Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 19–26

REACTION-DIFFUSION EQUATIONS FOR INFINITY LAPLACIAN

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ABSTRACT: We derive sharp regularity for viscosity solutions of an inhomogeneous infinity Laplace equation across the free boundary, when the right hand side of the equation does not change sign and satisfies a certain growth condition. We prove geometric regularity estimates for solutions and conclude that the free boundary is a porous set and hence has zero Lebesgue measure. Additionally, we derive a Liouville type theorem. When the right hand side is comparable to power function of degree three, we show that if a non-negative viscosity solution vanishes at a point, then it has to vanish everywhere.

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AMS Subject Classifications MSC 2010: 35B09, 35B53, 35B65, 35R35.

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1. Introduction

Reaction-diffusion equations arise naturally when modeling certain phenomena in biological, chemical and physical systems. In this paper we study reaction-diffusion equations for infinity Laplacian, which despite being too degenerate to realistically represent a physical diffusion process, has been studied in the framework of optimization and free boundary problems (see, for example, [3], [10], [12, 13, 14], just to cite a few). More precisely, we establish regularity and geometric properties of solutions of the problem

$$\Delta_\infty u = f(u) \quad \text{in} \quad \Omega, \tag{1.1}$$

Received July 17, 2019.

RT was partially supported by FCT– Fundação para a Ciência e a Tecnologia, I.P., through projects PTDC/MAT-PUR/28686/2017 and UTAP-EXPL/MAT/0017/2017, and by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese government through FCT and co-funded by the European Regional Development Fund through Partnership Agreement PT2020. NMLD was partially supported by Instituto Federal de Educação, Ciência e Tecnologia do Rio Grande do Sul. NMLD thanks the Analysis group at CMUC for fostering a pleasant and productive scientific atmosphere during his postdoctoral program.

where $\Omega \subset \mathbb{R}^n$, $f \in C(\mathbb{R}_+)$ and

$$f(\delta t) \geq M\delta^\gamma f(t) \geq 0, \tag{1.2}$$

with $M > 0$, $\gamma \in [0,3)$, $t > 0$ bounded, and $\delta > 0$ small enough. Additionally, we assume that

$$f \text{ is non-decreasing or } \inf f > 0. \tag{1.3}$$

Here, $\mathbb{R}_+$ is the set of non-negative numbers, and the infinity Laplacian is defined as follows:

$$\Delta_\infty u(x) := \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j},$$

with $u_{x_i} = \partial u/\partial x_i$. Note that the continuity of f provides that $f(u)$ is bounded once u is bounded. Note also that (1.2) is quite general in the sense that it needs to hold only for δ close to zero. For example, it holds for functions that are homogeneous of degree γ. Condition (1.3) is needed to guarantee the comparison principle. Solutions of (1.1) are understood in the viscosity sense according to the following definition:

Definition 1.1. *A function $u \in C(\Omega)$ is called a viscosity super-solution (resp. sub-solution) of* (1.1)*, and written as $\Delta_\infty u \leq f(u)$ (resp. $\geq$), if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local minimum at $x_0 \in \Omega$, with $\phi(x_0) = u(x_0)$, we have*

$$\Delta_\infty \phi(x_0) \leq f(\phi(x_0)). \quad \textit{(resp. } \geq\textit{)}$$

A function u is called a viscosity solution if it is both a viscosity super-solution and a viscosity sub-solution.

The infinity Laplace operator is related to the absolutely minimizing Lipschitz extension problem: for a given Lipschitz function on the boundary of a bounded domain, find its extension inside the domain in a way that has the minimal Lipschitz constant, [1]. It is known (see [7]) that such function u has to be an infinity harmonic one, i.e. $\Delta_\infty u = 0$ (in the viscosity sense). The regularity issue of infinity harmonic functions received extensive attention over the years. As was shown in [5], the infinity harmonic functions in the plane are $C^{1,\alpha}$, for a small α (it is conjectured that the optimal regularity is $C^{1,\frac{1}{3}}$). In higher dimensions infinity harmonic functions are known to be everywhere differentiable (see [6]).

