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A FREE BOUNDARY PROBLEM WITH FRACTIONAL DIFFUSION

DIEGO MARCON AND RAFAYEL TEYMURAZYAN

Abstract: We study a free boundary optimization problem for the fractional Laplacian with a volume constraint and a lower bound. We prove existence and optimal Hölder regularity of solutions, as well as derive geometric properties of solutions and of the corresponding exterior free boundary.

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1. Introduction

For a purely jump Lévy process, originated in a bounded domain $\Omega \subset \mathbb{R}^n$, the expected value of the function at the first exit point solves the non-local Dirichlet problem driven by the fractional Laplacian, for a prescribed (outside of the domain) "boundary data". The non-local nature of the operator requires the "boundary data" to be defined in the whole complement of Ω since, when exiting the domain, the jump can end up at any point of Ω^c.

When we minimize the corresponding energy functional, among functions that dictate a certain behavior of the process inside the domain, within an "insulation material" of a certain volume, we have to deal with an optimal design problem which is driven by a non-local operator. Such problems arise, for instance, in the study of best insulation devices.

If, instead of jump processes, one considers continuous processes, then optimal design problems actually arise from local operators, a variant of which can be stated as follows (see [10, 11]): with heating sources inside of a

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room, and with a fixed volume of an insulation material outside of the room, minimize the associated energy functional while keeping the temperature in the room above a given non-negative function (see the picture below).

In this work, we are interested in the non-local counterpart of such problems – minimizing the fractional energy under a volume constraint and a lower bound condition. More precisely, given a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, a smooth non-negative function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ which is compactly supported in Ω, and numbers $\alpha \in (0,1)$ and $\gamma > 0$, we search for a function $u : \mathbb{R}^n \to \mathbb{R}$ that minimizes the fractional energy functional

$$J(u) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \tag{P}$$

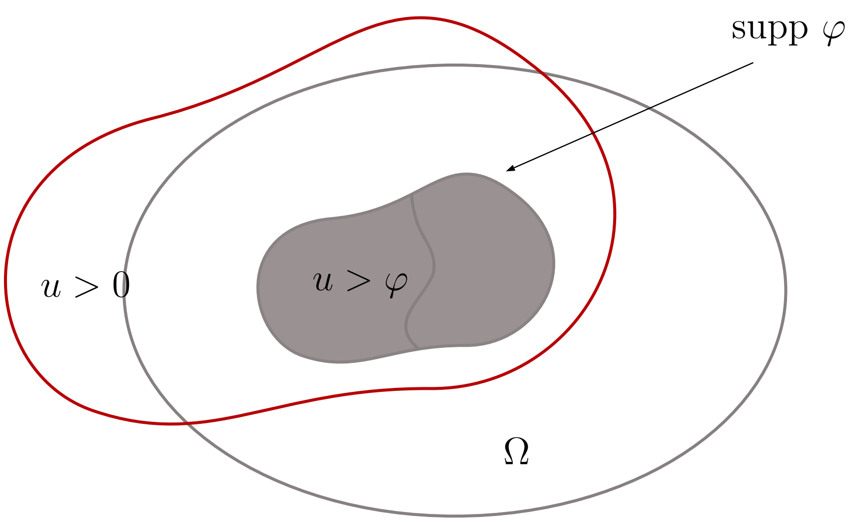
in the set $\mathbb{K}$ of functions $u \in H^\alpha(\mathbb{R}^n)$ for which

$$\begin{aligned} u &\geq \varphi, \\ (-\Delta)^\alpha u &\leq 0 \text{ in } \Omega, \\ (-\Delta)^\alpha u &= 0 \text{ in } \{u>0\}\setminus\Omega, \\ |\{u>0\}\setminus\Omega| &= \gamma. \end{aligned}$$

Here, $|E|$ denotes the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$ and $(-\Delta)^\alpha u$ is the fractional Laplacian, defined as follows:

$$(-\Delta)^\alpha u(x) := c_{n,\alpha}\ \mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}\,dy,$$

where PV is short for the Cauchy principal value of the integral, $c_{n,\alpha}$ is a normalization constant, and $\alpha \in (0,1)$.



Interior and exterior free boundaries.

We prove that solutions are locally non-degenerate and α-Hölder continuous (optimal regularity), and that the exterior free boundary, that is, the set $\partial(\{u > 0\})$, has finite $(n-1)$-dimensional Hausforff measure. Unlike the local case, we cannot infer a regularity information of the interior free boundary, that is, of the set $\partial(\{u > \varphi\} \cap \Omega)$, since, in Ω, the solution u cannot be interpreted as a solution of the usual fractional obstacle problem. This is due to the non-local nature of the fractional Laplacian. Nonetheless, in Ω, we obtain an interior Harnack inequality.

