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REPRESENTATION TYPE OF BOREL-SCHUR ALGEBRAS

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Abstract: In our previous work [8], we found all Borel-Schur algebra of finite representation type. In the present article, we determine which Borel-Schur algebras of infinite representation type are tame, and which are wild.

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Borel-Schur algebras occur as subalgebras of Schur algebras. They were introduced by J. A. Green in [13]. A main result, or even the motivation, of that work, is that the Schur algebra has a triangular decomposition with factors an upper and a lower Borel-Schur algebra.

Borel-Schur algebras have shown to be a powerful tool for the study of projective resolutions of Weyl modules for the general linear group [22, 23]. More recently they played a crucial role in the work of the last two authors [20] on this problem. Also, in the same paper, Borel-Schur algebras were used to prove the Boltje-Hartmann conjecture [2] on permutational resolutions of (co-)Specht modules.

We fix an infinite field $\mathbb{K}$. The Schur algebra $S(n, r)$ is the $\mathbb{K}$-algebra whose module category is equivalent to the category of r-homogeneous polynomial representations of the general linear group $\mathrm{GL}_n(\mathbb{K})$. This Schur algebra is finite-dimensional, and it has an explicit subalgebra $S^+(n, r)$, the (upper) Borel-Schur algebra, whose module category is equivalent to the category of r-homogeneous polynomial representations of B^+, the group of upper triangular matrices in $\mathrm{GL}_n(\mathbb{K})$. The parameters of $S^+(n, r)$ which we have to take into account are n, r, and, in addition, the characteristic of the field $\mathbb{K}$.

Borel-Schur algebras are basic. They have finite global dimension and a highest weight theory. There is an explicit formula for the multiplication,

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but except for small cases, this is not easy to use. In [19], the second author determined the Ext quiver of a Borel-Schur algebra. She also proved some results useful for the construction of almost split sequences of simple modules. More recently this was continued in [8], where it was also determined precisely which Borel-Schur algebras are of finite representation type. The answer is:

Theorem 0.1 ([8]). *Consider the Borel-Schur algebra $S^+(n,r)$ over an algebraically closed field $\mathbb{K}$. Then $S^+(n,r)$ has finite representation type if and only if*

1) $n = 2$ and one of the following alternatives holds:
(a) $\operatorname{char}(\mathbb{K}) = 0$*;*
(b) $\operatorname{char}(\mathbb{K}) = 2$ *and* $r \leq 3$*;*
(c) $\operatorname{char}(\mathbb{K}) = 3$ *and* $r \leq 4$*;*
(d) $\operatorname{char}(\mathbb{K}) = p \geq 5$ *and* $r \leq p$*;*
2) $n \geq 3$ and $r = 1$.

This leaves to identify when a Borel-Schur algebra has tame representation type, and this is answered completely in this paper. Our main result is as follows.

Theorem 0.2. *Consider the Borel-Schur algebra $S^+(n,r)$ over an algebraically closed field $\mathbb{K}$. Suppose that $S^+(n,r)$ is of infinite type. Then $S^+(n,r)$ is tame if*

(a) $n = 2$, $\operatorname{char}\mathbb{K} = 3$*, and $r = 5$;*
(b) $n = 3$ and $r = 2$.

Otherwise $S^+(n,r)$ is wild.

The above results can be visualized as follows:

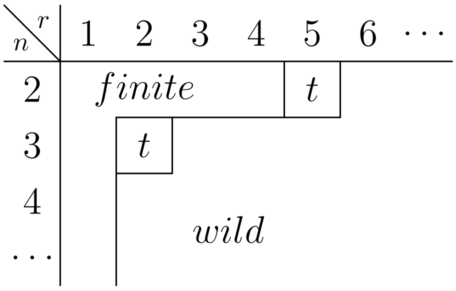
$\operatorname{char}\mathbb{K} = 0$

$n \backslash r$	1	2	3	$\cdots$
2	*finite*			
3		t		
4			*wild*	
$\cdots$				

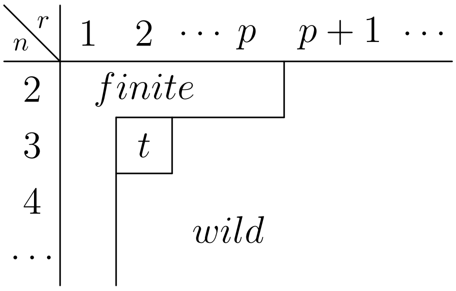
$\operatorname{char}\mathbb{K} = 2$

$n \backslash r$	1	2	3	4	$\cdots$
2	*finite*				
3		t			
4				*wild*	
$\cdots$					

$\operatorname{char} \mathbb{K} = 3$



$\operatorname{char} \mathbb{K} = p \geq 5$



To prove Theorem 0.2, we reduce the problem by idempotent methods: one has to show that only few algebras are wild, and that the algebras listed in the theorem are tame. To prove that an algebra is wild can be done by relating its module category to that of some known wild algebra. Our main method is based on coverings (see Section 2).

This leaves to prove that the two remaining algebras are tame. We show that $S^+(3,2)$ degenerates to a special biserial algebra, which is known to be tame. Then a result from [12] implies that the algebra $S^+(3,2)$ is tame. Our proof works for arbitrary characteristic, although the algebra structure for characteristic 2 is different.

Our proof that $S^+(2,5)$ in characteristic 3 is tame is very different from the proof of the previous case. This is done by exploiting representation theory of posets, and using the fact that the representation type of posets is completely understood. To follow this route, it is crucial that $S^+(2,r)$ is a one-point extension of $S^+(2,r-1)$ (see Section 6 for details).

In general, given a one-point extension $A[M] = \begin{pmatrix} A & M \\ 0 & \mathbb{K} \end{pmatrix}$, if A has finite type and M is suitable, one can construct a finite poset from the Auslander-Reiten quiver of A. Moreover, the representation type of this poset is the same as the representation type of the algebra $A[M]$. We use this when $A = S^+(2,4)$, $A[M] = S^+(2,5)$, and the characteristic of the base field is 3. It was proved in [8] that the algebra $S^+(2,4)$ has finite type by explicitly computing its Auslander-Reiten quiver. Now we take this quiver, compute the relevant poset, and then prove that it has tame type (see Section 6).

We note that the representation type of Schur algebras $S(n,r)$ (and also of their q-analogs) has been classified, but the methods used are different (see [3] and [7]), and also as far as we can see there is no connection with the techniques we use in the present article. This makes the problem we describe next rather intriguing. Besides Borel-Schur algebras, Green defined in [13] a subalgebra $S(G, r)$ of $S(n, r)$ for every subgroup G of $\mathrm{GL}_n(\mathbb{K})$. This subalgebra coincides with $S^+(n, r)$ if $G = B^+$ and with $S(n, r)$ if $G = \mathrm{GL}_n(\mathbb{K})$. Of course, B^+ and $\mathrm{GL}_n(\mathbb{K})$ are extremal elements of the family of parabolic subgroups P_λ in $\mathrm{GL}_n(\mathbb{K})$. It would be interesting to determine the representation type of the algebras $S(P_\lambda, r)$ with arbitrary P_λ.

The paper is organized as follows. In Section 2 we introduce the techniques we will use on the study of wild type. Section 3 is dedicated to Borel-Schur algebras. We introduce basic facts and prove that for every positive integer $m \leq n$ and $s \leq r$ there is an idempotent e in $S^+(n, r)$ such that $eS^+(n, r)e \cong S^+(m, s)$. This result will be crucial for our classification. In Section 4 we classify the Borel-Schur algebras of wild representation type using the covering techniques described in Section 2. Using degeneration techniques due to Gabriel [10] and Geiss [12], we prove in Section 5 that $S^+(3, 2)$ is tame. Section 6 contains the proof that $S^+(2, 5)$ is tame over fields of characteristic 3. As was mentioned above this is done using representation theory of posets.

For background on Schur algebras and Borel-Schur algebras we refer to [14] and [13]. Background on representation theory of algebras can be found in [18], or other text books.

We assume throughout that the field $\mathbb{K}$ is algebraically closed, and that all quivers are finite.

1. Preliminaries on wild representation type

Let A be a finite dimensional algebra over $\mathbb{K}$. We will write A-mod for the category of finite dimensional left A-modules. The algebra A is said to have *finite representation type* if there are only finitely many isomorphism classes of finite dimensional indecomposable modules. Otherwise A has *infinite representation type.* The famous Drozd Dichotomy Theorem, proved in [4], divides algebras of infinite representation type into two mutually exclusive classes: algebras of *tame type* and algebras of *wild type.* The algebra A is tame if it has infinite type and, for every dimension $d \geq 0$, all, but a finite number of, isomorphism classes of indecomposable A-modules of dimension d can be parametrised by a finite number of 1-parameter families.

To define wild we need a further notion.

Given another $\mathbb{K}$-algebra B, a functor $F\colon B\text{-mod} \to A\text{-mod}$ is called a *representation embedding* if it preserves indecomposability and isomorphism classes. More formally, F is a representation embedding if for every indecomposable object $X \in B$-mod the object $F(X)$ is indecomposable in A-mod, and if $F(Y) \cong F(Z)$ for some Y and Z in B-mod, then $Y \cong Z$.

An algebra A is *wild* if there is an A-$\mathbb{K}\langle u, v\rangle$-bimodule Z, free of finite rank as a right $\mathbb{K}\langle u, v\rangle$-module, such that the functor $Z \otimes_{\mathbb{K}\langle u,v\rangle} - \colon \mathbb{K}\langle u, v\rangle\text{-mod} \to A\text{-mod}$ is a representation embedding.

It is shown in [1, Proposition 22.4] that to prove that A is wild it is enough to see that the functor $Z \otimes_{\mathbb{K}\langle u,v\rangle} -$ preserves isomorphism classes. As an immediate corollary we get that if B is a wild algebra and there is an isomorphisms preserving functor B-mod $\to A$-mod, then A has wild representation type.

Since full and faithful functors preserve isomorphism classes the following result is obvious.

Proposition 1.1. *Let A be a finite dimensional algebra over $\mathbb{K}$. Suppose there is an ideal I of A such that A/I has wild representation type. Then A is a wild algebra.*

It is not in general true that if B is a wild subalgebra of an algebra A, then A is also wild. Nevertheless, the following partial result in this direction holds.

Proposition 1.2. *Let A be a finite dimensional algebra. Suppose there is an idempotent e in A such that eAe is wild. Then A has wild representation type.*

Proof: The induction functor $M \mapsto Ae \otimes_{eAe} M$ from eAe-mod to A-mod preserves isomorphism classes. In fact, suppose $Ae \otimes_{eAe} M \cong Ae \otimes_{eAe} N$ for some M, $N \in eAe$-mod. Then

$$M \cong eAe \otimes_{eAe} M = e(Ae \otimes_{eAe} M) \cong e(Ae \otimes_{eAe} N) \cong N. \qquad \blacksquare$$

We will now describe further sufficiency criteria for wildness of basic algebras in terms of their quivers. For this we need some notation.

For a (finite) quiver $\mathcal{Q}$ and a $\mathbb{K}$-algebra B, we denote by $\mathcal{Q}$-$\mathbf{rep}_B$ the category of representations $((V_x)_{x\in\mathcal{Q}_0}, (V_\alpha)_{\alpha\in\mathcal{Q}_1})$ such that V_x is a finitely generated free B-module for every $x \in \mathcal{Q}_0$. Given a collection P of paths in $\mathcal{Q}$ with common source and target, we say that $\sum_{p\in P} b_p p$, with $b_p \in B$, is a *relation defined over* B. For a collection R of relations defined over B, we write $(\mathcal{Q}, R)$-$\mathbf{rep}_B$ for the full subcategory of $\mathcal{Q}$-$\mathbf{rep}_B$ whose objects are the representations on which every relation in R vanishes. We suppress B when it coincides with the base field $\mathbb{K}$. The objects of $(\mathcal{Q}, R)$-**rep** are of course just *finite dimensional representations of* $(\mathcal{Q}, R)$. Evidently $(\mathcal{Q}, R)$-**rep** is equivalent to the category of finite dimensional modules over $\mathbb{K}\mathcal{Q}/\langle R\rangle$, where $\mathbb{K}\mathcal{Q}$ is the path algebra of $\mathcal{Q}$ and $\langle R\rangle$ the ideal generated by R in this algebra.