As for the inhomogeneous case of $\Delta_\infty u = f$, it is known that the Dirichlet problem has a unique viscosity solution, provided f does not change sign (see [9]). Moreover, as was shown in [8], for bounded right hand side, the Lipschitz estimate and everywhere differentiability of solutions remain true. The case of f not being bounded away from zero, mainly, when $f = u_+^\gamma$, where $u_+ := \max(u, 0)$ and $\gamma \in [0, 3)$ is a constant, was studied in [2] (dead-core problem). The authors show that for such right hand side (strong absorbtion) across the free boundary $\partial\{u > 0\}$ non-negative viscosity solutions are of class $C^{\frac{4}{3-\gamma}}$. The denominator $3 - \gamma$ is related to the degree of homogeneity of the operator, which is three, i.e., $\Delta_\infty(Cu) = C^3\Delta_\infty u$, for any constant C. Note that for $\gamma \in (0, 3)$ this regularity is more than the conjectured $C^{1,\frac{1}{3}}$, i.e., we obtain higher regularity across the free boundary. This result allows to establish Hausdorff dimension estimate for the free boundary $\partial\{u > 0\}$ and conclude that it has Lebesgue measure zero.

Our strategy is the following: by means of a flattening argument, we show that across the free boundary $\partial\{u > 0\} \cap \Omega$ non-negative viscosity solutions of (1.1) are of class $C^{\frac{4}{3-\gamma}}$, when (1.2) holds. This result is sharp in the sense that across the free boundary non-negative viscosity solutions grow exactly as $r^{\frac{4}{3-\gamma}}$ in the ball of radius r. We also analyze the borderline (critical) case, that is, when $\gamma = 3$. Unlike [2], f is not given explicitly, which makes it harder to construct a barrier function - needed for our analysis. Nevertheless, we are able to show that if f is comparable to power function of degree three, (which is also the degree of the homogeneity of the infinity Laplacian), then (1.1) has a viscosity sub-solution whose gradient has modulus separated from zero. We use this function to build up a suitable barrier to conclude that if a viscosity solution vanishes at a point, it has to vanish everywhere.

The paper is organized as follows: in Section 2, we prove an auxiliary result (flattening solutions) (Lemma 2.2), which we use in Section 3 to derive the main regularity result (Theorem 3.1), and as a consequence, in Section 4, we obtain Liouville type theorems (Theorem 4.1 and Theorem 4.2). In Section 5, we prove several geometric measure estimates (Theorem 5.1 (non-degeneracy) and Corollary 5.1 (porosity)), and conclude that the free boundary has Lebesgue measure zero (Corollary 5.2). In Section 6, for f comparable to power function of degree three, we show that the only non-negative viscosity solution that has zero, is the function that is identically zero (Theorem 6.1).

2. Preliminaries

In this section we list some preliminaries, as well as prove an auxiliary lemma for future reference. We start by the comparison principle, the proof of which can be found in [4, 9].

Lemma 2.1. *Let u, $v \in C(\overline{\Omega})$ be such that*

$$\Delta_\infty u - f(u) \leq 0, \;\; \Delta_\infty v - f(v) \geq 0 \;\; \text{in} \;\; \Omega$$

in the viscosity sense, and f satisfy (1.3). *If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.*

The comparison principle, together with Perron's method leads to the following result (for the proof we refer the reader to [4], for example). In fact, existence of solutions can be shown even without directly applying the comparison principle, as it was done, for example, in [11, Theorem 3.1].

Theorem 2.1. *If $\Omega \subset \mathbb{R}^n$ is bounded and $\varphi \in C(\partial\Omega)$ is a non-negative function, then there is a unique and non-negative function u that solves the Dirichlet problem*

$$\begin{cases} \Delta_\infty u = f(u) \;\; \text{in} \; \Omega, \\ \quad\;\; u = \varphi \;\; \text{on} \; \partial\Omega \end{cases} \tag{2.1}$$

in the viscosity sense.