The study of best insulation devices was boosted by the seminal work [1] of Alt and Caffarelli, later followed by many others, relatively recent examples of which include [3, 4, 7, 8], for functionals generated by divergence type operators and [9] for the fractional Laplacian. In these model problems, the temperature along the walls of the room is often prescribed. If, instead, we consider a minimal temperature profile in the interior of the room, we change from a boundary value problem in the bounded set Ω into a problem in the entire space $\mathbb{R}^n$. Then, new challenges arise mainly from the fact that the operator changes the sign. To handle the case for the Laplacian operator, in [11], several perturbed problems were studied. The rough idea is that these perturbed problems have regular enough solutions which converge to a solution of the original problem. This approach was later used in [10] to study the problem for the infinity Laplacian operator (as a limit of solutions from the divergence structured p-Laplacian operator). Here, in order to treat the problem (P), we follow this idea of penalization, and we consider three perturbed problems. However, since we are dealing with a non-local operator, we cannot expect higher regularity, which forces some adjustments. Also, due to the non-local nature of the fractional Laplacian, unlike [10, 11], solutions of (P) inside the domain cannot solve the obstacle problem. This fact, in turn, does not allow to conclude the corresponding regularity result for the interior free boundary, as in [10, 11]. Nevertheless, we obtain an interior Harnack inequality.

The paper is organized as follows: we start, in Section 2, with the mathematical set-up of a three parameter penalization problem and prove existence and boundedness of its minimizers (Proposition 2.1). In Section 3, we prove uniform, in one of the parameters, estimates, which allow to reduce the problem to the study of a two parameter penalization functional (Corollary 3.1). In Section 4, we prove uniform Hölder estimates in one of the remaining two parameters (Theorem 4.1) - reducing the problem to the study of a single parameter minimization problem, which is studied in Section 5. We show that when this last parameter is small enough (but fixed), then solutions of the penalized problem turn into solutions of the original problem (Theorem 5.2). This, in turn, implies α-Hölder regularity, which is optimal, non-degeneracy, and positive density results (Theorem 5.3). We conclude the paper with an interior Harnack inequality (Theorem 5.4) and an exterior free boundary regularity result (Theorem 5.5).

2. Preliminaries

Recall the definition of the functional J from (P) and let $H^\alpha(\mathbb{R}^n)$ be the fractional Sobolev space of order $\alpha \in (0,1)$ with usual norm

$$\|u\|_{H^\alpha(\mathbb{R}^n)} = \left(\|u\|^2_{L^2(\mathbb{R}^n)} + J(u)\right)^{\frac{1}{2}}.$$

For three parameters $\sigma, \delta, \varepsilon \in (0,1)$, we introduce the following penalized functional

$$I_{\sigma,\delta,\varepsilon}(u) = J(u) + g_\sigma(u-\varphi) + f_\varepsilon\left(\int_{\Omega^c} h_\delta(u(x))\,\mathrm{d}x\right), \tag{2.1}$$

where

(i) the function $g_\sigma : \mathbb{R} \to \mathbb{R}$ is smooth, non-negative, decreasing, convex, and such that

$$g_\sigma(t) = \begin{cases} -\frac{1}{\sigma}(t + \frac{\sigma}{2}), & t \le -\sigma, \\ \text{smooth}, & -\sigma \le t \le 0, \\ 0, & t \ge 0. \end{cases}$$

(ii) the function $h_\delta : \mathbb{R} \to \mathbb{R}$ is continuous and vanishes on $(-\infty, 0]$, it is linear on $[0, \delta]$, and it equals 1 on $[\delta, +\infty)$;

(iii) the function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}(t - \gamma) & \text{for } t \ge \gamma, \\ \varepsilon(t - \gamma) & \text{for } t \le \gamma. \end{cases}$$

The term $g_\sigma(v - \varphi)$ penalizes functions that do not lie above φ, the term h_δ regularizes the map $u \mapsto |\{u > 0\} \setminus \Omega|$, and f_ε penalizes functions whose positivity set does not have the desired volume γ (see [10, 11]).

We start by the following existence result.

Proposition 2.1. *The functional $I_{\sigma,\delta,\varepsilon} : H^\alpha(\mathbb{R}^n) \longrightarrow \mathbb{R}$, given by (2.1), has a minimizer. Moreover, if u is a minimizer, then*

$$0 \le u \le \|\varphi\|_\infty. \tag{2.2}$$

Proof: First, we show the existence of solutions. Observe that φ itself is an admissible function, and so we have

$$I_{\sigma,\delta,\varepsilon}(\varphi) \le J(\varphi) =: M < \infty,$$

where the constant M is independent of the parameters $\sigma, \delta, \varepsilon$. Let $\{u_k\}$ be a minimizing sequence such that $I_{\sigma,\delta,\varepsilon}(u_k) \le M$. As $I_{\sigma,\delta,\varepsilon} \ge -\varepsilon\gamma$, the sequence $\{u_k\}$ is bounded in $H^\alpha(\mathbb{R}^n)$. Thus, we can extract a weakly converging subsequence in $H^\alpha(\mathbb{R}^n)$, which we still denote by $\{u_k\}$. If u is the weak limit, by the lower semicontinuity of J and the definitions of the auxiliary functions, we have

$$J(u) \le J(u_k),$$
$$g_\sigma(u - \varphi) = \lim_{k\to\infty} g_\sigma(u_k - \varphi),$$
$$f_\varepsilon\left(\int_{\Omega^c} h_\delta(u(x))\,\mathrm{d}x\right) = \lim_{k\to\infty} f_\varepsilon\left(\int_{\Omega^c} h_\delta(u_k(x))\,\mathrm{d}x\right).$$

Hence,

$$I_{\sigma,\delta,\varepsilon}(u) \le \liminf_{k\to\infty} I_{\sigma,\delta,\varepsilon}(u_k) = \inf_{w\in H^\alpha(\mathbb{R}^n)} I_{\sigma,\delta,\varepsilon}(w).$$

Therefore, u is a minimizer of $I_{\sigma,\delta,\varepsilon}$.

