Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 19–42

A DESCRIPTION OF AD-NILPOTENT ELEMENTS IN LIE ALGEBRAS ARISING FROM SEMIPRIME RINGS WITH INVOLUTION

JOSE BROX, ESTHER GARCÍA, MIGUEL GÓMEZ LOZANO, RUBÉN MUÑOZ ALCÁZAR AND GUILLERMO VERA DE SALAS

Abstract: We study ad-nilpotent elements in Lie algebras arising from semiprime rings R without 2-torsion. In order to keep under control the torsion of R we introduce a more restrictive notion of ad-nilpotence, pure ad-nilpotence, which is a technical condition since every ad-nilpotent element can be expressed as an orthogonal sum of pure ad-nilpotent elements of decreasing indices. This allows the torsion inside the ring R to be more accurate. If R is a semiprime ring and $a \in R$ is a pure ad-nilpotent element of R of index n with R free of t and $\binom{n}{t}$-torsion for $t := [\frac{n+1}{2}]$, then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index t. If R is a semiprime ring with involution $*$ and a is a pure ad-nilpotent element of $\mathrm{Skew}(R, *)$ free of t and $\binom{n}{t}$-torsion for $t := [\frac{n+1}{2}]$, then either a is an ad-nilpotent element of R of the same index n (this may occur if $n \equiv 1, 3 \pmod 4$) or a is a nilpotent element of index $t + 1$ and R satisfies a nontrivial GPI (this may occur if $n \equiv 0, 3 \pmod 4$). The case $n \equiv 2 \pmod 4$ is not possible.

Keywords: semiprime rings, rings with involution, Lie algebras, ad-nilpotent elements.
Math. Subject Classification (2000): 16W10, 16N60, 17B60.

1. Introduction

Nilpotent derivations have been a topic of interest since the 1960's. In 1963, Herstein showed that any ad-nilpotent element a of index n in a simple ring of characteristic zero or greater than n gives rise to a nilpotent element $a - \lambda$ for some λ in the center of R. Moreover, he showed that the index of nilpotence of such element is not greater than $[\frac{n+1}{2}]$, see [12, Theorem page 84]. This result

Received December 16, 2019.
The first author was supported by the Portuguese Government through the FCT grant SFRH/BPD/118665/2016. This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2019, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.
The four last authors were partially supported by MTM2017-84194-P (AEI/FEDER, UE), and by the Junta de Andalucía FQM264.

of Herstein was generalized by Martindale and Miers in 1983 ([17, Corollary 1]) to prime rings of characteristic greater than n. This time the nilpotent element has the form $a - \lambda$ for an element λ in the extended centroid of R. In 1978, Kharchenko obtained in [14] an important result: all algebraic derivations of prime rings of characteristic zero are inner for certain elements in an overring; he extended this result to torsion-free semiprime rings in 1979, see [15]. In 1983, Chung and Luh stated that the index of nilpotence of a nilpotent derivation on a semiprime ring of zero characteristic is always odd (see [6] and [7]), and in 1984 Chung, Kobayashi and Luh ([8]) proved that if R is semiprime and $\operatorname{char} R = p > 2$ then the index of nilpotence of a nilpotent derivation is of the form $n = a_s p^s + a_{s+1}p^{s+1} + \cdots + a_l p^l$ where $0 \leq s \leq l$, the a_i are nonnegative integers less than p, a_s is odd, and $a_{s+1}, \ldots, a_l$ are even. Moreover, Chung in 1985 proved, for prime rings R of characteristic zero, that a nilpotent derivation is inner and induced by a nilpotent element of an overring, see [5]. In 1992, with different techniques, Grzeszczuk showed that any nilpotent derivation in a semiprime ring with minimal restrictions on its characteristic is an inner derivation in a semiprime subring of the right Martindale ring of quotients of R and is induced by a nilpotent element in such subring, see [11, Corollary 8] and its generalization by Chuang and Lee in [4, §3].

On the other hand, when dealing with rings with involution $*$, it is natural to study the Lie algebra of skew-symmetric elements $K := \operatorname{Skew}(R, *)$ and the derived Lie algebra $[K, K]/([K, K] \cap Z(R))$. The nilpotent derivations of the skew-symmetric elements of prime rings with involution were studied in the 1990's by Martindale and Miers, who showed that if R is a prime ring with involution of zero characteristic which is not an order in a 4-dimensional central simple Lie algebra and has some inner derivation ad_a with $\operatorname{ad}_a^n = 0$, then there exists an element λ in the extended centroid of R such that either $(a - \lambda)^{[\frac{n+1}{2}]} = 0$ or the involution is of the first kind and $a^{[\frac{n+1}{2}]+1} = 0$, see [18, Main Theorem]. This result was partially extended to semiprime rings by Lee in 2018. In his main result he proved that if R is semiprime with involution and has no $n!$-torsion, then for any $a \in K$ with $\operatorname{ad}_a^n(K) = 0$ there exist λ and a symmetric idempotent e in the extended centroid of R such that $(ea - \lambda)^{[\frac{n+1}{2}]+1} = 0$, see [16, Theorem 1.5].

The main goal of this paper is to deepen into the description of ad-nilpotent elements of K for semiprime rings. In the spirit of Martindale and Miers' result [18, Main Theorem], we will obtain different results about the form of an ad-nilpotent element of K of index n depending on the equivalence class of n modulo 4. To get such results in the semiprime context we introduce a new concept, that of pure ad-nilpotence. We say that an ad-nilpotent element a of index n in $L := R^-$ or K is pure if λa remains ad-nilpotent of the same index for every λ in the extended centroid such that $\lambda a \neq 0$. This is just a technical condition, since every ad-nilpotent element of R^- can be expressed as an orthogonal sum of pure ad-nilpotent elements of decreasing indices.

As a first step we focus on ad-nilpotent elements of R. In this case, under the hypothesis of pure ad-nilpotence, the condition on the torsion of the ring can be weakened when compared with the result of Lee in [16, Theorem 1.3]:

Theorem 4.4 *Let R be a centrally closed semiprime ring with no 2-torsion, and let $a \in R$ be a pure ad-nilpotent element of R of index n. Let $t := [\frac{n+1}{2}]$ and suppose that R is free of $\binom{n}{t}$-torsion and t-torsion. Then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index $\frac{n+1}{2}$.*

When dealing with ad-nilpotent elements of K, we can again split them into orthogonal sums of pure ad-nilpotent elements of decreasing indices. We study each of these pure pieces and get precise descriptions of them depending on the equivalence class of their indices of ad-nilpotence modulo 4.

