^{\mathrm{op}} \to \mathsf{Set}$$

$$\underline{\mathsf{Cat}}(1,a) \;\underset{\underline{\mathsf{Cat}}(d^1,a)}{\overset{\underline{\mathsf{Cat}}(d^0,a)}{\leftleftarrows}}\; \xrightarrow{\underline{\mathsf{Cat}}(s^0,a)} \underline{\mathsf{Cat}}(2,a) \;\overset{\underline{\mathsf{Cat}}(D^0,a)}{\longleftarrow}\; \overset{\underline{\mathsf{Cat}}(D^1,a)}{\longleftarrow}\; \overset{\underline{\mathsf{Cat}}(D^2,a)}{\longleftarrow}\; \underline{\mathsf{Cat}}(3,a)$$

in which, denoting by Δ the category of the finite non-empty ordinals and order preserving functions, $\mathrm{j} : \Delta_3 \to \mathsf{Cat}$ is the usual inclusion given by the composition of the inclusions $\Delta_3 \to \Delta \to \mathsf{Cat}$.

It is well known that there is a fully faithful functor $\Sigma : \mathsf{Mon}(\mathsf{Set}) \to \mathsf{Cat}(\mathsf{Set})$ between the category of monoids (internal monoids in Set) and the category of small categories (internal categories in Set) that associates each monoid with the corresponding single object category. The underlying precategory of Σm is given by

$$\Sigma m : \Delta_3^{\mathrm{op}} \to \mathsf{Set}$$

$$\{m\} \;\leftleftarrows\; \xrightarrow{\Sigma m(s_0)} m \;\overset{\Sigma m(D_0)}{\longleftarrow}\; \overset{\Sigma m(D_1)}{\longleftarrow}\; \overset{\Sigma m(D_2)}{\longleftarrow}\; m \times m$$

in which m is the underlying set of the monoid, $\{m\}$ is the singleton with m as element, $\Sigma m(D_2), \Sigma m(D_0) : m \times m \to m$ are the two product projections, $\Sigma m(D_1)$ is the operation of the monoid, and $\Sigma m(s_0)$ gives the unit. In this context, the objects and morphisms of the category m-Set can be described internally in Set as follows.

Since Set has pullbacks, we can consider the (basic) *indexed category*, that is to say, the pseudofunctor coming from the *basic* bifibration

$$\begin{aligned} \mathsf{Set}/- : \ \mathsf{Set}^{\mathrm{op}} &\to \mathsf{Cat} \\ w &\mapsto \mathsf{Set}/w \\ f &\mapsto f^* \end{aligned}$$

in which Set/w denotes the comma category, and f^* denotes the *change of base functor* (given by the pullback along f).

An m-set is a set w endowed with an endomorphism ξ of the projection $\mathrm{proj}_m : m \times w \to m$ in the comma category Set/m, subject to the equations

$$\mathfrak{p} \cdot m(s_0)^*(\xi) \cdot \mathfrak{p} = \mathrm{id}_{\mathsf{Set}}, \quad m(D_0)^*(\xi) \cdot \mathfrak{p} \cdot m(D_2)^*(\xi) = \mathfrak{p} \cdot m(D_1)^*(\xi) \cdot \mathfrak{p}$$

in which, by abuse of language, we denote by $\mathfrak{p}$ the appropriate *canonical isomorphisms* given by the pseudofunctor $\mathsf{Set}/-$ (induced by the universal properties of the pullbacks in each case). These equations correspond to the identity and associativity equations for the action. The morphisms $(w, \xi) \to (w', \xi')$ of m-sets are morphisms (functions) $w \to w'$ between the underlying sets respecting the structures ξ and ξ'.

This viewpoint gives m-Set precisely as the *lax descent category* of the composition of $\mathrm{op}\,(\Sigma m) : \Delta_3 \to \mathsf{Set}^{\mathrm{op}}$ with the pseudofunctor $\mathsf{Set}/- : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Cat}$. More generally, given a small category a, the lax descent category (see Definition 1.2) of

$$\mathsf{Set}/\underline{\mathsf{Cat}}(1,a) \underset{\underline{\mathsf{Cat}}(d^1,a)^*}{\overset{\mathsf{Set}/\underline{\mathsf{Cat}}(d^0,a)\,=\,\underline{\mathsf{Cat}}(d^0,a)^*}{\rightrightarrows}} \xleftarrow{\underline{\mathsf{Cat}}(s^0,a)^*} \mathsf{Set}/\underline{\mathsf{Cat}}(2,a) \underset{\mathsf{Cat}(D^2,a)^*}{\overset{\underline{\mathsf{Cat}}(D^0,a)^*}{\rightrightarrows}} \xrightarrow{\underline{\mathsf{Cat}}(D^1,a)^*} \mathsf{Set}/\underline{\mathsf{Cat}}(3,a)$$

is equivalent to the category $\mathsf{Cat}\,[a, \mathsf{Set}]$ of functors $a \to \mathsf{Set}$ and natural transformations, that is to say, the category of *actions of the small category* a *in* Set.

In order to reach the level of abstraction of [23], firstly it should be noted that the definitions above can be considered in any category $\mathbb{C}$ with pullbacks, using the basic indexed category $\mathbb{C}/- : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$. That is to say, we get the (basic) internal notion of the category of actions $a \to \mathbb{C}$ for each internal category a. Secondly, we can replace the pseudofunctor $\mathbb{C}/-$ by any other pseudofunctor (indexed category) $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ of interest. By definition, given an internal (pre)category $a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$ of $\mathbb{C}$, the lax descent category of

$$\mathcal{F}\,a(1) \underset{\mathcal{F}\,a(d_1)}{\overset{\mathcal{F}\,a(d_0)}{\rightrightarrows}} \mathcal{F}\,a(2), \quad \mathcal{F}\,a(s_0) : \mathcal{F}\,a(2) \to \mathcal{F}\,a(1), \qquad \mathcal{F}\,a(D_0),\ \mathcal{F}\,a(D_1),\ \mathcal{F}\,a(D_2) : \mathcal{F}\,a(2) \to \mathcal{F}\,a(3)$$

is the category of $\mathcal{F}$*-internal actions* of a in $\mathbb{C}$.

Recall that, if $\mathbb{C}$ has pullbacks, given a morphism $p : e \to b$, the kernel pair induces a precategory which is actually the underlying precategory of an *internal groupoid* of $\mathbb{C}$, denoted herein by $\mathsf{Eq}(p)$. Following the definition, given any pseudofunctor $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$, we have that the category of internal actions of $\mathsf{Eq}(p)$ is given by the lax descent category lax-$\mathcal{D}$esc $(\mathcal{F} \cdot \mathrm{op}\,(\mathsf{Eq}(p)))$. In this case, the universal property of the lax descent category induces a factorization (see [23, 34] or, in our context, Lemma 3.6)

$$\begin{array}{ccc} \mathcal{F}(b) & \xrightarrow{\quad\mathcal{F}(p)\quad} & \mathcal{F}(e) \\ & \searrow \qquad \nearrow & \\ & \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{F} \cdot \mathrm{op}\,(\mathsf{Eq}(p))) & \end{array}$$

($\mathcal{F}$-descent factorization of $\mathcal{F}(p)$)

in which lax-$\mathcal{D}$esc $(\mathcal{F}\,\mathsf{Eq}(p)) \to \mathcal{F}(e)$ is the forgetful functor that forgets descent data.

In this setting, Bénabou and Roubaud [3, 34] showed that, if $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ comes from a bifibration satisfying the so called *Beck-Chevalley condition* [34, 32], then the $\mathcal{F}$-descent factorization of $\mathcal{F}(p)$ is equivalent to the Eilenberg-Moore factorization of the adjunction $\mathcal{F}(p)! \dashv \mathcal{F}(p)$, that is to say, the semantic factorization of $\mathcal{F}(p)$ (see [16, 43, 36]). In particular, in this case, $\mathcal{F}(p)$ is monadic if and only if p is of *effective* $\mathcal{F}$*-descent* (which means that $\mathcal{F}(b) \to$ lax-$\mathcal{D}$esc $(\mathcal{F} \cdot \mathrm{op}\,(\mathsf{Eq}(p)))$ is an equivalence).

It should be observed that, without assuming the Beck-Chevalley condition, monadicity of $\mathcal{F}(p)$ does not imply that p is of effective $\mathcal{F}$-descent. This is shown for instance in Remark 7 of [42], where Sobral, considering Cat endowed with the fibration of op-fibrations, provides an example of a morphism that is not of effective descent but does induce a monadic functor.

The main result of the present paper is within the general context of the lax descent object of a truncated pseudocosimplicial object inside a 2-category $\mathfrak{A}$ (see [34, 36]). In the case of $\mathfrak{A} = \mathsf{Cat}$, the main result says that, for any given truncated pseudocosimplicial category

$$\mathcal{A} : \Delta_3 \to \mathsf{Cat}$$

$$\mathcal{A}(1) \;\substack{\xrightarrow{\mathcal{A}(d^0)} \\ \xleftarrow{\mathcal{A}(s^0)} \\ \xrightarrow{\mathcal{A}(d^1)}}\; \mathcal{A}(2) \;\substack{\xrightarrow{\mathcal{A}(D^0)} \\ \xrightarrow{\mathcal{A}(D^1)} \\ \xrightarrow{\mathcal{A}(D^2)}}\; \mathcal{A}(3)$$

the functor $\mathfrak{d}^{\mathcal{A}} : \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A}) \to \mathcal{A}$ that forgets descent data creates the right Kan extensions that are preserved by $\mathcal{A}(d^0)$ and $\mathcal{A}(D^0) \cdot \mathcal{A}(d^0)$. In particular, such forgetful functor creates absolute Kan extensions, and, hence, more particularly, it creates absolute limits and colimits.

The result sheds light to 2-*dimensional exact properties* of Cat and general 2-categories. For instance, it might suggest a conjecture towards the characterization of effective faithful functors in Cat (see [36] for the definition of effective faithful morphisms in a 2-category). Yet, in the present paper, we focus on the consequences within the context of [22, 23, 24] briefly described above.

The main result implies that, given any pseudofunctor $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$, the forgetful functor

$$\text{lax-}\mathcal{D}\text{esc}\,(\mathcal{F} \circ \mathrm{op}(a)) \to \mathcal{F}\, a(1)$$

between the $\mathcal{F}$-internal actions of a precategory $a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$ and the category of internal actions of the *underlying precategory* of a creates absolute limits and colimits. This generalizes the fact that, if a is actually a small category, the forgetful functor (restriction functor)

$$\mathsf{Cat}\,[a, \mathsf{Set}] \to \mathsf{Cat}\left[\overline{a(1)}, \mathsf{Set}\right]$$

creates absolute limits and colimits, in which, by abuse of language, $\overline{a(1)}$ denotes the *underlying discrete category* of a.

As a particular case of this conclusion, given any indexed category $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$, whenever p is of effective $\mathcal{F}$-descent, $\mathcal{F}(p)$ creates absolute limits and colimits. Therefore, by Beck's monadicity theorem [1, 15], assuming that $\mathcal{F}(p)$ has a left adjoint, if p is of effective $\mathcal{F}$-descent then $\mathcal{F}(p)$ is monadic.

This result shows that, if $\mathcal{F}$ comes from a bifibration, one of the implications of the Bénabou-Roubaud theorem does not depend on the Beck-Chevalley condition. Namely, in a bifibred category with pullbacks, effective descent morphisms always induce monadic functors.