The following auxiliary lemma is a variant of the flatness improvement technique introduced in [2, 15, 16] to study the regularity properties of solutions of dead-core problems.

Lemma 2.2. *Let $g \in L^\infty(B_1) \cap C(B_1)$ be a non-negative function. For any given $\mu > 0$ there exists a constant $\kappa_\mu = \kappa(\mu, n) > 0$ such that if in B_1 a continuous functions v, which vanishes at the origin and $v \in [0,1]$, satisfies, in viscosity sense,*

$$\Delta_\infty v - \kappa_\mu^4 g(v) = 0$$

for $0 < \kappa \leq \kappa_\mu$, then

$$\sup_{B_{1/2}} v \leq \mu.$$

Proof: We argue by contradiction assuming that there exist $\mu^* > 0$, $\{v_i\}_{i\in\mathbb{N}}$ and $\{\kappa_i\}_{i\in\mathbb{N}}$ with $v_i(0) = 0$, $0 \leq v_i \leq 1$, in B_1 satisfying in viscosity sense to

$$\Delta_\infty v_i - \kappa_i^4 g(v_i) = 0$$

where $\kappa_i = \mathrm{o}(1)$, while

$$\sup_{B_{1/2}} v_i > \mu^*. \tag{2.2}$$

By local Lipschitz regularity (see [8, Corollary 2], for example), the sequence $\{v_i\}_{i\in\mathbb{N}}$ is pre-compact in the $C^{0,1}(B_{3/4})$. Hence, by Arzelà-Ascoli theorem, v_i converges (up to a subsequence) to a function v_∞ locally uniformly in $B_{2/3}$. Moreover, $v_\infty(0) = 0$, $0 \le v_\infty \le 1$ and $\Delta_\infty v_\infty = 0$. The maximum principle for the infinity harmonic functions then yields $v \equiv 0$, which contradicts to (2.2) once i is big enough. ■

The following definition is for future reference.

Definition 2.1. *A function u is called an entire solution, if it is a viscosity solution of* (1.1) *in* $\mathbb{R}^n$*.*

We close this section by reminding the notion of porosity.

Definition 2.2. *The set $E \subset \mathbb{R}^n$ is called porous with porosity σ, if there is $R > 0$ such that $\forall x \in E$ and $\forall r \in (0, R)$ there exists $y \in \mathbb{R}^n$ such that*

$$B_{\sigma r}(y) \subset B_r(x) \setminus E.$$

A porous set of porosity σ has Hausdorff dimension not exceeding $n - c\sigma^n$, where $c > 0$ is a constant depending only on dimension. In particular, a porous set has Lebesgue measure zero (see [17], for instance).

3. Regularity across the free boundary

In this section we make use of Lemma 2.2 and derive regularity result for viscosity solutions of (1.1) across the free boundary $\partial\{u > 0\}$.

Theorem 3.1. *If u is a non-negative viscosity solution of* (1.1)*, where f satisfies* (1.2)*, and $x_0 \in \partial\{u > 0\} \cap \Omega$, then there exists a constant $C > 0$, depending only on γ, $\|u\|_\infty$ and* $\mathrm{dist}(x_0, \partial\Omega)$*, such that*

$$u(x) \le C|x - x_0|^{\frac{4}{3-\gamma}}$$

for $x \in \{u > 0\}$ near x_0.

Proof: The idea is to use an iteration argument and carefully choose sequence of functions that allows to make use of the Lemma 2.2. Observe that without loss of generality, we may assume that $x_0 = 0$ and $B_1 \subset \Omega$.

