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JOINS, EARS AND CASTELNUOVO–MUMFORD REGULARITY

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ABSTRACT: We introduce a new class of polynomial ideals associated to a simple graph, G. Let $K[E_G]$ be the polynomial ring on the edges of G and $K[V_G]$ the polynomial ring on the vertices of G. We associate to G an ideal, $I(X_G)$, defined as the preimage of $(x_i^2 - x_j^2 : i,j \in V_G) \subseteq K[V_G]$ by the map $K[E_G] \to K[V_G]$ which sends a variable, t_e, associated to an edge $e = \{i,j\}$, to the product $x_i x_j$ of the variables associated to its vertices. We show that $K[E_G]/I(X_G)$ is a one-dimensional, Cohen–Macaulay, graded ring, that $I(X_G)$ is a binomial ideal and that, with respect to a fixed monomial order, its initial ideal has a generating set independent of the field K. We focus on the Castelnuovo–Mumford regularity of $I(X_G)$ providing the following sharp upper and lower bounds:

$$\mu(G) \leq \operatorname{reg} I(X_G) \leq |V_G| - b_0(G) + 1,$$

where $\mu(G)$ is the maximum vertex join number of the graph and $b_0(G)$ is the number of its connected components. We show that the lower bound is attained for a bipartite graph and use this to derive a new combinatorial result on the number of even length ears of nested ear decomposition.

KEYWORDS: Castelnuovo–Mumford regularity, Binomial ideal, maximum vertex join number, ear decompositions.
MATH. SUBJECT CLASSIFICATION (2000): 13F20; 13P20; 05E40; 05C70.

1. Introduction

The study of polynomial ideals associated to combinatorial structures, exploring relations between algebraic and combinatorial invariants, has been a source for many new results. In the case of graphs, these ideals include, but are not limited to, the *toric ideal*, the *edge ideal* and *binomial edge ideal*, and, in this framework, one of the algebraic invariants that has been the object of growing interest is the Castelnuovo–Mumford regularity. Bounds for the regularity of the toric ideal have been obtained in [1, 10]. These

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bounds involve the number and sizes of families of disjoint induced complete bipartite subgraphs of the graph. By [16], the regularity of the edge ideal is bounded below by the induced matching number plus 1, and, by [35], it is bounded above by the co-chordal number plus 1. Several classes of graphs for which regularity of the edge ideal attains one, or both, of these bounds have been studied — see [2, 11, 12, 13, 14, 20, 22, 33, 35]. Refinements of these bounds were recently obtained in [29]. In [23], it is shown that the regularity of the binomial edge ideal is bounded below by the length of the longest induced path of the graph plus 1 and above by the number of its vertices. It is conjectured that the number of maximal cliques of the graph plus 1 is an upper bound for the regularity of the binomial edge ideal — see [17, 18]. See also [6, 18, 19] for values of the regularity of the binomial edge ideal for specific classes of graphs.

In this article we define a new class of polynomial ideals associated to graphs and study their Castelnuovo–Mumford regularity. Let G be a simple graph on a finite vertex set, $V_G \subseteq \mathbb{N}$, without isolated vertices. Let its edge set be denoted by E_G, let K be a field and let $K[V_G]$ and $K[E_G]$ denote the polynomial rings

$$K[V_G] = K[x_i : i \in V_G], \quad K[E_G] = K[t_e : e \in E_G],$$

associated to the vertex and edge set, respectively. Let $\theta\colon K[E_G] \to K[V_G]$ be the ring homomorphism defined by $t_e \mapsto x_i x_j$, for every $e = \{i, j\} \in E_G$. In particular, recall, the toric subring of G is the image of θ and the toric ideal of G is $\ker \theta$.

Definition 1.1. Let $I(X_G) \subseteq K[E_G]$ be given by

$$I(X_G) = \theta^{-1}(x_i^2 - x_j^2 : i, j \in V_G).$$

Since θ is graded and the ideal $(x_i^2 - x_j^2 : i, j \in V_G)$ is homogeneous, $I(X_G)$ is also a homogeneous ideal. In fact, we will show that $I(X_G)$ is generated by homogeneous binomials and that with respect to a fixed monomial order its initial ideal has a generating set independent of the field. The main results of this article reveal a strong connection between the Castelnuovo–Mumford regularity of $I(X_G)$ and the maximum vertex join number of G.

Definition 1.2. [31, 7] The maximum vertex join number of G is the maximum cardinality of $J \subseteq E_G$ satisfying $|J \cap E_C| \leq |E_C|/2$, for every circuit C in G.

Following [7], we will denote the maximum vertex join number by $\mu(G)$. Let $b_0(G)$ denote the number of connected components of G. Then, by Theorem 4.2 and Proposition 4.3, proved in this article, the following bounds hold, for any graph:

$$\mu(G) \leq \operatorname{reg} I(X_G) \leq |V_G| - b_0(G) + 1.$$

Moreover, by Theorem 4.5, if G is bipartite, then $\operatorname{reg} I(X_G) = \mu(G)$. This relation and the results of [24], on the regularity of the vanishing ideal over a graph endowed with *nested* ear decomposition, yield a new combinatorial result (see Corollary 4.12).

The motivation for the definition of $I(X_G)$ comes from the notion of vanishing ideal over a graph for a finite field, introduced by Renteria, Simis and Villarreal in [28]. We will see, in Proposition 2.9, that the two ideals coincide when $K = \mathbb{Z}_3$. It is this relation and the existence of a set of generators of the initial ideal of $I(X_G)$ independent of the field that allow transferring to $I(X_G)$ the known properties and values of the regularity of the vanishing ideal over a graph.

This article is organized as follows. In Section 2 we will study the basic properties of $I(X_G)$. We start by showing that $K[E_G]/I(X_G)$ is a one-dimensional Cohen–Macaulay graded ring (Proposition 2.1). We then show that $I(X_G)$ is a binomial ideal (Proposition 2.2 and Corollary 2.3) and we characterize binomials in $I(X_G)$ in terms of associated subgraphs of G (Proposition 2.5). Next we prove that, with respect to a given monomial order, the initial ideal of $I(X_G)$ has a generating set independent of the field (Proposition 2.8). We then show that $I(X_G)$ coincides with the vanishing ideal over the graph when $K = \mathbb{Z}_3$ (Proposition 2.9) and give a first application of these two results to the computation of the degree of the ideal (Proposition 2.11). In Section 3, using the fact that the regularity of $I(X_G)$ is independent of the field, we transfer from the context of the vanishing ideal over the graph known properties and values of the regularity (Proposition 3.2). We also describe two useful results in our approach to the computation of the regularity (Propositions 3.4 and 3.5). In Section 4 we describe the connection between $\operatorname{reg} I(X_G)$ and the maximum vertex join number, first establishing upper and lower bounds that hold for any graph (Theorem 4.2 and Proposition 4.3) and then proving equality between the regularity and the lower bound, $\mu(G)$, in the bipartite case (Theorem 4.5). We then use this theorem to deduce a new combinatorial result related to the number of even length ears of *nested* ear decompositions of a bipartite graph (Corollary 4.12).

2. The ideals

2.1. Assumptions and notation. The graphs considered in this work are finite simple graphs without isolated vertices. K is any field and, as in the introduction, $K[V_G]$ and $K[E_G]$ will denote the polynomial rings on the vertex and edge sets of the graph, respectively. Given an edge $e = \{i, j\}$, we will also use t_{ij} as an alternative notation to t_e. Monomials in $K[V_G]$ and $K[E_G]$ will be denoted using the multi-index notation. Namely, given $\alpha \in \mathbb{N}^{V_G}$ and $\beta \in \mathbb{N}^{E_G}$, the notations $\mathbf{x}^\alpha$ and $\mathbf{t}^\beta$ shall stand for the monomials

$$\mathbf{x}^\alpha = \prod\nolimits_{i\in V_G} x_i^{\alpha(i)} \quad \text{and} \quad \mathbf{t}^\beta = \prod\nolimits_{e\in E_G} t_e^{\beta(e)},$$

respectively.