This consequence can be seen as a generalization of an observation given in Remark 7 of [42]. Therein, Sobral suggested that, for the particular case of the fibration of op-fibrations in Cat, *descent gives more information than monadicity.*

In Section 1, we briefly show the basic definition of the lax descent category, and give the corresponding definition for a general 2-category. Namely, a 2-dimensional limit [44, 27, 33] called *the lax descent object.* We mostly follow the approach of [36] but, because of our setting, we start with pseudofunctors $(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathfrak{A}$, instead of using a strict replacement of the domain.

In Section 2, we establish our main theorems on the morphisms that forget descent data. In order to do so, we start by recalling the definitions on Kan extensions [16] (sometimes just called extensions [47]) inside a 2-category. Then, we prove the main theorem (Theorem 2.4) and show the main consequences, including the monadicity characterization (Theorem 2.8), proven as a consequence of Theorem 2.4 and the monadicity theorem of Section 5 of [36]. It says that *a right adjoint functor is monadic if and only if it is, up to the precomposition of an equivalence, a functor that forgets descent data.*

Section 3 establishes the setting of [19, 3, 22, 23], finishing with the definition of effective descent morphism. Finally, in Section 4, we discuss the main consequences of our Theorem 2.4 in the context of [3, 22, 23], including the result that *effective descent morphisms always induce monadic functors.*

1. The lax descent category

Let Cat be the cartesian closed category of categories in some universe (see, for instance, Section 1 of [33, 31, 36]). We denote the *internal hom* by

$$\mathsf{Cat}[-,-] : \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat} \to \mathsf{Cat},$$

which of course is a 2-functor (Cat-enriched functor). Moreover, we denote by

$$\underline{\mathsf{Cat}}(-,-) : \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat} \to \mathsf{Cat}$$

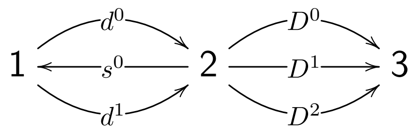
the composition of $\mathsf{Cat}[-,-]$ with the functor that gives the underlying discrete category. Finally, a *small category* is a category $\mathbb{S}$ such that the underlying discrete category, *i.e.* $\underline{\mathsf{Cat}}(\mathsf{1}, \mathbb{S})$, and the collection of morphisms, *i.e.* $\underline{\mathsf{Cat}}(\mathsf{2}, \mathbb{S})$, consist of sets. Equivalently, a *small category* is an internal category of Set.

A 2-category herein is the same as a Cat-enriched category. We denote the *enriched hom* of a 2-category $\mathfrak{A}$ by

$$\mathfrak{A}(-,-) : \mathfrak{A}^{\mathrm{op}} \times \mathfrak{A} \to \mathsf{Cat}$$

which, again, is of course a 2-functor. As usual, the composition of 1-cells (morphisms) are denoted by $\circ$, $\cdot$ or omitted whenever it is clear from the context. The vertical composition of 2-cells is denoted by $\cdot$ or omitted when it is clear, while the horizontal composition is denoted by $*$. From the vertical and horizontal compositions, we construct the fundamental operation of *pasting* [39, 45], introduced in [2, 26].

We denote by Δ the full subcategory of the underlying category of Cat whose objects are finite nonempty ordinals seen as posets (or thin categories). We are particularly interested in the subcategory Δ_3 of Δ with the objects $\mathsf{1}, \mathsf{2}$ and $\mathsf{3}$ generated by the morphisms



with the following relations:

$$D^2 d^0 = D^0 d^1; \qquad s^0 d^1 = \mathrm{id}_{\mathsf{1}};$$
$$D^1 d^0 = D^0 d^0; \qquad s^0 d^0 = \mathrm{id}_{\mathsf{1}}.$$
$$D^2 d^1 = D^1 d^1;$$

In order to fix notation, we briefly recall the definition of pseudofunctor [30] between a category $\mathbb{C}$ and a 2-category $\mathfrak{A}$ below. For the case of $\mathfrak{A} = \mathsf{Cat}$, this definition was originally introduced by Grothendieck [18, 19] in its contravariant form, while its further generalization for arbitrary bicategories was originally introduced by Bénabou [2] under the name *homomorphism of bicategories.*

Definition 1.1. Let $\mathbb{C}$ be a category (which can be seen as a locally discrete 2-category) and $\mathfrak{A}$ a 2-category. A *pseudofunctor* $\mathcal{F} : \mathbb{C} \to \mathfrak{A}$ is a pair $(\mathcal{F}, \mathfrak{f})$ with the following data:

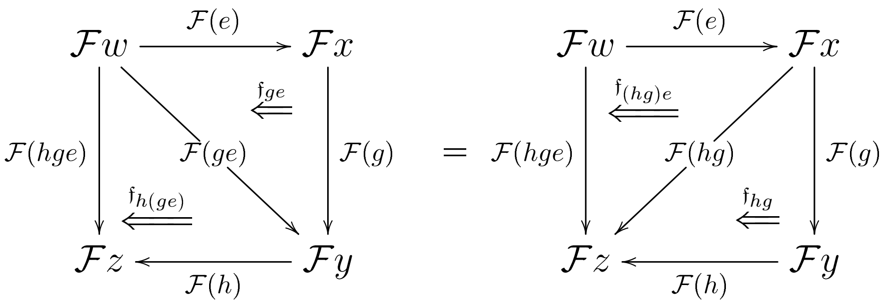
– A *function* $\mathcal{F} : \mathrm{obj}(\mathbb{C}) \to \mathrm{obj}(\mathfrak{A})$;
– For each pair (x, y) of objects in $\mathbb{C}$, functions

$$\mathcal{F}_{x,y} : \mathbb{C}(x, y) \to \mathfrak{A}(\mathcal{F}(x), \mathcal{F}(y));$$

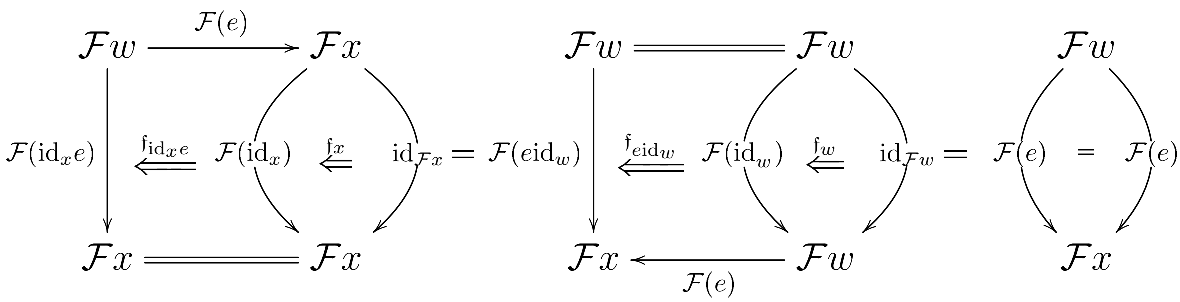
– For each pair $g : x \to y, h : y \to z$ of morphisms in $\mathbb{C}$, an invertible 2-cell
$$\mathfrak{f}_{hg} : \mathcal{F}(h)\mathcal{F}(g) \Rightarrow \mathcal{F}(hg);$$
– For each object x of $\mathbb{C}$, an invertible 2-cell
$$\mathfrak{f}_x : \mathrm{id}_{\mathcal{F}(x)} \Rightarrow \mathcal{F}(\mathrm{id}_x);$$

such that, if $g : x \to y, h : y \to z$ and $e : w \to x$ are morphisms of $\mathbb{C}$, the following equations hold in $\mathfrak{A}$:

(1) Associativity:



(2) Identity:



In this paper, we are going to be particularly interested in pseudofunctors of the type
$$(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathfrak{A},$$
also called truncated pseudocosimplicial objects. For simplicity, given such a truncated pseudocosimplicial category, we define:
$$\mathcal{A}(\sigma_{01}) = \mathfrak{a}^{-1}_{D^0 d^0} \cdot \mathfrak{a}_{D^1 d^0}; \qquad \mathcal{A}(\mathfrak{n}_0) = \mathfrak{a}^{-1}_{1} \cdot \mathfrak{a}_{s^0 d^0};$$
$$\mathcal{A}(\sigma_{02}) = \mathfrak{a}^{-1}_{D^0 d^1} \cdot \mathfrak{a}_{D^2 d^0}; \qquad \mathcal{A}(\mathfrak{n}_1) = \mathfrak{a}^{-1}_{1} \cdot \mathfrak{a}_{s^0 d^1}.$$
$$\mathcal{A}(\sigma_{12}) = \mathfrak{a}^{-1}_{D^1 d^1} \cdot \mathfrak{a}_{D^2 d^1};$$

Using this terminology, we recall the definition of the lax descent category of a pseudofunctor $\Delta_3 \to \mathsf{Cat}$ (see, for instance, [23, 33, 36]).

Definition 1.2. [Lax descent category] Given a pseudofunctor

$$(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathsf{Cat},$$

the *lax descent category* lax-$\mathcal{D}$esc $(\mathcal{A})$ of $\mathcal{A}$ is defined as follows:

(1) The objects are pairs (w, φ) in which w is an object of $\mathcal{A}(1)$ and $\varphi : \mathcal{A}(d^1)(w) \to \mathcal{A}(d^0)(w)$ is a morphism in $\mathcal{A}(2)$ satisfying the following equations:

Associativity:

$$\mathcal{A}(D^0)(\varphi) \cdot \mathcal{A}(\sigma_{02})_w \cdot \mathcal{A}(D^2)(\varphi) = \mathcal{A}(\sigma_{01})_w \cdot \mathcal{A}(D^1)(\varphi) \cdot \mathcal{A}(\sigma_{12})_w;$$

Identity:

$$\mathcal{A}(\mathfrak{n}_0)_w \cdot \mathcal{A}(s^0)(\varphi) = \mathcal{A}(\mathfrak{n}_1)_w.$$

If the pair (w, φ) satisfies the above, we say that φ is a *descent datum* for w w.r.t. $\mathcal{A}$, or just an $\mathcal{A}$-*descent datum*.

(2) A morphism $\mathfrak{m} : (w, \varphi) \to (w', \varphi')$ is a morphism $\mathfrak{m} : w \to w'$ in $\mathcal{A}(1)$ such that

$$\mathcal{A}(d^0)(\mathfrak{m}) \cdot \varphi = \varphi' \cdot \mathcal{A}(d^1)(\mathfrak{m}).$$

The composition of morphisms is given by the composition of morphisms in $\mathcal{A}(1)$.