Theorem 5.6 *Let R be a centrally closed semiprime ring with involution $*$ and no 2-torsion, and let $a \in K$ be a pure ad-nilpotent element of K of index $n > 1$. If R is free of $\binom{n}{t}$-torsion and t-torsion for $t := [\frac{n+1}{2}]$ then:*

- *(1) If $n \equiv 0 \,(mod\ 4)$ then $a^{t+1} = 0$, $a^t \neq 0$ and $a^t K a^t = 0$. Moreover, there exists an idempotent $e \in H(C(R), *)$ such that $ea = a$ and the ideal generated by a^t is essential in eR. In addition eR satisfies the GPI $a^t x a^t y a^t = a^t y a^t x a^t$ for every $x, y \in eR$.*
- *(2) If $n \equiv 1 \,(mod\ 4)$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(a - \lambda)^t = 0$ (a is an ad-nilpotent element of R of index n).*
- *(3) It is not possible that $n \equiv 2 \,(mod\ 4)$.*
- *(4) If $n \equiv 3 \,(mod\ 4)$ then there exists an idempotent $e \in H(C(R), *)$ making $a = ea + (1 - e)a$ such that:*
  - *(4.1) If $ea \neq 0$ then $ea^{t+1} = 0$, $ea^t \neq 0$ and $ea^t k ea^{t-1} = ea^{t-1} k ea^t$ for every $k \in K$. The ideal generated by ea^t is essential in eR and eR satisfies the GPI $a^t x a^t y a^t = a^t y a^t x a^t$ for every $x, y \in eR$.*

(4.2) If $(1-e)a \neq 0$ *then there exists* $\lambda \in \mathrm{Skew}(C(R), *)$ *such that* $((1-e)a-\lambda)^t = 0$ *(*$(1-e)a$ *is a pure ad-nilpotent element of* R *of index* n*).*

In particular, for all $n > 1$ *there exists* $\lambda \in \mathrm{Skew}(C(R), *)$ *such that* $(a-\lambda)^{t+1} = 0$*,* $(a-\lambda)^{t-1} \neq 0$*.*

From these two results describing pure ad-nilpotent elements of R and of K we easily recover Lee's results [16, Theorem 1.3 and Theorem 1.5]. Furthermore, we also describe ad-nilpotent elements of Lie algebras of the form $R/Z(R)$ and $K/(K \cap Z(R))$, and of their derived Lie algebras $[R, R]/([R, R] \cap Z(R))$ and $[K, K]/([K, K] \cap Z(R))$.

2. Preliminaries

In this paper we will be dealing with rings R with or without involution $*$, free of 2-torsion. When R has an involution $*$ we will consider the subsets of skew-symmetric elements $K := \mathrm{Skew}(R, *)$ and symmetric elements $H := H(R, *)$. We will also be dealing with Lie algebras. As usual, a Lie algebra L over a ring of scalars Φ is a Φ-module with an anticommutative bilinear product $[\ ,\ ]$ satisfying the Jacobi identity. Recall that the adjoint map determined by any $x \in L$ is $\mathrm{ad}_x(y) := [x, y]$ for every $y \in L$. Typical examples of Lie algebras come from the associative setting: if R is an associative algebra over a ring of scalars Φ, then R with product $[x, y] := xy - yx$ is a Lie algebra denoted by R^-, and if R has an involution $*$ then K is a Lie subalgebra of R^-.

2.1. A ring R is semiprime (resp. $*$-semiprime) if for every nonzero ideal (resp. $*$-ideal) I of R, $I^2 := \{\sum_i x_i y_i \mid x_i, y_i \in I\} \neq 0$, and it is prime (resp. $*$-prime) if $IJ := \{\sum_i x_i y_i \mid x_i \in I, y_i \in J\} \neq 0$ for every pair of nonzero ideals (resp. $*$-ideals) I, J of R. It is well known that a ring R is prime if and only if $aRb \neq 0$ for arbitrary nonzero elements $a, b \in R$, and it is semiprime if and only if it is nondegenerate, i.e., $aRa \neq 0$ for every nonzero element $a \in R$. Moreover, if R has an involution, the notions of semiprimeness and $*$-semiprimeness coincide.

An ideal I_α of a ring R (resp. with involution $*$) is prime (resp. $*$-prime) if R/I_α is a prime (resp. $*$-prime) ring. If R is a semiprime ring then there exists a family of prime ideals $\{I_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} I_\alpha = \{0\}$ and therefore R can be seen as a subdirect product of prime rings. Similarly, if R is a semiprime ring with involution $*$ there exists a family of $*$-prime ideals $\{I_\alpha\}_{\alpha\in\Delta}$ such that

$\bigcap_{\alpha\in\Delta} I_\alpha = \{0\}$ and therefore R can be seen as a subdirect product of $*$-prime rings.

Moreover, if R is semiprime and free of n-torsion then the intersection of all prime ideals I_α such that R/I_α is free of n-torsion is zero (notice that the intersection of all the prime ideals J_α such that R/J_α has n-torsion is zero since it contains the essential ideal nR of R). With the same argument we also have that semiprime rings without m and n-torsion are subdirect products of prime rings with no m nor n-torsion.

2.2. Given an ideal I of R, the annihilator of I in R is the set $\mathrm{Ann}_R(I) := \{z \in R \mid zI = Iz = 0\}$. The annihilator of an ideal I of R is an ideal of R. Moreover, when R is semiprime $\mathrm{Ann}_R(I) = \{z \in R \mid zIz = 0\}$ and an ideal I of R is essential (for every nonzero ideal J of R, $I \cap J \neq 0$) if and only if $\mathrm{Ann}_R(I) = 0$.

2.3. Recall that the elements of the symmetric Martindale ring of quotients $Q^s_m(R)$ can be seen as pairs $q = [\lambda, I]$ where I is an essential ideal of R and $\lambda : I \to R$ is a monomorphism of right R-modules (if R has an involution one can assume that I is an essential $*$-ideal). When R has an involution $*$, this involution can be extended to $Q^s_m(R)$ as follows: for any $q = [\lambda, I] \in Q^s_m(R)$, $q^* := [\lambda^*, I]$ where $\lambda^*(y) := (\lambda(y^*))^*$ for any y in the essential $*$-ideal I.