2.2. The Cohen–Macaulay property. For the sake of clarity and also for later use, we begin by dealing with the case when G is a single edge. Assume, without loss of generality, that $V_G = \{1, 2\}$ and $E_G = \{\{1, 2\}\}$. The map $\theta\colon K[E_G] \to K[V_G]$ is then defined by sending the unique variable in the domain, t_{12}, to the product $x_1x_2 \in K[V_G]$. Let $f \in K[E_G]$, which we write as:

$$f = a_0 + a_1t_{12} + \cdots a_dt_{12}^d,$$

for some $a_0, \ldots, a_d \in K$ and $d \in \mathbb{N}$. If $\theta(f) \in (x_1^2 - x_2^2)$ then, setting $x_2 = x_1$ in $\theta(f)$, we deduce that:

$$a_0 + a_1x_1^2 + a_2x_1^4 + \cdots + a_dx_1^{2d} = 0,$$

which implies that $a_0 = \cdots = a_d = 0$, i.e. that, $f = 0$. Therefore, if G consists of a single edge, $I(X_G) = (0)$. In this situation, $K[E_G]/I(X_G) \simeq K[t_{12}]$, which is clearly a one-dimensional Cohen–Macaulay graded ring.

Taking now G a general graph, if $\{i, j\}, \{k, \ell\} \in E_G$ are two edges in G, one can easily see that $t_{ij}^2 - t_{k\ell}^2 \in I(X_G)$. Indeed,

$$\theta(t_{ij}^2 - t_{k\ell}^2) = x_i^2x_j^2 - x_k^2x_\ell^2 = (x_i^2 - x_k^2)x_j^2 + x_k^2(x_j^2 - x_\ell^2).$$

Therefore $(t_{ij}^2 - t_{k\ell}^2 : {\scriptstyle \{i,j\},\{k,\ell\}} \in E_G) \subseteq I(X_G)$.

Proposition 2.1. *$K[E_G]/I(X_G)$ is one-dimensional and Cohen–Macaulay.*

Proof: We may assume that $|E_G| > 1$. Then, in view of the above, the zero set of $(I(X_G), t_{ij})$ in affine space is the singleton $\{(0, \dots, 0)\}$, for any $\{i, j\} \in E_G$. By [34, Proposition 8.3.22], we conclude that

$$\operatorname{ht} I(X_G) = |E_G| - 1.$$

Hence $K[E_G]/I(X_G)$ is a one-dimensional graded ring. To show that

$$K[E_G]/I(X_G)$$

is Cohen–Macaulay we will show that it contains a regular element. Consider an element of the form $\mathbf{t}^\delta + I(X_G)$, with $\delta \in \mathbb{N}^{E_G} \setminus 0$. Let us show that this element is regular. It suffices to consider the case $\mathbf{t}^\delta = t_{ij}$ for some $\{i, j\} \in E_G$. Without loss of generality, let this edge be $\{1, 2\}$. By Definition 1.1, showing that t_{12} is a regular element of $K[E_G]/I(X_G)$ can be achieved by showing that x_1x_2 is a regular element of $K[V_G]/(x_i^2 - x_j^2 : i, j \in V_G)$. To this end, by symmetry, it is enough to prove that x_1 is a regular element of $K[V_G]/(x_i^2 - x_j^2 : i, j \in V_G)$. Assume that $g \in K[V_G]$ is such that

$$x_1 g \in (x_i^2 - x_j^2 : i, j \in V_G). \tag{1}$$

Let $k \in V_G$ be a vertex different from 1. Then $x_i^2 - x_k^2$, when i varies in $V_G \setminus \{k\}$, yields a Gröbner basis for the ideal $(x_i^2 - x_j^2 : i, j \in V_G)$, with respect to a monomial order where x_k is the least variable. Since we want to show that g belongs to the ideal $(x_i^2 - x_j^2 : i, j \in V_G)$ we may assume that no term of g is divisible by x_i^2, for any $i \in V_G \setminus \{k\}$, and aim to show that $g = 0$. Assume that $g \neq 0$. Then, from (1), we deduce that at least one term of g must be divisible by x_1. If $c_\delta \mathbf{x}^\delta$, where $\delta \in \mathbb{N}^{V_G}$ and $c_\delta \in K$, is a term of g divisible by x_1 (and not by x_1^2) then the division of $x_1(c_\delta \mathbf{x}^\delta)$ by $x_1^2 - x_k^2$ yields

$$x_1(c_\delta \mathbf{x}^\delta) = c_\delta \tfrac{\mathbf{x}^\delta}{x_1}(x_1^2 - x_k^2) + c_\delta \tfrac{\mathbf{x}^\delta}{x_1} x_k^2.$$

If $\mathbf{x}^\delta$, where δ varies in some set $\Delta \subseteq \mathbb{N}^{V_G}$, are the supporting monomials for terms of g divisible by x_1 (and not by x_1^2) and $\mathbf{x}^\gamma$, where γ varies in $\Gamma \subseteq \mathbb{N}^{V_G}$, are those supporting terms of g that are not divisible by x_1, then it is clear that

$$\{\tfrac{\mathbf{x}^\delta}{x_1} x_k^2 : \delta \in \Delta\} \cup \{x_1 \mathbf{x}^\gamma : \gamma \in \Gamma\}$$

remains a linearly independent set of monomials. This implies that the remainder of $x_1 g$ by the division by the Gröbner basis of $(x_i^2 - x_j^2 : i, j \in V_G)$ is not zero, contradicting (1). Hence, we must have $g = 0$. ■

2.3. The binomial property. To prove that $I(X_G)$ is a binomial ideal, we shall use the next proposition, the proof of which follows closely the proof of [28, Theorem 2.1].

Proposition 2.2. *Let $\theta\colon K[y_1,\ldots,y_s]\to K[x_1,\ldots,x_n]$ be a ring homomorphism with $\theta(y_i)$ a monomial, for all $i=1,\ldots,s$. Let $I\subseteq K[x_1,\ldots,x_n]$ be an ideal generated by a finite number of homogeneous binomials. Then, $\theta^{-1}(I)$ is the intersection with $K[y_1,\ldots,y_s]$ of the ideal of $K[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ generated by*

$$\{y_i-\theta(y_i)z : i=1,\ldots,s\}\cup I. \tag{2}$$

Moreover, $\theta^{-1}(I)$ is an ideal generated by a finite number of homogeneous binomials.

Proof: Let us denote by $J\subseteq K[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ the ideal generated by (2). Since θ is a graded ring homomorphism, $\theta^{-1}(I)$ is a homogeneous ideal. Thus, to prove the inclusion $\theta^{-1}(I)\subseteq J\cap K[y_1,\ldots,y_s]$, it suffices to restrict to homogeneous polynomials. Assume that $f\in K[y_1,\ldots,y_s]$, homogeneous of degree d, is such that

$$\theta(f)=f(\theta(y_1),\ldots,\theta(y_s))\in I.$$

For each i consider the substitution of y_i in f by $(y_i-\theta(y_i)z)+\theta(y_i)z$. Using the binomial theorem, we deduce that

$$f=z^d f(\theta(y_1),\ldots,\theta(y_n))+\textstyle\sum_{i=1}^{s} h_i\cdot(y_i-\theta(y_i)z),$$

for some $h_i\in K[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$. Since, by assumption, $\theta(f)\in I$ we conclude that $f\in J\cap K[y_1,\ldots,y_s]$.

To prove the opposite inclusion, $J\cap K[y_1,\ldots,y_s]\subseteq\theta^{-1}(I)$, we will show first that the ideal $J\cap K[y_1,\ldots,y_s]$ is generated by binomials. As, by assumption, $\theta(y_i)$ are monomials, every element of the set $\{y_i-\theta(y_i)z : i=1,\ldots,s\}$ is a binomial. This is also true for the given generating set of I. We deduce that J is generated by a finite number of binomials. As $S(f,g)$, when f and g are binomials, if non-zero, is also a binomial, and the remainder of the division of a binomial by another binomial, if non-zero, is also a binomial, Buchberger's algorithm, for producing a Gröbner basis from the set of generators of J, will also yield a set of binomials. Using the elimination order (variables $y_1,\ldots,y_s$ as last variables) we deduce that $J\cap K[y_1,\ldots,y_s]$ has a Gröbner basis consisting of binomials and thus, in particular, it is generated by binomials. Accordingly, assume that $\mathbf{y}^\delta-\mathbf{y}^\gamma$, for some $\delta,\gamma\in\mathbb{N}^s$ belongs to $J \cap K[y_1,\dots,y_s]$. Then, there exist $h_i, g_j \in K[x_1,\dots,x_n,z,y_1,\dots,y_s]$ and $\ell_j \in I$ such that

$$\mathbf{y}^\delta - \mathbf{y}^\gamma = \textstyle\sum_{i=1}^s h_i \cdot (y_i - \theta(y_i)z) + \sum_{j=1}^k g_j \ell_j, \tag{3}$$