The lax descent category comes with an obvious *forgetful functor*

$$\begin{array}{rll} \mathfrak{d}^{\mathcal{A}} : & \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A}) & \to \mathcal{A}(1) \\ & (w, \varphi) & \mapsto w \\ & \mathfrak{m} & \mapsto \mathfrak{m} \end{array}$$

and a natural transformation $\psi : \mathcal{A}(d^1) \circ \mathfrak{d}^{\mathcal{A}} \Longrightarrow \mathcal{A}(d^0) \circ \mathfrak{d}^{\mathcal{A}}$ defined pointwise by

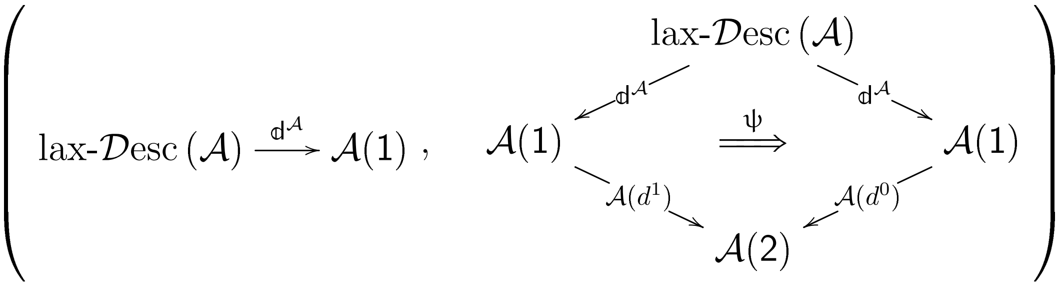
$$\psi_{(w,\varphi)} := \varphi : \mathcal{A}(d^1)(w) \to \mathcal{A}(d^0)(w).$$

Actually, the pair $\left(\mathfrak{d}^{\mathcal{A}} : \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A}) \to \mathcal{A}(1), \psi : \mathcal{A}(d^1) \circ \mathfrak{d}^{\mathcal{A}} \Rightarrow \mathcal{A}(d^0) \circ \mathfrak{d}^{\mathcal{A}}\right)$ is a two dimensional limit (see [44, 31, 33, 36]) of $\mathcal{A}$. Namely, the lax descent category of

$$(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathsf{Cat}$$

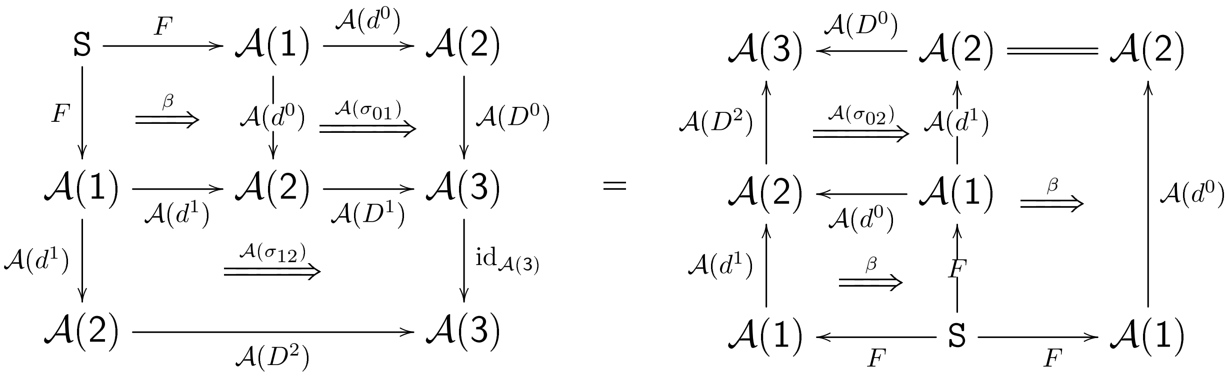
is the *lax descent object*, as defined below, of the pseudofunctor $\mathcal{A}$ in the 2-category Cat.

Definition 1.3. [Lax descent object [36]] Given a pseudofunctor $\mathcal{A} : \Delta_3 \to \mathfrak{A}$, the *lax descent object* lax-$\mathcal{D}$esc$(\mathcal{A})$ is an object lax-$\mathcal{D}$esc$(\mathcal{A})$ of $\mathfrak{A}$ together with a pair

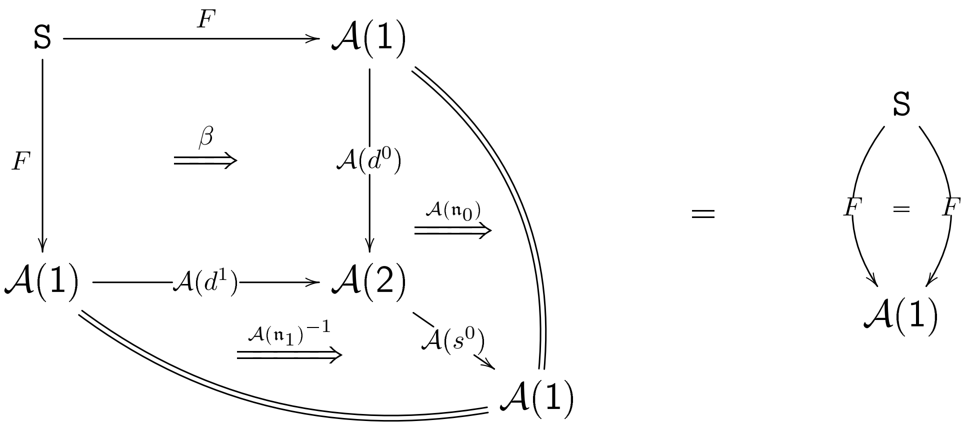


in which $\mathbb{d}^{\mathcal{A}}$: lax-$\mathcal{D}$esc$(\mathcal{A}) \to \mathcal{A}(1)$ is a morphism, called herein the *forgetful morphism* (of descent data), and ψ is a 2-cell satisfying the following universal property.

(1) For each pair $(F : \mathsf{S} \to \mathcal{A}(1), \beta : \mathcal{A}(d^1) \circ F \Rightarrow \mathcal{A}(d^0) \circ F)$ in which F is a morphism and β is a 2-cell such that the equations

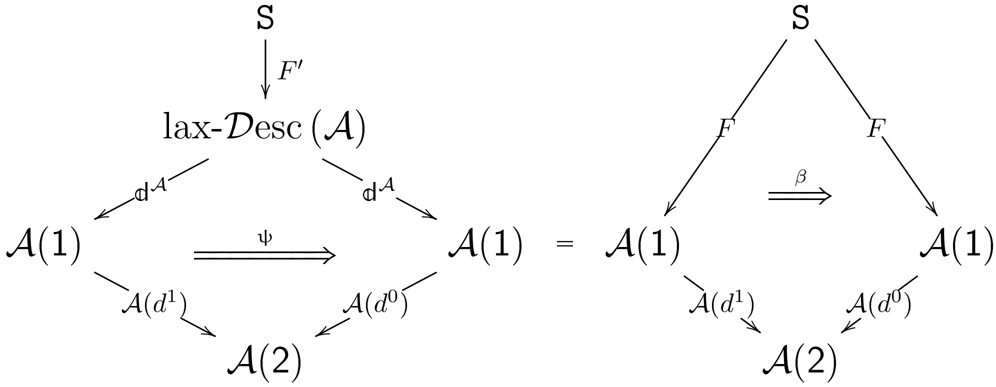


(*descent associativity*)



(*descent identity*)

hold in $\mathfrak{A}$, there is a unique morphism $F' : \mathsf{S} \to \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})$ in $\mathfrak{A}$ such that



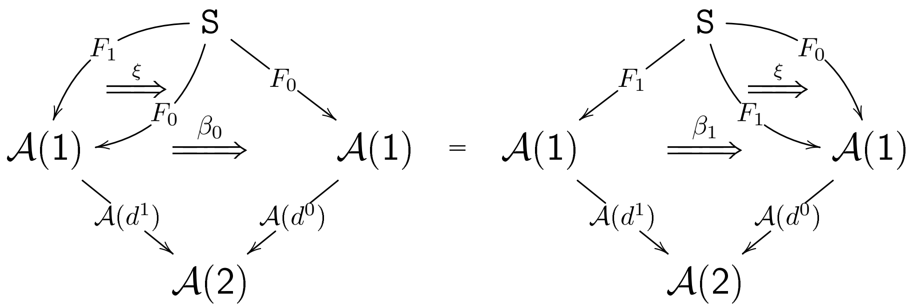
and $F = \mathbb{d}^{\mathcal{A}} \circ F'$. In this case, we say that the 2-cell β is an $\mathcal{A}$-*descent datum* for the morphism F.

(2) The pair $(\mathbb{d}^{\mathcal{A}}, \psi)$ satisfies the *descent associativity* and *descent identity* equations above. In this case, the unique morphism induced is clearly the identity on $\text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})$.

(3) Assume that (F_1, β_1) and (F_0, β_0) are pairs satisfying the *descent associativity* and *descent identity* equations, and that they induce respectively the morphisms

$$F_1', F_0' : \mathsf{S} \to \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})\,.$$

For each 2-cell $\xi : F_1 \Rightarrow F_0 : \mathsf{S} \to \mathcal{A}(1)$ satisfying the equation



there is a unique 2-cell

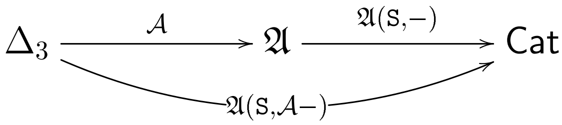
$$\xi' : F_1' \Rightarrow F_0' : \mathcal{A}(1) \to \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})$$

in $\mathfrak{A}$ such that $\mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \xi' = \xi$.

Lemma 1.4. *Let $\mathcal{A} : \Delta_3 \to \mathfrak{A}$ be a pseudofunctor. The pseudofunctor $\mathcal{A}$ has a lax descent object* $\text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})$ *if and only if there is an isomorphism*

$$\mathfrak{A}\,(\mathsf{S}, \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})) \cong \text{lax-}\mathcal{D}\text{esc}\,(\mathfrak{A}\,(\mathsf{S}, \mathcal{A}-))$$

2-*natural in* $\mathtt{S}$*, in which* $\mathfrak{A}(\mathtt{S}, \mathcal{A}-) : \Delta_3 \to \mathsf{Cat}$ *is the composition below.*



2. Forgetful morphisms and Kan extensions

Assuming the existence of the lax descent object of a pseudofunctor

$$(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathfrak{A},$$

the forgetful morphism $\mathtt{d}^{\mathcal{A}}$ has many properties that are direct consequences of the definition. Among them, the morphism $\mathtt{d}^{\mathcal{A}}$ is *faithful* and *conservative* (by which we mean that, for any object $\mathtt{S}$ of $\mathfrak{A}$, the functor $\mathfrak{A}(\mathtt{S}, \mathtt{d}^{\mathcal{A}})$ is faithful and reflects isomorphisms). In this section, we give the core observation of the present paper. Namely, we investigate the properties of creation of Kan extensions by $\mathtt{d}^{\mathcal{A}}$. We start by briefly recalling the basic definitions on preservation and creation of Kan extensions [16, 43, 47, 36].

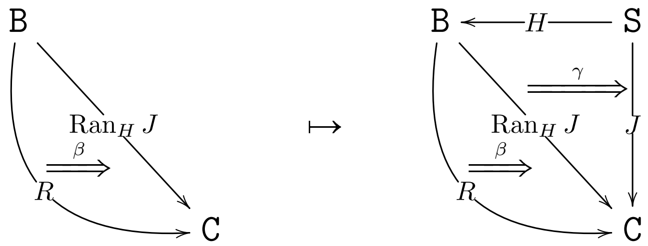
Let $J : \mathtt{S} \to \mathtt{C}$ and $H : \mathtt{S} \to \mathtt{B}$ be morphisms of a 2-category $\mathfrak{A}$. The right Kan extension of J along H is, if it exists, the right reflection $\mathrm{Ran}_H J$ of J along the functor

$$\mathfrak{A}(H, \mathtt{C}) : \mathfrak{A}(\mathtt{B}, \mathtt{C}) \to \mathfrak{A}(\mathtt{S}, \mathtt{C}).$$

This means that the right Kan extension is actually a pair

$$(\mathrm{Ran}_H J : \mathtt{B} \to \mathtt{C}, \gamma : (\mathrm{Ran}_H J) \circ H \Rightarrow J)$$

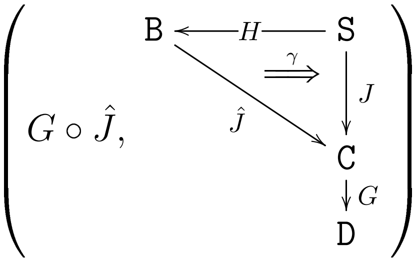
consisting of a morphism $\mathrm{Ran}_H J$ and a 2-cell γ, called the universal 2-cell, of $\mathfrak{A}$ such that, for each morphism $R : \mathtt{B} \to \mathtt{C}$ of $\mathfrak{A}$,



defines a bijection $\mathfrak{A}(\mathtt{B}, \mathtt{C})(R, \mathrm{Ran}_H J) \cong \mathfrak{A}(\mathtt{S}, \mathtt{C})(R \circ H, J)$.