Next, we prove (2.2). In order to prove the upper bound, we define $v \in H^\alpha(\mathbb{R}^n)$ by

$$v := \begin{cases} u & \text{if } u > \|\varphi\|_\infty \\ \frac{1}{2}\left(u + \|\varphi\|_\infty\right) & \text{if } u \le \|\varphi\|_\infty. \end{cases}$$

Clearly, $v \ge u$. In particular, $v - \varphi \ge u - \varphi$ and, since g_σ is decreasing,

$$g_\sigma(v-\varphi) \le g_\sigma(u-\varphi).$$

On the other hand, as h_δ is non-decreasing, then

$$h_\delta(v) \le h_\delta(u).$$

Since u is a minimizer, we can estimate

$$\begin{aligned} 0 &\le I_{\sigma,\delta,\varepsilon}(v) - I_{\sigma,\delta,\varepsilon}(u) \le J(v) - J(u) \\ &= \int_{u>\|\varphi\|_\infty}\int_{u>\|\varphi\|_\infty} \frac{|v(x)-v(y)|^2 - |u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\ &= -\frac{3}{4}\int_{u>\|\varphi\|_\infty}\int_{u>\|\varphi\|_\infty} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Then, $|\{u > \|\varphi\|_\infty\}| = 0$, that is, $u \le \|\varphi\|_\infty$ almost everywhere.

It remains to show that $u \ge 0$. For this purpose, we set

$$w := \begin{cases} u & \text{if } u \ge 0 \\ \frac{u}{2} & \text{if } u < 0 \end{cases}.$$

As before, we have $w \ge u$ so that $g_\sigma(w-\varphi) \le g_\sigma(u-\varphi)$ and $h_\delta(w) \le h_\delta(u)$. Thus, estimating as above, we have

$$0 \le -\frac{3}{4}\int_{u<0}\int_{u<0} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y,$$

which implies $u \ge 0$ almost everywhere. ■

Remark 2.1. *Clearly, a minimizer u of $I_{\sigma,\delta,\varepsilon}$ satisfies the following Euler-Lagrange equation*

$$2\,(-\Delta)^\alpha u = g_\sigma'(u-\varphi) + f_\varepsilon'\left(\int_{\Omega^c} h_\delta(u(x))\,dx\right) h_\delta'(u)\chi_{\Omega^c},$$

where χ_E is the characteristic function of the set E.

We end this section by recalling the following result [5, Proposition 2.9] for future reference (see also [6]).

Proposition 2.2. *Let $u \in L^\infty(\mathbb{R}^n)$ be such that $w := (-\Delta)^\alpha u \in L^\infty(\mathbb{R}^n)$.*

- *If $2\alpha \leq 1$, then, for any $\lambda < 2\alpha$,*
$$\|u\|_{C^{0,\lambda}(\mathbb{R}^n)} \leq C\left(\|u\|_\infty + \|w\|_\infty\right).$$
- *If $2\alpha > 1$, then, for any $\lambda < 2\alpha - 1$,*
$$\|u\|_{C^{1,\lambda}(\mathbb{R}^n)} \leq C\left(\|u\|_\infty + \|w\|_\infty\right).$$

In both cases, the constant $C > 0$ depends only on n, λ, and α.

3. Uniform estimates

In this section, we prove estimates for minimizers of $I_{\sigma,\delta,\varepsilon}$ that are uniform in the parameter σ. These allow us to pass to the limit as $\sigma \to 0$.

Lemma 3.1. *If $u_{\sigma,\delta,\varepsilon}$ is a minimizer of $I_{\sigma,\delta,\varepsilon}$, then*
$$\|g'_\sigma(u_{\sigma,\delta,\varepsilon} - \varphi)\|_\infty \leq C\left(C(\varphi) + \frac{1}{\varepsilon\delta}\right), \tag{3.1}$$
where $C(\varphi) > 0$ is a constant depending only on φ, and $C > 0$ is a constant independent of δ, σ, and ε.