The algebra $\mathbb{K}\mathcal{Q}/\langle R\rangle$ is wild if and only if there exists $Z \in (\mathcal{Q}, R)\text{-}\mathbf{rep}_{\mathbb{K}\langle u,v\rangle}$ such that the functor

$$Z \otimes_{\mathbb{K}\langle u,v\rangle} - \colon \mathbb{K}\langle u, v\rangle\text{-mod} \to (\mathcal{Q}, R)\text{-}\mathbf{rep}$$

defined by

$$\left(Z \otimes_{\mathbb{K}\langle u,v\rangle} V\right)_x = Z_x \otimes_{\mathbb{K}\langle u,v\rangle} V, \quad \left(Z \otimes_{\mathbb{K}\langle u,v\rangle} V\right)_a = Z_a \otimes_{\mathbb{K}\langle u,v\rangle} V, \text{ all } x \in \mathcal{Q}_0,\ \alpha \in \mathcal{Q}_1,$$

is a representation embedding. We say that $(\mathcal{Q}, R)$ is wild if the corresponding path algebra is wild.

Let $(\mathcal{Q}, R)$ be a quiver with relations defined over B and $\mathcal{Q}'$ a subquiver of $\mathcal{Q}$. Suppose $r = \sum_p a_p p$ is a relation in $\mathcal{Q}$. We define the restriction $r|_{\mathcal{Q}'}$ of r to $\mathcal{Q}'$ by $r|_{\mathcal{Q}'} = \sum\limits_{p \text{ in } \mathcal{Q}'} a_p p$. In particular, if the initial or the final vertex of r is not in $\mathcal{Q}'$ then $r|_{\mathcal{Q}'} = 0$. Denote by $R|_{\mathcal{Q}'}$ the collection $\left\{ r|_{\mathcal{Q}'} : r \in R,\ r|_{\mathcal{Q}'} \neq 0\right\}$ of relations in $\mathcal{Q}'$. If V is a representation of $(\mathcal{Q}', R|_{\mathcal{Q}'})$ over B then, following [6], we define the *extension-by-zero* representation $\widetilde{V}$ of $(\mathcal{Q}, R)$ over B by

$$\widetilde{V}_x = \begin{cases} V_x, & x \in \mathcal{Q}' \\ 0, & \text{otherwise} \end{cases} \qquad \widetilde{V}_a = \begin{cases} V_\alpha, & \alpha \in \mathcal{Q}' \\ 0, & \text{otherwise} \end{cases}$$

for every vertex x and every arrow α in $\mathcal{Q}$. It is not difficult to see that the correspondence $V \mapsto \widetilde{V}$ defines a full and faithful functor from $(\mathcal{Q}', R|_{\mathcal{Q}'})\text{-}\mathbf{rep}_B$ to $(\mathcal{Q}, R)\text{-}\mathbf{rep}_B$. This implies the following result.

Theorem 1.3. *Let $\mathcal{Q}$ be a quiver and R a set of relations defined over $\mathbb{K}$ in $\mathcal{Q}$. Suppose $\mathcal{Q}'$ is a subquiver of $\mathcal{Q}$ such that $(\mathcal{Q}', R|_{\mathcal{Q}'})$ is wild. Then $(\mathcal{Q}, R)$ is wild.*

Given $V \in \mathcal{Q}\text{-}\mathbf{rep}_B$, we define $\operatorname{supp}(V)$ to be the subquiver of $\mathcal{Q}$ containing all the vertices $x \in \mathcal{Q}_0$ and all the arrows $\alpha \in \mathcal{Q}_1$ such that $V_x \neq 0$ and $V_\alpha \neq 0$. Then the essential image of the extension-by-zero functor from $(\mathcal{Q}', R|_{\mathcal{Q}'})\text{-}\mathbf{rep}_B$ to $(\mathcal{Q}, R)\text{-}\mathbf{rep}_B$ coincides with the class of those representations V such that $\operatorname{supp}(V) \subset \mathcal{Q}'$. We will use this fact in the proof of Theorem 1.5.

Next we discuss the behaviour of representation type under coverings of quivers. Let $\mathcal{Q}$ be a quiver equipped with an action of a finite group G. Denote by ϕ the canonical projection $\mathcal{Q} \to \mathcal{Q}/G$. In that situation one says that $\mathcal{Q}$ is a *regular covering* of $\mathcal{Q}/G$. Given a representation V of $\mathcal{Q}$ we define the representation $\phi_* V$ of $\mathcal{Q}/G$ by

$$\phi_*(V)_{xG} := \bigoplus_{g \in G} V_{xg}, \qquad \phi_*(V)_{\alpha G} = \sum_{g \in G} \varepsilon_{yg} \circ V_{\alpha g} \circ \pi_{xg},$$

for every $xG \in \mathcal{Q}/G$ and every arrow $x \xrightarrow{\alpha} y$ in $\mathcal{Q}$, where π_z and ε_z are, respectively, the canonical projections and inclusions associated with the direct sum decomposition.

The group G induces an action on the category of finite dimensional representations of $\mathcal{Q}$ as follows. Given such a representation V and $g \in G$ we define

$$(g_* V)_x = V_{xg^{-1}}, \quad (g_* V)_\alpha = V_{\alpha g^{-1}} \colon (g_* V)_x = V_{xg^{-1}} \to (g_* V)_y = V_{yg^{-1}}.$$

For the convenience of the reader we restate [11, Lemma 3.5]

Theorem 1.4 ([11])**.** *Let $\mathcal{Q}$ be a quiver and G a group acting freely on $\mathcal{Q}$. Suppose that V is a finite dimensional indecomposable representation of $\mathcal{Q}$ over $\mathbb{K}$ such that $g_* V \ncong V$, for every $g \in G$, $g \neq 1_G$. Then $\phi_* V$ is indecomposable. Moreover, if $U \ncong V$ is a representation of $\mathcal{Q}$ such that $\phi_* U \cong \phi_* V$, then there is $g \in G$, $g \neq 1_G$, such that $g_* V \cong U$.*

Let G be a finite group acting freely on $\mathcal{Q}$. Then for every vertices x, $y \in \mathcal{Q}$, path $p \colon xG \to yG$ in $\mathcal{Q}/G$, and $g \in G$ there is a unique element $g' \in G$ such that there is a (unique) path $p_g \colon xg \to yg'$ in $\mathcal{Q}$ that lifts p. Notice that the correspondence $g \mapsto g'$ defines a bijective map $\sigma_p \colon G \to G$. For each relation $r = \sum_{p \in \mathcal{J}} a_p p$ in $\mathcal{Q}/G$ we define the set of relations r_G in $\mathcal{Q}$ by

$$r_G := \Big\{ \sum_{p \in \mathcal{J}} a_p p_g \,\Big|\, g \in G \Big\}.$$

It is not difficult to see that if V is a representation of $(\mathcal{Q}, r_G)$ then $\phi_* V$ is a representation of $(\mathcal{Q}/G, r)$. More generally, if R is a set of relations in $\mathcal{Q}/G$ and we define $R_G = \bigcup_{r \in R} r_G$, then for every representation V of $(\mathcal{Q}, R_G)$, the representation $\phi_* V$ of $\mathcal{Q}/G$ satisfies the relations in R.

The next theorem will be used to prove wildness of some quivers with relations. We say that a wild quiver with relations $(\mathcal{Q}, S)$ is *minimal wild* if $(\mathcal{Q}', S|_{\mathcal{Q}'})$ is not wild for every proper subquiver $\mathcal{Q}'$ of $\mathcal{Q}$.

Theorem 1.5. *Let $\mathcal{Q}$ be a finite quiver, G a group acting freely on $\mathcal{Q}$, and S a set of relations in $\mathcal{Q}/G$. Suppose there is a subquiver $\mathcal{Q}'$ of $\mathcal{Q}$ such that*
- *$(\mathcal{Q}', S_G|_{\mathcal{Q}'})$ is minimal wild;*
- *there is no non-trivial $g \in G$ that fixes $\mathcal{Q}'$;*

Then the quiver $(\mathcal{Q}/G, S)$ has wild representation type.

Proof: Let Z be a representation of $(\mathcal{Q}', S_G|_{\mathcal{Q}'})$ over $\mathbb{K}\langle u, v\rangle$ such that the functor $Z \otimes_{\mathbb{K}\langle u,v\rangle} -$ is a representation embedding. Recall that we denote by $\widetilde{\cdot}$ the extension-by-zero functor. It is clear that the functors $\phi_* \circ (\widetilde{Z} \otimes_{\mathbb{K}\langle u,v\rangle} -)$ and $(\phi_* \widetilde{Z}) \otimes_{\mathbb{K}\langle u,v\rangle} -$ are naturally isomorphic. Thus to show that $(\mathcal{Q}/G, S)$ is wild it is enough to check that $\phi_* \circ (\widetilde{Z} \otimes_{\mathbb{K}\langle u,v\rangle} -)$ is a representation embedding.

Since $(\mathcal{Q}', S_G|_{\mathcal{Q}'})$ is minimal wild, the support of Z, and thus also the support of $\widetilde{Z}$, coincides with $\mathcal{Q}'$.

Let X be an indecomposable $\mathbb{K}\langle u, v\rangle$-module. Denote $\widetilde{Z} \otimes_{\mathbb{K}\langle u,v\rangle} X$ by $\overline{X}$. Then $\operatorname{supp}\left(\overline{X}\right) = \operatorname{supp}\left(\widetilde{Z}\right) = \mathcal{Q}'$ and so $\operatorname{supp}(g_* \overline{X}) = \operatorname{supp}(\overline{X}) g^{-1} \neq \operatorname{supp}(\overline{X})$ unless $g = e_G$. In particular, $g_* \overline{X} \not\cong \overline{X}$. Thus, by Theorem 1.4, the representation $\phi_* \overline{X}$ of $(\mathcal{Q}/G, S)$ is indecomposable.

Now let Y be another indecomposable $\mathbb{K}\langle u, v\rangle$-module not isomorphic to X. We write $\overline{Y}$ for $\widetilde{Z} \otimes_{\mathbb{K}\langle u,v\rangle} Y$. Then $\overline{X} \not\cong \overline{Y}$ since $\widetilde{Z} \otimes_{\mathbb{K}\langle u,v\rangle} -$ is a representation embedding. Therefore, by Theorem 1.4, the representations $\phi_* \overline{X}$ and $\phi_* \overline{Y}$ of $(\mathcal{Q}/G, S)$ are isomorphic if and only if there is $g \neq e_G$ such that $g_* \overline{X} \cong \overline{Y}$. But then

$$\operatorname{supp}(\widetilde{Z}) = \operatorname{supp}(\overline{Y}) = \operatorname{supp}(g_* \overline{X}) = \operatorname{supp}(\overline{X}) g^{-1} = \operatorname{supp}(\widetilde{Z}) g^{-1},$$

which contradicts the already proved assertion that $\operatorname{supp}(\widetilde{Z}) g \neq \operatorname{supp}(\widetilde{Z})$ for $g \neq e_G$. This finishes the proof of the theorem. ■

Let $(\mathcal{Q}, R)$ be a quiver with relations and A the corresponding basic algebra. Suppose $\mathcal{Q}'$ is a subquiver of $\mathcal{Q}$. Denote by e the idempotent of A given by $\sum_{x \in \mathcal{Q}'} e_x$. Then eAe is a basic algebra. It is not true in general that the quiver of eAe is $\mathcal{Q}'$. We say that $\mathcal{Q}'$ is *convex* if every path in $\mathcal{Q}$ connecting two vertices in $\mathcal{Q}'$ completely lies in $\mathcal{Q}'$. Notice, that a convex subquiver is always full.

Proposition 1.6. *Let $(\mathcal{Q}, R)$ be a quiver with relations defined over $\mathbb{K}$, and $\mathcal{Q}'$ a convex subquiver of $\mathcal{Q}$. Then $\mathcal{Q}'$ is the quiver of the basic algebra eAe, where $A = \mathbb{K}\mathcal{Q}/\langle R \rangle$ and $e = \sum_{x \in \mathcal{Q}'} e_x$.*

Proof: The algebra eAe is $\mathbb{K}$-spanned by the paths in $\mathcal{Q}$ that start and end in $\mathcal{Q}'$. As $\mathcal{Q}'$ is convex all these paths lie inside $\mathcal{Q}'$. Thus eAe is generated as a ring by arrows in $\mathcal{Q}'$. Every arrow of $\mathcal{Q}'$ lies in the radical of A and so is nilpotent. Hence it also lies in the radical of eAe. Similarly every path of length no less than two with start and end in $\mathcal{Q}'$ lies in the $\operatorname{rad}^2(eAe)$. This shows that $\operatorname{rad}(eAe)/\operatorname{rad}^2(eAe)$ is generated by the arrows in $\mathcal{Q}'$, i.e. that $\mathcal{Q}'$ is the quiver of the basic algebra eAe. ■

2. Borel-Schur algebras

In this section we introduce Schur and Borel-Schur algebras and establish some basic facts about these algebras.