for some $k \geq 0$. Substituting above each y_i by $\theta(y_i) \in K[x_1,\dots,x_n]$ and the variable z by 1 we deduce that

$$\theta(\mathbf{y}^\delta - \mathbf{y}^\gamma) = \textstyle\sum_{j=1}^k g'_j \ell_j,$$

for some $g'_j \in K[x_1,\dots,x_n]$. This proves the inclusion

$$J \cap K[y_1,\dots,y_s] \subseteq \theta^{-1}(I).$$

We have shown that $\theta^{-1}(I) = J \cap K[y_1,\dots,y_s]$. Hence, in particular, $\theta^{-1}(I)$ is generated by a finite number of polynomials of the form $\mathbf{y}^\delta - \mathbf{y}^\gamma$. To see that each of these must be homogeneous, we go back to (3) and substitute all the variables $x_1,\dots,x_n$ by 1. Then, since I is generated by binomials we get $\ell_j(1,\dots,1) = 0$. Moreover, $\theta(y_i)(1,\dots,1) = 1$, as, by assumption, $\theta(y_i)$ are monomials. We deduce:

$$\mathbf{y}^\delta - \mathbf{y}^\gamma = \textstyle\sum_{i=1}^s h'_i \cdot (y_i - z),$$

for some $h'_i \in K[z,y_1,\dots,y_s]$. Substituting in the above y_i by z we get:

$$z^{\delta_1+\cdots+\delta_s} - z^{\gamma_1+\cdots+\gamma_s} = 0,$$

which implies that $\delta_1+\cdots+\delta_s = \gamma_1+\cdots+\gamma_s$, i.e., that $\mathbf{y}^\delta - \mathbf{y}^\gamma$ is homogeneous. ∎

Corollary 2.3. *$I(X_G)$ is generated by homogeneous binomials.*

Proof: Apply Proposition 2.2 with

$$K[y_1,\dots,y_s] = K[E_G], \quad K[x_1,\dots,x_n] = K[V_G],$$

$\theta(t_{ij}) = x_i x_j$ and $I = (x_i^2 - x_j^2 : i,j \in V_G)$. ∎

Remark 2.4. Since, as was shown in the proof of Proposition 2.1, any monomial is regular on $K[E_G]/I(X_G)$, from a generating set of $I(X_G)$ consisting of binomials we obtain one in which all binomials $\mathbf{t}^\alpha - \mathbf{t}^\beta$ satisfy $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$.

2.4. Binomials and subgraphs. In Section 4, we will use the following characterization of homogeneous binomials in $I(X_G)$.

Proposition 2.5. *Let* $\mathbf{t}^\alpha - \mathbf{t}^\beta$ *a homogeneous binomial with* $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$ *and let* H *be the subgraph of* G *the edge set of which is in bijection with the variables that occur in either* $\mathbf{t}^\alpha$ *or* $\mathbf{t}^\beta$ *raised to an odd power. Then* $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G)$ *if and only if the degree of* v *in* H *is even, for all* $v \in V_H$*. In particular, if* $\{i,k\} \neq \{j,\ell\}$ *are two edges then,* $t_{ik} - t_{j\ell} \notin I(X_G)$*.*

Proof: Let $\mathbf{x}^\delta - \mathbf{x}^\gamma \in K[V_G]$, with $\delta, \gamma \in \mathbb{N}^{V_G}$ be a homogeneous binomial of degree > 1. We claim that $\mathbf{x}^\delta - \mathbf{x}^\gamma \in (x_i^2 - x_j^2 : i,j \in V_G)$ if, and only if, $\delta(i) + \gamma(i)$ is even, for every $i \in V_G$.

To prove this claim, assume first that $\delta(i)+\gamma(i)$ is even, for every $i \in V_G$ and let us show that $\mathbf{x}^\delta + \mathbf{x}^\gamma \in (x_i^2 - x_j^2 : i,j \in V_G)$. We will argue by induction on the degree of $\mathbf{x}^\delta - \mathbf{x}^\gamma$. Since $\mathbf{x}^\delta - \mathbf{x}^\gamma \neq 0$ there exists $i \in V_G$ such that $\delta(i) \neq \gamma(i)$. If the degree of $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is two and $\delta(i) + \gamma(i)$ is even, one of $\delta(i)$ or $\gamma(i)$ must be equal to 2 and the other equal to 0. Assume, without loss of generality that $\delta(i) = 2$ and $\gamma(i) = 0$. Then, there exists $j \neq i$ such that $\delta(j) = 0$ and $\gamma(j) = 2$. In other words, $\mathbf{x}^\delta - \mathbf{x}^\gamma = x_i^2 - x_j^2$. Assume now that the degree of $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is > 2 and, without loss of generality, that $\delta(i) \geq \gamma(i)+2$. Let $j \in V_G$ be such that $\gamma(j) > 0$ and let $\delta', \gamma' \in \mathbb{N}^{V_G}$ be such that $\mathbf{x}^{\delta'} = \mathbf{x}^\delta / x_i^2$ and $\mathbf{x}^{\gamma'} = \mathbf{x}^\gamma / x_j$. Then

$$\mathbf{x}^\delta - \mathbf{x}^\gamma = (x_i^2 - x_j^2)\mathbf{x}^{\delta'} + x_j(x_j \mathbf{x}^{\delta'} - \mathbf{x}^{\gamma'}). \tag{4}$$

Write $x_j \mathbf{x}^{\delta'} - \mathbf{x}^{\gamma'} = \mathbf{x}^\mu - \mathbf{x}^\nu$ for some $\mu, \nu \in \mathbb{N}^{V_G}$. Then $\mathbf{x}^\mu + \mathbf{x}^\nu$ has degree one less than $\mathbf{x}^\delta + \mathbf{x}^\gamma$. Additionally

$$\mu(i) + \nu(i) = \delta(i) - 2 + \gamma(i), \quad \mu(j) + \nu(j) = 1 + \delta(j) + \gamma(j) - 1$$

are even, and so are $\mu(k) + \nu(k) = \delta(k) + \gamma(k)$, for every $k \in V_G \setminus \{i,j\}$. By induction hypothesis, $\mathbf{x}^\mu - \mathbf{x}^\nu \in (x_i^2 - x_j^2 : i,j \in V_G)$ and then, by (4),

$$\mathbf{x}^\delta - \mathbf{x}^\gamma \in (x_i^2 - x_j^2 : i,j \in V_G).$$

Conversely, let us assume that $\mathbf{x}^\delta - \mathbf{x}^\gamma \in (x_i^2 - x_j^2 : i,j \in V_G)$. We want to show that $\delta(i) + \gamma(i)$ is even, for every $i \in V_G$. Write

$$\mathbf{x}^\delta - \mathbf{x}^\gamma = \textstyle\sum_{ij} f_{ij}(x_i^2 - x_j^2),$$

for some $f_{ij} \in K[V_G]$ and, fixing $i \in V_G$, substitute in the above all x_j by 1, for all $j \neq i$. Then, there exists $g \in K[x_i]$ such that

$$x_i^{\delta(i)} - x_i^{\gamma(i)} = g(x_i)(x_i^2 - 1).$$

Without loss of generality we may assume that $\delta(i) > \gamma(i)$. Then we deduce that

$$g(x_i) = x_i^{\delta(i)-2} + x_i^{\delta(i)-4} + \cdots + x_i^{\delta(i)-2m},$$

for some $m > 0$, which, in particular, implies that $\gamma(i) = \delta(i) - 2m$ and, therefore, that $\delta(i) + \gamma(i)$ is even. We have proved our claim.