The extended centroid $C(R)$ of a semiprime ring R is defined as the center of $Q^s_m(R)$. The extended centroid of a prime ring is a field (see [2, page 70]), the set of symmetric elements of the extended centroid of a $*$-prime ring is again a field (see [1, Theorem 4(a)]), and the extended centroid of a semiprime ring is a commutative and unital von Neumann regular ring (see [2, Theorem 2.3.9(iii)]). In particular, if R is semiprime, $C(R)$ is a semiprime ring without nilpotent elements.

The central closure of R, denoted by $\hat{R}$, is defined as the subring of $Q^s_m(R)$ generated by R and $C(R)$, i.e., $\hat{R} := C(R)R + C(R)$, and can be seen as a $C(R)$-algebra. Therefore we can consider R contained in $\hat{R}$. Moreover, since $\hat{R}$ contains R and is contained in $Q^s_m(R)$, if R is semiprime then $\hat{R}$ is semiprime. The ring $\hat{R}$ is centrally closed, i.e., it coincides with its central closure. In particular its center equals its extended centroid, $Z(\hat{R}) = C(\hat{R})$.

If R is a centrally closed semiprime ring then R^- is a Lie algebra over the ring of scalars $C(R)$; if in addition R has an involution $*$, then K is a Lie algebra over $H(C(R), *)$.

If R is centrally closed without 2-torsion and $\mathrm{Skew}(C(R), *) \neq 0$ (for example, this occurs when R is $*$-prime but not prime), then for any $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$ we have $R = H + K = \lambda^2 H + K \subseteq \lambda K + K \subseteq R$ because $0 \neq \lambda^2$ is invertible, so $R = \lambda K + K$ for every $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$.

2.4. Since the extended centroid $C(R)$ of a semiprime ring R is von Neumann regular, given an element $\lambda \in C(R)$ there exists $\lambda' \in C(R)$ such $\lambda\lambda'\lambda = \lambda$ and $\lambda' = \lambda'\lambda\lambda'$. Let us define $e_\lambda := \lambda\lambda'$. Then e_λ is an idempotent of $C(R)$ satisfying $e_\lambda \lambda = \lambda$. If R has no k-torsion for some $k \in \mathbb{N}$, then for $k = k \cdot 1 \in C(R)$ there exists a unique $k' \in C(R)$ such that $kk'k = k$, so $k(k'k - 1) = 0$ and $k'k = 1$, i.e, $k' = \frac{1}{k} \in C(R)$. In particular, throughout this paper $\frac{1}{2} \in C(R)$ because R will always be a semiprime ring without 2-torsion.

Moreover, if R is a semiprime ring without 2-torsion with involution $*$ and $\lambda \in \mathrm{Skew}(C(R), *)$, then $-\lambda = \lambda^* = (\lambda\lambda'\lambda)^* = \lambda\lambda'^*\lambda$, which implies that λ' can be taken in $\mathrm{Skew}(C(R), *)$ (indeed, replace λ' by $\frac{1}{2}(\lambda' - \lambda'^*)$). In this case, $e_\lambda = \lambda\lambda' \in H(C(R), *)$ is a symmetric idempotent of $C(R)$.

Lemma 2.5. *Let $(R, *)$ be a semiprime ring with involution free of 2-torsion and let $a \in R$. If there exist λ and $\mu \in C(R)$ such that $a - \lambda$ and $a - \mu$ are nilpotent then $\lambda = \mu$. Moreover, if $a \in K$ and $\lambda \in C(R)$ is such that $a - \lambda$ is nilpotent, then $\lambda \in \mathrm{Skew}(C(R), *)$.*

Proof: If $a - \lambda$ and $a - \mu$ are nilpotent elements of the central closure $\hat{R}$ of R, $a - \lambda - (a - \mu) = \mu - \lambda$ is a nilpotent element in the semiprime commutative ring $C(R)$. Therefore $\lambda = \mu$. Now, if $a \in K$ and $a - \lambda$ is nilpotent then $(a - \lambda)^* = -(a + \lambda^*)$ is nilpotent and therefore $a + \lambda^*$ is nilpotent, which implies that $\lambda = -\lambda^* \in \mathrm{Skew}(C(R), *)$. ∎

We will use the following two results due to Beidar, Martindale and Mikhalev.

Theorem 2.6. ([19, Theorem 2(a)]) *Let R be a prime ring. Let $a_i, b_i \in R$ for $i = 1, 2, \ldots, n$ with $b_1 \neq 0$ be such that $\sum_{i=1}^n a_i x b_i = 0$ for every $x \in R$. Then there exist $\lambda_i \in C(R)$ for $i = 2, \ldots, n$ such that $a_1 = \sum_{i=2}^n \lambda_i a_i$ in $\hat{R}$.*

Theorem 2.7. ([2, Theorem 2.3.3]) *Let R be a semiprime ring and let $a_1, a_2, \ldots, a_n \in R$. If $a_1 \notin \sum_{i=2}^n C(R) a_i$ in $\hat{R}$ then there exist $r_j, s_j \in R$*

for $j = 1, 2, \ldots, m$ *such that* $\sum_{j=1}^{m} r_j a_1 s_j \neq 0$ *and* $\sum_{j=1}^{m} r_j a_k s_j = 0$ *for* $k = 2, \ldots, n$.

The next corollary can be found in [3]. For the sake of completeness we include its proof here.

Corollary 2.8. *Let* R *be a semiprime ring. Let* $a_i, b_i \in R$ *for* $i = 1, 2, \ldots, n$ *be such that* $\mathrm{Id}_R(a_1) \subset \mathrm{Id}_R(b_1)$ *and* $\sum_{i=1}^{n} a_i x b_i = 0$ *for every* $x \in R$. *Then there exist* $\lambda_i \in C(R)$ *for* $i = 2, \ldots, n$ *such that* $a_1 = \sum_{i=2}^{n} \lambda_i a_i$ *in* $\hat{R}$.

Proof: By Theorem 2.7, if $a_1 \notin \sum_{i=2}^{n} C(R) a_i$ there exist $r_j, s_j \in R$, $j = 1, \ldots, m$, such that $\sum_{j=1}^{m} r_j a_1 s_j \neq 0$ and $\sum_{j=1}^{m} r_j a_k s_j = 0$ for $k = 2, 3, \ldots, n$. Replace x by $s_j x$ and multiply $\sum_{i=1}^{n} a_i x b_i = 0$ on the left by r_j. We have

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{m} r_j a_i s_j x b_i = \sum_{j=1}^{m} r_j a_1 s_j x b_1,$$

which implies that the ideal generated by $\sum_{j=1}^{m} r_j a_1 s_j$ is orthogonal to the ideal generated by b_1 and therefore, since $\mathrm{Id}_R(a_1) \subset \mathrm{Id}_R(b_1)$, the ideal generated by $\sum_{j=1}^{m} r_j a_1 s_j$ has zero square, a contradiction because R is semiprime. ■

The following proposition is an easy generalization of [2, Theorem 2.3.9(i)].