Let n and r be arbitrary fixed positive integers. Consider the general linear group $\mathrm{GL}_n(\mathbb{K})$ and denote by B^+ the Borel subgroup of $\mathrm{GL}_n(\mathbb{K})$ consisting of all upper triangular invertible matrices. The general linear group $\mathrm{GL}_n(\mathbb{K})$ acts on $\mathbb{K}^n$ by multiplication. So $\mathrm{GL}_n(\mathbb{K})$ acts on the r-fold tensor product $(\mathbb{K}^n)^{\otimes r}$ by the rule $g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r$, all $g \in \mathrm{GL}_n(\mathbb{K})$, $v_1, \dots, v_r \in \mathbb{K}^n$. Let

$$\rho_{n,r} \colon \mathbb{K}\mathrm{GL}_n(\mathbb{K}) \to \operatorname{End}_{\mathbb{K}}((\mathbb{K}^n)^{\otimes r})$$

be the representation afforded by $(\mathbb{K}^n)^{\otimes r}$ as a $\mathbb{K}\mathrm{GL}_n(\mathbb{K})$-module. Then $\rho_{n,r}(\mathbb{K}\mathrm{GL}_n(\mathbb{K}))$ is a subalgebra of $\operatorname{End}_{\mathbb{K}}((\mathbb{K}^n)^{\otimes r})$.

Definition 2.1. The algebra $\rho_{n,r}(\mathbb{K}\mathrm{GL}_n(\mathbb{K}))$ is called the Schur algebra for n, r and $\mathbb{K}$ and is denoted by $S_{\mathbb{K}}(n,r)$, or simply $S(n,r)$. The Borel-Schur algebra $S^+(n,r) = S^+_{\mathbb{K}}(n,r)$ is the subalgebra $\rho_{n,r}(\mathbb{K}\mathrm{B}^+)$ of the Schur algebra.

To describe a standard basis for $S(n,r)$ we need some combinatorics. We start by summarizing the terms we use.

- Σ_r is the symmetric group on $\{1, \dots, r\}$.
- $I(n,r) = \{ i = (i_1, \dots, i_r) \mid i_s \in \mathbb{Z},\ 1 \le i_s \le n, \text{ for all } s \}$. The elements of $I(n,r)$ are called multi-indices.
- $\Lambda(n,r) = \{ \lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_t \in \mathbb{Z},\ 0 \le \lambda_t\ (t = 1, \dots, n),\ \sum_{t=1}^n \lambda_t = r \}$ is the set of all compositions of r into n parts.
- $i \in I(n,r)$ has *weight* $\lambda \in \Lambda(n,r)$ if $\lambda_t = \#\{ 1 \le s \le r \mid i_s = t \}$, $t = 1, \dots, n$.

- $\trianglelefteq$ denotes the *dominance order* on $\Lambda(n,r)$, that is $\alpha \trianglelefteq \beta$ if $\sum_{t=1}^{s} \alpha_t \leq \sum_{t=1}^{s} \beta_t$, for all $1 \leq s \leq n$.
- For $i, j \in I(n,r)$, $i \leq j$ means $i_s \leq j_s$, $s = 1, \ldots, r$, and $i < j$ means $i \leq j$ and $i \neq j$. Obviously, if i has weight μ and j has weight λ, then $i \leq j$ implies $\lambda \trianglelefteq \mu$.

The symmetric group Σ_r acts on the right of $I(n,r)$ and of $I(n,r) \times I(n,r)$, respectively, by

$$i\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(r)}) \text{ and } (i,j)\sigma = (i\sigma, j\sigma), \text{ all } i, j \in I(n,r), \sigma \in \Sigma_r.$$

Note that $i, j \in I(n,r)$ are in the same Σ_r-orbit if and only if they have the same weight. Therefore the Σ_r-orbits on $I(n,r)$ are identified with the elements of $\Lambda(n,r)$. We denote by Σ_i the stabilizer of i in Σ_r, that is $\Sigma_i = \{\, \sigma \in \Sigma_r \mid i\sigma = i \,\}$. We write $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j$, all $i, j \in I(n,r)$.

To each pair $(i,j) \in I(n,r) \times I(n,r)$ one can associate an element $\xi_{i,j}$ of $S(n,r)$ (see [14]). These elements satisfy $\xi_{i,j} = \xi_{k,\ell}$ if and only if (i,j) and (k,ℓ) are in the same Σ_r-orbit of $I(n,r) \times I(n,r)$. Fix a transversal $\Omega(n,r)$ for the action of Σ_r on $I(n,r) \times I(n,r)$. Then the set $\{\, \xi_{i,j} \mid (i,j) \in \Omega(n,r)\}$ is a basis of $S(n,r)$ over $\mathbb{K}$. It is also well known (see [13]) that $S^+(n,r) = \mathbb{K}\{\, \xi_{i,j} \mid i \leq j,\ (i,j) \in \Omega(n,r)\}$.

A formula for the product of two basis elements is the following (see [13]): $\xi_{i,j}\xi_{k,h} = 0$, unless j and k are in the same Σ_r-orbit, and

$$\xi_{i,j}\xi_{j,h} = \sum_{\sigma} [\Sigma_{i\sigma,h} : \Sigma_{i\sigma,j,h}]\, \xi_{i\sigma,h} \tag{2.1}$$

where the sum is over a transversal $\{\sigma\}$ of the set of all double cosets $\Sigma_{i,j}\sigma\Sigma_{j,h}$ in Σ_j, and $\Sigma_{i\sigma,j,h} = \Sigma_{i\sigma,h} \cap \Sigma_j$.

If i has weight $\lambda \in \Lambda(n,r)$, we write $\xi_{i,i} = \xi_\lambda$. Then $1_{S(n,r)} = \sum_{\lambda \in \Lambda(n,r)} \xi_\lambda$ is an orthogonal idempotent decomposition of $1_{S(n,r)}$.

It was shown in [19] that the algebra $S^+(n,r)$ is a basic algebra. The idempotents ξ_λ , $\lambda \in \Lambda(n,r)$, are primitive in $S^+(n,r)$. The quiver $\mathcal{Q}$ of $S^+(n,r)$ was *de facto* determined in [19, Theorem 5.4]. The vertices of $\mathcal{Q}$ correspond to the primitive idempotents ξ_λ, and so $\mathcal{Q}_0$ can be identified with $\Lambda(n,r)$. If $\operatorname{char}\mathbb{K} = 0$, the quiver $\mathcal{Q}$ contains an arrow from the vertex λ to the vertex μ if and only if, for some positive integer s, we have $\mu - \lambda = \gamma_s$, where $\gamma_s = (0, \ldots, 1, -1, \ldots, 0)$ with 1 at the sth position. If $\operatorname{char}\mathbb{K} = p$, such an arrow exists if and only if there are integers $s \geq 1$ and $d \geq 0$ such that $\mu - \lambda = p^d\gamma_s$. Notice that for such λ and μ the vector space $\xi_\mu S^+(n,r)\xi_\lambda$ is one-dimensional and is spanned by the element $\xi_{\ell(\lambda(s,p^d)),\ell(\lambda)}$, where $\ell(\lambda)$ and $\ell(\lambda(s,p^d)) \in I(n,r)$ are the standard elements

$$\ell(\lambda) = (\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{n,\dots,n}_{\lambda_n}),$$

$$\ell(\lambda(s,p^d)) = (\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{s,\dots,s}_{\lambda_s+p^d},\underbrace{s+1,\dots,s+1}_{\lambda_{s+1}-p^d},\dots,\underbrace{n,\dots,n}_{\lambda_n}).$$

A similar result holds in characteristic 0, with p^d replaced by 1. It should also be mentioned that the sets

$$\begin{aligned} &\left\{ \xi_{\ell(\lambda(s,1)),\ell(\lambda)} \;\middle|\; 1 \le s \le n-1 \right\}, \text{ if } \operatorname{char}\mathbb{K} = 0, \\ &\left\{ \xi_{\ell(\lambda(s,p^d)),\ell(\lambda)} \;\middle|\; 1 \le s \le n-1;\ 1 \le p^d \le \lambda_{s+1} \right\}, \text{ if } \operatorname{char}\mathbb{K} = p, \end{aligned} \tag{2.2}$$

are minimal sets of $S^+(n,r)$-generators of rad $S^+(n,r)\xi_\lambda$ (see [19, Theorem 4.5]). In [8] we determined the relations for the quiver $\mathcal{Q}$ of $S^+(2,r)$. If $\operatorname{char}\mathbb{K} = 0$, $\mathcal{Q}$ is of type A_{r+1} and $\mathbb{K}\mathcal{Q} \simeq S^+(2,r)$. Suppose now that $\operatorname{char}\mathbb{K} = p$. For every λ, $\mu \in \Lambda(2,r)$ such that $\mu - \lambda = (p^d, -p^d)$, we denote by $\alpha_{d,\lambda}$ the arrow from λ to μ in $\mathcal{Q}$. We say that $\alpha_{d,\lambda}$ is of type α_d. Notice that every vertex in $\mathcal{Q}$ has at most one incoming and at most one outgoing arrow of type α_d. This implies that to specify a path in $\mathcal{Q}$ it is enough to indicate the starting vertex and the types of arrows in the path. For example $(\alpha_0\alpha_1)_{(a,b)}$ will denote the path $\alpha_{0,(a+p,b-p)}\alpha_{1,(a,b)}$. It is also natural to abbreviate the repeated types with the usual power notation. With these conventions, the relations for $S^+(2,r)$ as quotient algebra of the path algebra of $\mathcal{Q}$ can be written as

$$(\alpha_s\alpha_t)_\lambda = (\alpha_t\alpha_s)_\lambda,\ \lambda_2 \ge p^s + p^t,\ s \ne t$$

$$(\alpha_s^p)_\lambda = 0,\ \lambda_2 \ge p^{s+1}.$$

Proposition 2.2. *For every $m \le n$ and $s \le t$ there is an idempotent e in $S^+(n,t)$ such that*

$$eS^+(n,t)e \cong S^+(m,s).$$

Proof: It is enough to consider the cases $n = m+1$, $t = s$ and $n = m$, $t = s+1$. The case $n = m+1$, $t = s$ was treated in Section 5 of [8]. For

$n = m$ and $t = s + 1$, we define

$$e = \sum_{\lambda \in \Lambda(n,s+1):\ \lambda_1 \geq 1} \xi_\lambda .$$

For each $i = (i_1, \ldots, i_s) \in I(n,s)$, write $\bar{\imath} = (1, i_1, \ldots, i_s) \in I(n, s+1)$. Note that, for $i, j, k, \ell \in I(n,s)$, we have that (i,j) and (k,ℓ) are in the same Σ_s-orbit of $I(n,s) \times I(n,s)$ if and only if $(\bar{\imath}, \bar{\jmath})$ and $(\bar{k}, \bar{\ell})$ are in the same Σ_{s+1}-orbit of $I(n,s+1) \times I(n,s+1)$. So there is an injective linear map $\phi\colon S^+(n,s) \to S^+(n,s+1)$ defined by $\phi(\xi_{ij}) = \xi_{\bar{\imath},\bar{\jmath}}$. It is easy to see that the image of ϕ coincides with $eS^+(n,s+1)e$, and that $\phi(1) = \phi(\sum_{\lambda \in \Lambda(n,s)} \xi_\lambda) = e$. Next we verify that ϕ preserves products.