Let $\mathbf{t}^\alpha - \mathbf{t}^\beta$ be a homogeneous binomial with $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. Write

$$\theta(\mathbf{t}^\alpha - \mathbf{t}^\beta) = \mathbf{x}^\delta - \mathbf{x}^\gamma,$$

for some $\delta, \gamma \in \mathbb{N}^{V_G}$. Then, since $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$, we deduce that $\delta(i) + \gamma(i)$ differs from $\deg_H(i)$ by an even number, for every $i \in V_G$. If $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is zero then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G)$ and $\delta(i) = \gamma(i)$, for every $i \in V_G$, which implies that $\delta(i)+\gamma(i)$ is even. Assume that $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is non-zero, and, thus, a homogeneous binomial of degree ≥ 2. Then, $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G)$ if and only if, by definition, $\mathbf{x}^\delta - \mathbf{x}^\gamma \in (x_i^2 - x_j^2 : i, j \in V_G)$ which, by our claim and previous observation, is equivalent to $\deg_H(i)$ being even, for every $i \in V_G$. ■

Example 2.6. Consider the graph, G, in Figure 1. Then, by Proposition 2.5,

$$\begin{array}{lll} t_{13}t_{45}t_{56} - t_{12}t_{23}t_{46}, & t_{23}t_{45}t_{56} - t_{12}t_{13}t_{46}, & t_{12}t_{45}t_{56} - t_{23}t_{13}t_{46}, \\ t_{23}t_{13}t_{56} - t_{12}t_{45}t_{46}, & t_{12}t_{13}t_{56} - t_{23}t_{45}t_{46}, & t_{12}t_{23}t_{56} - t_{13}t_{45}t_{46}, \\ t_{23}t_{13}t_{45} - t_{12}t_{56}t_{46}, & t_{12}t_{13}t_{45} - t_{23}t_{56}t_{46}, & t_{12}t_{23}t_{45} - t_{13}t_{56}t_{46}, \\ t_{12}t_{23}t_{13} - t_{45}t_{56}t_{46}. & & \end{array}$$

are binomials belonging to $I(X_G)$, as they are all associated to the subgraph given by the two triangles of G. Using [15, Macaulay2] one can show that these binomials together with $\left\{t_{ij}^2 - t_{46}^2 : \{i,j\} \in E_G \setminus \{\{4,6\}\}\right\}$ give a reduced Gröbner basis for $I(X_G)$, with respect to the graded reverse lexicographic order induced by $t_{12} \succ t_{23} \succ t_{13} \succ t_{34} \succ t_{45} \succ t_{56} \succ t_{46}$.

Corollary 2.7. *Let $H \subseteq G$ be a subgraph without isolated vertices. Consider $I(X_H)$ as a subset of $K[E_G]$ under the inclusion $K[E_H] \subseteq K[E_G]$. Then*

$$I(X_H) = I(X_G) \cap K[E_H].$$

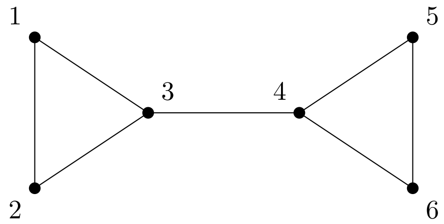

FIGURE 1. A non-bipartite graph.

Proof: Since both $I(X_H)$ and $I(X_G)$ are generated by homogeneous binomials, we may restrict to checking that $I(X_H)$ and $I(X_G)\cap K[E_H]$ contain the same homogeneous binomials $\mathbf{t}^\alpha - \mathbf{t}^\beta \in K[E_H]$, with $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. This follows from Proposition 2.5. ■

2.5. Independence of the field. The construction of Proposition 2.2 can be used to show that, for a fixed monomial order, there exists a set of generators of the initial ideal of $I(X_G)$ which is independent of the field.

Proposition 2.8. *For a fixed monomial order, there exists a set of generators of the initial ideal of $I(X_G)$ which is independent of the field. Moreover, if the characteristic of the field is not 2, then there exists a Gröbner basis of $I(X_G)$ independent of the field.*

Proof: Consider a monomial order in $K[E_G]$. Let us apply Proposition 2.2 with

$$K[y_1,\dots,y_s] = K[E_G], \quad K[x_1,\dots,x_n] = K[V_G],$$

$\theta(t_{ij}) = x_ix_j$ and $I = (x_i^2 - x_j^2 : i,j \in V_G)$. Fix a monomial order in

$$K[\{x_i : i \in V_G\} \cup \{z\} \cup \{t_e : e \in E_G\}]$$

extending the monomial order of $K[E_G]$ in which the variables in

$$\{t_e : e \in E_G\}$$

are the least variables. Then, by the argument in the proof of Proposition 2.2, a Gröbner basis of $I(X_G)$ can be obtained by first applying Buchberger's algorithm to the set

$$\{t_e - \theta(t_e)z : e \in E_G\} \cup \left\{x_i^2 - x_j^2 : i,j \in V_G\right\},$$

and then taking only the elements of the output belonging $K[E_G]$. Assume that the field has characteristic $\neq 2$. Then, since the S-polynomial of the difference of two monomials, if it is not zero, is again a difference of two monomials, since the remainder in the standard expression of a binomial with respect to a list of binomials, if it is not zero, is again a binomial, the result of Buchberger's algorithm will be independent of the field. If the characteristic of K is 2, in which case one replaces "difference of monomials" by "sum of monomials" in the previous argument, the Gröbner basis obtained yields a set of leading terms with coefficient 1. The ones obtained in the previous case may differ, possibly, by the multiplication with -1. ■

2.6. Relation to the vanishing ideal over graphs. Let us now recall from [28] the notion of vanishing ideal over a graph. Assume that K is a finite field, let $\mathbb{P}^{|V_G|-1}$ and $\mathbb{P}^{|E_G|-1}$ be the projective spaces with coordinates indexed by V_G and E_G, respectively and let

$$\vartheta\colon \mathbb{P}^{|V_G|-1} \dashrightarrow \mathbb{P}^{|E_G|-1}$$

be the rational map defined by $t_e \mapsto x_i x_j$, for every edge $e = \{i,j\}$. The *projective toric subset parameterized by* G is the subset of $\mathbb{P}^{|E_G|-1}$ given by the image by ϑ of the projective torus,

$$\mathbb{T}^{|V_G|-1} = \left\{(x_i)_{i\in V_G} \in \mathbb{P}^{|V_G|-1} : \textstyle\prod_{i\in V_G} x_i \neq 0\right\} \subseteq \mathbb{P}^{|V_G|-1}.$$

The vanishing ideal over G, for the finite field K, is, by definition, the vanishing ideal of this set. These ideals were defined and studied in [28] and, since then, appeared in the literature in [8, 9, 21, 24, 25, 26, 27, 30, 32]. We know that the vanishing ideal over a graph is a binomial ideal and that the quotient of $K[E_G]$ by it is a Cohen–Macaulay, reduced, one-dimensional graded ring. However the invariants of the vanishing ideal and its minimal generating sets are not independent of the field. For example, if G is a cycle of length 4 on the vertex set $V_G = \{1,2,3,4\}$, with K a field with q elements, then, by [26, Theorem 5.9], we know that the vanishing ideal over G is generated by:

$$\begin{aligned}
&t_{12}t_{34} - t_{23}t_{14},\\
&t_{12}^{q-1} - t_{14}^{q-1},\ t_{23}^{q-1} - t_{14}^{q-1},\ t_{34}^{q-1} - t_{14}^{q-1},\\
&t_{12}^{q-2}t_{23} - t_{34}t_{14}^{q-2},\ t_{12}^{q-3}t_{23}^{2} - t_{34}^{2}t_{14}^{q-3}, \dots,\ t_{12}t_{23}^{q-2} - t_{34}^{q-2}t_{14}\\
&t_{12}^{q-2}t_{14} - t_{34}t_{23}^{q-2},\ t_{12}^{q-3}t_{14}^{2} - t_{34}^{2}t_{23}^{q-3}, \dots,\ t_{12}t_{14}^{q-2} - t_{34}^{q-2}t_{23}.
\end{aligned}$$

By [26, Theorems 3.2 and 6.2], this ideal has degree $(q-1)^2$ and regularity $q-1$.

The link between the ideals $I(X_G)$ and the vanishing ideals occurs when $K=\mathbb{Z}_3$. As we show below, in this situation, both ideals coincide.

Proposition 2.9. *Assume that $K=\mathbb{Z}_3$. Then $I(X_G)$ is the vanishing ideal of the projective toric subset parameterized by G*

Proof: Since $I(X_G)$ is a homogeneous ideal, it will suffice to check that $I(X_G)$ and the vanishing ideal of the projective toric subset parameterized by G have the same homogeneous elements. Let $f\in K[E_G]$ be homogeneous. Then f vanishes on the projective toric subset parameterized by G if and only if $\theta(f)$ vanishes on $\mathbb{T}^{|V_G|-1}$. Now, if $K=\mathbb{Z}_3$, the vanishing ideal of $\mathbb{T}^{|V_G|-1}$ is

$$(x_i^2-x_j^2 : i,j\in V_G).$$

We deduce that $\theta(f)$ vanishes on $\mathbb{T}^{|V_G|-1}$ if and only if

$$\theta(f)\in (x_i^2-x_j^2 : i,j\in V_G)$$

which, by definition, is equivalent to $f\in I(X_G)$. ■

Remark 2.10. One can define a subset, $X_G\subseteq \mathbb{P}^{|E_G|-1}$ over any field K, by the image of a map ϑ, defined exactly as above, of a subset $D\subseteq \mathbb{P}^{|V_G|-1}$ consisting of all points in $\mathbb{P}^{|V_G|-1}$ with homogeneous coordinates in

$$\{-1,1\}\subseteq K.$$

One can then show that, if the characteristic of K is not equal to 2, $I(X_G)$ is the vanishing ideal of X_G. Note, however, that this is definitely not the case if the characteristic of K is equal to 2, as then D consists of a single point.