Proposition 2.9. *Let* R *be a centrally closed semiprime ring free of 2-torsion. For any subset* $V \subset R$ *there exists a unique idempotent* $e \in C(R)$ *such that* $ev = v$ *for all* $v \in V$, *the annihilator in* $C(R)$ *of* V *is* $\mathrm{Ann}_{C(R)}(V) = (1 - e)C(R)$, *the annihilator in* R *of the ideal generated by* V *is* $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1 - e)R$, *and the ideal generated by* V *is essential in* eR. *Moreover, when* R *has an involution* $*$ *and* $V \subset H$ *or* $V \subset K$, *then* $e \in H(C(R), *)$.

Proof: The first part of the proof follows as in [2, Theorem 2.3.9(i)] with the obvious changes. Let $V \subset H \cup K$ and consider the unique idempotent $e \in C(R)$ such that $ev = v$ for all $v \in V$, the annihilator in $C(R)$ of V is $\mathrm{Ann}_{C(R)}(V) = (1 - e)C(R)$ and the annihilator in R of the ideal generated by V is $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1 - e)R$. When R has an involution we can decompose $e = e_k + e_h$ with $e_k \in \mathrm{Skew}(C(R), *)$ and $e_h \in H(C(R), *)$. We have that $ev = v$ implies $e_k v = 0$. Therefore, $e_k \in \mathrm{Ann}_{C(R)}(V) = (1 - e)C(R)$, i.e., $e_k e = 0$ and $e_k^2 = e_k e_h = 0$ and therefore $e = e^2 = (e_k + e_h)^2 = e_h^2 \in H(C(R), *)$. ■

Lemma 2.10. *Let R be a centrally closed semiprime algebra and let $\{u_i\}_{i\in I}$ be a family of idempotent elements in $C(R)$. Suppose there exists a family $\{\lambda_i\}_{i\in I}$ of elements in $C(R)$ such that for every $i, j \in I$, $\lambda_i u_i u_j = \lambda_j u_i u_j$. Then there exists $\lambda \in C(R)$ such that $\lambda u_i = \lambda_i u_i$ for every $i \in I$. Moreover, if the ideal generated by the family $\{u_i\}_{i\in I}$ is essential in R, such λ is unique.*

Proof: Let us consider the ideal $S = \sum Ru_i$ generated by the family of idempotents $\{u_i\}_{i\in I}$ and the essential ideal $T = S \oplus \mathrm{Ann}_R(S)$. Define $\lambda : T \to R$ by

$$\lambda(\sum x_i u_i + z) := \sum \lambda_i x_i u_i.$$

Let us prove that λ is well defined and an element in $C(R)$. If $\sum x_i u_i + z = 0$ then $\sum x_i u_i = 0 = z$ and for every u_k we have

$$\left(\sum \lambda_i x_i u_i\right) u_k = \sum \lambda_k x_i u_i u_k = \lambda_k \left(\sum x_i u_i\right) u_k = 0.$$

Therefore $\sum \lambda_i x_i u_i \in S \cap \mathrm{Ann}_R(S) = 0$ which proves that λ is well defined. By construction $[\lambda, S \oplus \mathrm{Ann}_R(S)] \in C(R)$. Moreover, if the ideal S generated by the family $\{u_i\}_{i\in I}$ is essential, $\mathrm{Ann}_R(S) = 0$ and $[\lambda, S] \in C(R)$ is uniquely defined. ■

3. Pure ad-nilpotent elements

Recall that an element a in a Lie algebra L is ad-nilpotent of index n if $\mathrm{ad}_a^n(L) = 0$ and $\mathrm{ad}_a^{n-1}(L) \neq 0$.

3.1. In the particular case of $L = R^-$ (resp. $L = K$ for a ring R with involution $*$), we say that an element a is a pure ad-nilpotent element of L of index n if for every $\lambda \in C(R)$ (resp., $\lambda \in H(C(R)), *)$) with $\lambda a \neq 0$, λa is again ad-nilpotent of the same index n in $\hat{R}^-$ where $\hat{R}$ is the central closure of R.

Lemma 3.2. *If R is a semiprime ring and a is an ad-nilpotent element of R of index n, the following conditions are equivalent:*

(i) *a is a pure ad-nilpotent element of R^-.*
(ii) $\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))$ *is an essential ideal of* $\mathrm{Id}_R(a)$.
(iii) $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a))$.

Proof: Suppose that R is semiprime and centrally closed (otherwise, substitute R by its central closure $\hat{R}$).

(i) $\Rightarrow$ (ii). Let us consider $V = \{\mathrm{ad}_a^{n-1} x \mid x \in R\}$. By Proposition 2.9 there exists $e \in C(R)$ such that $ev = v$ for every $v \in V$ and $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1-e)R$. Suppose that $(1-e)a \neq 0$. By hypothesis $(1-e)a$ is ad-nilpotent of index n, hence $0 \neq \mathrm{ad}_{(1-e)a}^{n-1}(R) = (1-e)\,\mathrm{ad}_a^{n-1}(R) = 0$, a contradiction. So $ea = a$ and $\mathrm{Ann}_{\mathrm{Id}_R(ea)}(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) \subset \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = (1-e)R$ must be zero, i.e., $\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))$ is essential in $\mathrm{Id}_R(ea)$.

(ii)$\Rightarrow$ (iii). This holds in general if I and J are ideals of R with I essential in J: $0 = \mathrm{Ann}_J(I) = \mathrm{Ann}_R(I) \cap J$ implies $\mathrm{Ann}_R(I)J = 0$, so $\mathrm{Ann}_R(I) \subset \mathrm{Ann}_R(J)$.