If j, $k \in I(n,s)$ are not in the same Σ_s-orbit, then $\bar{\jmath}$ and $\bar{k}$ are not in the same Σ_{s+1}-orbit. Therefore $\xi_{i,j}\xi_{k,\ell} = 0$ and $\xi_{\bar{\imath},\bar{\jmath}}\xi_{\bar{k},\bar{\ell}} = 0$. Now fix multi-indices $i \leq j \leq \ell$ in $I(n,s)$, and consider the products $\xi_{i,j}\xi_{j,\ell}$ and $\xi_{\bar{\imath},\bar{\jmath}}\xi_{\bar{\jmath},\bar{\ell}}$. For each multi-index h, let $h(t) = \{u \mid h_u = t\}$, $t = 1, \ldots, n$. Then the stabilizer of h is $\Sigma_{h(1)} \times \cdots \times \Sigma_{h(n)}$. Since we know that $i \leq j \leq \ell$, we have $\ell(1) \subseteq j(1) \subseteq i(1)$, and $\Sigma_{i,j} = \Sigma_{j(1)} \times \ldots$. Therefore, we can choose a transversal $\{\sigma\}$ of the set of double cosets $\Sigma_{i,j}\sigma\Sigma_{j,\ell}$ in Σ_j so that the restriction of σ to $\Sigma_{j(1)}$ is the identity. Now that we have such a transversal, we construct from it a transversal $\bar{\sigma}$ of the set of double cosets $\Sigma_{\bar{\imath},\bar{\jmath}}\bar{\sigma}\Sigma_{\bar{\jmath},\bar{\ell}}$ in $\Sigma_{\bar{\jmath}}$ in the following way: $\sigma \mapsto \bar{\sigma}$, where $\bar{\sigma}_{|_{\bar{\jmath}(t)}} = \sigma_{|_{j(t)}}$, for $t \neq 1$, and $\bar{\sigma}_{|_{\bar{\jmath}(1)}} = id$. This can be done because $j(t) = \bar{\jmath}(t)$, for $t \neq 1$, and $\Sigma_{\bar{\imath},\bar{\jmath}} = \Sigma_{\bar{\jmath}(1)} \times \ldots$. Now we have

$$\phi\left(\xi_{i,j}\xi_{j,\ell}\right) = \sum_\sigma \left[\Sigma_{i\sigma,\ell} : \Sigma_{i\sigma,j,\ell}\right] \xi_{\overline{i\sigma},\bar{\ell}}\,, \quad \text{and} \quad \phi\left(\xi_{i,j}\right)\phi\left(\xi_{j,\ell}\right) = \sum_\sigma \left[\Sigma_{\bar{\imath}\,\bar{\sigma},\bar{\ell}} : \Sigma_{\bar{\imath}\,\bar{\sigma},\bar{\jmath},\bar{\ell}}\right] \xi_{\bar{\imath}\,\bar{\sigma},\bar{\ell}}.$$

But $\bar{\imath}\bar{\sigma} = (1, i_{\sigma(1)}, \ldots, i_{\sigma(s)}) = \overline{i\sigma}$. Also, if we write $\Sigma_{i\sigma,\ell} = \Sigma_{\ell(1)} \times X$ and $\Sigma_{i\sigma,j,\ell} = \Sigma_{\ell(1)} \times Y$, then $\Sigma_{\overline{i\sigma},\bar{\ell}} = \Sigma_{\bar{\ell}(1)} \times X$ and $\Sigma_{\overline{i\sigma},\bar{\jmath},\bar{\ell}} = \Sigma_{\bar{\ell}(1)} \times Y$. Therefore $\left[\Sigma_{i\sigma,\ell} : \Sigma_{i\sigma,j,\ell}\right] = \left[\Sigma_{\overline{i\sigma},\bar{\ell}} : \Sigma_{\overline{i\sigma},\bar{\jmath},\bar{\ell}}\right]$ for all σ in the transversal.

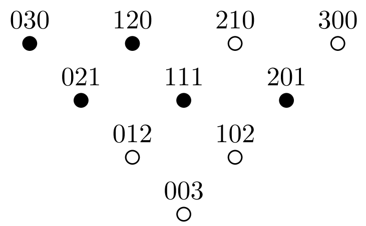
■

3. Borel-Schur algebras of wild representation type

In this section we will show that all the Borel-Schur algebras of infinite type, excluding the algebras $S^+(2,5)$ over fields of characteristic 3 and the algebras $S^+(3,2)$ over fields of arbitrary characteristic, are indeed wild.

3.1. The algebra $S^+(3,3)$. In this subsection we prove that the algebra $S^+(3,3)$ has wild representation type. By Propositions 1.2 and 2.2, this implies that the algebras $S^+(n,r)$ are wild for all $n \geq 3$ and $r \geq 3$.

Consider the subset X of $\Lambda(3,3)$ marked by the black dots below



Let $\mathcal{Q}$ denote the quiver of $S^+(3,3)$ and write $\mathcal{Q}'$ for the subquiver of $\mathcal{Q}$ spanned by X. Since existence of an arrow $\alpha\colon \lambda \to \mu$ in $\mathcal{Q}$ implies that μ dominates λ, we get that the subquiver $\mathcal{Q}'$ is convex. Hence by Proposition 1.6, $\mathcal{Q}'$ is the quiver of the basic algebra $eS^+(3,3)e$ for $e = \sum_{\lambda \in X} \xi_\lambda$.

If $\operatorname{char} \mathbb{K} = 2$, $\mathcal{Q}'$ has the form

$$\begin{tikzcd}
030 \arrow[rr, "\xi_{122,222}"] & & 120 & & & \\
& 021 \arrow[ul, "\xi_{222,223}"] \arrow[rr, "\xi_{123,223}"] \arrow[rrrr, bend right, "\xi_{113,223}"'] & & 111 \arrow[ul, "\xi_{122,123}"'] \arrow[rr, "\xi_{113,123}"] & & 201
\end{tikzcd}, \tag{3.1}$$

where we labelled arrows by the corresponding basis elements of $eS^+(3,3)e$. Notice that (3.1) does not contain any path of length greater than or equal to 3 . By direct computation we have that $\xi_{122,123}\xi_{123,223} \neq \xi_{122,222}\xi_{222,223}$ and $\xi_{113,123}\xi_{123,223} = 0$ in $S^+(3,3)$, and so also in $eS^+(3,3)e$. Therefore the category $eS^+(3,3)e$-mod is equivalent to the category $(\mathcal{Q}', \xi_{113,123}\xi_{123,223})$-**rep**. Denote by $\mathcal{Q}''$ the quiver obtained from $\mathcal{Q}'$ by removing the arrow $\xi_{113,123}$. Since the path $\xi_{113,123}\xi_{123,223}$ does not belong to $\mathcal{Q}''$, we get that $\mathcal{Q}''$-**rep** embeds into $(\mathcal{Q}', \xi_{113,123}\xi_{123,223})$-**rep**. Now $\mathcal{Q}''$ is a quiver without relations and oriented cycles. Moreover it is neither a Dynkin nor an extended Dynkin diagram. Therefore $\mathcal{Q}''$ is wild. Hence also $(\mathcal{Q}', \xi_{113,123}\xi_{123,223})$ and $eS^+(3,3)e$ are wild. This shows that $S^+(3,3)$ is wild when $\operatorname{char} \mathbb{K} = 2$.

When $\operatorname{char}\mathbb{K} \geq 3$ the quiver $\mathcal{Q}'$ defined above has the form

$$
\begin{tikzcd}
030 \arrow[rr, "\xi_{122,222}"] & & 120 & & & \\
& 021 \arrow[ul, "\xi_{222,223}"] \arrow[rr, "\xi_{123,223}"] & & 111 \arrow[ul, "\xi_{122,123}"'] \arrow[rr, "\xi_{113,123}"] & & 201
\end{tikzcd}
. \tag{3.2}
$$

We have again $\xi_{122,123}\xi_{123,223} \neq \xi_{122,222}\xi_{222,223}$ in $S^+(3,3)$. Therefore $eS^+(3,3)e$ is hereditary in this case. As (3.2) is not of Dynkin type we conclude that $eS^+(3,3)e$ and, hence, also $S^+(3,3)$ have wild representation type.

3.2. The algebra $S^+(4,2)$. We consider now the algebra $S^+(4,2)$ and show that it has wild representation type if the characteristic of the base field $\mathbb{K}$ is $p \geq 3$. In view of Proposition 1.2 and Proposition 2.2, this will imply that the algebras $S^+(n,r)$ are wild for all $n \geq 4$, $r \geq 2$, and $\operatorname{char}\mathbb{K} \geq 3$.

We consider the idempotent

$$e = \xi_{24,24} + \xi_{23,23} + \xi_{12,12}$$

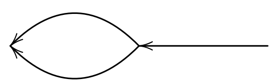
in $S^+(4,2)$, and write $A = eS^+(4,2)e$. We are going to compute the quiver of A. It has three vertices corresponding to the primitive idempotents $\xi_{12,12}$, $\xi_{23,23}$, $\xi_{24,24}$. Now, to obtain a basis of rad A, note that

$$
\begin{aligned}
\xi_{23,23}A\xi_{24,24} &= \mathbb{K}\xi_{23,24}, \quad \xi_{12,12}A\xi_{24,24} = \mathbb{K}\xi_{12,24} \oplus \mathbb{K}\xi_{12,42},\\
\xi_{12,12}A\xi_{23,23} &= \mathbb{K}\xi_{12,23} \oplus \mathbb{K}\xi_{12,32}.
\end{aligned}
$$

So $\{\, \xi_{23,24}, \xi_{12,24}, \xi_{12,42}, \xi_{12,23}, \xi_{12,32} \,\}$ is a basis of rad A. On the other hand,

$$\xi_{12,23}\xi_{23,24} = \xi_{12,24}, \quad \xi_{12,32}\xi_{23,24} = \xi_{12,42},$$

which implies that $\operatorname{rad}^2 A$ has basis $\{\xi_{12,24}, \xi_{12,42}\,\}$. Thus A is the path algebra of the quiver



(3.3)

without relations. Since (3.3) is neither a Dynkin nor an extended Dynkin diagram, it has wild representation type. Hence the algebra $S^+(4,2)$ is wild.

3.3. The algebras $S^+(2,r)$. Let p be the characteristic of the base field. In this section we show that $S^+(2,p+1)$ has wild representation type if $p \geq 5$, and that the same result holds for $S^+(2,4)$ if $p=2$, and for $S^+(2,6)$ if $p=3$. In Section 5 we will prove that $S^+(2,5)$ has tame representation type if $p=3$. By Propositions 1.2 and 2.2, this completes the classification of all algebras $S^+(2,r)$ of infinite representation type.

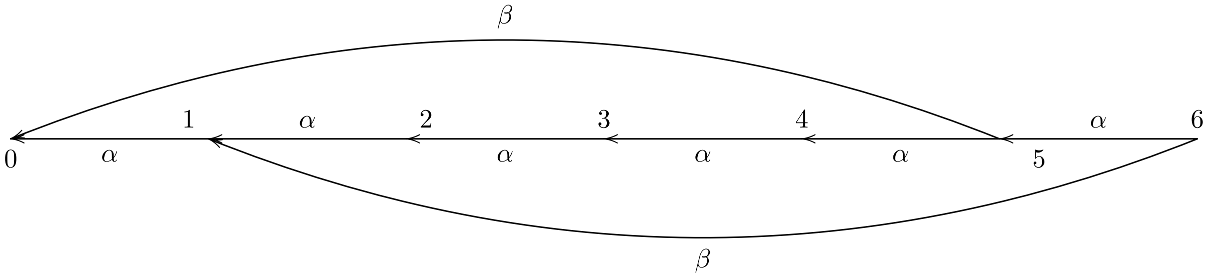
We change slightly the notation introduced in Section 2, namely we write α for α_0, β for α_1, and γ for α_2, for the types of arrows in the quiver of $S^+(2,r)$. This is convenient since the types α_s with $s \geq 3$ will not appear in this section. Further, we will identify compositions $\lambda = (\lambda_1, \lambda_2) \in \Lambda(2,r)$ with λ_2. This will not create an ambiguity, since r will be fixed in each particular case and $\lambda_1 = r - \lambda_2$.

3.3.1. *The algebra $S^+(2,p+1)$ over a field of characteristic $p \geq 7$.* We described the quiver $\mathcal{Q}$ of $S^+(2,p+1)$ and the corresponding relations R in Section 2. Denote by $\mathcal{Q}'$ the subquiver

$$\begin{tikzcd}
p-2 & p-1 \arrow[l, "\alpha"] & p \arrow[l, "\alpha"] \arrow[d, "\beta"'] & p+1 \arrow[l, "\alpha"] \arrow[d, "\beta"] & & & \\
 & & 0 & 1 \arrow[l, "\alpha"'] & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\alpha"'] & 4 \arrow[l, "\alpha"']
\end{tikzcd} \tag{3.4}$$

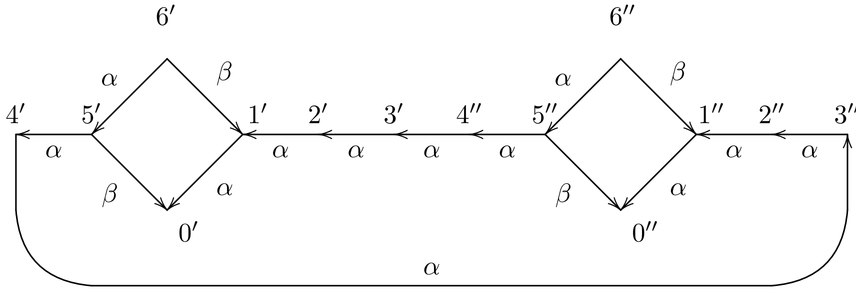
of $\mathcal{Q}$. We used here the fact that $\operatorname{char} \mathbb{K} \geq 7$, as otherwise different vertices in (3.4) would have the same labels in $\Lambda(2,p+1)$, and so (3.4) would not be a subquiver of $\mathcal{Q}$. Now, $\operatorname{char} \mathbb{K} \geq 7$ implies that the sets of relations $(\alpha)^p_\lambda|_{\mathcal{Q}'}$ and $(\beta)^p_\lambda|_{\mathcal{Q}'}$ are empty for all $\lambda \in \mathcal{Q}'_0$. Hence $R|_{\mathcal{Q}'} = \{(\beta\alpha)_{p+1} - (\alpha\beta)_{p+1}\}$. The pair $(\mathcal{Q}', R|_{\mathcal{Q}'})$ is isomorphic to the XXVIIIth quiver in Ringel's list in [17] of minimal wild quivers with one relation. By Proposition 1.3, we get that $S^+(2,p+1)$ is wild over a field of characteristic no less than 7.