2.7. Degree of the ideal. The next result relates the degree of $I(X_G)$ with $|V_G|$ and $b_0(G)$, the number of connected components of G. Recall that we are assuming that G is a simple graph without isolated vertices.

Proposition 2.11. *The degree of $I(X_G)$ is $2^{|V_G|-b_0(G)}$, if G is non-bipartite, and it is $2^{|V_G|-b_0(G)-1}$ if G is bipartite.*

Proof: Fix a monomial order in $K[E_G]$. By Proposition 2.8 there exists a generating set for the initial ideal of $I(X_G)$ independent of the field. Hence the multiplicity degree of $K[E_G]/I(X_G)$ is independent of the field. Consider then $K=\mathbb{Z}_3$. By Proposition 2.9, $I(X_G)$ is the vanishing ideal over the graph G. The result now follows from [26, Theorem 3.2]. ■

3. Regularity

The Castelnuovo–Mumford regularity of a graded finitely generated module, M, over a polynomial ring is, by definition,

$$\operatorname{reg} M = \max_{i,j} \{j - i : \beta_{ij} \neq 0\},$$

where β_{ij} are the graded Betti numbers of M. The index of regularity of M is, by definition,

$$\operatorname{ir} M = \min \{k \in \mathbb{N} : \varphi_M(d) = P_M(d),\ \forall\, d \geq k\},$$

where φ_M and P_M are the Hilbert function and the Hilbert Polynomial of M, respectively. By [5, Corollary 4.8], if M is Cohen–Macaulay,

$$\operatorname{reg} M = \operatorname{ir} M + \dim M - 1,$$

and hence the Castelnuovo–Mumford regularity and the index of regularity of the module $K[E_G]/I(X_G)$ coincide. Since

$$\operatorname{reg} I(X_G) = \operatorname{reg} K[E_G]/I(X_G) + 1,$$

and since the Hilbert function of $K[E_G]/I(X_G)$ and the Hilbert function of the quotient of $K[E_G]$ by the initial ideal of $I(X_G)$ coincide, using Proposition 2.8, the proof of the following result is straightforward.

Proposition 3.1. *The regularity of $I(X_G)$ is independent of the field.*

Together with Proposition 2.9, this result enables the transfer to our setting of some results about the regularity of the vanishing ideal of a graph.

Proposition 3.2. *Let G be a simple graph without isolated vertices.*

(i) *If $H \subseteq G$ is a spanning subgraph with the same number of connected components such that either H and G are both non-bipartite or both bipartite, then $\operatorname{reg} I(X_G) \leq \operatorname{reg} I(X_H)$.*

(ii) *If G is bipartite and $H_1, \ldots, H_c$ are its blocks, then*

$$\operatorname{reg} I(X_G) = \textstyle\sum_{i=1}^{c} \operatorname{reg} I(X_{H_i}).$$

(iii) *If $G = \mathcal{K}_{a,b}$ is a complete bipartite graph, then $\operatorname{reg} I(X_G) = \max\{a, b\}$.*

(iv) *If G is bipartite and Hamiltonian, then $\operatorname{reg} I(X_G) = \frac{|V_G|}{2}$.*

(v) *If G is a forest, then $\operatorname{reg} I(X_G) = |V_G| - b_0(G)$.*

(vi) *If G has a single cycle and this cycle is odd, then*

$$\operatorname{reg} I(X_G) = |V_G| - b_0(G) + 1.$$

Proof: By Proposition 3.1, we may consider $K = \mathbb{Z}_3$, in which case, by Proposition 2.9, $I(X_G)$ is the vanishing ideal of X_G, the projective toric subset parameterized by G.

(i) Since $V_H = V_G$ and either H and G are both non-bipartite or both bipartite, we get, from Proposition 2.11, $|X_H| = |X_G|$. Using [32, Lemma 2.13],

$$\operatorname{reg} I(X_G) = K[E_G]/I(X_G) + 1 \leq \operatorname{reg} K[E_H]/I(X_H) + 1 = \operatorname{reg} I(X_H).$$

(ii) Using [27, Theorem 7.4],

$$\operatorname{reg} I(X_G) = \textstyle\sum_{i=1}^{c} \operatorname{reg} K[E_{H_i}]/I(X_{H_i}) + (c-1) + 1 = \sum_{i=1}^{c} \operatorname{reg} I(X_{H_i}).$$

(iii) Using [34, Corollary 5.1.9] and [8, Corollary 5.4],

$$\operatorname{reg} I(X_G) = \operatorname{reg} K[E_G]/I(X_G) + 1 = \max\{a, b\} - 1 + 1 = \max\{a, b\}.$$

(iv) If G is an even cycle then this follows from [26, Theorem 6.2]. Consider the general case. Then G is connected and its vertex set has even cardinality. Let a be equal to $\frac{|V_G|}{2}$ and let C denote an (even) Hamiltonian cycle. Then $C \subseteq G \subseteq \mathcal{K}_{a,a}$. By (i), (iii) and the even cycle case described before:

$$a \leq \operatorname{reg} I(X_G) \leq \tfrac{|V_G|}{2} = a.$$

(v) Suppose G is an edge. Then $I(X_G) = (0)$ and hence

$$\operatorname{reg} I(X_G) = 1 = |V_G| - 1.$$

If G is a forest, then G is bipartite and each edge is a block of G, hence, by (ii), we get $\operatorname{reg} I(X_G) = |E_G| = |V_G| - b_0(G)$.

(vi) If G has a single odd cycle then, from Proposition 2.11, we get

$$|X_G| = 2^{|V_G| - b_0(G)} = 2^{|E_G|-1}.$$

We deduce that X_G coincides with the projective torus, $\mathbb{T}^{|E_G|-1} \subseteq \mathbb{P}^{|E_G|-1}$ and, accordingly, $I(X_G) = (t_e^2 - t_f^2 : e, f \in E_G)$, which is a complete intersection of $|E_G| - 1$ binomials of degree two. The Hilbert series of the quotient $K[E_G]/I(X_G)$ is then equal to

$$\tfrac{(1-T^2)^{|E_G|-1}}{(1-T)^{|E_G|}}.$$

By [34, Corollary 5.1.9], $\operatorname{reg} I(X_G) = 2|E_G| - 2 - |E_G| + 2 = |V_G| - b_0(G) + 1$. ∎

Remark 3.3. If G is bipartite, it is straightforward from Proposition 3.2 that the regularity of $I(X_G)$ is additive on the connected components of G. This does not hold without the bipartite assumption. In fact, even if G is connected, additivity along the blocks of G does not hold without the bipartite assumption. A counter-example is given by the graph in Figure 1. The three blocks of the graph yield regularities 3 (twice) and 1. However, using [15, Macaulay2] one checks that the regularity of $I(X_G)$ is 4.