Proof: This follows from the Euler-Lagrange equation, taking into account Proposition 2.1, and by the fact that $|f'_\varepsilon| \leq \frac{1}{\varepsilon}$ and $|h'_\delta| \leq \frac{1}{\delta}$. More precisely, if $u = u_{\sigma,\delta,\varepsilon}$ is a minimizer of $I_{\sigma,\delta,\varepsilon}$, then from (2.1) one has
$$2\left(-\Delta\right)^\alpha u = g'_\sigma(u-\varphi) + f'_\varepsilon\left(\int_{\Omega^c} h_\delta(u(x))\,dx\right) h'_\delta(u)\chi_{\Omega^c}. \tag{3.2}$$
Set $\tilde{u} = u - \varphi$. Since φ is supported in Ω, then $u = \tilde{u}$ in Ω^c, and (3.2) can be rewritten as
$$2\left(-\Delta\right)^\alpha(\tilde{u}+\varphi) = g'_\sigma(\tilde{u}) + f'_\varepsilon\left(\int_{\Omega^c} h_\delta(\tilde{u})\,dx\right) h'_\delta(\tilde{u})\chi_{\Omega^c}. \tag{3.3}$$
Taking $[g'_\sigma(\tilde{u})]^k$ as a test function in (3.3), we obtain
$$\int 2\left(-\Delta\right)^\alpha(\tilde{u}+\varphi)[g'_\sigma(\tilde{u})]^k + [g'_\sigma(\tilde{u})]^{k+1}$$
$$+ f'_\varepsilon\left(\int_{\Omega^c} h_\delta(\tilde{u})\right) h'_\delta(\tilde{u})\chi_{\Omega^c}[g'_\sigma(\tilde{u})]^k = 0. \tag{3.4}$$

Recall that

$$\int_{\mathbb{R}^n} \zeta(x)\,(-\Delta)^{\alpha}\,\zeta(x)\,\mathrm{d}x = \|\zeta\|^2_{\dot{H}^s} \geq 0,\ \forall\,\zeta \in \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$. By linearity,

$$\int_{\mathbb{R}^n} (\xi - \eta)\,[(-\Delta)^{\alpha}\,\xi - (-\Delta)^{\alpha}\,\eta]\,\mathrm{d}x \geq 0,\ \ \forall \xi, \eta \in \mathcal{S}.$$

From (3.4), using the last inequality, we have

$$\int 2\,(-\Delta)^{\alpha}\,\varphi[g'_\sigma(\tilde{u})]^k + [g'_\sigma(\tilde{u})]^{k+1} + f'_\varepsilon\left(\int_{\Omega^c} h_\delta(\tilde{u})\right) h'_\delta(\tilde{u})\chi_{\Omega^c}[g'_\sigma(\tilde{u})]^k \geq 0.$$

or

$$-\int [g'_\sigma(\tilde{u})]^{k+1} \leq \int f'_\varepsilon\left(\int_{\Omega^c} h_\delta(\tilde{u})\right) h'_\delta(\tilde{u})\chi_{\Omega^c}[g'_\sigma(\tilde{u})]^k - 2[g'_\sigma(\tilde{u})]^k\,(-\Delta)^{\alpha}\,\varphi.$$

Note that $u \geq \varphi$ implies that $g'_\sigma(\tilde{u})$ is supported in Ω, and hence the last inequality leads to

$$\begin{aligned}\int_{\Omega} |g'_\sigma(\tilde{u})|^{k+1} &\leq \int_{\Omega}\left[f'_\varepsilon\left(\int_{\Omega^c} h_\delta(\tilde{u})\right) h'_\delta(\tilde{u})\chi_{\Omega^c}[g'_\sigma(\tilde{u})]^k - [g'_\sigma(\tilde{u})]^k\,(-\Delta)^{\alpha}\,\varphi\right] \\ &\leq \left[\int_{\Omega}\left|f'_\varepsilon\left(\int_{\Omega^c} h_\delta(\tilde{u})\right) h'_\delta(\tilde{u})\chi_{\Omega^c} + (-\Delta)^{\alpha}\,\varphi\right|^k\right]^{\frac{1}{k}}\left[\int_{\Omega} |g'_\sigma(\tilde{u})|^{k+1}\right]^{\frac{k}{k+1}}.\end{aligned}$$

Therefore,

$$\|g'_\sigma(\tilde{u})\|_{L^{k+1}(\Omega)} \leq \left(C(\varphi) + \frac{1}{\varepsilon\delta}\right)|\Omega|^{\frac{1}{k}}. \tag{3.5}$$

Here, we used the fact that if a function is smooth, then its fractional Laplacian is also smooth (see [5]). Since φ is compactly supported, then $(-\Delta)^{\alpha}\,\varphi$ is a smooth compactly supported function, thus bounded, and

$$|(-\Delta)^{\alpha}\,\varphi| \leq C(\varphi),$$

for a constant $C(\varphi) > 0$ depending only on φ. Letting $k \to +\infty$ in (3.5), we obtain (3.1). ■

As a consequence of Lemma 3.1, we have the following result.

Theorem 3.1. *Let $u_{\sigma,\delta,\varepsilon}$ be a minimizer of $I_{\sigma,\delta,\varepsilon}$.*

- *If $2\alpha \leq 1$, then, for any $\lambda < 2\alpha$,*

$$\|u_{\sigma,\delta,\varepsilon}\|_{C^{0,\lambda}(\mathbb{R}^n)} \leq C\left(C(\varphi) + \frac{1}{\varepsilon\delta}\right).$$

- *If $2\alpha > 1$, then, for any $\lambda < 2\alpha - 1$,*

$$\|u_{\sigma,\delta,\varepsilon}\|_{C^{1,\lambda}(\mathbb{R}^n)} \leq C\left(C(\varphi) + \frac{1}{\varepsilon\delta}\right).$$

In both cases, the constant $C > 0$ depends only on n, λ, and α.