For $\mu = 2^{-\frac{4}{3-\gamma}}$, let now $\kappa_\mu > 0$ be as in Lemma 2.2. We then construct the first member of the sequence by setting

$$w_1(x) := \tau u(\rho x) \quad \text{in} \quad B_1,$$

where

$$\tau := \min\left\{1, \|u\|_\infty^{-1}\right\} \quad \text{and} \quad \rho := \kappa_\mu \tau^{-\frac{3-\gamma}{4}}.$$

Note that $\tau^3\rho^4 = \kappa_\mu^4\tau^\gamma$, $w_1(0) = 0$ and in $w_1 \in [0,1]$. Since u is a viscosity solution of (1.1), then

$$\Delta_\infty w_1(x) - \tau^3\rho^4 f(\tau^{-1}w_1(x)) = 0$$

or, equivalently,

$$\Delta_\infty w_1(x) - \kappa_\mu^4\tau^\gamma f(\tau^{-1}w_1(x)) = 0. \tag{3.1}$$

From (1.2) we have for $0 < \tau \leq \delta_0$, for δ_0 small enough

$$0 \leq \tau^\gamma f(\tau^{-1}w_1) \leq M^{-1} f(w_1),$$

hence $g(w_1) = \tau^\gamma f(\tau^{-1}w_1)$ is bounded. Otherwise, if $\delta_0 < \tau \leq 1$, g remains bounded. Then from Lemma 2.2, we obtain

$$\sup_{B_{1/2}} w_1 \leq 2^{-\frac{4}{3-\gamma}}.$$

For $i \in \mathbb{N}$, we then define

$$w_i(x) := 2^{-\frac{4}{3-\gamma}} w_{i-1}(2^{-1}x).$$

and observe that $w_i(0) = 0$, $w_i \in [0,1]$ and w_i satisfies (3.1). Hence, once again applying Lemma 2.2, one gets

$$\sup_{B_{1/2}} w_i \leq 2^{-\frac{4}{3-\gamma}},$$

or in other terms,

$$\sup_{B_{1/4}} w_{i-1} \leq 2^{-2\frac{4}{3-\gamma}}.$$

Continuing this way, for w_1 we obtain

$$\sup_{B_{2^{-i}}} w_1 \leq 2^{-i\frac{4}{3-\gamma}}. \tag{3.2}$$

Next, for a fixed $0 < r \leq \frac{\rho}{2}$, by choosing $i \in \mathbb{N}$ such that

$$2^{-(i+1)} < \frac{r}{\rho} \leq 2^{-i},$$

and using (3.2), we estimate

$$\begin{aligned}\sup_{B_r} u &\le \sup_{B_{\rho 2^{-i}}} u = \tau^{-1} \sup_{B_{\rho 2^{-i}}} w_1 \\ &\le \tau^{-1} 2^{-i\frac{4}{3-\gamma}} = 2^{\frac{4}{3-\gamma}} \tau^{-1} 2^{-(i+1)\frac{4}{3-\gamma}} \\ &\le \left(\tau^{-1} 2 \rho^{-1}\right)^{\frac{4}{3-\gamma}} r^{\frac{4}{3-\gamma}} \\ &= C r^{\frac{4}{3-\gamma}}.\end{aligned}$$

■

Geometrically Theorem 3.1 means that no matter how "bad" the function u is in $\{u > 0\}$, it touches the free boundary $\partial\{u > 0\}$ smoothly. In other words, a non-negative viscosity solution of (1.1) may have cusp singularities in its positivity set, and yet it is smooth near its free boundary.

4. Liouville type results

Despite the regularity information being available only across the free boundary, it is enough to derive the following Liouville type theorem.

Theorem 4.1. *If u is an entire solution,* (1.2) *holds and $u(x_0) = 0$ for a $x_0 \in \mathbb{R}^n$ with*

$$u(x) = o\left(|x|^{\frac{4}{3-\gamma}}\right), \quad \text{as} \quad |x| \to \infty, \tag{4.1}$$

then $u \equiv 0$.