The next results reflect our approach to the computation of the regularity of $I(X_G)$ or, equivalently, the regularity of the quotient $K[E_G]/I(X_G)$. We will resort to an Artinian reduction of $K[E_G]/I(X_G)$, by quotienting the polynomial ring by the ideal generated by $I(X_G)$ and an arbitrary monomial $\mathbf{t}^\delta$ of degree d. As we saw in the proof of Proposition 2.1, $\mathbf{t}^\delta$ is $K[E_G]/I(X_G)$-regular and therefore multiplication by $\mathbf{t}^\alpha$ yields the following short exact sequence:

$$0 \to \frac{K[E_G]}{I(X_G)}[-d] \xrightarrow{\mathbf{t}^\delta} \frac{K[E_G]}{I(X_G)} \to \frac{K[E_G]}{(I(X_G), \mathbf{t}^\alpha)} \to 0. \tag{5}$$

Proposition 3.4. *Let $\mathbf{t}^\delta \in K[E_G]$ be a monomial of degree d. Then the quotient $K[E_G]/(I(X_G), \mathbf{t}^\delta)$ is zero in degree k if and only if*

$$k \geq \operatorname{reg} K[E_G]/I(X_G) + d.$$

Proof: Let φ denote the Hilbert function of $K[E_G]/I(X_G)$. Then the quotient $K[E_G]/(I(X_G), \mathbf{t}^\alpha)$ is zero in degree k if and only if $K[E_G]/(I(X_G), \mathbf{t}^\alpha)$ is zero in every degree $i \geq k$. By (5), this is equivalent to $\varphi(i-d) = \varphi(i)$, for every $i \geq k$, which holds if and only if φ is constant from $k-d$ and on, i.e., if and only if, $\operatorname{reg} K[E_G]/I(X_G) \leq k-d$. ∎

Proposition 3.5. *Let $\mathbf{t}^\delta \in K[E_G]$ be a monomial. Then, $\mathbf{t}^\alpha \in (I(X_G), \mathbf{t}^\delta)$ if and only if there exists a monomial $\mathbf{t}^\beta \in K[E_G]$ such that $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is homogeneous, belongs to $I(X_G)$, and $\mathbf{t}^\delta \mid \mathbf{t}^\beta$.*

Proof: Fix a monomial order on $K[E_G]$. Let $\mathscr{G}$ be a Gröbner basis of $I(X_G)$ obtained as in the proofs of Proposition 2.2 or Proposition 2.8. Then each element of $\mathscr{G}$ is a homogeneous binomial, $\mathbf{t}^\alpha - \mathbf{t}^\beta$. Furthermore, we may assume, without loss of generality, that $\operatorname{lt}(\mathbf{t}^\alpha - \mathbf{t}^\beta) = \mathbf{t}^\alpha$.

We claim that $(I(X_G), \mathbf{t}^\delta)$ has a Gröbner basis of the form $\mathscr{G} \cup \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_r}\}$, where, for each $i = 1, \ldots, r$, there exists $\mathbf{t}^{\beta_i} \in K[E_G]$ such that $\mathbf{t}^{\mu_i} - \mathbf{t}^{\beta_i}$ is homogeneous, belongs to $I(X_G)$ and $\mathbf{t}^\delta \mid \mathbf{t}^{\beta_i}$. To prove this claim it suffices to show that there exists an application of Buchberger's algorithm which, starting from $\mathscr{G} \cup \left\{\mathbf{t}^{\delta}\right\}$, produces in step i a set

$$\mathscr{G}_i = \mathscr{G} \cup \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_{i+1}}\}$$

with the stated properties. We prove this by induction on $i \geq 0$. If $i = 0$, the algorithm is in the initialization step and hence $\mathscr{G}_0 = \mathscr{G} \cup \left\{\mathbf{t}^{\delta}\right\}$. It suffices to set $\mu_1 = \beta_1 = \delta$. Now fix $i > 0$ and assume Buchberger's algorithm has not finished in the step $i - 1$. Then there is an S-polynomial which does not reduce to zero modulo $\mathscr{G}_{i-1} = \mathscr{G} \cup \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_i}\}$. Since the S-polynomial of two monomials is zero and the S-polynomial of two elements of $\mathscr{G}$ reduces to zero modulo $\mathscr{G}$, the S-polynomial in question must be of some $\mathbf{t}^{\mu_k} \in \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_i}\}$ and $\mathbf{t}^{\alpha} - \mathbf{t}^{\beta} \in \mathscr{G}$. Let us write

$$S(\mathbf{t}^{\mu_k}, \mathbf{t}^{\alpha} - \mathbf{t}^{\beta}) = \mathbf{t}^{\mu'}\mathbf{t}^{\mu_k} - \mathbf{t}^{\alpha'}(\mathbf{t}^{\alpha} - \mathbf{t}^{\beta}) = \mathbf{t}^{\alpha'}\mathbf{t}^{\beta}, \tag{6}$$

where $\mathbf{t}^{\mu'}$ and $\mathbf{t}^{\alpha'}$ are such that $\mathbf{t}^{\mu'}\mathbf{t}^{\mu_k} = \mathbf{t}^{\alpha'}\mathbf{t}^{\alpha} = \operatorname{lcm}(\mathbf{t}^{\mu_k}, \mathbf{t}^{\alpha})$. Let r be the remainder of $S(\mathbf{t}^{\mu_k}, \mathbf{t}^{\alpha} - \mathbf{t}^{\beta})$ in its standard expression with respect to $\mathscr{G}_{i-1}$. Since $S(\mathbf{t}^{\mu_k}, \mathbf{t}^{\alpha} - \mathbf{t}^{\beta})$ is a monomial and $r \neq 0$, to obtain r only division by the elements of $\mathscr{G}$ is carried out. Since division of a monomial by a binomial yields a monomial of the same degree, we deduce that $r = \mathbf{t}^{\mu_{i+1}}$, for some monomial $\mathbf{t}^{\mu_{i+1}} \in K[E_G]$ with $\deg(\mathbf{t}^{\mu_{i+1}}) = \deg(\mathbf{t}^{\alpha'}\mathbf{t}^{\beta})$, and that there exists $g \in I(X_G)$ such that

$$S(\mathbf{t}^{\mu_k}, \mathbf{t}^{\alpha} - \mathbf{t}^{\beta}) = g + r \iff \mathbf{t}^{\alpha'}\mathbf{t}^{\beta} = g + \mathbf{t}^{\mu_{i+1}} \iff \mathbf{t}^{\mu_{i+1}} - \mathbf{t}^{\alpha'}\mathbf{t}^{\beta} \in I(X_G).$$

Using (6) and the induction hypothesis,

$$\mathbf{t}^{\mu_{i+1}} - \mathbf{t}^{\mu'}\mathbf{t}^{\mu_k} \in I(X_G) \iff \mathbf{t}^{\mu_{i+1}} - \mathbf{t}^{\mu'}\mathbf{t}^{\beta_k} \in I(X_G),$$

where $\deg(\mathbf{t}^{\mu_k}) = \deg(\mathbf{t}^{\beta_k})$. As

$$\deg(\mathbf{t}^{\mu_{i+1}}) = \deg(\mathbf{t}^{\alpha'}\mathbf{t}^{\beta}) = \deg(\mathbf{t}^{\alpha'}\mathbf{t}^{\alpha}) = \deg(\mathbf{t}^{\mu'}\mathbf{t}^{\mu_k}) = \deg(\mathbf{t}^{\mu'}\mathbf{t}^{\beta_k}),$$

if we set $\beta_{i+1} = \mu' + \beta_k$, we see that $\mathscr{G}_i = \mathscr{G} \cup \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_{i+1}}\}$, obtained in this step, satisfies the properties of the claim.

Let us now use the Gröbner basis $\mathscr{G} \cup \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_r}\}$ of the ideal $(I(X_G), \mathbf{t}^{\delta})$ to prove this proposition. Let $\mathbf{t}^{\alpha} \in K[E_G]$ belong to this ideal. Then the remainder in its standard expression with respect to $\mathscr{G} \cup \{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_r}\}$ is zero. As the remainder of the division of $\mathbf{t}^{\alpha}$ by a binomial is a monomial of the same degree, the division of $\mathbf{t}^{\alpha}$ by the elements of the Gröbner basis finishes the first time we use an element of the set $\{\mathbf{t}^{\mu_1}, \ldots, \mathbf{t}^{\mu_r}\}$. This means that there exists $k \in \{1, \ldots, r\}$, $\mathbf{t}^{\alpha'} \in K[E_G]$ and $g \in I(X_G)$ such that

$$\mathbf{t}^{\alpha} = g + \mathbf{t}^{\alpha'}\mathbf{t}^{\mu_k} \iff \mathbf{t}^{\alpha} - \mathbf{t}^{\alpha'}\mathbf{t}^{\mu_k} \in I(X_G), \tag{7}$$

with $\mathbf{t}^{\alpha} - \mathbf{t}^{\alpha'}\mathbf{t}^{\mu_k}$ homogeneous. Let $\mathbf{t}^{\beta_k} \in K[E_G]$ be such that $\mathbf{t}^{\delta} \mid \mathbf{t}^{\beta_k}$ and such that $\mathbf{t}^{\mu_k} - \mathbf{t}^{\beta_k}$ is homogeneous and belongs to $I(X_G)$. Then, setting $\mathbf{t}^{\beta} = \mathbf{t}^{\alpha'}\mathbf{t}^{\beta_k}$, we see that $\mathbf{t}^{\alpha} - \mathbf{t}^{\beta}$ is homogeneous, by (7) that it belongs to $\mathbf{t}^{\alpha} - \mathbf{t}^{\beta} \in I(X_G)$, and that $\mathbf{t}^{\delta} \mid \mathbf{t}^{\beta}$. We have proved one implication in the statement of the proposition, the other is trivial. ■

4. Joins and ears of graphs

4.1. Regularity and the maximum vertex join number. We now derive the bounds for the regularity of $I(X_G)$ mentioned in the introduction of this article. The lower bound, which is the maximum vertex join number of the graph, gives the value of $\operatorname{reg} I(X_G)$ in the bipartite case. The proofs of this section rely on Propositions 3.4 and 3.5.