Proof: Using (3.1), the right hand side in the Euler-Lagrange equation (see Remark 2.1) can be estimated uniformly in σ, that is,

$$|(-\Delta)^{\alpha} u_{\sigma,\delta,\varepsilon}| \leq C\left(C(\varphi) + \frac{1}{\varepsilon\delta}\right).$$

By Proposition 2.2, if $2\alpha \leq 1$ and $\lambda < 2\alpha$, one has

$$\|u_{\sigma,\delta,\varepsilon}\|_{C^{0,\lambda}} \leq C\left(\|u_{\sigma,\delta,\varepsilon}\|_{\infty} + C(\varphi) + \frac{1}{\varepsilon\delta}\right).$$

By taking into account (2.2), we obtain the first statement of the theorem. Similarly, the second part of the theorem again follows from Proposition 2.2. ■

Corollary 3.1. *Up to a subsequence, as $\sigma \to 0$, the function $u_{\sigma,\delta,\varepsilon}$ converges to a function $u_{\delta,\varepsilon}$ locally uniformly on $\mathbb{R}^n$ and weakly in $H^{\alpha}(\mathbb{R}^n)$. Moreover, $u_{\delta,\varepsilon} \geq \varphi$.*

Proof: The convergence follows immediately from the boundedness of $u_{\sigma,\delta,\varepsilon}$ in $H^{\alpha}(\mathbb{R}^n)$, Theorem 3.1, and the Arzelà-Ascoli Theorem. To show that $u_{\delta,\varepsilon} \geq \varphi$, take any $c > 0$ and any compact set $K \subset \mathbb{R}^n$. Then

$$\{u_{\delta,\varepsilon} - \varphi < -c\} \cap K \subset \{u_{\sigma,\delta,\varepsilon} - \varphi < -c/2\} \cap K \tag{3.6}$$

for sufficiently small $\sigma > 0$. On the other hand, by the construction of g_{σ} and inequality $I_{\sigma,\delta,\varepsilon}(u_{\sigma,\delta,\varepsilon}) \leq M$ (see Step 1 of the proof of Proposition 2.1), we have

$$\frac{c}{2\sigma}\,|\{u_{\sigma,\delta,\varepsilon} - \varphi < -c/2\} \cap K| \leq \int_{\mathbb{R}^n} g_{\sigma}(u_{\sigma,\delta,\varepsilon} - \varphi) \leq M < \infty.$$

This, together with (3.6), yields $|\{u_{\delta,\varepsilon} - \varphi < -c\} \cap K| = 0$, since otherwise we would have a contradiction in the last inequality once $\sigma > 0$ is small enough.

As the number $c > 0$ and the compact $K \subset \mathbb{R}^n$ are arbitrary, we conclude that $u_{\delta,\varepsilon} \geq \varphi$. ∎

4. Uniform Hölder regularity of solutions

The aim of this section is to pass to the limit as $\delta \to 0$, and derive uniform Hölder estimates for solutions. For this purpose, we need the following.

Lemma 4.1. *If $w \in H^\alpha(\mathbb{R}^n)$, $w \geq \varphi$ and $u_{\delta,\varepsilon}$ is as in Corollary 3.1, then*

$$\begin{aligned} 2J(w) + 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[w(x)-w(y)][u_{\delta,\varepsilon}(y)-u_{\delta,\varepsilon}(x)]}{|x-y|^{n+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\ + f'_\varepsilon\left(\int_{\Omega^c} h_\delta(u_{\delta,\varepsilon})\right)\int_{\Omega^c} h'_\delta(u_{\delta,\varepsilon})(w-u_{\delta,\varepsilon}) \geq 0. \end{aligned} \tag{4.1}$$

Proof: Since $u_{\sigma,\delta,\varepsilon}$ is a minimizer of $I_{\sigma,\delta,\varepsilon}$, the function

$$F(t) := I_{\sigma,\delta,\varepsilon}(u_{\sigma,\delta,\varepsilon} + t(w - u_{\sigma,\delta,\varepsilon}))$$

has a minimum at $t = 0$ and so $F'(t) \geq 0$. Thus,

$$2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u_{\sigma,\delta,\varepsilon}(x)-u_{\sigma,\delta,\varepsilon}(y))\left([w(x)-w(y)]-[u_{\sigma,\delta,\varepsilon}(x)-u_{\sigma,\delta,\varepsilon}(y)]\right)}{|x-y|^{n+2\alpha}}$$
$$+g'_\sigma(u-\varphi)(w-u_{\sigma,\delta,\varepsilon}) + f'_\varepsilon\left(\int_{\Omega^c} h_\delta(u_{\sigma,\delta,\varepsilon})\right)\int_{\Omega^c} h'_\delta(u_{\sigma,\delta,\varepsilon})(w-u_{\sigma,\delta,\varepsilon}) \geq 0,$$

which, by the monotonicity of g'_σ (recall that g_σ is convex) and the elementary inequality $A(B-A) \leq B(B-A)$ for any numbers A and B, yields:

$$\begin{aligned} 2J(w) + 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[w(x)-w(y)][u_{\sigma,\delta,\varepsilon}(y)-u_{\sigma,\delta,\varepsilon}(x)]}{|x-y|^{n+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\ + g'_\sigma(w-\varphi)(w-u_{\sigma,\delta,\varepsilon}) \\ + f'_\varepsilon\left(\int_{\Omega^c} h_\delta(u_{\sigma,\delta,\varepsilon})\right)\int_{\Omega^c} h'_\delta(u_{\sigma,\delta,\varepsilon})(w-u_{\sigma,\delta,\varepsilon}) \geq 0. \end{aligned} \tag{4.2}$$