3.3.2. *The algebra* $S^+(2,6)$ *over a field of characteristic* 5. The quiver $\mathcal{Q}$ of the algebra $S^+(2,6)$ over a field of characteristic 5 is given by



(3.5)

and the corresponding relations are $R = \{ \ (\alpha^5)_6\,, \ (\alpha^5)_5\,, \ (\alpha\beta - \beta\alpha)_6 \ \}$. Next we consider the quiver $\widetilde{\mathcal{Q}}$

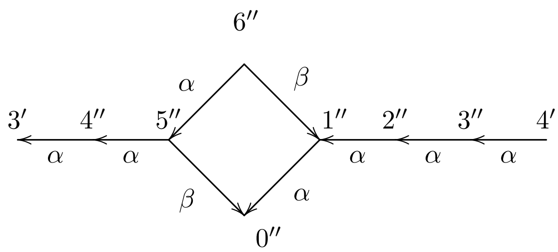


(3.6)

with a free action of the symmetric group Σ_2 given by interchanging $'$ with $''$. The quiver (3.5) is the quotient of (3.6) under this action when we identify the orbit $\{s', \ s''\}$ with the vertex s of the quiver (3.5). As the quotient map preserves the types of arrows, we get that

$$R_{\Sigma_2} = \{ \ (\alpha^5)_{6'}\,, (\alpha^5)_{6''}\,, (\alpha^5)_{5'}\,, (\alpha^5)_{5''}\,, (\alpha\beta - \beta\alpha)_{6'}\,, \ (\alpha\beta - \beta\alpha)_{6''} \ \}\,.$$

Denote by $\widetilde{\mathcal{Q}}'$ the subquiver



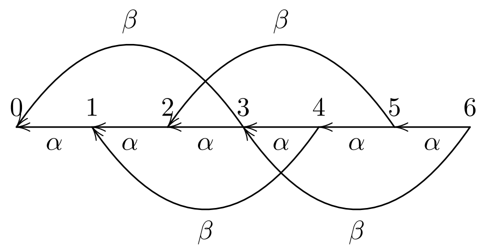
(3.7)

of $\widetilde{\mathcal{Q}}$. As $\widetilde{\mathcal{Q}}'$ does not contain paths of type α^5 and $6' \notin \widetilde{\mathcal{Q}}'_0$, we get that

$$R_{\Sigma_2}\big|_{\widetilde{\mathcal{Q}}'} = \{ \ (\alpha\beta - \beta\alpha)_{6''}\} \,.$$

The pair $(\widetilde{\mathcal{Q}}', R_{\Sigma_2}|_{\widetilde{\mathcal{Q}}'})$ is isomorphic to the XVIIIth quiver in Ringel's list in [17] of minimal wild quivers with one relation. By Theorem 1.5, the quiver $(\mathcal{Q}, R)$ is wild. This shows that $S^+(2,6)$ has wild representation type over fields of characteristic 5.

3.3.3. *The algebra* $S^+(2,6)$ *over a field of characteristic* 3. In this section we show that if the base field has characteristic 3 then the algebra $S^+(2,6)$ is wild. The quiver $\mathcal{Q}$ of $S^+(2,6)$ is

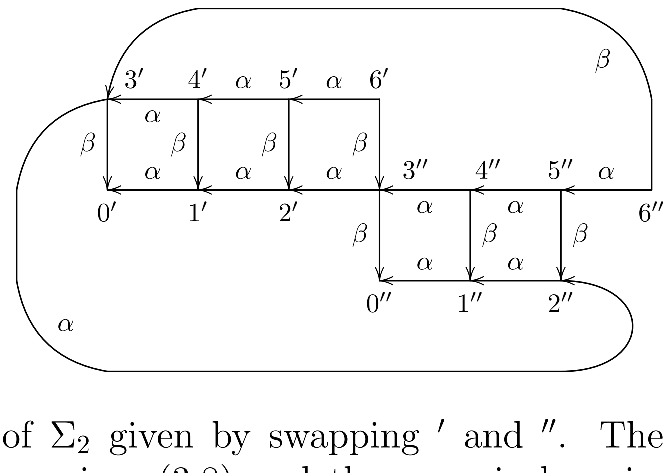


(3.8)

and the corresponding relations are

$$R := \{(\alpha^3)_6\,,\ (\alpha^3)_5\,,\ (\alpha^3)_4\,,\ (\alpha^3)_3\,,$$
$$(\alpha\beta - \beta\alpha)_6\,,\ (\alpha\beta - \beta\alpha)_5\,,\ (\alpha\beta - \beta\alpha)_4\ \}\,.$$

Let us consider the quiver $\widetilde{\mathcal{Q}}$

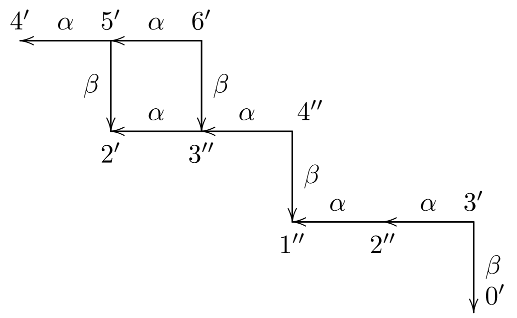


(3.9)

with the action of Σ_2 given by swapping $'$ and $''$. The quotient $\widetilde{\mathcal{Q}}/\Sigma_2$ is isomorphic to the quiver (3.8) and the canonical projection preserves the types of arrows. Therefore

$$R_{\Sigma_2} = \left\{\ (\alpha^3)_{s'}\,,\ (\alpha^3)_{s''}\,,\ (\alpha\beta - \beta\alpha)_{t'}\,,\ (\alpha\beta - \beta\alpha)_{t''} \,\middle|\, 3 \le s \le 6,\ 4 \le t \le 6\right\}.$$

We consider the following subquiver $\widetilde{\mathcal{Q}}'$ of (3.9)



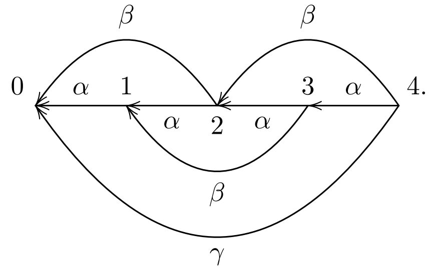
(3.10)

As $\widetilde{\mathcal{Q}}'$ does not contain any path of type α^3 and contains only one square with the border paths of types $\alpha\beta$ and $\beta\alpha$, we get that $R|_{\widetilde{\mathcal{Q}}'} = \{\ (\alpha\beta - \beta\alpha)_{6'}\ \}$. The quiver with relations $(\widetilde{\mathcal{Q}}', R|_{\widetilde{\mathcal{Q}}'})$ is isomorphic to the XXIXth quiver in Ringel's list of minimal wild quivers with one relation provided in [17]. From Theorem 1.5, we conclude that $(\mathcal{Q}, R)$ is wild. This implies that $S^+(2,6)$ is wild over a field of characteristic 3.

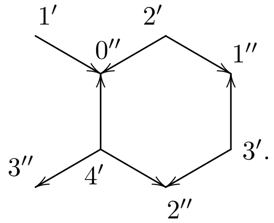
3.3.4. *The algebra* $S^+(2,4)$ *over a field of characteristic* 2*.* The last algebra to be analyzed in this section is $S^+(2,4)$ over a base field of characteristic 2. We will show that this algebra is wild.

Given a quiver $\mathcal{Q}$, the associated *separated quiver* $\mathrm{sp}(\mathcal{Q})$ is the quiver with the set of vertices $\{\, v',\ v'' \mid v \in \mathcal{Q}_0\}$ and arrows $\bar{\varepsilon}\colon v' \to w''$ for every $\varepsilon\colon v \to w$ in $\mathcal{Q}$. Let R be any set of relations for $\mathcal{Q}$. According to Gabriel (see [9]) the category of representations of $\mathrm{sp}(\mathcal{Q})$ is equivalent to the category of representations of $(\mathcal{Q}, R)$ whose Loewy length does not exceed 2. From this and from the classification of tame hereditary quivers it follows that if $(\mathcal{Q}, R)$ is tame then $\mathrm{sp}(\mathcal{Q})$ is a union of Dynkin and extended Dynkin diagrams.

The quiver of the algebra $S^+(2,4)$ over a base field of characteristic 2 is



The corresponding separated quiver has three connected components: two isolated vertices $0'$, $4''$, and



Since the above quiver is neither Dynkin nor extended Dynkin diagram, we conclude that $S^+(2,4)$ is wild if the characteristic of the base field is 2.

4. Tame representation type: the degeneration technique and $S^+(3,2)$.

We proved in [8] that the Borel-Schur algebra $S^+(3,2)$ has infinite representation type (independently of the characteristic of the ground field $\mathbb{K}$). The aim of this section is to show that $S^+(3,2)$ is tame. For this we will use degeneration techniques developed by Gabriel in [10] and by Geiss in [12].

Given a vector space V, denote by $\mathrm{alg}(V)$ the variety of associative algebra structures with identity on V. Each such structure is determined by the multiplication map $\mu\colon V\otimes V\to V$. Hence we can consider $\mathrm{alg}(V)$ as a subset of the affine space $\mathrm{Hom}_{\mathbb{K}}(V\otimes V,V)$. The group $\mathrm{GL}(V)$ acts on $\mathrm{Hom}_{\mathbb{K}}(V\otimes V,V)$ by $g\mu = g\circ\mu\circ(g^{-1}\otimes g^{-1})$, all $g\in\mathrm{GL}(V)$. This action preserves $\mathrm{alg}(V)$.

A product $\mu_0\in\mathrm{alg}(V)$ is called a *degeneration* of $\mu\in\mathrm{alg}(V)$ if μ_0 lies in the Zariski closure of the $\mathrm{GL}(V)$-orbit of μ.

We will use the following result proved in [12].

Theorem 4.1 ([12]). *Let μ_0 be a degeneration of μ in $\mathrm{alg}(V)$. If μ_0 is not wild, then the same holds for μ.*

One way to construct a degeneration of $\mu\in\mathrm{alg}(V)$ is as follows. Fix a basis $\{v_1,\dots,v_m\}$ of V, such that v_1 is the identity element for μ. Then μ determines the multiplication constants γ^k_{hl} by

$$\mu(v_h\otimes v_l)=\sum_{k=1}^{m}\gamma^k_{hl}v_k.$$

Suppose we have a function $\phi\colon\{1,\dots,m\}\to\mathbb{N}$ such that $\phi(k)-\phi(h)-\phi(l)\geq 0$, for every triple (h,l,k) satisfying $\gamma^k_{hl}\neq 0$. We also assume that

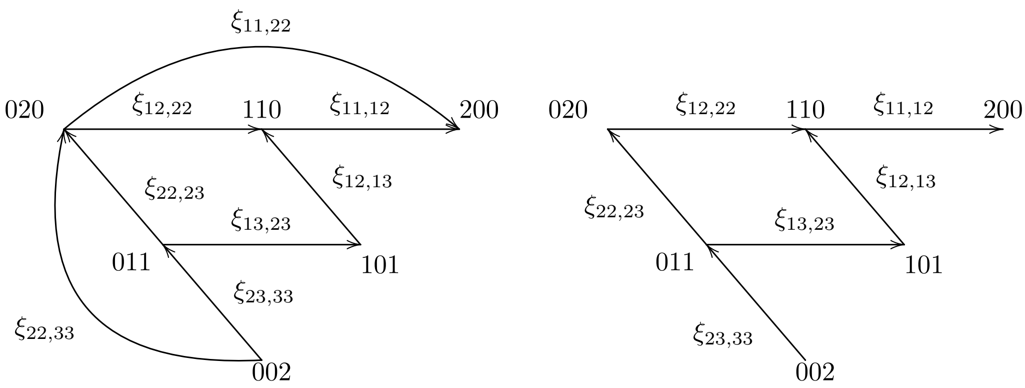
$\phi(1) = 0$. For each $t \in \mathbb{K}^*$, define the linear isomorphism $g_t\colon V \to V$ by $g_t(v_h) = t^{\phi(h)}v_h$, $h = 1, \ldots, m$. Then we obtain another algebra with the product $\mu_t := g_t\mu$ satisfying

$$(\mu_t)(v_h \otimes v_l) = g_t\left(\mu(g_t^{-1}v_h \otimes g_t^{-1}v_l)\right) = \sum_{k=1}^{m} t^{\phi(k)-\phi(h)-\phi(l)}\gamma_{hl}^k v_k.$$

Since $\phi(k) - \phi(h) - \phi(l) \geq 0$ if $\gamma_{hl}^k \neq 0$, we can substitute t by 0 in this formula. We obtain a new product μ_0 which is well defined. It follows from the associativity of the products μ_t for $t \neq 0$ that μ_0 is associative. Since $\phi(1) = 0$, it is easy to check that v_1 is the identity for μ_0. Let G be the subgroup $\{g_t | t \in \mathbb{K}^*\}$ of $\mathrm{GL}(V)$. Then μ_0 is in the Zariski closure of $G\mu$. Henceforth it is also in the Zariski closure of $\mathrm{GL}(V)\mu$.