Definition 4.1. [7, 31] A *join* of a graph, G, is a set of edges, $J \subseteq E_G$, such that, for every circuit C in G, $|J \cap E_C| \leq |E_C|/2$.

Recall from Definition 1.2, that the maximum cardinality of a join of G is called the maximum vertex join number and is denoted by $\mu(G)$.

Theorem 4.2. $\operatorname{reg} I(X_G) \geq \mu(G)$.

Proof: Let J be a join of G and let us show that

$$\operatorname{reg} K[E_G]/I(X_G) \geq |J| - 1.$$

Fix $e \in J$. By Proposition 3.4 it suffices to show that there exists a monomial of degree $|J| - 1$ which does not belong to $(I(X_G), t_e)$. We will show that the product of variables corresponding to the edges in any subset of J that does not include e satisfies this property. We argue by induction. Starting with the base case, let $f \in J \setminus \{e\}$. Then, by Proposition 3.5,

$$t_f \in (I(X), t_e) \iff t_f - t_e \in I(X),$$

which, by Proposition 2.5 is not true.

Assume now that $J' \subseteq J \setminus \{e\}$ is a subset of k edges, with $k \geq 2$, and let $\mathbf{t}^{\alpha}$ be the product of all variables corresponding to edges of J'. We want to show that $\mathbf{t}^{\alpha} \notin (I(X_G), t_e)$. By the induction hypothesis, if $\mathbf{t}^{\gamma}$ is the product of variables corresponding to $k-1$ or fewer edges of $J \setminus \{e\}$ then $\mathbf{t}^{\gamma} \notin (I(X_G), t_e)$. We argue by contradiction. Suppose that $\mathbf{t}^\alpha \in (I(X_G), t_e)$. By Proposition 3.5, there exists a monomial $\mathbf{t}^\beta$ such that $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is homogeneous, $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G)$, and $t_e \mid \mathbf{t}^\beta$. Let $\mathbf{t}^\gamma$, $\mathbf{t}^\mu$ be such that $\mathbf{t}^\alpha = \mathbf{t}^\gamma \gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta)$ and $\mathbf{t}^\beta = \mathbf{t}^\mu \gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta)$. Since any monomial is a regular element of $K[E_G]/I(X_G)$, we get

$$\mathbf{t}^\gamma - \mathbf{t}^\mu \in I(X_G).$$

Since t_e still divides $\mathbf{t}^\mu$, we deduce that $\mathbf{t}^\gamma \in (I(X_G), t_e)$. But $\mathbf{t}^\gamma$ cannot be the product of fewer than k of $J \setminus \{e\}$, for otherwise we would have a contradiction with our induction hypothesis. Therefore, $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$.

Let H be the subgraph of G the edges of which correspond to variables occurring in $\mathbf{t}^\alpha$ or $\mathbf{t}^\beta$ raised to an odd power. Notice that $J' \subseteq E_H$. Then, by Proposition 2.5, every vertex of H has even degree. By [3, Theorem 1], we conclude that H decomposes into a union of edge disjoint cycles. Let $C_l \subseteq H \subseteq G$, for $l = 1, \ldots, r$, be the cycles satisfying $E_H = \sqcup_l E_{C_l}$. Since $J' \subseteq E_H$ and J' is a join, we get:

$$\deg(\mathbf{t}^\alpha) = |J'| = \textstyle\sum_l |J' \cap E_{C_l}| \leq \frac{1}{2} \sum_l |E_{C_l}| = \frac{1}{2}|E_H| \,. \tag{8}$$

But, as $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is homogeneous, we know that $|E_H| \leq 2\deg(\mathbf{t}^\alpha)$. By (8) this implies that $|E_H| = 2\deg(\mathbf{t}^\alpha)$ from which we deduce that all variables in $\mathbf{t}^\beta$ occur raised to 1 and that, therefore, E_H contains all edges corresponding to variables dividing $\mathbf{t}^\beta$. In particular, $e \in E_H$. Considering now the join $J' \cup \{e\} \subseteq J$, we get

$$\deg(\mathbf{t}^\alpha)+1 = |J' \cup \{e\}| = \textstyle\sum_l |(J' \cup \{e\}) \cap E_{C_l}| \leq \frac{1}{2} \sum_l |E_{C_l}| = \frac{1}{2}|E_H| = \deg(\mathbf{t}^\alpha),$$

which is a contradiction. We conclude that $\mathbf{t}^\alpha \notin (I(X_G), t_e)$, and, thus, finish the proof of the induction step. ■

Proposition 4.3.

$$\operatorname{reg} I(X_G) \leq \begin{cases} |V_G| - b_0(G), & \textit{if } G \textit{ is bipartite} \\ |V_G| - b_0(G) + 1, & \textit{if } G \textit{ is non-bipartite.} \end{cases} \tag{9}$$

Proof: Suppose G is bipartite. Let H be a subgraph of G consisting of a spanning tree for each connected component of G. Then, by Proposition 3.2,

$$\operatorname{reg} I(X_G) \leq \operatorname{reg} I(X_H) = |V_G| - b_0(G).$$

Suppose now that G is non-bipartite and take H, as before, given by a spanning tree for every connected component of G, except for one of the non-bipartite components in which we take a spanning connected graph containing a single odd cycle. Then H and G have the same number of connected components, they are both non-bipartite and H is a graph with a single odd cycle. According to Proposition 3.2,

$$\operatorname{reg} I(X_G) \leq \operatorname{reg} I(X_H) = |V_H| - b_0(H) - 1 = |V_G| - b_0(G) - 1. \quad \blacksquare$$

Remark 4.4. The graphs H in the proof of Proposition 4.3, a forest in the bipartite case and a graph with a unique odd cycle in the non-bipartite case, are examples of graphs for which the upper bounds (9) are attained. From this observation and the next theorem, we deduce that the bounds for $\operatorname{reg} I(X_G)$ are sharp.

Theorem 4.5. *If G is a bipartite graph, then* $\operatorname{reg} I(X_G) = \mu(G)$.

Proof: We must show that

$$\operatorname{reg} K[E_G]/I(X_G) \leq \mu(G) - 1. \tag{10}$$

Fix $t_e \in E_G$. According to Proposition 3.4, to prove (10) it suffices to show that any monomial $\mathbf{t}^\alpha \in K[E_G]$ of degree $\mu(G)$ belongs to $(I(X_G), t_e)$. If $t_e \mid \mathbf{t}^\alpha$, we are done. Assume that $t_e \nmid \mathbf{t}^\alpha$. Suppose now that $t_f^2 \mid \mathbf{t}^\alpha$, for some $f \neq e$. Then, setting $\mathbf{t}^\gamma = \mathbf{t}^\alpha / t_f^2$,

$$\mathbf{t}^\alpha = t_f^2 \mathbf{t}^\gamma - t_e^2 \mathbf{t}^\gamma + t_e^2 \mathbf{t}^\gamma = (t_f^2 - t_e^2)\mathbf{t}^\gamma + t_e^2 \mathbf{t}^\gamma \in (I(X_G), t_e).$$