(iii) $\Rightarrow$ (i). Let $\lambda \in C(R)$ be such that $\lambda a \neq 0$. Clearly $\mathrm{ad}_{\lambda a}^n(R) = 0$. Suppose that $\mathrm{ad}_{\lambda a}^{n-1}(R) = 0$: then $\lambda^{n-1}\,\mathrm{ad}_a^{n-1}(R) = 0$, so

$$\lambda^{n-1} \in \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a)),$$

which is not possible because R is semiprime and $\lambda a \neq 0$. ■

Lemma 3.3. *Let R be a centrally closed semiprime ring with involution $*$ and no 2-torsion, and let $a \in K$ be a pure ad-nilpotent element of K of index n. If there exists $\lambda \in H(C(R), *)$ such that λa is ad-nilpotent of R of index n, then λa is a pure ad-nilpotent element of R of index n.*

Proof: Suppose on the contrary that there exists

$$\mu \in H(C(R), *) \bigcup \mathrm{Skew}(C(R), *)$$

such that $\mathrm{ad}_{\mu\lambda a}^{n-1} R = 0$. Then $\mathrm{ad}_{\mu\mu\lambda a}^{n-1} R = 0$ implies $\mathrm{ad}_{\mu\mu\lambda a}^{n-1} K = 0$, a contradiction. ■

The next proposition shows that every ad-nilpotent of R^- and of K can be expressed as an orthogonal sum of pure ad-nilpotent elements of decreasing indices.

Proposition 3.4. *Let R be a centrally closed semiprime ring and let $a \in R$ be an ad-nilpotent element of R^- of index n. There exists a family of orthogonal idempotents $\{e_i\}_{i=1}^k \subset C(R)$ such that $a = \sum_{i=1}^k e_i a_i$ with $e_i a$ a pure ad-nilpotent element of index n_i in $e_i R$ for $n = n_1 > n_2 > \cdots > n_k$.*

Similarly, if R has an involution $*$ and a is an ad-nilpotent element of K of index n, then there exists a family of orthogonal idempotents $\{e_i\}_{i=1}^k \subset$

$H(C(R),*)$ such that $a = \sum_{i=1}^{k} e_i a_i$ with $e_i a$ a pure ad-nilpotent element of index n_i in $\mathrm{Skew}(e_i R, *)$ *for $n = n_1 > n_2 > \cdots > n_k$.*

Proof: Let us prove the result for Lie algebras of skew-symmetric elements. We will proceed by induction on n. If $n = 2$ there is nothing to prove. Let us suppose that the result is true for every ad-nilpotent element of index less than n and let $a \in K$ be an ad-nilpotent element of index $n \geq 3$. Let us consider $V = \{\mathrm{ad}_a^{n-1} x \mid x \in K\}$. By Proposition 2.9 there exists $e \in H(C(R),*)$ such that $ev = v$ for every $v \in V$ and $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1-e)R$. Then $a = ea + (1-e)a$.

Clearly, by construction $(1-e)a$ is ad-nilpotent of index less than n in K: for every $x \in K$, $\mathrm{ad}_{(1-e)a}^{n-1} x = (1-e)\,\mathrm{ad}_a^{n-1} x = \mathrm{ad}_a^{n-1} x - e\,\mathrm{ad}_a^{n-1} x = 0$.

Let us prove that ea is pure ad-nilpotent of index n in $\mathrm{Skew}(eR,*)$. For any $\lambda \in H(C(R),*)$ such that $\lambda ea \neq 0$, λea is ad-nilpotent of index n: clearly $\mathrm{ad}_{\lambda ea}^{n}(\mathrm{Skew}(eR,*)) = 0$ and if $\mathrm{ad}_{\lambda ea}^{n-1}(\mathrm{Skew}(eR,*)) = 0$ then $\lambda^{n-1} e \in \mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1-e)R$, which leads to a nilpotent ideal generated by the nonzero element λea, a contradiction with the semiprimeness of R.

Apply now the induction hypothesis to $(1-e)a$ and the Lie algebra of skew-symmetric elements $\mathrm{Skew}((1-e)R,*)$. ∎

4. Ad-nilpotent elements of R

In this section we are going to prove that every nilpotent inner derivation is induced by a nilpotent element, generalizing to semiprime rings Herstein's result [12, Theorem page 84] for simple rings. This result was already proved by Grzeszczuk ([11, Corollary 8]). Our techniques are rather elementary and, by adding the hypothesis of pure ad-nilpotence, we can describe such elements with less restrictions on the torsion of the ring.

Lemma 4.1. *Let R be a semiprime ring and let $a \in R$ be a nilpotent element. Suppose that there exist some $\lambda_i \in \mathbb{Z}$, $i = 0, \ldots, n$, such that*

$$\sum_{i=0}^{n} \lambda_i a^i [x,y] a^{n-i} = 0$$

for all $x, y \in R$. Then for every $i = 0, \ldots, n$ we have $\lambda_i a^{\max(i,n-i)} = 0$. In particular, each term in the identity above is zero.

Proof: First, let us suppose that R is prime and suppose that $a \neq 0$ has index of nilpotence s. If the lemma is not satisfied, there exists some k with $\lambda_k a^{\max(k,n-k)} \neq 0$. In particular, $\max(k, n-k) < s$. Let us multiply the expression $\sum_{i=0}^{n} \lambda_i a^i [x, y] a^{n-i}$ by a^{s-1-k} on the left and by $a^{s-1-(n-k)}$ on the right, so that

$$0 = a^{s-1-k} \left(\sum_{i=0}^{n} \lambda_i a^i [x, y] a^{n-i} \right) a^{s-1-(n-k)} = \lambda_k a^{s-1} [x, y] a^{s-1}$$

for every $x, y \in R$. Hence $\lambda_k a^{s-1} xya^{s-1} = \lambda_k a^{s-1} yxa^{s-1}$ for every $x, y \in R$. Since $a^{s-1} \neq 0$ for every $x \in R$ we have by Theorem 2.6 that there exists $\alpha_x \in C(R)$ such that $\lambda_k a^{s-1} x = \alpha_x \lambda_k a^{s-1}$. Multiplying this last expression by a on the right we get $\lambda_k a^{s-1} xa = 0$ for every $x \in R$. By primeness of R we get that either $a^{s-1} = 0$ or $\lambda_k a = 0$, leading to a contradiction.