Let $J : \mathtt{S} \to \mathtt{C}$, $H : \mathtt{S} \to \mathtt{B}$ and $G : \mathtt{C} \to \mathtt{D}$ be morphisms in $\mathfrak{A}$. On the one hand, if $(\hat{J}, \gamma)$ is the right Kan extension of J along H, we say that G

preserves the right Kan extension $\mathrm{Ran}_H J$ if the pair



is the right Kan extension $\mathrm{Ran}_H GJ$ of GJ along H. Equivalently, G preserves $\mathrm{Ran}_H J$ if $\mathrm{Ran}_H GJ$ exists and, in addition to that, the unique 2-cell

$$G \circ \hat{J} \Rightarrow \mathrm{Ran}_H GJ,$$

induced by the pair $(G \circ \hat{J}, \mathrm{id}_G * \gamma)$ and the universal property of $\mathrm{Ran}_H GJ$, is invertible [37, 10, 35].

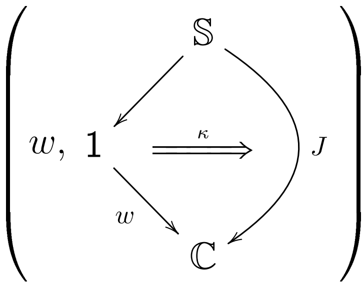
On the other hand, we say that G *reflects the right Kan extension of* J *along* H if, whenever $(G \circ \hat{J}, \mathrm{id}_G * \gamma)$ is the right Kan extension of GJ along H, $(\hat{J}, \gamma)$ is the right Kan extension of J along H.

Finally, assuming the existence of $\mathrm{Ran}_H GJ$, we say that $G : \mathtt{C} \to \mathtt{D}$ *creates the right Kan extension of* $GJ : \mathtt{S} \to \mathtt{D}$ *along* H if we have that (1) G reflects $\mathrm{Ran}_H GJ$ and (2) $\mathrm{Ran}_H J$ exists and is preserved by G.

Remark 2.1. [Coduality] The dual notion of that of a right Kan extension is called *right lifting* (see [47] or [36]), while the codual notion is called the *left Kan extension*, denoted herein by $\mathrm{Lan}_H J$. Finally, of course, we also have the codual notion of the right lifting, the *left lifting*.

Remark 2.2. [Conical (co)limits [37, 10, 35]] For $\mathfrak{A} = \mathsf{Cat}$, right Kan extensions along functors of the type $\mathbb{S} \to \mathsf{1}$ give the notion of conical limits. This is the most elementary and well known relation between Kan extensions and conical limits [37, 16, 41], which give the most elementary examples of right Kan extensions. We briefly recall this fact below.

Let $J : \mathbb{S} \to \mathbb{C}$ be a functor in which $\mathbb{S}$ is a small category. Firstly, recall that a cone over J is a pair

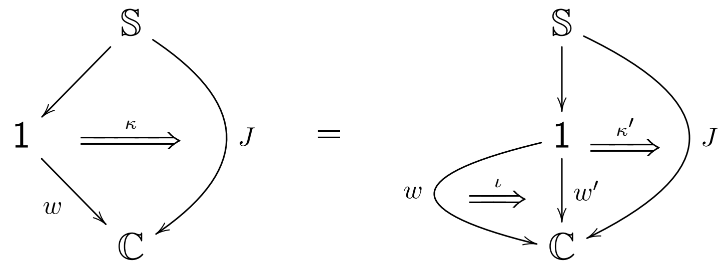


in which $\mathsf{1}$ is the terminal category, $w : \mathsf{1} \to \mathbb{C}$ denotes the functor whose image is the object w, and κ is a natural transformation.

Secondly, denoting the composition of

$$\mathbb{S} \longrightarrow \mathsf{1} \xrightarrow{w} \mathbb{C}$$

by $\overline{w}$, a morphism $\iota : w \to w'$ of $\mathbb{C}$ defines a morphism between the cones $(w, \kappa : \overline{w} \Rightarrow J)$ and $(w', \kappa' : \overline{w'} \Rightarrow J)$ over J if the equation



holds, in which, by abuse of language, ι denotes the natural transformation defined by the morphism $\iota : w \to w'$.

Thirdly, of course, the above defines a category of cones over J. If it exists, the *conical limit of* J is the terminal object of the category of cones over J. This is clearly equivalent to say that the conical limit of J, denoted herein by $\mathsf{lim} J$, is the right Kan extension $\mathrm{Ran}_{\mathbb{S}\to\mathsf{1}} J$ in the 2-category of categories Cat, either one existing if the other does. In this context, the definitions of *preservation*, *reflection* and *creation* of conical limits coincide with those coming from the respective notions in the case of right Kan extensions along $\mathbb{S} \to \mathsf{1}$ [37, 29, 41].

Codually, the notion of *conical colimit* of $J : \mathbb{S} \to \mathbb{C}$ coincides with the notion of left Kan extension of J along the unique functor $\mathbb{S} \to \mathsf{1}$ in the 2-category Cat. Again, the notions of *preservation*, *reflection* and *creation* of conical colimits coincide with those coming from the respective notions in the case of left Kan extensions along $\mathbb{S} \to \mathsf{1}$.

It is well known that there is a deeper relation between conical (and weighted) limits and Kan extensions for much more general contexts. For instance, in the case of 2-categories endowed with Yoneda structures [47], the concept of pointwise Kan extensions [16, 47] encompasses this relation. Although this concept plays a fundamental role in the theory of Kan extensions, we do not give further comment or use to this concept in the present paper.

In order to prove our main theorem, we present an elementary result below, whose version for limits and colimits is well known.

Lemma 2.3. *Let* $\mathfrak{A}$ *be a* 2*-category and* H, J, G *morphisms of* $\mathfrak{A}$*. Assume that* $\mathrm{Ran}_H J : \mathtt{B} \to \mathtt{C}$ *exists and is preserved by* $G : \mathtt{C} \to \mathtt{D}$*. If* G *is conservative, then* G *creates the right Kan extension of* GJ *along* H*.*

Proof: By hypothesis, $(G \cdot \mathrm{Ran}_H J, \mathrm{id}_G * \gamma)$ is the right Kan extension of GJ along H. If $(G \cdot \check{J}, \mathrm{id}_G * \gamma')$ is also the right Kan extension of GJ along H, on the one hand, we get a (unique) induced invertible 2-cell $G \cdot \check{J} \Rightarrow G \cdot \mathrm{Ran}_H J$. On the other hand, by the uniqueness property of the universal properties, this induced invertible 2-cell should be the image by $\mathfrak{A}(\mathtt{S}, G)$ of the 2-cell $\check{J} \Rightarrow \mathrm{Ran}_H J$ induced by the universal property of $\mathrm{Ran}_H J$ and the 2-cell γ'. Since $\mathfrak{A}(\mathtt{S}, G)$ reflects isomorphisms, the proof is complete. ∎

Theorem 2.4 (Main Theorem)**.** *Assume that the lax descent object of the pseudofunctor* $(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathfrak{A}$ *exists. Given morphisms* $J : \mathtt{S} \to \text{lax-}\mathcal{D}\mathrm{esc}\,(\mathcal{A})$ *and* $H : \mathtt{S} \to \mathtt{B}$ *of* $\mathfrak{A}$*, the forgetful morphism* $\mathbb{d}^{\mathcal{A}} : \text{lax-}\mathcal{D}\mathrm{esc}\,(\mathcal{A}) \to \mathcal{A}(1)$ *creates the right Kan extension of* $\mathbb{d}^{\mathcal{A}} J : \mathtt{S} \to \mathcal{A}(1)$ *along* H*, provided that* $\mathrm{Ran}_H\, \mathbb{d}^{\mathcal{A}} J$ *exists and is preserved by the morphisms* $\mathcal{A}(d^0)$ *and* $\mathcal{A}(D^0) \cdot \mathcal{A}(d^0)$*.*

Proof: By Lemma 2.3, since $\mathbb{d}^{\mathcal{A}}$ is conservative, in order to prove that $\mathbb{d}^{\mathcal{A}}$ creates the right Kan extension of $\mathbb{d}^{\mathcal{A}} J : \mathtt{S} \to \mathcal{A}(1)$ along H, it is enough to prove that $\mathrm{Ran}_H\, J$ exists and is preserved by $\mathbb{d}^{\mathcal{A}}$.

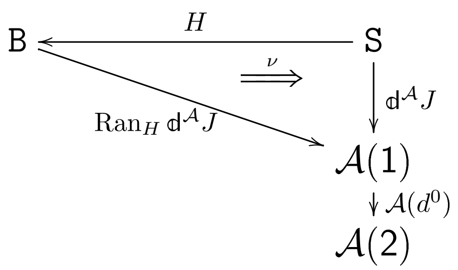
Let $\left(\mathbb{d}^{\mathcal{A}}, \psi\right)$ be the universal pair that gives the lax decent object $\text{lax-}\mathcal{D}\mathrm{esc}\,(\mathcal{A})$. We assume that $J : \mathtt{S} \to \text{lax-}\mathcal{D}\mathrm{esc}\,(\mathcal{A})$ is a functor satisfying the hypotheses above. We denote by

$$\left(\mathrm{Ran}_H\, \mathbb{d}^{\mathcal{A}} J, \quad \nu : \left(\mathrm{Ran}_H\, \mathbb{d}^{\mathcal{A}} J\right) \circ H \Rightarrow \mathbb{d}^{\mathcal{A}} J\right)$$

the right Kan extension of $\mathbb{d}^{\mathcal{A}} J$ along H.