Proof: Without loss of generality we may assume that $x_0 = 0$. For $k \in \mathbb{N}$, set

$$u_k(x) := k^{\frac{-4}{3-\gamma}} u(kx), \quad x \in B_1,$$

where B_1 is the ball of radius one centered at the origin. Note that $u_k(0) = 0$. Since u is an entire solution, for $x \in B_1$ one has

$$\Delta_\infty u_k(x) - k^{\frac{-4\gamma}{3-\gamma}} f\left(k^{\frac{4}{3-\gamma}} u_k(x)\right) = 0.$$

Also, from (1.2) we have

$$0 \le k^{\frac{-4\gamma}{3-\gamma}} f\left(k^{\frac{4}{3-\gamma}} u_k(x)\right) \le M^{-1} f(u_k(x))$$

From Theorem 3.1, we then deduce that if $x_k \in \overline{B}_r$ is such that

$$u_k(x_k) = \sup_{\overline{B}_r} u_k,$$

where $r > 0$ is small, then in B_r one has

$$\|u_k\|_\infty \to 0, \quad \text{as} \quad k \to \infty. \tag{4.2}$$

In fact, if $|kx_k|$ remains bounded as $k \to \infty$, then applying Theorem 3.1 to u_k we obtain

$$u_k(x_k) \leq C_k |x_k|^{\frac{4}{3-\gamma}}, \tag{4.3}$$

where $C_k > 0$ and $C_k \to 0$. This implies that $u(kx_k)$ remains bounded as $k \to \infty$, and therefore $u_k(x_k) \to 0$, as $k \to \infty$, and (4.2) is true. It remains true also in the case when $|kx_k| \to \infty$, as $k \to \infty$, since then from (4.1) we get

$$u_k(x_k) \leq |kx_k|^{-\frac{4}{3-\gamma}} k^{-\frac{4}{3-\gamma}} \to 0, \quad \text{as} \quad k \to \infty.$$

Now, if there exists $y \in \mathbb{R}^n$ such that $u(y) > 0$, by choosing $k \in \mathbb{N}$ large enough so $y \in B_{kr}$ and using (4.2) and (4.3), we estimate

$$\frac{u(y)}{|y|^{\frac{4}{3-\gamma}}} \leq \sup_{B_{kr}} \frac{u(x)}{|x|^{\frac{4}{3-\gamma}}} = \sup_{B_r} \frac{u_k(x)}{|x|^{\frac{4}{3-\gamma}}} \leq \frac{u(y)}{2|y|^{\frac{4}{3-\gamma}}},$$

which is a contradiction. ■

In fact, once the comparison principle holds, the condition (4.1) can be weakened in the following sense (Theorem 4.2 below). Let $x_0 \in \mathbb{R}^n$ and $r > 0$ be fixed, and let $u \geq 0$ be the unique solution of (2.1) in $B_r(x_0)$ with $\varphi \equiv \alpha_r > 0$ constant, guaranteed by Theorem 2.1. Note that u is a viscosity sub-solution of

$$\begin{cases} \Delta_\infty v = \lambda v_+^\gamma & \text{in } B_r(x_0), \\ \quad v = \alpha_r & \text{on } \partial B_r(x_0), \end{cases} \tag{4.4}$$

where

$$\lambda := M\beta^{-\gamma} f(\beta), \tag{4.5}$$

and $\beta > \|u\|_\infty$ is a constant big enough so (1.2) holds. Then the condition (4.1) can be weakened and substituted by

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x - x_0|^{\frac{4}{3-\gamma}}} < \left(\lambda \frac{(3-\gamma)^4}{64(1+\gamma)}\right)^{\frac{1}{3-\gamma}}, \tag{4.6}$$

where λ is defined by (4.5), and Theorem 4.1 can be improved to the following variant (see Theorem 4.2 below). The choice of the right hand side of (4.6) comes from the explicit structure of the unique solution of (4.4), which, as observed in [2], is given by

$$v(x) := \Upsilon \left(|x - x_0| - r + \left(\frac{\alpha_r}{\Upsilon} \right)^{\frac{3-\gamma}{4}} \right)_+^{\frac{4}{3-\gamma}}, \tag{4.7}$$

where

$$\Upsilon := \left(\lambda \frac{(3-\gamma)^4}{64(1+\gamma)} \right)^{\frac{1}{3-\gamma}}. \tag{4.8}$$

Theorem 4.2. *Let* (1.2), (1.3) *hold. If u is an entire solution and satisfies* (4.6)*, then $u \equiv 0$.*