Now we apply this technique to the algebra $S^+(3,2)$. Its quiver depends on the characteristic of the base field:

(a) char $\mathbb{K} = 2$ $\qquad$ (b) char $\mathbb{K} \neq 2$



(4.1)

Next we compute the multiplication table for $S^+(3,2)$ with respect to the ξ-basis

Multiplication at $(1,0,1)$

	$\xi_{13,23}$	$\xi_{13,33}$
$\xi_{11,13}$	$\xi_{11,23}$	$2\xi_{11,33}$
$\xi_{12,13}$	$\xi_{12,23}$	$\xi_{12,33}$

Multiplication at $(0,1,1)$

	$\xi_{23,33}$
$\xi_{11,23}$	$2\ \xi_{11,33}$
$\xi_{12,32}$	$\xi_{12,33}$
$\xi_{12,23}$	$\xi_{12,33}$
$\xi_{13,23}$	$\xi_{13,33}$
$\xi_{22,23}$	$2\ \xi_{22,33}$

Multiplication at $(0,2,0)$

	$\xi_{22,33}$	$\xi_{22,23}$
$\xi_{11,22}$	$\xi_{11,33}$	$\xi_{11,23}$
$\xi_{12,22}$	$\xi_{12,33}$	$\xi_{12,23}\ +\ \xi_{12,32}$

Multiplication at $(1,1,0)$

	$\xi_{12,23}$	$\xi_{12,33}$	$\xi_{12,13}$	$\xi_{12,22}$	$\xi_{12,32}$
$\xi_{11,12}$	$\xi_{11,23}$	$2\xi_{11,33}$	$\xi_{11,13}$	$2\xi_{11,22}$	$\xi_{11,23}$

Note that we do not list the trivial products $\xi_{i,j}\xi_{i',j'} = 0$ in case j and i' are not in the same Σ_r-orbit, and $\xi_{i,i}\xi_{i,j} = \xi_{i,j} = \xi_{i,j}\xi_{j,j}$. Given $t \in \mathbb{K}^*$, we define the diagonal automorphism g_t of $S^+(3,2)$ by multiplying

$$\xi_{11,22},\ \xi_{22,33},\ \xi_{11,13},\ \xi_{12,13},\ \xi_{13,23},\ \xi_{13,33},$$

with t,

$$\xi_{11,33},\ \xi_{11,23},\ \xi_{12,23},\ \xi_{12,33}$$

with t^2, and by fixing all the other basis elements. The induced product $*_t$ coincides with the original product for all pairs of elements, except the following ones:

$$\xi_{12,32} *_t \xi_{23,33} = t^2\xi_{12,33},\quad \xi_{22,23} *_t \xi_{23,33} = 2t\xi_{22,33},$$

$$\xi_{11,22} *_t \xi_{22,23} = t\xi_{11,23},\quad \xi_{12,22} *_t \xi_{22,33} = t\xi_{12,33}$$

$$\xi_{11,12} *_t \xi_{12,22} = 2t\xi_{11,22},\quad \xi_{11,12} *_t \xi_{12,32} = t^2\xi_{11,23}$$

and

$$\xi_{12,22} *_t \xi_{22,23} = \xi_{12,32} + t^2\xi_{12,23}. \tag{4.2}$$

We shaded the corresponding cells in the multiplication table for $S^+(3,2)$. Now, by setting $t = 0$, we get a new product $*_0$ on the vector space $S^+(3,2)$.

Denote the resulting algebra by B. We can identify from the multiplication table for $S^+(3,2)$ a basis of $\operatorname{rad}^2 B$. Namely, every non-shaded cell in which the basis element has coefficient 1 give an element of $\operatorname{rad}^2 B$ independently of the characteristic of $\mathbb{K}$, i.e.

$$\xi_{11,23},\ \xi_{12,23},\ \xi_{12,33},\ \xi_{13,33},\ \xi_{11,33},\ \xi_{11,13} \tag{4.3}$$

are in $\operatorname{rad}^2 B$. Now, the non-shaded cells in which the basis element has coefficient 2 could give extra elements of $\operatorname{rad}^2 B$ if $\operatorname{char}\mathbb{K} \neq 2$. But only $\xi_{11,33}$ appears in these cells and it is already in the list (4.3). One more element of $\operatorname{rad}^2 B$ comes from (4.2), namely $\xi_{12,32}$. We obtain in this way a basis of $\operatorname{rad}^2 B$. Taking the complement of the computed basis of $\operatorname{rad}^2 B$ inside the basis of $\operatorname{rad} B$, we get that the images of

$$\xi_{11,12},\ \xi_{11,22},\ \xi_{12,22},\ \xi_{12,13},\ \xi_{22,23},\ \xi_{22,33},\ \xi_{13,23},\ \xi_{23,33}$$

by the canonical epimorphism $\operatorname{rad} B \to \operatorname{rad} B/\operatorname{rad}^2 B$ form a basis of $\operatorname{rad} B/\operatorname{rad}^2 B$. Now, it is easy to verify, that the quiver of B coincides with the quiver (4.1)(a) of $S^+(3,2)$ over a field of characteristic 2. From the multiplication table we know that the following products vanish

$$\xi_{22,23} *_0 \xi_{23,33}, \quad \xi_{12,22} *_0 \xi_{22,33}, \quad \xi_{11,22} *_0 \xi_{22,23}, \quad \xi_{11,12} *_0 \xi_{12,22}.$$

Therefore, by inspecting the quiver of B, we see that B is a special biserial algebra. By a classification result of Wald and Waschbüsch [21], we get that the representation type of B is either finite or tame. In other words B is not wild. Hence by Proposition 4.1, the Borel-Schur algebra $S^+(3,2)$ does not have wild representation type. We proved in [8] that $S^+(3,2)$ has infinite representation type, hence $S^+(3,2)$ is tame.

5. Tame representation type: the algebra $S^+(2,5)$ over a field of characteristic 3.

To finish classifying the representation type of Borel-Schur algebras, we need to study $S^+(2,5)$ over a field $\mathbb{K}$ of characteristic 3. In this section we show that this algebra has tame representation type. We will use a combination of Auslander-Reiten theory and poset representation theory as described by Ringel in [17].

Given a finite dimensional algebra A and an A-module M, the *one-point extension* algebra $A[M]$ is the matrix algebra $\left(\begin{smallmatrix} A & M \\ 0 & \mathbb{K} \end{smallmatrix}\right)$. This is relevant since any Borel-Schur algebra $S^+(2,r)$ is a one-point extension algebra $A[M]$, where A is isomorphic to $S^+(2,r-1)$. To see this, let $S := S^+(2,r)$. Take the idempotent $e = \xi_{(0,r)}$ of S, and let $\overline{e} = 1 - e$. From the relations for S (see also [8, Lemma 6.2]), it is immediate that $\overline{e}S\overline{e} \cong S^{+}(2, r-1)$. As a left S-module S can be decomposed into the direct sum $S\overline{e} \oplus Se$. Note that, for any $(0,r) \neq \lambda \in \Lambda(2,r)$, λ dominates $(0,r)$. Therefore $eS\overline{e} = 0$ and hence $S\overline{e} = \overline{e}S\overline{e}$, and $Se = M \oplus \mathbb{K}e$, where $M = \mathrm{rad}Se = \overline{e}Se$. With this, the product of two elements in S has precisely the form of the product in $A[M]$ with $A = \overline{e}S\overline{e}$.

The representation theory of a one-point extension algebra is often closely related to representations of a certain poset. Given a poset $(\mathcal{P}, \leq)$, a $\mathcal{P}$-space (V, V_p) is a vector space V together with subspaces V_p, $p \in \mathcal{P}$, such that $p \leq q$ implies $V_p \subset V_q$. A homomorphism between $\mathcal{P}$-spaces (V, V_p) and (W, W_p) is a linear map $f \colon V \to W$ such that $f(V_p) \subset W_p$ for all $p \in \mathcal{P}$.

Given a natural number k, we also denote by k the ordinal with k elements. Given posets $\mathcal{P}_1$, ..., $\mathcal{P}_s$, we write $(\mathcal{P}_1, \ldots, \mathcal{P}_s)$ for their disjoint union. We denote by N the poset consisting of four elements t_1, t_2 , b_1, b_2 with relations $t_1 < b_i$, $t_i < b_2$ for $i = 1$, 2. This poset can be visualized as follows

$$\begin{tikzcd} t_1 \arrow[d] \arrow[dr] & t_2 \arrow[d] \\ b_1 & b_2. \end{tikzcd}$$

Like for algebras, it is possible to define the representation type of a poset $\mathcal{P}$, with the category of $\mathcal{P}$-spaces taking the place of the category of modules. It was proved by Nazarova in [15] that every poset has either finite, or tame, or wild representation type, and that these possibilities are mutually exclusive. Moreover, she characterized the wild posets.

Theorem 5.1 (Nazarova, [15])**.** *The poset $\mathcal{P}$ is of wild representation type if and only if $\mathcal{P}$ contains as a full subposet one of the sets* $(1,1,1,1,1)$*,* $(1,1,1,2)$*,* $(2,2,3)$*,* $(1,3,4)$*,* $(1,2,6)$*, or* $(N,5)$.

We call the six posets listed in this theorem *Nazarova posets.*

For a finite dimensional algebra A, denote by Γ_A the Auslander-Reiten quiver of A. Now consider a finite dimensional A-module M, and the functor $h_M := \mathrm{Hom}_A(M, -)$. Define Γ_M to be the subquiver of Γ_A whose vertices are given by the indecomposable A-modules N with $h_M(N) \neq 0$, and arrows are given by the irreducible morphisms f in Γ_A such that $h_M(f) \neq 0$.

Proposition 5.2. *Let A be a finite dimensional algebra of finite representation type and M an indecomposable A-module. If $\dim h_M(N) \leq 1$ for all indecomposable A-modules N, then Γ_M is the Hasse diagram of a poset $\mathcal{P}_M$. In this case, the representation type of $A[M]$ coincides with the representation type of $\mathcal{P}_M$.*

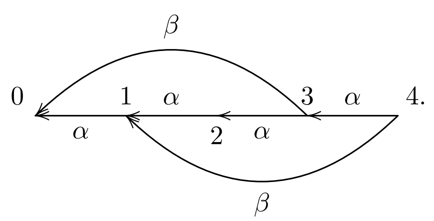
For a proof we refer to Section 2 of [17], see also Section 2 of [18], which discusses this in more detail. The underlying set of $\mathcal{P}_M$ consists of the isomorphism classes $[U]$ of indecomposable A-modules U such that $h_M(U) \neq 0$. The partial order is defined by $[U] \leq [V]$ provided there is an irreducible map $f : U \to V$ such that $h_M(f)$ is non-zero. With our assumptions, for Γ_A without multiple arrows, $h_M(f) \neq 0$ if and only if h_M induces an injective map $h_M(U) \to h_M(V)$. To minimize the number of symbols we write U instead of $[U]$ for an element of the poset.

We apply these results to $S = S^+(2,5)$ for $\mathbb{K}$ of characteristic 3. We have seen that this is the one-point extension $A[M]$ with $A = S^+(2,4)$ and where M is the radical of Se, as described above. We have proved in [8] that $A = S^+(2,4)$ has finite representation type, and computed its Auslander-Reiten quiver. We reproduce it in Figure 1. It is 90° clockwise rotated for typographical reasons. The module M is framed with a circle. We also labelled the modules in Γ_M with encircled numbers for future reference, and we will write M_t for the module in Γ_M labelled with t. In Subsection 5.1 we will prove the following:

- *The modules not in Γ_M are framed with dots.*
- *The arrows in Γ_M are solid and all other arrows are dotted.*

The right side of the quiver is glued to the left side along the dashed line. The resulting poset is redrawn in Figure 2. Later we will show that this poset has tame representation type.