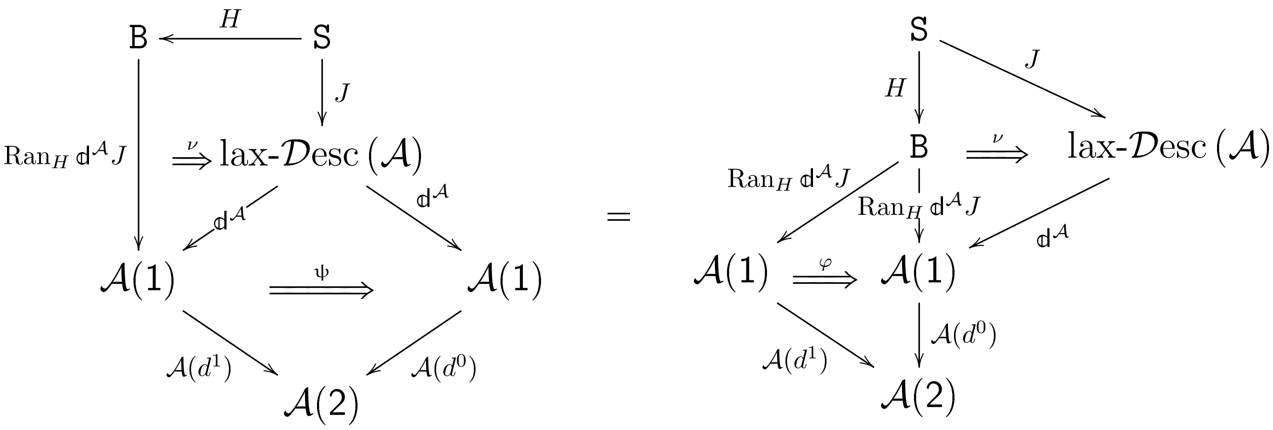
– By the universal property of the right Kan extension

$\Bigg(\mathcal{A}(d^0) \cdot \mathrm{Ran}_H\, \mathbb{d}^{\mathcal{A}} J,$



$\Bigg)$

we get that there is a unique 2-cell $\varphi : \mathcal{A}(d^1) \cdot \mathrm{Ran}_H \, \mathbb{d}^{\mathcal{A}} J \Rightarrow \mathcal{A}(d^0) \cdot \mathrm{Ran}_H \, \mathbb{d}^{\mathcal{A}} J$ in $\mathfrak{A}$ such that the equation

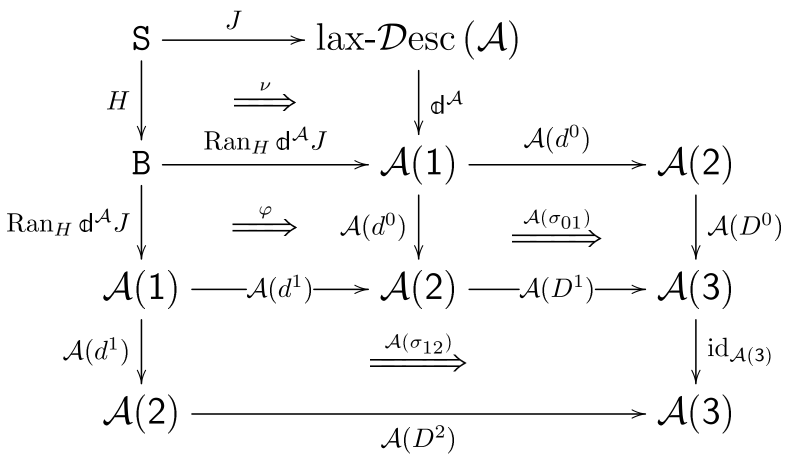


(definition of φ)

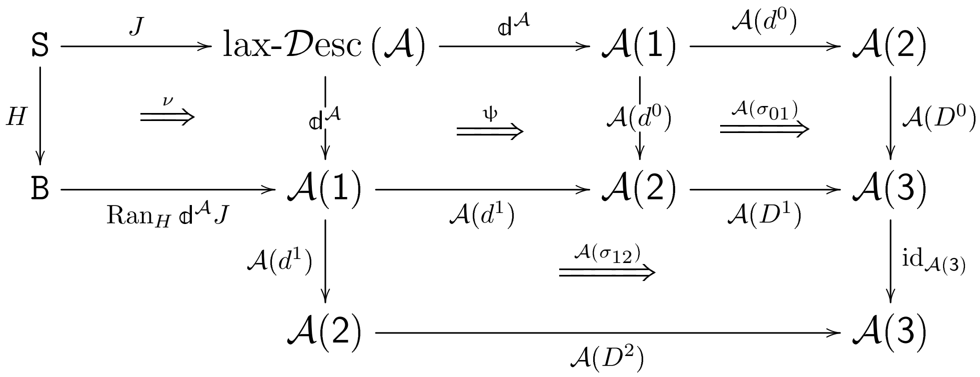
holds. We prove below that $(\mathrm{Ran}_H \, \mathbb{d}^{\mathcal{A}} J, \varphi)$ satisfies the *descent identity* and *descent associativity* equations w.r.t. $\mathcal{A}$.

By the definition of φ, we have that

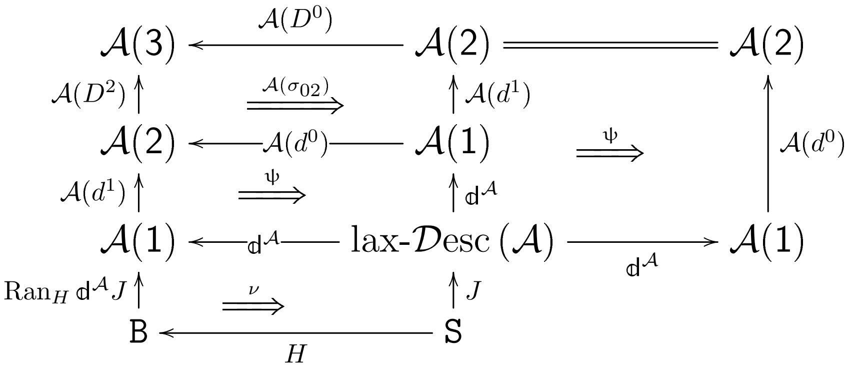
$\varphi' \quad :=$



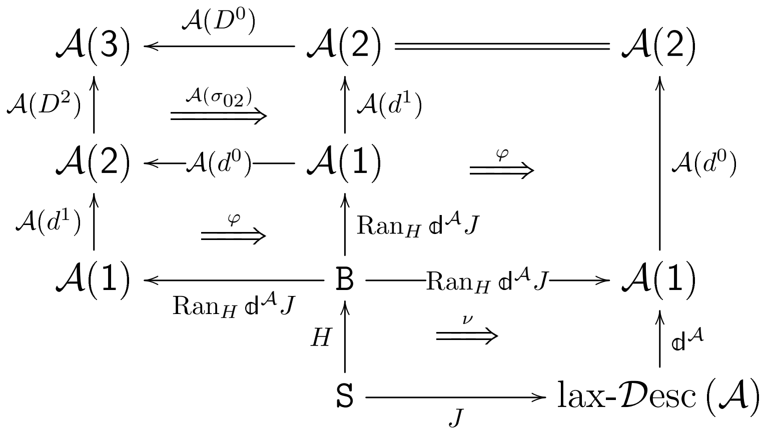
is equal to



Since ψ is an $\mathcal{A}$-descent datum for $\mathbb{d}^{\mathcal{A}}$, we have that the 2-cell above (and hence φ') is equal to



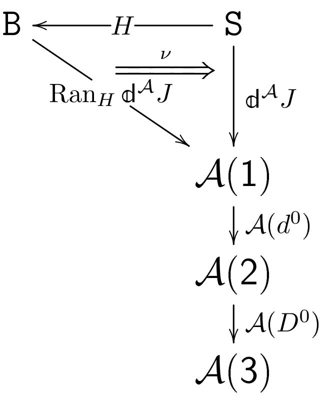
which, by the definition of φ, is equal to the 2-cell



denoted by φ''. It should be noted that we proved that $\varphi' = \varphi''$.

By the universal property of the right Kan extension

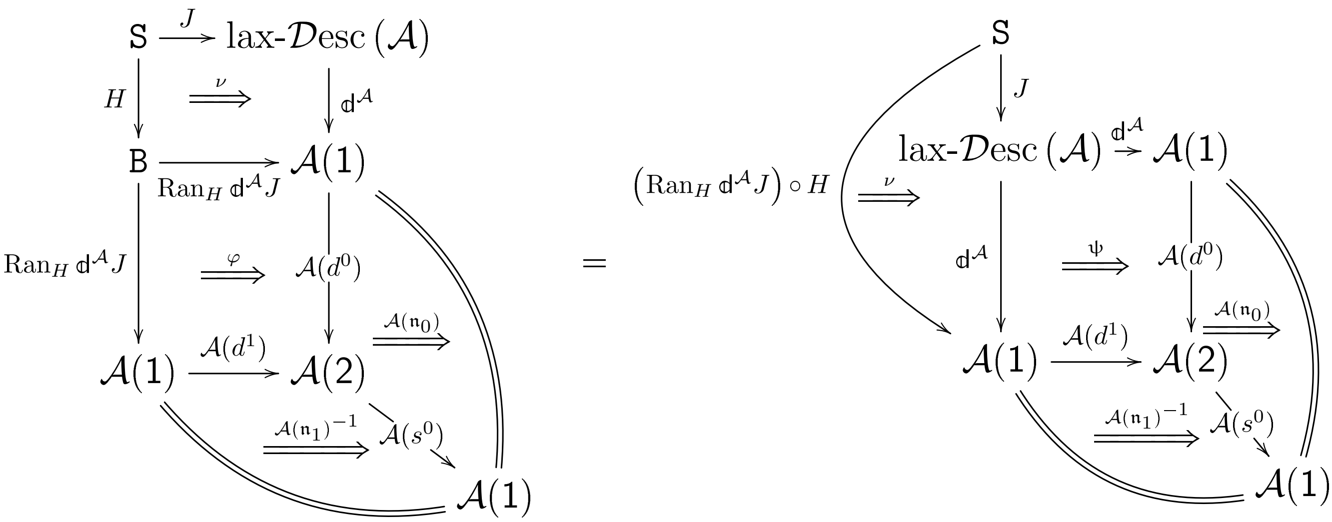
$$\left(\mathcal{A}(D^0) \cdot \mathcal{A}(d^0) \cdot \operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J, \right.$$



$$\left. \right)$$

the equality $\varphi' = \varphi''$ implies that the *descent associativity* equation w.r.t. $\mathcal{A}$ for the pair $(\operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J, \varphi)$ holds.

Analogously, we have that, by the definition of φ, the equation



holds. Moreover, by the *descent identity* equation w.r.t. $\mathcal{A}$ for the pair $\left(\mathbb{d}^{\mathcal{A}}, \psi\right)$, the right side (hence both sides) of the equation above is equal to ν.

Therefore, by the universal property of the right Kan extension $(\operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J, \nu)$, we conclude that the the *descent identity* equation w.r.t. $\mathcal{A}$ for the pair $(\operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J, \varphi)$ holds.

This completes the proof that φ is an $\mathcal{A}$-*descent datum* for $\operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J$.

– By the universal property of the lax descent object, we conclude that there is a unique morphism $\check{J} : \mathsf{B} \to \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})$ of $\mathfrak{A}$ such that

$$\psi * \mathrm{id}_{\check{J}} = \varphi \quad \text{and} \quad \mathbb{d}^{\mathcal{A}} \cdot \check{J} = \operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J.$$

Moreover, by the universal property of the lax descent object and the definition of φ, it follows that there is a unique 2-cell $\tilde{\mathfrak{v}} : \check{J} \cdot H \Rightarrow J$ in $\mathfrak{A}$ such that

$$\mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \tilde{\mathfrak{v}} = \nu.$$

We prove below that the pair $(\check{J}, \tilde{\mathfrak{v}})$ is in fact the right Kan extension of J along H.