Proof: Once $r > 0$ is large enough, then (4.6) guarantees, with $\Upsilon > 0$ defined by (4.8),

$$\sup_{\partial B_r} \frac{u(x)}{r^{\frac{4}{3-\gamma}}} \leq \theta \Upsilon,$$

for some $\theta < 1$. On the other hand, using (1.2), one has that the unique solution of (4.4), with $\alpha_r = \sup\limits_{\partial B_r(x_0)} u$, given by (4.7), is a viscosity super-solution of (1.1). The comparison principle, Lemma 2.1, then implies that $u \leq v$ in $B_r(x_0)$. Letting $r \to \infty$, we conclude that $u \equiv 0$. ■

Remark 4.1. *As can be seen from* (4.7)*, the plateau of v, i.e., the set $\{v = 0\}$, is the ball $\overline{B}_R(x_0)$, where*

$$0 < R := r - \left(\frac{\alpha_r}{\Upsilon} \right)^{\frac{3-\gamma}{4}}.$$

Since $0 \leq u \leq v$, the plateau of u contains the $\overline{B}_R(x_0)$.

Remark 4.2. *Note that the inequality* (4.6) *has to be strict. For example, if*

$$w(x) := \Upsilon |x - x_0|^{\frac{4}{3-\gamma}}$$

then

$$\limsup_{|x| \to \infty} \frac{w(x)}{|x - x_0|^{\frac{4}{3-\gamma}}} = \Upsilon,$$

but w is not identically zero.

5. Non-degeneracy and porosity

In this section we show that the result of Theorem 3.1 is sharp in the sense that across the free boundary non-negative viscosity solutions of (1.1) grow exactly as $r^{\frac{4}{3-\gamma}}$ in the ball B_r, for $r > 0$ small enough. As a consequence, we conclude that the touching ground surface is a porous set, which implies that it has Hausdorff dimension less than n, and so its Lebesgue measure is zero (see [17]). We start by the following non-degeneracy theorem.

Theorem 5.1. *Let* (1.2), (1.3) *hold. If u is a non-negative viscosity solution of* (1.1)*, then there exists a universal constant $c > 0$, depending only on dimension and γ, such that*

$$\sup_{B_r(x_0)} u \geq cr^{\frac{4}{3-\gamma}},$$

where $x_0 \in \overline{\{u > 0\}} \cap \Omega$ and $0 < r < \operatorname{dist}(x_0, \partial\Omega)$.

Proof: Since u is continuous, it is enough to prove the theorem for points $x_0 \in \{u > 0\} \cap \Omega$ such that $u(x_0) > 0$. Set

$$v(x) := c|x - x_0|^{\frac{4}{3-\gamma}},$$

with a constant $c \in (0, \Upsilon)$, where $\Upsilon > 0$ is defined by (4.8). Using (1.2), direct computation reveals that the choice of c makes v a viscosity supersolution of (1.1) in $B_r(x_0)$, where $r > 0$ is such that $B_r(x_0) \subset \Omega$. If $v \geq u$ on $\partial B_r(x_0)$, then the comparison principle, Lemma 2.1, would imply $v \geq u$ in $B_r(x_0)$, contradicting to the fact that $0 = v(x_0) < u(x_0)$. Hence, there is a point $y \in \partial B_r(x_0)$ such that $v(y) < u(y)$. We then estimate

$$\sup_{B_r(x_0)} u \geq u(y) \geq v(y) = cr^{\frac{4}{3-\gamma}}.$$

■

As a consequence, we obtain that the free boundary is a porous set, therefore it has Hausdorff dimension strictly less than n, hence its Lebesgue measure is zero.