As for $w \geq \varphi$ one has $g'_\sigma(w-\varphi) = 0$. We can pass to the limit, as $\sigma \to 0$, in the last term of (4.2), as in [10, 11, Proof of Lemma 4.1]. ∎

Corollary 4.1. *The function $u_{\delta,\varepsilon}$ satisfies*

$$2(-\Delta)^\alpha u_{\delta,\varepsilon} = f'_\varepsilon\left(\int_{\Omega^c} h_\delta(u_{\delta,\varepsilon})\right) h'_\delta(u_{\delta,\varepsilon})\chi_{\Omega^c} \tag{4.3}$$

in $\{u_{\delta,\varepsilon} > \varphi\}$ *and*

$$-C\left(C(\varphi)+\frac{1}{\varepsilon\delta}\right) \leq 2(-\Delta)^\alpha u_{\delta,\varepsilon} \leq f'_\varepsilon\left(\int_{\Omega^c} h_\delta(u_{\delta,\varepsilon})\right) h'_\delta(u_{\delta,\varepsilon})\chi_{\Omega^c}. \tag{4.4}$$

Proof: Remark 2.1 provides (4.3). The first inequality of (4.4) follows from (3.1); the second, from (4.1). ■

To pass to the limit, as $\delta \to 0$, we need uniform in δ estimates.

Theorem 4.1. *There exists a constant* $C > 0$ *such that*

$$\|u_{\delta,\varepsilon}\|_{C^{0,\alpha}} \leq C\left((1+\delta)C(\varphi)+\frac{1}{\varepsilon}\right).$$

Proof: We define

$$v(x) := u_{\delta,\varepsilon}(\delta^{\frac{1}{n-1+2\alpha}}x)$$

and notice that

$$(-\Delta)^\alpha v(x) = \delta(-\Delta)^\alpha u(\delta^{\frac{1}{n-1+2\alpha}}x).$$

Recall $f'_\varepsilon \leq 1/\varepsilon$ and $h'_\delta \leq 1/\delta$. From (4.4), we deduce

$$-C\left(\frac{\delta C(\varphi)}{2}+\frac{1}{2\varepsilon}\right) \leq (-\Delta)^\alpha v \leq \frac{1}{2\varepsilon},$$

which together with Proposition 2.2 and Proposition 2.1 provides the desired result. ■

As a consequence of Proposition 2.1, the Arzelà-Ascoli Theorem and Theorem 4.1, we obtain the next result.

Corollary 4.2. *If* $u_{\sigma,\delta,\varepsilon}$ *is a minimizer of* $J_{\sigma,\delta,\varepsilon}$, *then* $u_{\sigma,\delta,\varepsilon}$ *converges weakly (up to a subsequence as* $\sigma,\delta \to 0$*) in* $H^\alpha(\mathbb{R}^n)$ *to a function* u_ε. *This convergence is locally uniform. Moreover, there exists a constant* $C > 0$ *such that*

$$\|u_\varepsilon\|_{C^{0,\alpha}} \leq C\left(C(\varphi)+\frac{1}{\varepsilon}\right).$$

5. Back to the original problem

Here we show that the function u_ε from Corollary 4.2 is a minimizer for a certain functional. This, in turn, provides information on the regularity of the exterior free boundary. Furthermore, we show that for $\varepsilon > 0$ small enough (but fixed), the desired volume is attained automatically, which means that solutions of the penalized problems turn into solutions to our original problem inheriting all the properties.

Theorem 5.1. *The function u_ε from Corollary 4.2 is a local minimizer of*

$$J_\varepsilon(u) := J(u) + f_\varepsilon(|\{u>0\} \setminus \Omega|)$$

over the functions in $H^\alpha(\mathbb{R}^n)$ which lie above φ.

Proof: We argue by contradiction and assume that $\inf J_\varepsilon < J_\varepsilon(u_\varepsilon)$. Hence, for given $\theta > 0$ there exists $v \in H^\alpha(\mathbb{R}^n)$ with $v \geq \varphi$ such that $J_\varepsilon(v) < J_\varepsilon(u_\varepsilon) - 2\theta$. Since $J(u)$ and $J(v)$ are finite, then

$$\int_{B_r^c}\int_{B_r^c}\left(\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2\alpha}} - \frac{|v(x)-v(y)|^2}{|x-y|^{n+2\alpha}}\right) < \frac{\theta}{2},$$

when $r>0$ is big enough. Also, we note that both $\{u>0\}\setminus\Omega$ and $\{v>0\}\setminus\Omega$ have finite measure, since otherwise f_ε would be infinity on the corresponding function. For $r>0$ sufficiently big, the sets $\{u>0\}\cap B_r^c \setminus \Omega$ and $\{v>0\}\cap B_r^c\setminus\Omega$ have arbitrarily small volume. The continuity of f_ε implies

$$|f_\varepsilon\left(|\{u>0\}\cap B_r^c\setminus\Omega|\right) - f_\varepsilon\left(|\{v>0\}\cap B_r^c\setminus\Omega|\right)| < \frac{\theta}{2}.$$