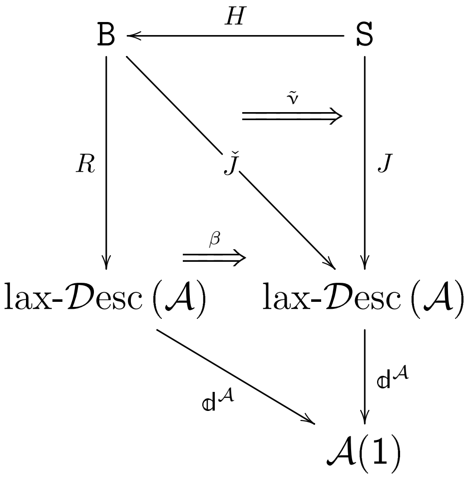
Given any morphism $R : \mathsf{B} \to \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A})$ and any 2-cell

$$\omega : R \circ H \Rightarrow J$$

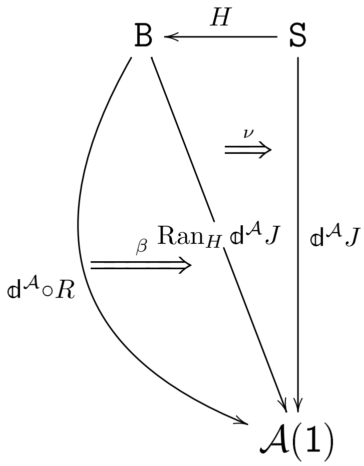
of $\mathfrak{A}$, by the universal property of $\left(\operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J, \nu\right)$, there is a unique 2-cell

$$\beta : \mathbb{d}^{\mathcal{A}} \cdot R \Rightarrow \operatorname{Ran}_H \mathbb{d}^{\mathcal{A}} J$$

in $\mathfrak{A}$ such that



$=$



$= \quad \mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \omega.$

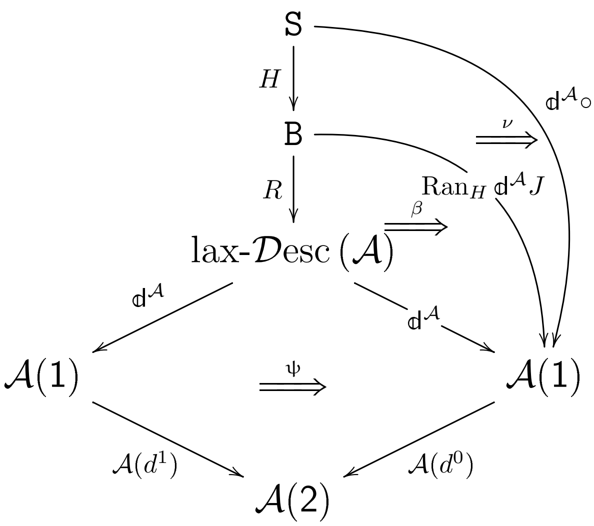
Thus, since the 2-cell $\mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \omega$ is in the image of $\mathfrak{A}(\mathtt{S}, \mathbb{d}^{\mathcal{A}})$, we have that

$$\begin{aligned}
\left(\mathrm{id}_{\mathcal{A}(d^0)} * \nu\right) \cdot \left(\mathrm{id}_{\mathcal{A}(d^0)} * \beta * \mathrm{id}_H\right) \cdot (\psi * \mathrm{id}_{R\circ H}) &= \left(\mathrm{id}_{\mathcal{A}(d^0)} * (\nu \cdot (\beta * \mathrm{id}_H))\right) \cdot (\psi * \mathrm{id}_{R\circ H}) \\
&= \left(\mathrm{id}_{\mathcal{A}(d^0)} * \mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \omega\right) \cdot (\psi * \mathrm{id}_{R\circ H}) \\
&= (\psi * \mathrm{id}_J) \cdot \left(\mathrm{id}_{\mathcal{A}(d^1)} * \mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \omega\right) \\
&= (\psi * \mathrm{id}_J) \cdot \left(\mathrm{id}_{\mathcal{A}(d^1)} * (\nu \cdot (\beta * \mathrm{id}_H))\right) \\
&= (\psi * \mathrm{id}_J) \cdot \left(\mathrm{id}_{\mathcal{A}(d^1)} * \nu\right) \cdot \left(\mathrm{id}_{\mathcal{A}(d^1)} * \beta * \mathrm{id}_H\right).
\end{aligned}$$

By the definition of φ,

$$(\psi * \mathrm{id}_J)\cdot\left(\mathrm{id}_{\mathcal{A}(d^1)} * \nu\right)\cdot\left(\mathrm{id}_{\mathcal{A}(d^1)} * \beta * \mathrm{id}_H\right) = \left(\mathrm{id}_{\mathcal{A}(d^0)} * \nu\right)\cdot(\varphi * \mathrm{id}_H)\cdot\left(\mathrm{id}_{\mathcal{A}(d^1)} * \beta * \mathrm{id}_H\right)$$

and, hence, the equation



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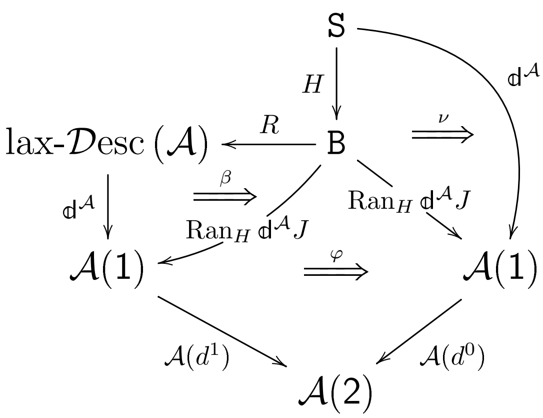


holds. Thus, by the universal property of the right Kan extension

$$\left(\mathcal{A}(D^0)\cdot\mathcal{A}(d^0)\cdot\mathrm{Ran}_H\,\mathbb{d}^{\mathcal{A}}J, \mathrm{id}_{\mathcal{A}(D^0)\cdot\mathcal{A}(d^0)} * \nu\right),$$

we get that

$$\left(\mathrm{id}_{\mathcal{A}(d^0)} * \beta\right)\cdot(\psi * \mathrm{id}_R) = \varphi\cdot\left(\mathrm{id}_{\mathcal{A}(d^1)} * \beta\right),$$

which, by the universal property of lax-$\mathcal{D}\mathrm{esc}\,(\mathcal{A})$, proves that there is a unique 2-cell $\tilde{\beta} : R \Rightarrow \check{J}$ in $\mathfrak{A}$ such that

$$\mathrm{id}_{\mathbb{d}^{\mathcal{A}}} * \tilde{\beta} = \beta.$$

By the faithfulness of $\mathbb{d}^{\mathcal{A}}$, it is clear then that $\tilde{\beta}$ is the unique 2-cell such that

$$\tilde{\nu}\cdot\left(\tilde{\beta} * \mathrm{id}_H\right) = \omega.$$

This completes the proof that $(\check{J}, \tilde{\nu})$ is the right Kan extension of J along H.

– Finally, from the definition of $\mathrm{Ran}_H J = (\check{J}, \tilde{\nu})$, it is clear that $\mathrm{Ran}_H J$ is indeed preserved by $\mathbb{d}^{\mathcal{A}}$.

■

It should be noted that, including the result itself, the Theorem 2.4 has four duals. We state below the most important one to the present work.

Corollary 2.5. *Assume that the lax descent object of the pseudofunctor* $(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathfrak{A}$ *exists. Given morphisms* $J : \mathsf{S} \to$ lax-$\mathcal{D}\mathrm{esc}\,(\mathcal{A})$ *and* $H : \mathsf{S} \to \mathsf{B}$ *of* $\mathfrak{A}$*, the forgetful morphism* $\mathbb{d}^{\mathcal{A}}$ *creates the left Kan extension of* $\mathbb{d}^{\mathcal{A}}J : \mathsf{S} \to \mathcal{A}(1)$ *along* H*, provided that* $\mathrm{Lan}_H\,\mathbb{d}^{\mathcal{A}}J$ *exists and is preserved by the morphisms* $\mathcal{A}(d^1)$ *and* $\mathcal{A}(D^2)\cdot\mathcal{A}(d^1)$*.*

2.1. Creation of absolute Kan extensions. In a 2-category $\mathfrak{A}$, we say that a right Kan extension $\mathrm{Ran}_H J$ is *absolute* if it is preserved by any morphism whose domain is the codomain of $\mathrm{Ran}_H J$.

Moreover, we say that a morphism G *creates absolute right Kan extensions* if, whenever $\mathrm{Ran}_H GJ$ is an absolute right Kan extension, G creates it. Finally, we say that G *creates absolute Kan extensions* if it creates both absolute right Kan extensions and absolute left Kan extensions.

The following is an immediate consequence of Theorem 2.4 and Corollary 2.5.

Corollary 2.6. *Assume that the lax descent object of the pseudofunctor* $(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathfrak{A}$ *exists. The forgetful morphism* $\mathbb{d}^{\mathcal{A}} : \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A}) \to \mathcal{A}(1)$ *creates absolute Kan extensions.*
Consequently, if a morphism F *of* $\mathfrak{A}$ *is equal to* $\mathbb{d}^{\mathcal{A}}$ *composed with any equivalence, then* F *creates absolute Kan extensions.*

Finally, as a consequence of Remark 2.2 and Corollary 2.6, since the notion of absolute limits/colimits of diagrams $J : \mathbb{S} \to \mathbb{C}$ coincide with the notion of absolute right/left Kan extensions along $\mathbb{S} \to \mathsf{1}$, we get:

Corollary 2.7. *Let* $(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathsf{Cat}$ *be a pseudofunctor. The forgetful functor* $\mathbb{d}^{\mathcal{A}} : \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{A}) \to \mathcal{A}(1)$ *creates absolute limits and colimits.*
Consequently, if a functor F *is equal to* $\mathbb{d}^{\mathcal{A}}$ *composed with any equivalence, then* F *creates absolute limits and colimits.*

By the result above, Beck's monadicity theorem [1], and the monadicity theorem of [36], we get:

Theorem 2.8 (Monadicity Theorem)**.** *A functor* $G : \mathbb{B} \to \mathbb{C}$ *is monadic if and only if* G *has a left adjoint and it is, up to the precomposition with an equivalence, a functor* $\mathbb{d}^{\mathcal{A}}$ *that forgets the descent data w.r.t. a pseudofunctor* $\mathcal{A}$*.*

Proof: Assume that G has a left adjoint.

By the monadicity theorem of Section 5 of [36], if G is monadic then it is an *effective faithful functor*, which means in particular that it is the forgetful functor (possibly composed with an equivalence) of the descent data w.r.t. the *higher cokernel* of G.

Reciprocally, if there is a pseudofunctor $(\mathcal{A}, \mathfrak{a}) : \Delta_3 \to \mathsf{Cat}$ such that $G = \mathbb{d}^{\mathcal{A}} \circ K$ for an equivalence K, then G creates absolute coequalizers by Corollary 2.7. By Beck's monadicity theorem, we conclude that G is monadic. ■

3. Descent theory

We briefly establish the setting of descent theory w.r.t. fibrations [18, 19, 34], within the context of [23]. Instead of considering fibrations, we start with a pseudofunctor

$$\mathcal{F} : \mathbb{C}^{\text{op}} \to \mathsf{Cat}$$

which can be also called an *indexed category* [25, 23].

A *precategory* [23, 5] in $\mathbb{C}$ is a functor $a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$. Hence, each internal category or groupoid of $\mathbb{C}$ has an underlying precategory. In particular, internal groups and monoids w.r.t. the cartesian monoidal structure also have underlying precategories [5, 20]. By abuse of language, whenever a precategory a is the underlying precategory of an internal category/groupoid/monoid/group, we say that the precategory a is an internal category/groupoid/monoid/group.