If R is semiprime then R is a subdirect product of prime quotients R/I_α with $\bigcap_\alpha I_\alpha = 0$. For any α and any i, by the prime case $\lambda_i a^{\max(i,n-i)} \in I_\alpha$, so $\lambda_i a^{\max(i,n-i)} = 0$. ∎

Lemma 4.2. *Every nilpotent element of a ring R is ad-nilpotent. If a has index of nilpotence s and index of ad-nilpotence n then $n \leq 2s - 1$. If R is semiprime then $n \geq s$, and if in addition R is free of $\binom{2s-2}{s-1}$-torsion, then the index of ad-nilpotence of a is $n = 2s - 1$.*

Proof: Since $a^s = 0$, for every $x \in R$ we have

$$\operatorname{ad}_a^{2s-1} x = \sum_{i=0}^{2s-1} \binom{2s-1}{i} (-1)^{2s-1-i} a^i x a^{2s-1-i} = 0$$

because if $i < s$ then $2s - 1 - i \geq s$. Therefore $n \leq 2s - 1$.

Suppose now that R is semiprime and let us see that $n \geq s$: if on the contrary

$$\operatorname{ad}_a^{s-1} x = \sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^{s-1-i} a^i x a^{s-1-i} = 0$$

for every $x \in R$, focusing on the first summand of this expression $((-1)^{s-1} x a^{s-1})$ we get that $a^{s-1} = 0$ by Lemma 4.1, a contradiction.

Now suppose that R is semiprime and free of $\binom{2s-2}{s-1}$-torsion. If $\mathrm{ad}_a^{2s-2}(R) = 0$, then for every $x \in R$

$$0 = \mathrm{ad}_a^{2s-2} x = \binom{2s-2}{s-1} (-1)^{s-1} a^{s-1} x a^{s-1}$$

since $a^s = 0$. By semiprimeness of R we get $\binom{2s-2}{s-1} a^{s-1} = 0$ and hence $a^{s-1} = 0$ by the lack of $\binom{2s-2}{s-1}$-torsion, a contradiction. Therefore the index of ad-nilpotence of a is $n = 2s - 1$. ■

The next example shows that all possible cases in the lemma above can be realized: Let p be an odd prime number and R a prime ring with characteristic p. If $a \in R$ is a nilpotent element of index $s \in \{\frac{p+1}{2}, \dots, p\}$ then a is ad-nilpotent of index p. In particular there are no ad-nilpotent elements of index between $p + 1$ and $2p - 1$, and a nilpotent element of index p is ad-nilpotent of the same index p.

Proposition 4.3. *Let R be a prime ring and let $a \in R$ be an ad-nilpotent element of R^- of index n. Then:*

(1) If $\overline{\mathbb{F}}$ denotes the algebraic closure of the field $\mathbb{F} = C(R)$, there exists $\mu \in \overline{\mathbb{F}}$ such that $a - \mu$ is a nilpotent element.

(2) If R is free of $\binom{n}{t}$-torsion for $t := [\frac{n+1}{2}]$ then n is odd and the index of nilpotence of $a - \mu$ is $\frac{n+1}{2}$. If in addition R is free of t-torsion then $\mu \in C(R)$.

Proof: (1) Since R is prime, $\mathbb{F} = C(R)$ is a field. From

$$0 = \mathrm{ad}_a^n x = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a^i x a^{n-i}$$

for every $x \in R$ we have, by Theorem 2.6, that a is an algebraic element of R over $\mathbb{F}$. Let us consider the minimal polynomial $p(X) \in \mathbb{F}(X)$ of a. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and let $\mu_1, \dots, \mu_t \in \overline{\mathbb{F}}$ be the roots of $p(X)$ in $\overline{\mathbb{F}}$, i.e., $p(X) = (X - \mu_1)^{k_1} \cdots (X - \mu_t)^{k_t} \in \overline{\mathbb{F}}[X]$.

Let us prove that $p(X)$ has only one root in $\overline{\mathbb{F}}$ and therefore $p(x) = (x - \mu)^k \in \mathbb{F}[X]$, whence $a - \mu$ is nilpotent in $\overline{\mathbb{F}}$: Suppose on the contrary that $p(X)$ has different roots $\mu_1, \dots, \mu_t$, $t > 1$, and define $q_i(X) := p(X)/(X - \mu_i)$ for every i. Since $p(X)$ is the minimal polynomial of a, $q_i(a) \neq 0$ in $R \otimes \overline{\mathbb{F}}$. Note that $(a - \mu_i) q_i(a) = p(a) = 0$ and therefore $a q_i(a) = \mu_i q_i(a)$. Now, since we are in the prime case, there exists $y \in R$ such that $q_1(a)yq_2(a) \neq 0$ and therefore $\mathrm{ad}_a(q_1(a)yq_2(a)) = aq_1(a)yq_2(a) - q_1(a)yq_2(a)a = (\mu_1 - \mu_2)q_1(a)yq_2(a) \neq 0$. This means that $q_1(a)yq_2(a)$ is an eigenvector of the linear map ad_a associated to the eigenvalue $\mu_1 - \mu_2$, hence it is an eigenvector of ad_a^2 associated to $(\mu_1 - \mu_2)^2$, etc. This is a contradiction because both $q_1(a)yq_2(a)$ and each power of $(\mu_1 - \mu_2)$ are nonzero, while ad_a is nilpotent. Therefore $t = 1$, $p(X) = (X - \mu)^k \in \mathbb{F}[X]$ and $(a - \mu)^k = 0$.

(2) Let us consider $b := a - \mu \in R \otimes \overline{\mathbb{F}}$, which is ad-nilpotent of index n. Let us see that n is odd: Suppose on the contrary that $n = 2m$. Then

$$0 = \mathrm{ad}_a^n x = \mathrm{ad}_b^n x = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b^i x b^{n-i}$$

implies by Lemma 4.1 that $\binom{n}{m} b^m = 0$ and, since $R \otimes \overline{\mathbb{F}}$ is free of $\binom{n}{m}$-torsion, that $b^m = 0$. Substituting in $\mathrm{ad}_b^{n-1} x = \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{n-1-i} b^i x b^{n-1-i}$ we get that $\mathrm{ad}_b^{n-1} x = 0$ for every $x \in R$, a contradiction.

Therefore n is odd and $a - \mu$ is nilpotent of index $t = \frac{n+1}{2}$ by Lemma 4.2. Moreover, since the coefficient of degree $t - 1$ of $p(X) = (X - \mu)^t \in \mathbb{F}[X]$ is $-t\mu \in \mathbb{F}$, if R is free of t-torsion then $\mu \in \mathbb{F}$, i.e., there exists $\mu \in C(R)$ such that $a - \mu$ is nilpotent of index $t = \frac{n+1}{2}$. ■

In the following theorem we get the description of the pure ad-nilpotent elements of R^-. In its proof, Proposition 4.3 is primarily used to find that any ad-nilpotent element $a \in R$ of index n forces $[a, [\mathrm{ad}_a^{n-1} x, [\mathrm{ad}_a^{n-1} x, y]]] = 0$ for every $x, y \in R$. If $2, 3, \ldots, r$ were invertible in R for $r \geq n + [\frac{n}{2}] + 1$, this identity would directly follow from the proof of [10, Theorem 2.3].