Corollary 5.1. *Let* (1.2), (1.3) *hold. If u is a bounded non-negative viscosity solution of* (1.1)*, then $\partial\{u > 0\}$ is a porous set.*

Proof: Let $x \in \partial\{u > 0\}$ and $y \in \overline{B}_r(x)$ be such that

$$u(y) = \sup_{B_r(x)} u.$$

By Theorem 5.1, $u(y) \geq cr^{\frac{4}{3-\gamma}}$. On the other hand, Theorem 3.1 provides

$$u(y) \leq C\,[d(y)]^{\frac{4}{3-\gamma}},$$

where $d(y) := \operatorname{dist}(y, \partial\{u > 0\})$. Therefore,

$$\left(\frac{c}{C}\right)^{\frac{3-\gamma}{4}} r \leq d(y).$$

Hence, if $\sigma := \frac{1}{2}\left(\frac{c}{C}\right)^{\frac{3-\gamma}{4}}$, one has

$$B_{2\sigma r}(y) \subset \{u > 0\}.$$

We now choose $\xi \in (0,1)$ such that for the point $z := \xi y + (1-\xi)x$ we have $|y - z| = \sigma r$. Then

$$B_{\sigma r}(z) \subset B_{2\sigma r}(y) \cap B_r(x).$$

Moreover, we have

$$B_{2\sigma r}(y) \cap B_r(x) \subset \{u > 0\},$$

which together with the previous inclusion implies

$$B_{\sigma r}(z) \subset B_{2\sigma r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{u > 0\},$$

that is, the set $\partial\{u > 0\}$ is porous with porosity σ. ■

Corollary 5.2. *If* (1.2), (1.3) *hold, and u is a viscosity solution of* (1.1), *then Lebesgue measure of the set $\partial\{u > 0\}$ is zero.*

6. The borderline case

Although, in general, one cannot expect more than $C^{1,\alpha}$ regularity for viscosity solutions of (1.1), Theorem 3.1 provides higher and higher regularity across the free boundary, as $\gamma \in [0,3)$ gets closer to 3. In this section we analyze the limit case of $\gamma = 3$. The scaling property of the operator plays an essential role here, as $\gamma = 3$ is also the degree of homogeneity of the infinity Laplacian, meaning that $\Delta_\infty(Cu) = C^3\Delta_\infty u$, for any constant C. Observe that Theorem 3.1 cannot be applied directly, since the estimates deteriorate as $\gamma \to 3$. Our first observation states as follows.

Lemma 6.1. *Let* (1.2) *hold. If u is a non-negative viscosity solution of* (1.1) *with $\gamma = 3$, then its every zero is of infinite order.*

Proof: This is a consequence of Theorem 3.1. To see that it is enough to rewrite (1.2), for $\gamma = 3$, as

$$f(\delta t) \geq M_\delta \delta^{3-\beta} f(t),$$

where $M_\delta := M\delta^\beta$ and $\beta > 0$. An application of Theorem 3.1 with $M = M_\delta$ leads to the conclusion that if $u(z) = 0$ for $z \in \Omega$, then $D^n u(z) = 0$, $\forall n \in \mathbb{N}$. ■

Furthermore, we show that in a particular case, when f is comparable to power function of degree three, i.e., for constants $N \geq M > 0$

$$N\delta^3 f(t) \geq f(\delta t) \geq M\delta^3 f(t) \geq 0, \tag{6.1}$$

with $t > 0$ bounded and $\delta > 0$ small, if a non-negative viscosity solution of (1.1) vanishes at a point, then it must vanish everywhere. In the particular case, when $N = M$, f is homogeneous of degree three, that is, $f(t) = Mt^3$. This case was studied in [2], where by means of a suitable barrier function, was concluded that if non-negative viscosity solution vanishes in an inner point, then it has to vanish everywhere. Unlike [2], our function f is not given explicitly, which makes the construction of a suitable barrier function more complicated. Observe that (6.1) implies $f(0) = 0$, hence $\inf f = 0$, so to use the comparison principle, we need to assume that f is non-decreasing.

Theorem 6.1. *Let u be a non-negative viscosity solution of* (1.1)*, where f is non-decreasing and satisfies* (6.1)*. If $\{u = 0\} \cap \Omega \neq \emptyset$, then $u \equiv 0$.*

Proof: We argue by contradiction, assuming that there is $x \in \Omega$ such that $u(x) = 0$, but $u(y) > 0$ for a point $y \in \Omega$. Without loss of generality we may assume that

$$r := \operatorname{dist}(y, \{u = 0\}) < \frac{1}{10} \operatorname{dist}(y, \partial\Omega).$$

We aim is to construct a sub-solution of (1.1) which stays below u on $\partial B_r(y)$.