Remark 3.1. [Composition of pseudofunctors] Let $a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$ be a precategory. Firstly, we can consider the functor $\mathrm{op}(a) : \Delta_3 \to \mathbb{C}^{\mathrm{op}}$, also denoted by a^{op}, which is the image of a by the usual dualization (invertible) 2-functor

$$\mathrm{op} : \mathsf{Cat}^{\mathrm{co}} \to \mathsf{Cat}.$$

Secondly, we can consider that $\mathrm{op}(a) : \Delta_3 \to \mathbb{C}^{\mathrm{op}}$ is actually a pseudofunctor between locally discrete 2-categories. Therefore we can define the composition

$$\mathcal{F} \circ \mathrm{op}(a) : \Delta_3 \to \mathsf{Cat}$$

as a particular case of composition of pseudofunctors/homomorphisms of bicategories/2-categories [2, 30]. Namely, the composition is defined by

$$\begin{array}{rccl} \mathcal{F} \circ \mathrm{op}(a) := \mathcal{B} : & \Delta_3 & \to & \mathsf{Cat} \\ & x & \mapsto & \mathcal{F}\left(a(x)\right) \\ & g : x \to y & \mapsto & \mathcal{F}\left(a^{\mathrm{op}}(g : x \to y)\right) \end{array}$$

$$\begin{array}{rccl} \mathfrak{b}_x := \mathfrak{f}_{a(x)} : & \mathrm{id}_{\mathcal{F}(a(x))} & \Rightarrow & \mathcal{F}\left(\mathrm{id}_{a(x)}\right) \\ \mathfrak{b}_{hg} := \mathfrak{f}_{a^{\mathrm{op}}(h) a^{\mathrm{op}}(g)} : & \mathcal{F}\, a^{\mathrm{op}}(h) \cdot \mathcal{F}\, a^{\mathrm{op}}(g) & \Rightarrow & \mathcal{F}\, a^{\mathrm{op}}(hg). \end{array}$$

By definition, the category of $\mathcal{F}$*-internal actions* of a precategory

$$a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$$

(actions $a \to \mathbb{C}$) is the lax descent object of the composition $\mathcal{F} \circ \mathrm{op}(a) : \Delta_3 \to \mathsf{Cat}$. That is to say,

$$\mathcal{F}\text{-IntAct}\,(a) := \text{lax-}\mathcal{D}\mathrm{esc}\,(\mathcal{F} \circ \mathrm{op}(a))\,.$$

As briefly mentioned in the introduction, the definition above generalizes the well known definitions of categories of actions. For instance, taking $\mathbb{C} = \mathsf{Set}$ and $\mathcal{F} = \mathsf{Set}/- : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Cat}$, if $a : \Delta_3^{\mathrm{op}} \to \mathsf{Set}$ is an internal category, the category of $(\mathsf{Set}/-)$-internal actions of a coincides up to equivalence with the category $\mathsf{Cat}\,[a, \mathsf{Set}]$ of functors $a \to \mathsf{Set}$ and natural transformations (see [23, 5] for further details). This shows that the definition above has as particular cases the well known categories of m-sets (or g-sets) for a monoid m (or a group g).

Analogously, given a topological group g, we can consider the category of g-Top of the Eilenberg-Moore algebras of the monad $g \times -$ with the multiplication $g\times g\times - \to g\times -$ given by the operation of g, that is to say, the category of g-spaces. This again coincides with the category of $(\mathsf{Top}/-)$-IntAct(g), in which g, by abuse of language, is the underlying precategory of g.

A precategory is discrete if it is naturally isomorphic to a constant functor $\overline{w} : \Delta_3^{\mathrm{op}} \to \mathbb{C}$ for an object w of $\mathbb{C}$. Clearly, we have:

Lemma 3.2. *The category of $\mathcal{F}$-internal actions of a discrete precategory $\overline{w}$ is equivalent to $\mathcal{F}(w)$.*

Given a precategory $a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$, the *underlying discrete precategory* of the precategory a is the precategory constantly equal to $a(\mathsf{1})$, which we denote by $\overline{a(\mathsf{1})} : \Delta_3^{\mathrm{op}} \to \mathbb{C}$. We have, then, that the functor

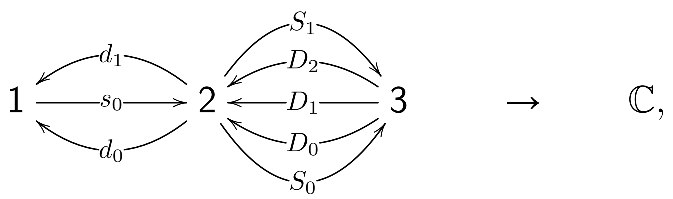
$$\text{lax-}\mathcal{D}\text{esc}\,(\mathcal{F}\circ a^{\mathrm{op}}) \to \mathcal{F}\circ a(\mathsf{1})$$

that forgets the descent data is the forgetful functor

$$\mathcal{F}\text{-IntAct}\,(a) \to \mathcal{F}\text{-IntAct}\left(\overline{a(\mathsf{1})}\right)$$

between the category of $\mathcal{F}$-internal actions of a and the category of $\mathcal{F}$-internal actions of the underlying discrete precategory of a.

Remark 3.3. [Underlying discrete precategory] The definition of the *underlying discrete precategory* of a precategory is motivated by the special case of internal categories, and/or the case of precategories that can be extended to cosimplicial objects $\underline{\Delta_3}^{\mathrm{op}} \to \mathbb{C}$,



in which $\underline{\Delta}_3$ is the full subcategory of Δ with the objects $\mathsf{1}$, $\mathsf{2}$ and $\mathsf{3}$. Namely, we actually get an adjunction

$$\mathsf{Cat}\,[\underline{\Delta}_3^{\mathrm{op}}, \mathbb{C}] \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{Cat}\,[\mathsf{1}, \mathbb{C}] \cong \mathbb{C}$$

in which the left adjoint is given by the usual functor $w \mapsto \overline{w}$ that associates each object to the constant functor $\overline{w} : \underline{\Delta}_3 \to \mathbb{C}$. Of course, the right adjoint is given by the conical limit, which, in this case, coincides with $a(\mathsf{1})$, since $\mathsf{1}$ is the initial object of $\underline{\Delta_3}^{\mathrm{op}}$. The underlying discrete precategory, in this case, is given by the monad induced by this adjunction.

Remark 3.4. [Forgetful functor] As particular case of Remark 3.3, in the case of $\mathbb{C} = \mathsf{Set}$ and $\mathcal{F} = \mathsf{Set}/-$, if $a : \Delta_3^{\mathrm{op}} \to \mathsf{Set}$ is an internal category, the forgetful functor

$$(\mathsf{Set}/-)\,\text{-IntAct}\,(a) \to (\mathsf{Set}/-)\,\text{-IntAct}\left(\overline{a(\mathsf{1})}\right)$$

coincides with the usual forgetful functor $\mathsf{Cat}[a, \mathsf{Set}] \to \mathsf{Set}^{a(\mathsf{1})} \simeq \mathsf{Set}/a(\mathsf{1})$ between the category of functors $a \to \mathsf{Set}$ and the category of *functions* between the set $a(\mathsf{1})$ of objects of a and the collection of objects of Set. In particular, this shows that, if a is a monoid, we get that this forgetful functor coincides with the usual forgetful functor a-$\mathsf{Set} \to \mathsf{Set}$. Analogously, taking $\mathbb{C} = \mathsf{Top}$ and $\mathcal{F} = (\mathsf{Top}/-)$, if $g : \Delta_3^{\mathrm{op}} \to \mathsf{Top}$ is an internal group (topological group), then the forgetful functor

$$(\mathsf{Top}/-)\,\text{-IntAct}\,(g) \to (\mathsf{Top}/-)\,\text{-IntAct}\left(\overline{g(\mathsf{1})}\right)$$

coincides with the usual forgetful functor g-$\mathsf{Top} \to \mathsf{Top}$ between the category of g-spaces and Top.

Corollary 3.5. *Given an indexed category $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ and a precategory $a : \Delta_3^{\mathrm{op}} \to \mathbb{C}$, the forgetful functor*

$$\mathcal{F}\text{-}\mathit{IntAct}\,(a) \to \mathcal{F}\text{-}\mathit{IntAct}\left(\overline{a(\mathsf{1})}\right)$$

creates absolute Kan extensions and, hence, in particular, it creates absolute limits and colimits.

Henceforth, we assume that $\mathbb{C}$ has pullbacks, and a pseudofunctor $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ is given. Every morphism $p : e \to b$ of $\mathbb{C}$ induces an internal groupoid whose underlying precategory, denoted herein by $\mathsf{Eq}(p)$, is given by

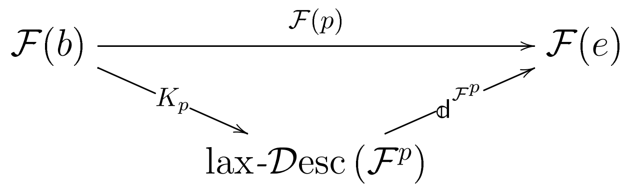
$$e \underset{\pi_e}{\overset{\pi^e}{\leftleftarrows}} \!\!\to e \times_b e \leftleftarrows\!\!\!\leftarrow e \times_b e \times_b e$$

in which $e \times_b e$ denotes the pullback of p along itself, and the arrows are given by the projections and the diagonal morphisms (see, for instance, Section 3 of [23], or Section 8 of [34]). For short, we denote by

$$\mathcal{F}^p : \Delta_3 \to \mathsf{Cat}$$

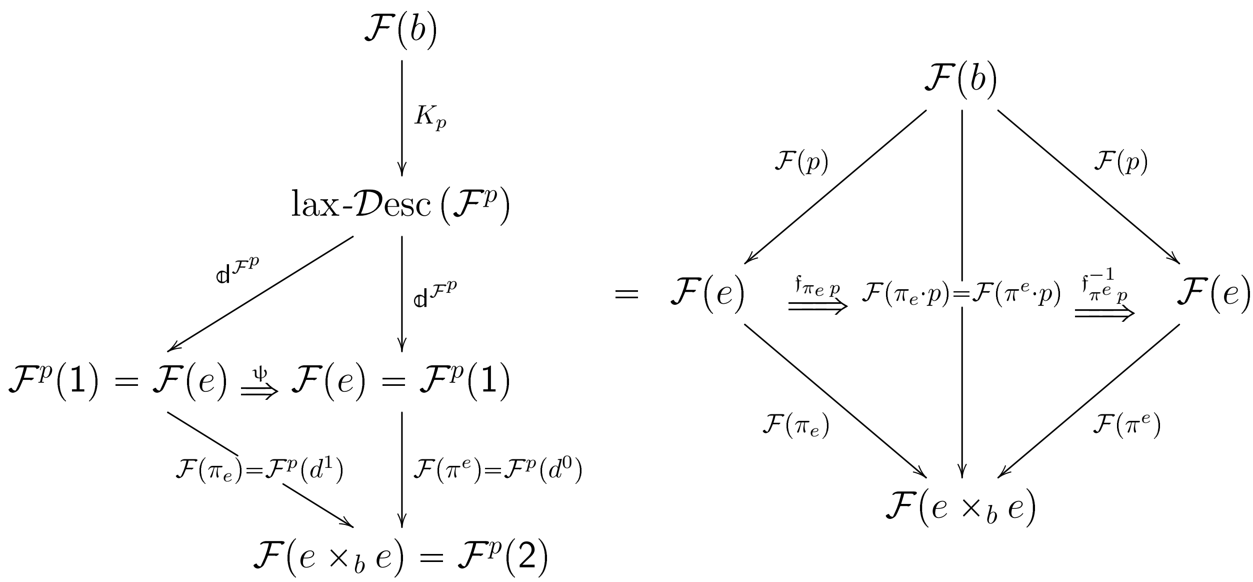
the composition pseudofunctor $\mathcal{F} \circ \mathsf{Eq}(p)^{\mathrm{op}}$.

Lemma 3.6. *Let* $\left(\mathbb{d}^{\mathcal{F}^p}, \psi\right)$ *be the universal pair that gives the lax descent category of* $\mathcal{F}^p$*. For each morphism* $p : e \to b$ *of* $\mathbb{C}$*, we get a factorization*



($\mathcal{F}$-descent factorization of $\mathcal{F}(p)$)

in which $\mathbb{d}^{\mathcal{F}^p} : \text{lax-}\mathcal{D}\text{esc}\,(\mathcal{F}^p) \to \mathcal{F}(e) = \mathcal{F}^p(e)$ *denotes the functor that forgets descent data, and* K_p *the unique functor such that the diagram above is commutative and the equation*



holds.