Therefore,

$$\begin{aligned}
&\int_{B_r}\int_{B_r}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Omega^c\cap B_r}\chi_{\{v>0\}}\right)\\
&< \int_{B_r}\int_{B_r}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Omega^c\cap B_r}\chi_{\{u_\varepsilon>0\}}\right) - \theta.
\end{aligned} \tag{5.1}$$

Since $h_\delta(v) \to \chi_{\{v>0\}}$, as $\delta\to 0$, and $g_\sigma(v-\varphi)=0$, then

$$\begin{aligned}
&\int_{B_r}\int_{B_r}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Omega^c\cap B_r}\chi_{\{v>0\}}\right)\\
&= \int_{B_r}\int_{B_r}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2\alpha}} + g_\sigma(v-\varphi) + \lim_{\delta\to 0} f_\varepsilon\left(\int_{\Omega^c\cap B_r} h_\delta(v)\right).
\end{aligned} \tag{5.2}$$

Note that if $\tau > 0$ is small, then $h_\delta(u_\delta) = \chi_{\{u_\delta>0\}}$ on $\{u_\varepsilon > \tau\}$ for δ small. Denoting $\Gamma := \Omega^c \cap B_r \cap \{u_\varepsilon \geq \tau\}$ and using Fatou lemma we then estimate

$$\begin{aligned}
&\int_{B_r}\int_{B_r} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Omega^c\cap B_r} \chi_{\{u_\varepsilon>0\}}\right) - \theta \\
&\leq \int_{B_r}\int_{B_r} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Gamma} \chi_{\{u_\varepsilon>0\}}\right) - \frac{\theta}{2} \\
&\leq \liminf_{\delta\to 0}\left[\int_{B_r}\int_{B_r} \frac{|u_{\delta,\varepsilon}(x)-u_{\delta,\varepsilon}(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Gamma} \chi_{\{u_{\delta,\varepsilon}>0\}}\right)\right] - \frac{\theta}{2} \\
&\leq \liminf_{\delta\to 0}\left[\int_{B_r}\int_{B_r} \frac{|u_{\delta,\varepsilon}(x)-u_{\delta,\varepsilon}(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Gamma} h_\delta(u_{\delta,\varepsilon})\right)\right] - \frac{\theta}{2} \qquad (5.3)\\
&\leq \liminf_{\sigma,\delta\to 0}\left[\int_{B_r}\int_{B_r} \frac{|u_{\sigma,\delta,\varepsilon}(x)-u_{\sigma,\delta,\varepsilon}(y)|^2}{|x-y|^{n+2\alpha}} + f_\varepsilon\left(\int_{\Gamma} h_\delta(u_{\sigma,\delta,\varepsilon})\right)\right] - \frac{\theta}{2} \\
&\leq \liminf_{\sigma,\delta\to 0}\left[\int_{B_r}\int_{B_r} \frac{|u_{\sigma,\delta,\varepsilon}(x)-u_{\sigma,\delta,\varepsilon}(y)|^2}{|x-y|^{n+2\alpha}} + g_\sigma(u_{\sigma,\delta,\varepsilon}-\varphi)\right. \\
&\qquad \left. + f_\varepsilon\left(\int_{\Omega^c\cap B_r} h_\delta(u_{\sigma,\delta,\varepsilon})\right)\right] - \frac{\theta}{2}.
\end{aligned}$$

From (5.1)-(5.3), we obtain

$$J_{\sigma,\delta,\varepsilon}(v) < J_{\sigma,\delta,\varepsilon}(u_{\sigma,\delta,\varepsilon}) - \frac{\theta}{4} < J_{\sigma,\delta,\varepsilon}(u_{\sigma,\delta,\varepsilon}),$$

which is a contradiction, since $u_{\sigma,\delta,\varepsilon}$ is a minimizer of $J_{\sigma,\delta,\varepsilon}$. ■

Corollary 5.1. *The Euler-Lagrange equation for u_ε is*

$$\begin{cases}
(-\Delta)^\alpha u_\varepsilon \leq 0 & \text{in } \Omega, \\
(-\Delta)^\alpha u_\varepsilon = 0 & \text{in } \Omega \cap \{u_\varepsilon > \varphi\}, \\
(-\Delta)^\alpha u_\varepsilon \geq 0 & \text{in } \Omega^c, \\
(-\Delta)^\alpha u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \setminus \Omega.
\end{cases}$$

The previous theorem puts us in the framework of [9], where the authors analyze properties of minimizers of J_ε. This leads to our main result, stated below.