Let w be an infinity sub-harmonic function in $B_r(y)$ such that $|\nabla w| \geq \eta$ for $\eta \geq 0$ constant to be chosen later. Such function can be built up as a limit, as $p \to \infty$, of p-super-harmonic functions with modulus of gradient separated from zero by η. We refer the reader for details to [7].

Now if g is a smooth function and $v = g(w)$, direct computation reveals that

$$\Delta_\infty v = [g'(w)]^3 \Delta_\infty w + [g'(w)]^2 g''(w) |\nabla w|^4.$$

Thus, for $g(t) = e^t + t$,

$$\Delta_\infty v \geq [g'(w)]^2 g''(w)|\nabla w|^4, \tag{6.2}$$

since $g' \geq 1$ and $\Delta_\infty w \geq 0$. Also $g'' \geq e^{-\|w\|_\infty} > 0$, and (6.2) yields (recall that $|\nabla w| \geq \eta$)

$$\Delta_\infty v \geq \mu\eta, \tag{6.3}$$

where $\mu := e^{-\|w\|_\infty} > 0$. Choosing

$$\eta > \frac{N}{\mu} \max_{[0,\|v\|_\infty]} f,$$

from (6.3) we obtain

$$\Delta_\infty v - Nf(v) \geq \Delta_\infty v - \mu\eta \geq 0,$$

i.e., v is a sub-solution of (1.1). The latter together with (6.1) gives, for any small constant $\delta > 0$,

$$\Delta_\infty (\delta v) - f(\delta v) \geq \delta^3 (\Delta_\infty v - Nf(v)) \geq 0,$$

that is, the function δv is also a sub-solution of (1.1). We choose $\delta > 0$ small enough to guarantee

$$\delta v(x) \leq u(x), \ \ x \in \partial B_r(y),$$

and by the comparison principle, Lemma 2.1,

$$\delta v(x) \leq u(x), \ \ x \in B_r(y). \tag{6.4}$$

Observe, that writing the lower bound of (6.1) as

$$f(\delta t) \geq M\delta\delta^2 f(t),$$

and applying Theorem 3.1 with $\widetilde{M} = M\delta$, we arrive at

$$\sup_{B_d(z)} u \leq Cd^4, \tag{6.5}$$

where $z \in \partial B_r(y) \cap \partial\{u > 0\}$, and $d > 0$ is small. In fact, we choose $0 < d < \left(\frac{\delta\eta}{4C}\right)^{\frac{1}{3}}$. Using the fact that $|\nabla v| = g'|\nabla w| \geq \eta$, recalling (6.4) and (6.5) and the choice of d, we estimate

$$\delta\eta d \leq \sup_{B_d(z)} \delta|v(x) - v(z)| \leq \sup_{B_d(z)} \delta v \leq \sup_{B_d(z)} u \leq Cd^4 \leq \frac{1}{4}\delta\eta d,$$

which is a contradiction. ∎

Remark 6.1. *Note that in order to use the regularity result across the free boundary, the barrier argument forces the right hand side to be comparable to power function of degree three. In that sense, condition* (6.1) *cannot be broadened, and Theorem 6.1 is optimal.*

Acknowledgments. RT was partially supported by FCT– Fundação para a Ciência e a Tecnologia, I.P., through projects PTDC/MAT-PUR/28686/2017 and UTAP-EXPL/MAT/0017/2017, and by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese government through FCT and co-funded by the European Regional Development Fund through Partnership Agreement PT2020. NMLD was partially supported by Instituto Federal de Educação, Ciência e Tecnologia do Rio Grande do Sul. NMLD thanks the Analysis group at CMUC for fostering a pleasant and productive scientific atmosphere during his postdoctoral program.

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