Proof: This factorization can be found, for instance, in Section 3 of [23] or Section 8 of [34]. In our context, in order to prove this result, it is enough to verify that

$$\mathfrak{f}^{-1}_{\pi^e p} \cdot \mathfrak{f}_{\pi_e p} : \mathcal{F}^p(d^1) \cdot \mathcal{F}(p) \Rightarrow \mathcal{F}^p(d^0) \cdot \mathcal{F}(p)$$

is an $\mathcal{F}^p$-descent datum for $\mathcal{F}(p)$, which follows directly from the fact that $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ is a pseudofunctor. ∎

Definition 3.7. [Effective descent morphism] A morphism p of $\mathbb{C}$ is of *effective $\mathcal{F}$-descent* if the comparison K_p above is an equivalence.

Remark 3.8. By definition, if p is of effective $\mathcal{F}$-descent, this means in particular that $\mathcal{F}(p) : \mathcal{F}(b) \to \mathcal{F}(e)$ is, up to the composition with a canonical equivalence, the forgetful functor between the category of $\mathcal{F}$-internal actions of the internal groupoid $\mathsf{Eq}(p)$ and the category of $\mathcal{F}$-internal actions $\overline{e}$.

Only with the interpretation above, then, it is easy to see that the effective $(\mathsf{Set}/-)$-descent morphisms are precisely the surjections (epimorphisms in Set).

4. Effective descent morphisms and monadicity

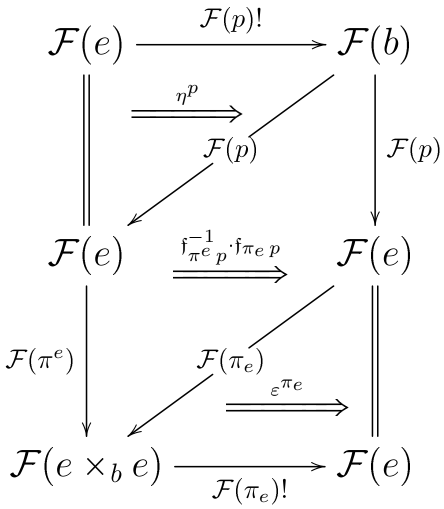
The celebrated Bénabou-Roubaud theorem [3, 34] gives an insightful connection between monad theory and descent theory. Namely, the theorem says that the $\mathcal{F}$-descent factorization of $\mathcal{F}(p)$ coincides up to equivalence with the Eilenberg-Moore/semantic factorization [17, 36] of the right adjoint functor $\mathcal{F}(p)$, provided that $\mathcal{F}$ comes from a bifibration satisfying the so called *Beck-Chevalley condition* (see, for instance, [22, 21, 32, 34] for the Beck-Chevalley condition).

The result motivates what is often called *monadic approach to descent* [4, 34], and it is useful to the characterization of effective descent morphisms in in several cases of interest [40, 22, 28, 5, 11, 12, 13, 14].

More precisely, in our context, the result can be stated as follows. Assuming that $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ is a pseudofunctor such that, for every morphism p of $\mathbb{C}$,

– there is an adjunction $(\mathcal{F}(p)! \dashv \mathcal{F}(p), \varepsilon^p, \eta^p) : \mathcal{F}(b) \to \mathcal{F}(e)$, and

– the 2-cell obtained from the pasting



is invertible.

We have that, denoting by T^p the monad

$$\left(\mathcal{F}(p)\cdot\mathcal{F}(p)!, \mathrm{id}_{\mathcal{F}(p)} * \varepsilon^p * \mathrm{id}_{\mathcal{F}(p)!}, \eta^p\right),$$

the Eilenberg-Moore factorization

$$\begin{array}{ccc} \mathcal{F}(b) & \xrightarrow{\quad\mathcal{F}(p)\quad} & \mathcal{F}(e) \\ & \searrow \qquad \nearrow & \\ & \mathcal{F}(e)^{T^p} & \end{array}$$

is *pseudonaturally* equivalent to the $\mathcal{F}$-descent factorization of $\mathcal{F}(p)$. In particular, we get that, assuming the above, a morphism *p is of effective $\mathcal{F}$-descent if and only if $\mathcal{F}(p)$ is monadic.*

Remark 4.1. [Basic bifibration] If $\mathbb{C}$ has pullbacks, the basic indexed category

$$\mathbb{C}/- : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$$

satisfies the Beck-Chevalley condition. Therefore, in this case, by the Bénabou-Roubaud theorem, one reduces the problem of characterization of effective descent morphisms to the problem of characterization of the morphisms p for which the change of base functor $\mathbb{C}/p$ is monadic.

For instance, if $\mathbb{C}$ is locally cartesian closed and has coequalizers, one can easily prove that $\mathbb{C}/p$ is monadic if and only if p is a universal regular epimorphism [22]. On the one hand, this result can be seen as a generalization of the case of Set. On the other hand, this result is an important of the usual framework to study effective $(\mathbb{C}/-)$-descent morphisms of non-locally cartesian closed categories via embedding results [40, 21, 34].

4.1. Non-effective descent morphisms inducing monadic functors.

On the one hand, the Bénabou-Roubaud theorem answers the question of comparison of the Eilenberg-Moore factorization with the $\mathcal{F}$-descent factorization of $\mathcal{F}(p)$ in the case of $\mathcal{F}$ coming from a bifibration and satisfying the Beck-Chevalley condition. On the other hand, one might ask what it is possible to prove in this direction without assuming the Beck-Chevalley condition.

Firstly, it should be noted that it is well known that there are indexed categories $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ (coming from bifibrations that do not satisfy the Beck-Chevalley condition) for which there are non-effective descent morphisms inducing monadic functors.

For instance, in her master's thesis [38], Melo gives a detailed proof in Example 3.2.3 of page 67 (*Exemplo* 3.2.3) that the so called *fibration of points* of the category of groups does not satisfy the Beck-Chevalley condition (in particular, w.r.t. the morphism $0 \to \mathsf{S}_3$). It is known that, denoting by Pt the corresponding indexed category, $\mathsf{Pt}(0 \to \mathsf{S}_3)$ is monadic but $0 \to \mathsf{S}_3$ is not of effective Pt-descent.

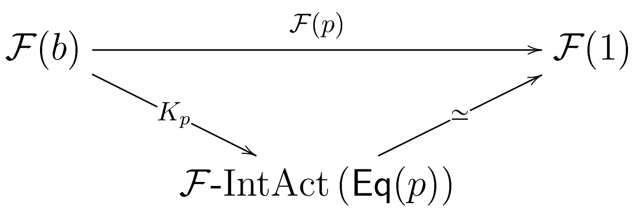
We can produce easy examples of non-effective descent morphisms inducing monadic functors as above, once we observe the results below.

Proposition 4.2. *If the domain of p is the terminal object of $\mathbb{C}$, then p is of effective $\mathcal{F}$-descent if and only if $\mathcal{F}(p)$ is an equivalence.*

Proof: Indeed, if the domain of p is the terminal object 1 of $\mathbb{C}$, $\mathsf{Eq}(p)$ is discrete, naturally isomorphic to the precategory $\overline{1} : \Delta_3^{\mathrm{op}} \to \mathbb{C}$ constantly equal to 1. Thus

$$\mathcal{F}\text{-IntAct}\,(\mathsf{Eq}(p)) \simeq \mathcal{F}(1).$$

Therefore the result follows, since the $\mathcal{F}$-descent factorization of $\mathcal{F}(p)$, in this case, is given by

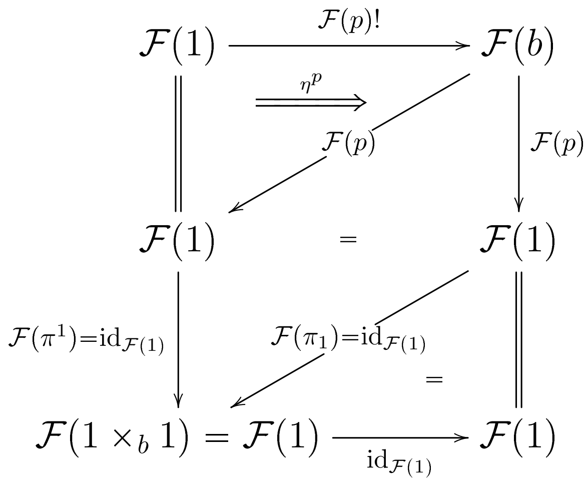


■

Remark 4.3. The Example 3.2.3 presented in [38] can be studied using Proposition 4.2. In an *exact protomodular* category [7, 8], on the one hand, denoting again by Pt the indexed category corresponding to the fibration of points, whenever $\mathsf{Pt}(p)$ has a left adjoint, it is monadic (see Theorem 3.4 of [8]).

On the other hand, by Proposition 4.2, $1 \to b$ is of effective Pt-descent if and only if $\mathsf{Pt}(1 \to b)$ is an equivalence. In the case of the category of groups, $\mathsf{Pt}(0 \to \mathsf{S}_3)$ has a left adjoint but it is not an equivalence.

Remark 4.4. It should be noted that, if $p : 1 \to b$ is a morphism of $\mathbb{C}$ satisfying the hypothesis of Proposition 4.2, the pasting



is invertible if and only if η^p is invertible. That is to say, if and only if $\mathcal{F}(p)!$ is fully faithful. In other words, $p : 1 \to b$ *satisfies the Beck-Chevalley condition w.r.t.* $\mathcal{F}$ *if and only if* $\mathcal{F}(p)!$ *is fully faithful.* In this case, in fact, if $\mathcal{F}(p)$ is (pre)monadic, then it is an equivalence and, hence, by Proposition 4.2, p is of effective $\mathcal{F}$-descent.

The most elementary examples of non-effective $\mathcal{F}$-descent morphisms inducing monadic functors can be constructed from Lemma 4.5. Namely, in order to get our desired example, it is enough to consider a pseudofunctor $\mathcal{G} : \mathsf{2}^{\mathrm{op}} \to \mathsf{Cat}$ whose image of d is a monadic functor which is not an equivalence. In this case, by Lemma 4.5, we conclude that, despite $\mathcal{G}(d)$ being monadic, d is not of effective $\mathcal{G}$-descent.

Lemma 4.5. *Consider the category* $\mathsf{2}$ *with the only non-trivial morphism* $d : \mathsf{0} \to \mathsf{1}$. *Given a pseudofunctor* $\mathcal{G} : \mathsf{2}^{\mathrm{op}} \to \mathsf{Cat}$, d *is of effective* $\mathcal{G}$*-descent if and only if* $\mathcal{G}(d)$ *is an equivalence.*

Proof: Again, in this case, $\mathsf{Eq}(d)$ is discrete. We have that

$$\mathcal{G}\text{-IntAct}\left(\mathsf{Eq}(d)\right) \simeq \mathcal{G}(\mathsf{0}),$$

and, hence, we get the result. ■

Finally, in Remark 7, Sobral [42], considering the indexed category

$$\mathcal{E} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{Cat}$$

of discrete op-fibrations, gives an example of a morphism p in Cat such that $\mathcal{E}(p)$ is monadic but p is not of effective $\mathcal{E}$-descent. She also suggests that, for the indexed category $\mathcal{E}$, *descent gives "more information" than monadicity.* We finish this article showing, as an immediate consequence of Theorem 2.8, that this is in fact the case for any indexed category.

Theorem 4.6 (Effective descent implies monadicity)**.** *Let* $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Cat}$ *be any pseudofunctor. If* p *is of effective* $\mathcal{F}$*-descent and* $\mathcal{F}(p)$ *has a left adjoint, then* $\mathcal{F}(p)$ *is monadic.*

Proof: It is clearly a particular case of Theorem 2.8. ∎

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