Hence we may assume that $t_e \nmid \mathbf{t}^\alpha$ and that $t_f^2 \nmid \mathbf{t}^\alpha$ for all $f \in E_G$. Let us denote by H the subgraph of G on the set of edges corresponding to the variables occurring in the monomial $\mathbf{t}^\alpha$. Then

$$|\, E_H \cup \{e\} \,| = \mu(G) + 1$$

and hence $E_H \cup \{e\}$ is not a join of G. I.e., there exists a circuit, C, in G such that

$$|\, (E_H \cup \{e\}) \cap E_C \,| > \tfrac{|E_C|}{2}\,. \tag{11}$$

Since G is bipartite, C decomposes into an edge disjoint union of even cycles. Without loss of generality, we may then assume that C is an even cycle. We will consider two cases. In the first case, $e \in E_C$. Then

$$|E_H \cap E_C| \geq \tfrac{|E_C|}{2}.$$

Since, by Proposition 3.2, $\operatorname{reg} I(X_C) = \frac{|E_C|}{2}$, we get

$$\operatorname{reg} K[E_C]/I(X_C) \leq |E_H \cap E_C| - 1.$$

We deduce, by Proposition 3.4, that any monomial in $K[E_C]$ of degree $|E_H \cap E_C|$, belongs to the ideal $(I(X_C), t_e)$. Let $\mathbf{t}^\beta$ be the monomial in $K[E_C]$ given by the multiplication of the variables corresponding to the edges of $E_H \cap E_C$. Then, $\mathbf{t}^\beta \in (I(X_C), t_e)$. Since, by Corollary 2.7,

$$I(X_C) = I(X_G) \cap K[E_C]$$

and $\mathbf{t}^\beta | \, \mathbf{t}^\alpha$, we conclude that $\mathbf{t}^\alpha \in (I(X_G), t_e)$, as desired.
In the second case, $e \notin E_C$. Then, from (11), we get

$$|E_H \cap E_C| \geq \tfrac{|E_C|}{2} + 1 = \operatorname{reg} K[E_C]/I(X_C) + 2. \tag{12}$$

Consider the graph $C' = C \cup \{e\} \subseteq G$. If C and $\{e\}$ have two vertices in common, then C' is Hamiltonian and, by Proposition 3.2,

$$\operatorname{reg} K[E_{C'}]/I(X_{C'}) = \operatorname{reg} K[E_C]/I(X_C).$$

If C and $\{e\}$ have either no vertex in common or just one vertex in common, then they are blocks of C' and, by Proposition 3.2,

$$\operatorname{reg} K[E_{C'}]/I(X_{C'}) = \operatorname{reg} K[E_C]/I(X_C) + 1.$$

In both cases, using (12), we get:

$$|E_H \cap E_C| \geq \operatorname{reg} K[E_{C'}]/I(X_{C'}) + 1.$$

By Proposition 3.4, this implies that any monomial in $K[E_{C'}]$ of degree $|E_H \cap E_C|$, belongs to the ideal $(I(X_{C'}), t_e)$. Let $\mathbf{t}^\beta \mid \mathbf{t}^\alpha$ be the monomial given by the multiplication of the variables corresponding to the edges of $E_H \cap E_C$. Then

$$\mathbf{t}^\beta \in (I(X_{C'}), t_e) \subseteq (I(X_G), t_e),$$

and thus $\mathbf{t}^\alpha \in (I(X_G), t_e)$, as we wanted. ∎

Remark 4.6. Proposition 4.3 and Theorem 4.5 yield $\mu(G) \leq |V_G| - b_0(G)$, for any bipartite graph. A better bound can be achieved using [31, Corollary 3.5] and the additivity of $\operatorname{reg} I(X_G)$ along connected components. Let $G_1, \ldots, G_r$ be the connected components of a bipartite graph and, for each i, let c_i be the length of the longest circuit in G_i, i.e., the *circumference* of G_i. Then

$$\operatorname{reg} I(X_G) = \mu(G) \leq |V_G| - \textstyle\sum_i \frac{c_i}{2}.$$

4.2. Regularity and Nested Ear decompositions. The notion of ear decomposition of a graph is involved in Whitney's Theorem, which states that their existence is equivalent to the 2-connectedness of the graph. Let us recall the definition of ear decomposition.

Definition 4.7. An ear decomposition of G is of a collection of $r > 0$ subgraphs $\mathcal{P}_0$, $\mathcal{P}_1, \ldots, \mathcal{P}_r$, the edge sets of which form a partition of E_G, such that $\mathcal{P}_0$ is a vertex and, for all $1 \leq i \leq r$, $\mathcal{P}_i$ is a path with end-vertices in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$ and with *none* of its inner vertices in $\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{i-1}$.

The paths $\mathcal{P}_1, \ldots, \mathcal{P}_r$ are called *ears* of the decomposition. Their number, for distinct decompositions of a graph, does not change, as each new ear increases the genus of the construction by one. However the number of ears of even length, and therefore the number of odd length ears, may change, as we show in the following example.

Example 4.8. Consider the graph, G, in Figure 2. It is a Hamiltonian bipartite graph, with Hamiltonian cycle $(1, 2, 5, 4, 3, 6, 1)$. This cycle can be

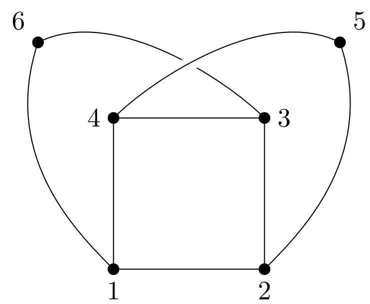

FIGURE 2. A Hamiltonian bipartite graph.

taken as the ear $\mathcal{P}_1$ of an ear decomposition starting from $\mathcal{P}_0 = 1$. The remaining ears are all edges, $\mathcal{P}_2 = \{2, 3\}$ and $\mathcal{P}_3 = \{1, 4\}$, for example, in this order. Another ear decomposition of G is given by:

$$\mathcal{P}_0 = 1, \quad \mathcal{P}_1 = (1, 2, 3, 4, 1), \quad \mathcal{P}_2 = (2, 5, 4), \quad \mathcal{P}_3 = (3, 6, 1).$$

Whereas the first ear decomposition has a single even length ear, the second has three.

Definition 4.9. The minimum number of even length ears in an ear decomposition of a graph G is denoted by $\varphi(G)$.

This definition was given in [7] and is related to Lòvasz characterization of factor-critical graphs. In [7, Theorem 4.5] it is shown that for a connected graph,

$$\mu(G) = \tfrac{\varphi(G)+|V_G|-1}{2}. \tag{13}$$

A subclass of the class of graphs endowed with an ear decomposition, i.e., by Whitney's Theorem, a subclass of the class of 2-connected graphs, consists of those graphs that admit a special type of ear decomposition, called *nested ear decomposition*. This definition was given in [4] where it was shown that this class consists of all two-terminal series parallel graphs. They are interesting because of the recent work [24] on the regularity of vanishing ideals. Let us recall the definition of nested ear decomposition.

Definition 4.10. Let $\mathcal{P}_0$, $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be an ear decomposition of a graph, G. If a path $\mathcal{P}_i$ has both its end-vertices in $\mathcal{P}_j$ we say that $\mathcal{P}_i$ is nested in $\mathcal{P}_j$ and we define the corresponding *nest interval* to be the subpath of $\mathcal{P}_j$ determined by the end-vertices of $\mathcal{P}_i$. An ear decomposition of G is *nested* if, for all $1 \leq i \leq r$, the path $\mathcal{P}_i$ is nested in a previous subgraph of the decomposition, $\mathcal{P}_j$, with $j < i$, and, in addition, if two paths $\mathcal{P}_i$ and $\mathcal{P}_l$ are nested in $\mathcal{P}_j$, then either the corresponding nest intervals in $\mathcal{P}_j$ have disjoint edge sets or one is contained in the other.

It is easy to construct graphs endowed with nested ear decompositions. One can check that none of the ear decompositions given in Example 4.8 is nested. In [24, Theorem 4.4] it is shown that the regularity of the vanishing ideal over a bipartite graph endowed with a nested ear decomposition with ϵ even ears is a function of $|V_G|$, ϵ and the order of the field. From this result we derive the following.

Proposition 4.11. *Let G be a bipartite graph endowed with nested ear decomposition with ϵ even ears. Then* $\operatorname{reg} I(X_G) = \frac{|V_G|+\epsilon-1}{2}$.

Proof: By Proposition 3.1 and Proposition 2.9 we may take the value of the regularity, given in [24, Theorem 4.4], for the field $K = \mathbb{Z}_3$, i.e., setting $q = 3$. To the result we add 1, since in [24] $\operatorname{reg} G$ is the regularity of the quotient of the polynomial ring by the vanishing ideal over G. ■

If follows from this result that for a bipartite graph endowed with a nested ear decomposition the number of ears of even length does not change among all nested ear decompositions of the graph. This conclusion was already drawn in [24, Corollary 4.5]. But now, by Theorem 4.5, we know that $\operatorname{reg} I(X_G) = \mu(G)$. This, together with (13), yields the following stronger result.

Corollary 4.12. *If G is a bipartite graph endowed with a nested ear decomposition, then the number of even length ears in any such decomposition coincides with $\varphi(G)$, the minimum number of even length ears in any ear decomposition of G.*

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