Now we explain the notation we use for modules in Figure 1 as much as we need. Once more we will write α for α_0, β for α_1, and λ_2 for $(\lambda_1, \lambda_2) \in \Lambda(2,4)$. As proved in [8], the quiver Q of A is of the form



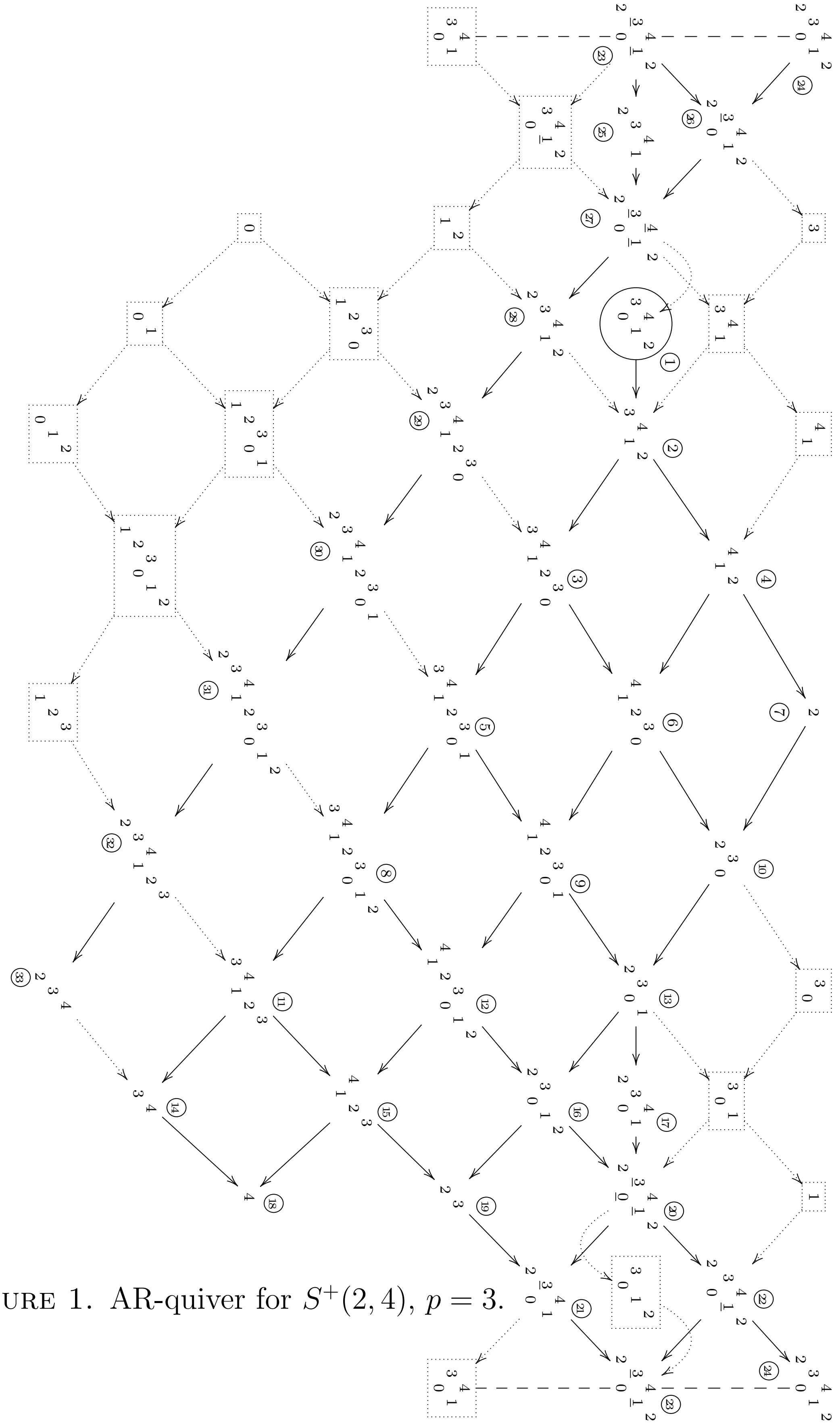


FIGURE 1. AR-quiver for $S^+(2,4)$, $p = 3$.

$$
\begin{tikzcd}[column sep=small, row sep=small]
1 \arrow[d] & & & & & \\
2 \arrow[d] \arrow[dr] & & & & & \\
3 \arrow[d] \arrow[dr] & 4 \arrow[d] \arrow[dr] & & & & \\
5 \arrow[d] \arrow[dr] & 6 \arrow[d] \arrow[dr] & 7 \arrow[d] & & & \\
8 \arrow[d] \arrow[dr] & 9 \arrow[d] \arrow[dr] & 10 \arrow[d] & & & \\
11 \arrow[d] \arrow[dr] & 12 \arrow[d] \arrow[dr] & 13 \arrow[d] \arrow[dr] & & & \\
14 \arrow[dr] & 15 \arrow[d] \arrow[dr] & 16 \arrow[d] \arrow[dr] & 17 \arrow[d] & & \\
& 18 & 19 \arrow[dr] & 20 \arrow[d] \arrow[dr] & & \\
& & & 21 \arrow[dr] & 22 \arrow[d] \arrow[dr] & \\
& & & & 23 \arrow[d] \arrow[dr] & 24 \arrow[d] \\
& & & & 25 \arrow[dr] & 26 \arrow[d] \\
& & & & & 27 \arrow[d, dotted, no head] \\
& & & & & 33
\end{tikzcd}
\tag{5.1}
$$

FIGURE 2. Poset $\mathcal{P}_M$

and A is defined by the relations $\alpha^3 = 0$, $(\beta\alpha)_4 = (\alpha\beta)_4$. Consider the factor algebra $\bar{A} := A/(\beta\alpha)_4$. Note that the $\bar{A}$-modules are precisely those A-modules on which $(\beta\alpha)_4$ (hence $(\alpha\beta)_4$) acts as zero. The algebra $\bar{A}$ is a special biserial algebra, of finite type. Thus indecomposable $\bar{A}$-modules (and the corresponding A-modules) are string modules and can be described by quivers as in [16] or in [5, Chapter II]. Consider for example $4\;{}_{1}\;2\;{}^{3}\;0$. This stands for the 5-dimensional module with basis $\{v_1, \dots, v_5\}$. The canonical idempotents of A act as $\xi_{(4-t,t)}v_t = v_t$, $t = 1, \dots, 4$, the arrows act as follows

$$\beta_4 v_4 = v_1, \quad \alpha_2 v_2 = v_1, \quad \alpha_3 v_3 = v_2, \quad \beta_3 v_3 = v_0,$$

and anything else acts as zero. This module has minimal generators v_4, v_3, and its largest semisimple submodule is spanned by v_1 and v_0.

The above string modules account for most vertices in Figure 1. We do not need explicit descriptions of the other modules, except the module ${}_{3}\overset{4}{\underset{0}{}}{}_{1}$. It has a basis $\{ v_i \mid 0 \leq i \leq 4\}$, the canonical idempotents of A act again as $\xi_{(4-t,t)} v_t = v_t$, the arrows act as follows

$$\beta_4 v_4 = v_1, \quad \alpha_4 v_4 = v_3, \quad \alpha_1 v_1 = v_0, \quad \beta_3 v_3 = v_0,$$

and anything else acts as zero.

5.1. Computing Γ_M. To compute Γ_M we use the following reduction that simplifies calculations. Let L_0 denote the (simple) socle of M and write $\bar{M} = M/L_0$. Notice that $\bar{M} \cong M_2$. We claim that the canonical projection $\pi\colon M \twoheadrightarrow \bar{M}$ induces an isomorphism $\operatorname{Hom}_A(\bar{M}, X) \cong \operatorname{Hom}_A(M, X)$, for every indecomposable A-module X which is not isomorphic to M. Since $M \to \bar{M}$ is an epimorphism, the map

$$\begin{aligned} \operatorname{Hom}_A(\bar{M}, X) &\to \operatorname{Hom}_A(M, X) \\ f &\mapsto f \circ \pi \end{aligned}$$

is an inclusion, and its image can be identified with those $\theta\colon M \to X$ such that $\theta(L_0) = 0$. Suppose this map is not surjective. Then there is $\theta\colon M \to X$ such that $\theta(L_0) \neq 0$. As L_0 is the socle of M, the map θ is injective, and it splits, since M is injective as an A-module. As we assumed that X is indecomposable, we must have $X \cong M$, which contradicts our assumption that $X \ncong M$.

On the part of Γ_A with $\bar{A}$-modules one can compute $h := \operatorname{Hom}_A(\bar{M}, -)$ using the string module presentation (recall that $\bar{A} = A/(\beta\alpha)_4)$. We deal now with the part involving non-string modules. For convenience, we draw the relevant part of Γ_A (recall that the labelling refers to Figure 1). It is also convenient to include several string modules, in particular, M_{13}, M_{16}, M_{19}, and M_{28}.

$$
\begin{tikzcd}[column sep=small]
 & & \boxed{1} \arrow[dr, dotted] & & M_{24} \arrow[dr] & & \boxed{3} \arrow[dr, dotted] & \\
 & \boxed{\begin{smallmatrix}3&&1\\&0&\end{smallmatrix}} \arrow[ur, dotted] \arrow[dr, dotted] & & M_{22} \arrow[ur] \arrow[dr] & & M_{26} \arrow[ur, dotted] \arrow[dr] & & \boxed{\begin{smallmatrix}&4&\\3&&1\end{smallmatrix}} \\
M_{13} \arrow[ur, dotted] \arrow[r] \arrow[dr] & M_{17} \arrow[r] & M_{20} \arrow[ur] \arrow[r, dotted] \arrow[dr] & \boxed{\begin{smallmatrix}&&&2\\3&&1&\\&0&&\end{smallmatrix}} \arrow[r, dotted] & M_{23} \arrow[ur] \arrow[r] \arrow[dr, dotted] & M_{25} \arrow[r] & M_{27} \arrow[ur, dotted] \arrow[r, dotted] \arrow[dr] & M \\
 & M_{16} \arrow[ur] \arrow[dr] & & M_{21} \arrow[ur] \arrow[dr, dotted] & & \boxed{\begin{smallmatrix}&4&&2\\3&&\underline{1}&\\&0&&\end{smallmatrix}} \arrow[ur, dotted] \arrow[dr, dotted] & & M_{28} \\
 & & M_{19} \arrow[ur] & & \boxed{\begin{smallmatrix}&4&\\3&&1\\&0&\end{smallmatrix}} \arrow[ur, dotted] & & \boxed{\begin{smallmatrix}&2\\1&\end{smallmatrix}} \arrow[ur, dotted] &
\end{tikzcd}
$$

(1) We exploit Auslander-Reiten sequences which have terms U, where U is a string module and $h(U) = 0$. In each case, isomorphisms are given by $h(f)$ for a relevant irreducible map, using the fact that h is a functor.

We start with the path from M_{20} via M_{24} to M_{28}. Applying h to the Auslander-Reiten sequence ending in M_{22} gives $h(M_{20}) \cong h(M_{22})$, and applying it to the Auslander-Reiten sequence ending in M_{24} shows that $h(M_{22}) \cong h(M_{24})$. The same type of argument shows that

$$h(M_{24}) \cong h(M_{26}) \cong h(M_{27}).$$

We claim that $h(M_{27}) \cong h(M_{28})$. Consider the Auslander-Reiten sequence $0 \to U \to Z \to V \to 0$ ending in the module $V = {}_1{}^2$. Note that $Z \cong \tau(M_{28})$. Then $h(U) = 0$, as the (simple) socle of U is not a composition factor of $\bar{M}$. As well, $h(V) = 0$, and hence $h(Z) = 0$. This implies that $h(M_{27}) \cong h(M_{28})$, by applying h to the Auslander-Reiten sequence ending in M_{28}. Using the fact that M_{28} is a string module, we check that $h(M_{28}) \cong \mathbb{K}$.

(2) Considering the Auslander-Reiten sequence starting with $M_{19} = {}_2{}^3$ gives $\mathbb{K} \cong h(M_{19}) \cong h(M_{21})$. The Auslander-Reiten sequence starting with M_{21} gives $h(M_{21}) \cong h(M_{23})$, which therefore also is 1-dimensional.

(3) The Auslander-Reiten sequence ending with M_{25} has middle term M_{23}. Since h takes the end term to zero we get $h(M_{23}) \cong h(M_{25})$. Similarly

the Auslander-Reiten sequence starting with M_{17} has indecomposable middle term M_{20}, and h maps the end term to zero. Hence $h(M_{17}) \cong h(M_{20}) \cong \mathbb{K}$.