Theorem 5.2. *For $\varepsilon > 0$ small, the function u_ε from Corollary 4.2, solves the problem (P). Moreover, u_ε is Hölder continuous with exponent α, and that regularity is optimal.*

Proof: As observed in [9, Theorem 5.1], when $\varepsilon > 0$ is small (but fixed), then $|\{u_\varepsilon > 0\} \setminus \Omega| = \gamma$. The latter implies that (see Corollary 5.1) $u_\varepsilon \in \mathbb{K}$ and additionally that $f_\varepsilon(|\{u_\varepsilon > 0\} \setminus \Omega|) = 0$. Therefore, the function u_ε solves (P). In other words, for $\varepsilon > 0$ small enough we have the desired volume, and minimizers of J_ε turn into minimizers of J, i.e., solutions of the original problem. The α-Hölder regularity of u_ε is observed in Corollary 4.2, and it is optimal, [9, Theorem 2.1]. ∎

Theorem 5.2 implies non-degeneracy and positive density results for solutions, [9, Lemma 2.2] and [9, Theorem 2.3] respectively, as stated in the next theorem.

Theorem 5.3. *If u is a solution of* (P)*, and $x_0 \in \partial\{u > 0\} \cap \Omega$, then there exists a constant $C > 0$ such that*

$$\sup_{B_r(x_0)} u \geq C r^\alpha,$$

for $0 < r < \frac{1}{2} dist(x_0, \partial\Omega)$. Furthermore, there extists a constant $c > 0$ such that

$$|\{u = 0\} \cap B_r(x_0)| \geq c r^n \quad and \quad |\{u > 0\} \cap B_r(x_0)| \geq c r^n.$$

Note that unlike [10, 11], the function u_ε from Corollary 4.2, which solves (P), does not solve the obstacle problem in Ω due to the non-local nature of the fractional Laplacian (see [5]). Hence, we cannot infer interior free boundary regularity from that of the obstacle problem. Nevertheless, in Ω we have the following interior Harnack inequality, as well as free boundary regularity result from [9, Theorem 3.1], concerning the exterior free boundary.

Theorem 5.4. *If u is a minimizer of* (P) *and $D' \subset\subset D$, where*

$$D := [\Omega \cap \{u > \varphi\}] \cup [\{u > 0\} \setminus \Omega],$$

then there exists a $C > 0$ constant, depending only on D', D and α, such that

$$\sup_{D'} u \leq C \inf_{D'} u.$$

Proof: From Corollary 5.1 and Theorem 5.2 we conclude that $u > 0$ is a fractional harmonic function in D. This implies the interior Harnack inequality (see [6, Theorem 10]). ∎

Theorem 5.5. *If u is a minimizer of* (P)*, then*

- $\mathcal{H}^{n-1}(\mathcal{K} \cap \partial\{u > 0\} \cap \mathbb{R}^n) < \infty$, *for every compact set* $\mathcal{K} \subset \Omega$.
- *The reduced free boundary* $\partial^*\{u > 0\} \cap \mathbb{R}^n$ *is locally a* $C^{1,\beta}$ *surface.*

Proof: This follows from Theorem 5.1, since it puts one in the framework of [9], where the result is true (see [9, Theorem 3.1]). ■

Remark 5.1. *As in* [11, Theorem 6.4] *(see also* [10, Lemma 6.2]*), the positivity set is well localized in a bounded set, meaning that the optimization is in fact in a big (but bounded) domain rather than the whole space.*

References

[1] H.W. Alt and L. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. 325 (1981), 105-144.

[2] L. Caffarelli, S. Salsa and L. Silvestre, *Reuglarity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. 171 (2008), 425-461.

[3] J.F. Bonder, S. Martínez and N. Wolanski, *An optimization problem with volume constraint for a degenerate quasilinear operator*, J. Differential Equations 227 (2006), 80-101.

[4] K. Oliveira and E. Teixeira, *An optimization problem with free boundary governed by a degenerate quasilinear operator*, Differential Integral Equations 19 (2006), 1061-1080.

[5] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 60 (2007), 67-112.

[6] P.R. Stinga, *User's guide to the fractional Laplacian and the method of semigroups*, In Anatoly Kochubei, Yuri Luchko (Eds.), Fractional Differential Equations (2019), 235-266, Berlin, Boston: De Gruyter.

[7] E. Teixeira, *The nonlinear optimization problem in heat conduction*, Calc. Var. Partial Differential Equations 24 (2005), 21-46.

[8] E. Teixeira, *Optimal design problems in rough inhomogeneous media. Existence theory*, Amer. J. Math. 132 (2010), 1445-1492.

[9] E. Teixeira and R. Teymurazyan, *Optimal design problems with fractional diffusions*, J. Lond. Math. Soc. (2) 92 (2015), 338-352.

[10] R. Teymurazyan and J.M. Urbano, *A free boundary optimization problem for the ∞-Laplacian*, J. Differential Equations 263 (2017), 1140-1159.

[11] H. Yu, *An optimization problem in heat conduction with minimal temperature constraint, interior heating and exterior insulation*, Calc. Var. Partial Differential Equations 55 (2016), Art. 130, 15.

Diego Marcon
Departamento de Matemática Pura e Aplicada, Universidade Federal do Rio Grande do Sul, Porto Alegre - RS, Brazil.
E-mail address: diego.marcon@ufrgs.br

Rafayel Teymurazyan
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: rafayel@utexas.edu