Theorem 4.4. *Let R be a centrally closed semiprime ring with no 2-torsion and let $a \in R$ be a pure ad-nilpotent element of R^- of index n. Put $t := [\frac{n+1}{2}]$, and suppose that R is free of $\binom{n}{t}$-torsion and t-torsion. Then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index $\frac{n+1}{2}$.*

Proof: Let us suppose that R is a prime ring and consider $\mu \in C(R)$ as given by Proposition 4.3. Putting $b := a - \mu$, we know that $b^t = 0$ for $t = \frac{n+1}{2}$, hence for every $x, y \in R$ we have

$$\begin{aligned}(\mathrm{ad}_a^{n-1} x)(\mathrm{ad}_a^{n-1} x) = (\mathrm{ad}_b^{n-1} x)(\mathrm{ad}_b^{n-1} x) = 0, \text{ and} \\ [a, [\mathrm{ad}_a^{n-1} x, [\mathrm{ad}_a^{n-1} x, y]]] = [b, [\mathrm{ad}_b^{n-1} x, [\mathrm{ad}_b^{n-1} x, y]]] \\ = -2\binom{n-1}{t-1}\binom{n-1}{t-1}[b, b^{t-1}xb^{t-1}yb^{t-1}xb^{t-1}] = 0.\end{aligned}$$

If R is semiprime, R is a subdirect product of prime rings (without $\binom{n}{t}$ and t-torsion) and in any of these prime quotients

$$\overline{(\mathrm{ad}_a^{n-1} x)(\mathrm{ad}_a^{n-1} x)} = \overline{0} \text{ and } \overline{[a, [\mathrm{ad}_a^{n-1} x, [\mathrm{ad}_a^{n-1} x, y]]]} = \overline{0},$$

which imply that

$$(\mathrm{ad}_a^{n-1} x)(\mathrm{ad}_a^{n-1} x) = 0, \text{ and } [a, [\mathrm{ad}_a^{n-1} x, [\mathrm{ad}_a^{n-1} x, y]]] = 0$$

for every $x, y \in R$. For every $x \in R$, let $z_x := \mathrm{ad}_a^{n-1} x$. By the identity above,

$$0 = \frac{1}{2}[a, [z_x, [z_x, y]]] = -az_xyz_x + z_xyz_xa.$$

Therefore, since $\mathrm{Id}_R(z_xa) \subset \mathrm{Id}_R(z_x)$, by Corollary 2.8 there exists $\lambda_x \in C(R)$ such that $z_xa = \lambda_x z_x$ and by Proposition 2.9 there exists $e_x \in H(C(R), *)$ such that $e_xz_x = z_x$ and $\mathrm{Ann}_R(\mathrm{Id}_R(z_x)) = (1 - e_x)R$. Therefore

$$\begin{aligned}0 = z_x \mathrm{ad}_a^n y = z_x \left(\sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i}a^iya^{n-i}\right) = \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i}z_xa^iya^{n-i} \\ = \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i}z_x\lambda_x^iya^{n-i} = z_xy\left(\sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i}\lambda_x^ia^{n-i}\right) = z_xy(a - \lambda_x)^n\end{aligned}$$

for every $y \in R$, whence $(a - \lambda_x)^n \in \mathrm{Ann}_R(\mathrm{Id}_R(z_x))$. So $e_x(a - \lambda_x)^n = 0$. Now, for every $x, x' \in R$ there exist $\lambda_x, \lambda_{x'} \in C(R)$ and idempotents $e_x, e_{x'} \in H(C(R), *)$ such that $0 = (e_xe_{x'}a - e_xe_{x'}\lambda_x)^n = (e_xe_{x'}a - e_xe_{x'}\lambda_{x'})^n$, so $e_xe_{x'}\lambda_x = e_xe_{x'}\lambda_{x'}$ by Lemma 2.5. By Lemma 2.10 there exists $\lambda \in C(R)$ such that $e_x\lambda = e_x\lambda_x$ for every $x \in R$. Then for every $x \in R$ we have $z_x(a - \lambda)^n = e_xz_x(a - \lambda_x)^n = 0$, so $(a - \lambda)^n \in \bigcap_{x\in R} \mathrm{Ann}_R(z_x) = \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a))$, because a is pure. Finally, let $e \in C(R)$ be such that $ea = a$ and $\mathrm{Ann}_R(\mathrm{Id}_R(a)) = (1 - e)R$. Then $e(a - \lambda)^n = (a - e\lambda)^n = 0$ because it is contained in $(1 - e)R$.

Hence $a - e\lambda$ is nilpotent in addition to being ad-nilpotent of index n. Put $t := [\frac{n+1}{2}]$ and take any prime quotient without t and $\binom{n}{t}$-torsion in which $\overline{a - e\lambda}$ is still ad-nilpotent of index n. By Proposition 4.3(2) we get that n must be odd and $\overline{a - e\lambda}$ is nilpotent of index t. Since in any prime quotient $(\overline{a - e\lambda})^t = \bar{0}$ by Proposition 4.3(2), we have that t is the index of nilpotence of $a - e\lambda$. ■

Lee's description of ad-nilpotent elements of R^- is recovered when the hypothesis of being pure is removed.

Corollary 4.5. ([16, Theorem 1.3]) *Let R be a centrally closed semiprime ring, let $a \in R$ be an ad-nilpotent element of R^- of index n, and suppose that R is free of $n!$-torsion. Then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index $\frac{n+1}{2}$.*

Proof: By Proposition 3.4 there exists a family of orthogonal idempotents $\{e_i\}_{i=1}^k \subset C(R)$ such that $a = \sum_{i=1}^k e_i a$ with $e_i a$ a pure ad-nilpotent element of index n_i $(n = n_1 > n_2 > \cdots)$ of Re_i. Then by Theorem 4.4 there exists a family of scalars $\{\lambda_i\}_{i=1}^k \subset C(R)$ such that $(e_i a - \lambda_i)^{t_i} = 0$ for $t_i := [\frac{n_i+1}{2}]$. Hence $\lambda = \sum_{i=1}^n \lambda_i$ satisfies the claim. ■

Interesting Lie algebras associated to simple rings R are the quotient algebras $[R, R]/([R, R] \cap Z(R))$, which are simple unless R has 2-torsion and is 4-dimensional over its center ([13, Theorem 1.13]). Let us study ad-nilpotent elements in these algebras.