(4) Considering the Auslander-Reiten sequence ending in $\bar{M}$, we conclude that $h(M) \cong h(\bar{M}) \cong \mathbb{K}$.

In total, for each non-string module W which is not framed, we have that $h(W)$ is one-dimensional and most of the irreducible maps are taken to isomorphisms.

The remaining irreducible maps can be dealt with, either they are injective between modules which are taken to $\mathbb{K}$ by h, or one can use the mesh relations to deduce that h takes them to isomorphisms.

5.2. Γ_M is tame. From now, we only work with the poset Γ_M, and we use the labels for elements as shown in Figure 2. Note that, since $S^+(2,5)$ has infinite representation type by Theorem 0.1, we know by Proposition 5.2 that the poset Γ_M is not of finite type.

Now we prove that Γ_M is not wild. By Theorem 5.1, we must show that it does not contain a Nazarova subposet. We can see directly that Γ_M does not contain five incomparable points, that is, $(1,1,1,1,1)$ does not occur. To exclude the other five Nazarova subposets we use some reductions.

We describe first the strategy. More generally, let $\mathcal{P}$ be any poset. We wish to show that some disjoint unions (Y, Z), with Z of width no less than 2, are not full subposets of $\mathcal{P}$.

For any subposet W of $\mathcal{P}$, we write

$$C_W := \{s \in \mathcal{P} \mid s \text{ is not comparable with any } w \in W\}.$$

If (Y, Z) is a full subposet of $\mathcal{P}$, then Z is contained in C_Y. Denote by U the convex hull of Y, then $C_Y = C_U$. So we get that $Z \subset C_U$. Moreover, if X is any subposet of U then $C_U \subseteq C_X$ and therefore C_X should contain the subposet Z. Thus if Z is of width no less than 2, we get that the same property holds for C_X. This suggests to have a list of *test subsets* X with C_X of width no less than 2.

So let $\mathcal{P} = \Gamma_M$. We will determine all subposets X of size 3 of the form $X = \{x, y, z\}$ with $x < y < z$ where x, y, z are neighbours with respect to $<$, and we will use this list for the strategy as above. We refer to these as minimal triples.

Take such X. Then, referring to the diagram, X is either "vertical", or X is "diagonal", or X is at the edge, by this we mean one of the posets

$$\{10, 13, 17\}, \quad \{17, 20, 22\}, \ \{21, 23, 25\},$$

or X is not convex and its convex hull is a set $V = X \cup \{w\}$ with $x < w < z$. We list now all such posets X and the corresponding C_X for which C_X has width no less than 2.

(1) There are six such X which are vertical:

$$\begin{tikzcd} 5 \arrow[d] \\ 8 \arrow[d] \\ 11 \end{tikzcd} \qquad \begin{tikzcd} 6 \arrow[d] \\ 9 \arrow[d] \\ 12 \end{tikzcd} \qquad \begin{tikzcd} 7 \arrow[d] \\ 10 \arrow[d] \\ 13 \end{tikzcd} \qquad \begin{tikzcd} 8 \arrow[d] \\ 11 \arrow[d] \\ 14 \end{tikzcd} \qquad \begin{tikzcd} 9 \arrow[d] \\ 12 \arrow[d] \\ 15 \end{tikzcd} \qquad \begin{tikzcd} 10 \arrow[d] \\ 13 \arrow[d] \\ 16 \end{tikzcd}$$

The corresponding subsets C_X are

$$\begin{tikzcd} 4 \arrow[d] \arrow[dr] & \\ 6 \arrow[dr] & 7 \arrow[d] \\ & 10 \end{tikzcd}, \quad 7 \ \begin{tikzcd} 11 \arrow[d] \\ 14 \end{tikzcd}, \quad \begin{tikzcd} 8 \arrow[d] \arrow[dr] & \\ 11 \arrow[d] \arrow[dr] & 12 \arrow[d] \\ 14 \arrow[dr] & 15 \arrow[d] \\ & 18 \end{tikzcd}, \quad \begin{tikzcd} 4 \arrow[d] \arrow[dr] & & \\ 6 \arrow[d] \arrow[dr] & 7 \arrow[d] & \\ 9 \arrow[dr] & 10 \arrow[d] & \\ & 13 \arrow[dr] & \\ & & 17 \end{tikzcd}, \quad 14 \ \begin{tikzcd} 7 \arrow[d] \\ 10 \end{tikzcd}, \quad \begin{tikzcd} 11 \arrow[d] \arrow[dr] & \\ 14 \arrow[dr] & 15 \arrow[d] \\ & 18 \end{tikzcd}$$

(2) There are four such X which are diagonal:

$$\begin{tikzcd} 15 \arrow[dr] & & \\ & 19 \arrow[dr] & \\ & & 21 \end{tikzcd} \qquad \begin{tikzcd} 16 \arrow[dr] & & \\ & 20 \arrow[dr] & \\ & & 22 \end{tikzcd} \qquad \begin{tikzcd} 20 \arrow[dr] & & \\ & 22 \arrow[dr] & \\ & & 24 \end{tikzcd} \qquad \begin{tikzcd} 19 \arrow[dr] & & \\ & 21 \arrow[dr] & \\ & & 23 \end{tikzcd}$$

The corresponding subsets C_X are

$$14 \ \begin{tikzcd} 22 \arrow[d] \\ 24 \end{tikzcd}, \quad \begin{tikzcd} 11 \arrow[d] \arrow[dr] & \\ 14 \arrow[dr] & 15 \arrow[d] \\ & 18 \end{tikzcd}, \quad \begin{tikzcd} 11 \arrow[d] \arrow[dr] & & \\ 14 \arrow[dr] & 15 \arrow[d] \arrow[dr] & \\ & 18 & 19 \end{tikzcd}, \quad 24 \ \begin{tikzcd} 14 \arrow[d] \\ 18 \end{tikzcd}$$

(3) There are three posets X at the edge of $\mathcal{P}$ with C_X of width 2:

$$
\begin{tikzcd}[row sep=small, column sep=small]
10 \arrow[d] & \\
13 \arrow[dr] & \\
& 17
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
17 \arrow[d] & \\
20 \arrow[dr] & \\
& 22
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
21 \arrow[dr] & \\
& 23 \arrow[d] \\
& 25
\end{tikzcd}
$$

The corresponding subsets C_X are

$$
\begin{tikzcd}[row sep=small, column sep=small]
8 \arrow[d] \arrow[dr] & \\
11 \arrow[d] \arrow[dr] & 12 \arrow[d] \\
14 \arrow[dr] & 15 \arrow[d] \\
& 18
\end{tikzcd}
,\qquad
\begin{tikzcd}[row sep=small, column sep=small]
11 \arrow[d] \arrow[dr] & & \\
14 \arrow[dr] & 15 \arrow[d] \arrow[dr] & \\
& 18 & 19
\end{tikzcd}
,\qquad
\begin{tikzcd}[row sep=small, column sep=small]
& 14 \arrow[dd] \\
24 & \\
& 18
\end{tikzcd}
$$

(4) There are three sets $V_t = X_t \cup \{w_t\}$, for $t = 1, 2, 3$, for which $C_{X_t}(= C_{V_t})$ has width no less than 2:

$$
V_1 = \begin{tikzcd}[row sep=small, column sep=small]
11 \arrow[d] \arrow[dr] & \\
14 \arrow[dr] & 15 \arrow[d] \\
& 18
\end{tikzcd}
,\quad
V_2 = \begin{tikzcd}[row sep=small, column sep=small]
13 \arrow[d] \arrow[dr] & \\
16 \arrow[dr] & 17 \arrow[d] \\
& 20
\end{tikzcd}
,\quad
V_3 = \begin{tikzcd}[row sep=small, column sep=small]
12 \arrow[d] \arrow[dr] & \\
15 \arrow[dr] & 16 \arrow[d] \\
& 19
\end{tikzcd}
$$

For these, the corresponding subsets C_{V_t} are, respectively,

$$
\begin{tikzcd}[row sep=small, column sep=small]
7 \arrow[d] & & & \\
10 \arrow[d] & & & \\
13 \arrow[d] \arrow[dr] & & & \\
16 \arrow[dr] & 17 \arrow[d] & & \\
& 20 \arrow[dr] & & \\
& & 22 \arrow[dr] & \\
& & & 24
\end{tikzcd}
,\qquad
\begin{tikzcd}[row sep=small, column sep=small]
11 \arrow[d] \arrow[dr] & \\
14 \arrow[dr] & 15 \arrow[d] \\
& 18
\end{tikzcd}
,\qquad
14 \qquad 17 .
$$

With this preparation, we can exclude Nazarova posets.

(I) We claim that $\mathcal{P}$ does not have a subposet isomorphic to $(1,3,4)$. Assume for a contradiction there is a subposet (Y,Z) where $Y \cong (3)$ and $Z \cong (1,4)$. Let U be the convex hull of Y, so $C_U = C_Y$ and $Z \subseteq C_U$. Then U contains a minimal triple X and then $C_U \subseteq C_X$. Therefore C_X contains a subposet isomorphic to $(1,4)$. The only sets X in our list such that C_X contains a subposet (4) are $X = \{8,11,14\}$, $X = \{7,10,13\}$, $X = \{10,13,17\}$, and $X \subset V_1$, but then C_X does not contain a subposet $(1,4)$, a contradiction.

(II) We claim that $\mathcal{P}$ does not have a subposet isomorphic to $(2,2,3)$. Suppose there is a subposet (Y,Z) with $Y \cong (3)$ and $Z \cong (2,2)$. Let U be the convex hull of Y, so $C_U = C_Y$ and $Z \subseteq C_U$. Then U contains a minimal triple X and then $C_U \subseteq C_X$ and $Z \subseteq C_X$ so that C_X contains a subposet isomorphic to $(2,2)$. Our list does not contain such a minimal triple, a contradiction.

(III) We claim that $\mathcal{P}$ does not have a subposet isomorphic to $(N,5)$. Assume for a contradiction that there is a subposet (Y,Z) with Y isomorphic to (5) and Z isomorphic to N. Let U be the convex hull of Y, so that $C_U = C_Y$. Then Y (and U) contains at least three different minimal triples X, and $C_U \subseteq C_X$. Each of these C_X must contain the same copy of N as a subposet. From our list, the only subposets isomorphic to N which occur as subsets of more than one of such C_X are

$$N_1 := \begin{array}{ccc} 14 & 15 & \\ \searrow & \downarrow \ \searrow & \\ & 18 & 19 \end{array} \qquad N_2 := \begin{array}{ccc} 11 & & 12 \\ \downarrow & \searrow & \downarrow \\ 14 & & 15. \end{array}$$

We see $C_{N_1} = \{17,20,22,24\}$ and $C_{N_2} = \{7,10,13,17\}$, which are too small to contain the subposet (5), and we have a contradiction.

(IV) We claim that $\mathcal{P}$ does not contain a subposet isomorphic to $(1,2,6)$. Suppose we have a subposet (Y,Z) with $Y \cong (6)$ and $Z \cong (1,2)$. Let U be the convex hull of Y, so that $C_Y = C_U$. Assume first that U has a subposet V of size four which is the union of two minimal triples $x < y < z$ and $x < w < z$. Then $Z \subseteq C_U \subseteq C_V$. Now, from part (4) of the list, C_V has width ≥ 2 for V_1, V_2 and V_3 but in each case C_V does not contain $(1,2)$, a contradiction. This shows that U (and Y) does not contain such a V. This implies that Y is either "vertical" or "diagonal" (using that Y has size 6). If Y is vertical then it can only be $\{4 < 6 < \ldots < 18\}$ but then $C_Y = \emptyset$. If

Y is diagonal then its smallest element is 8 or 11, and then $C_Y = \emptyset$ as well. This shows that no subposet isomorphic to $(1, 2, 6)$ exists.

(V) We claim that $\mathcal{P}$ does not contain a subposet Y isomorphic to $(1, 1, 1, 2)$. If so then this contains the unique subposet of $\mathcal{P}$ isomorphic to $(1, 1, 1, 1)$, which is $Z = \{14, 15, 16, 17\}$. Then Y is the union of Z together with precisely one element s not in this set. In each case there are two distinct elements of Z comparable with s, so that Y is not the disjoint union of (2) with $(1, 1, 1)$, a contradiction. $\square$

We proved that Γ_M does not contain any Nazarova subposet. So, by Theorem 5.1, we have that Γ_M is not wild. Since, by Proposition 5.2 and Theorem 0.1, we know that it is not of finite type, we conclude that Γ_M is tame.

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