Lemma 4.6. *Let R be a semiprime ring and let $a \in R$ be such that $\mathrm{ad}_a^n(R) \subset Z(R)$. Then $\mathrm{ad}_a^n(R) = 0$.*

Proof: For every $x \in R$ we have

$$0 = [\mathrm{ad}_a^n(xa), x] = [(\mathrm{ad}_a^n\, x)a, x] = (\mathrm{ad}_a^n\, x)[a, x].$$

Therefore $0 = \mathrm{ad}_a^{n-1}((\mathrm{ad}_a^n\, x)[a, x]) = (\mathrm{ad}_a^n\, x)^2$ which implies, since R is semiprime and $\mathrm{ad}_a^n\, x \in Z(R)$, that $\mathrm{ad}_a^n\, x = 0$. ■

Lemma 4.7. *Let R be a centrally closed semiprime associative ring, let $L := [R, R]/([R, R] \cap Z(R))$ and let $\overline{a} := a + ([R, R] \cap Z(R)) \in L$ be an ad-nilpotent element of L of index n. Then a is an ad-nilpotent element of index n in R^-.*

Proof: For every $x \in R$, $\mathrm{ad}_a^{n+1} x = \mathrm{ad}_a^n([a,x]) \in \mathrm{ad}_a^n([R,R]) \subset Z(R)$ so, by Lemma 4.6, $\mathrm{ad}_a^{n+1} x = 0$ for every $x \in R$, i.e., a is ad-nilpotent in R^- of index n or $n+1$.

Let us suppose that R is prime. Then, by Proposition 4.3, there exists $\mu \in \overline{\mathbb{F}}$, the algebraic closure of $\mathbb{F} := C(R)$, such that $a - \mu$ is nilpotent in $R \otimes \overline{\mathbb{F}}$ of some index s. Moreover, by Lemma 4.2, $s \leq n+1$. Put $b := a - \mu$. Then

$$0 = \mathrm{ad}_a^n([x,y]) = \mathrm{ad}_b^n([x,y]) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b^i [x,y] b^{n-i}$$

for every $x, y \in R$. By Lemma 4.1, for every $k \in \{0, 1, \ldots, [\frac{n+1}{2}]\}$ we have $\binom{n}{k} b^{\max(k,n-k)} = 0$, so

$$\mathrm{ad}_a^n x = \mathrm{ad}_b^n x = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b^i x b^{n-i} = 0,$$

i.e., a is an ad-nilpotent element of R^- of index n.

Finally, since $\overline{a}$ is ad-nilpotent of index not greater than n in any prime quotient, a is an ad-nilpotent element of R^- of index n when R is semiprime. ■

In particular, from these last two lemmas we get that if R is semiprime then $[R,R]/([R,R] \cap Z(R))$ and $R/Z(R)$ are nondegenerate Lie algebras.

Corollary 4.8. *Let R be a centrally closed semiprime associative ring and let $L := [R,R]/([R,R] \cap Z(R))$ or $L := R/Z(R)$. If $\overline{a} \in L$ is an ad-nilpotent element of L of index n and R is free of $n!$-torsion, then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index $\frac{n+1}{2}$.*

Proof: If $L = [R,R]/([R,R] \cap Z(R))$ the result follows by Lemma 4.7 and Theorem 4.5. If $L = R/Z(R)$ the result follows by Lemma 4.6 and Theorem 4.5. ■

5. Ad-nilpotent elements of K

In this section we focus on semiprime rings R with involution $*$ and their set of skew-symmetric elements K. As in the previous section, we will first describe the pure ad-nilpotent elements of K, and then remove the hypothesis of being pure by decomposing each ad-nilpotent element into a sum of pure ad-nilpotent elements of decreasing indices.

The following lemma collects some results about $*$-identities. Item (1) is [13, Remark on page 43] (with a different proof), item (2) is a generalization of [18, Lemma 5], and item (3) is a generalization of [3, Lemma 5.2].

Lemma 5.1. *Let R be a centrally closed semiprime ring with involution $*$ and free of 2-torsion. Let $k \in K$ and $h \in H$. Then:*

(1) $kKk = 0$ implies $k = 0$.
(2) $hKh = 0$ implies $hRh \subset H(C(R), *)h$. In particular, R satisfies

$$hxhyh = hyhxh \quad \text{for every } x, y \in R,$$

and if $\mathrm{Id}_R(h)$ is essential then $\mathrm{Skew}(C(R), *) = 0$.
(3) $hKh = 0$ and $hKk = 0$ imply $hRk = 0$. In particular, if $\mathrm{Id}_R(h)$ is essential then $k = 0$, while if $h \in \mathrm{Id}_R(k)$ then $h = 0$ (resp. if $k \in \mathrm{Id}_R(h)$ then $k = 0$).

Proof: (1) Take $x \in R$. Note that $k(x - x^*)k = 0$, so that $kxk = kx^*k$. Then

$$k(xkx)k = k(xkx)^*k = -kx^*kx^*k = -(kx^*k)x^*k = -kxkx^*k$$
$$= -kx(kx^*k) = -kxkxk$$

and so we have $kxkxk = 0$ since R is free of 2-torsion. Therefore $kxkxkyk = 0$ for every $y \in R$, hence

$$0 = -kxk(xky)k = -kxk(xky)^*k = kxky^*kx^*k = kxkykxk,$$

so $(kxk)R(kxk) = 0$ and $kxk = 0$ since R is semiprime. Now $kRk = 0$ implies, again by semiprimeness, that $k = 0$.

(2) If $h = 0$ then the claim is trivially fulfilled, so assume $h \neq 0$. Take $x, y \in R$. Note that $h(x - x^*)h = 0$ and therefore $hxh = hx^*h$. Then

$$0 = h(xhy - (xhy)^*)h = hxhyh - hy^*hx^*h = hxhyh - (hy^*h)x^*h =$$

$$= hxhyh - hy(hx^*h) = hxhyh - hyhxh = (hxh)yh - hy(hxh),$$

i.e., $hxhyh = hyhxh$. By Corollary 2.8, since $h \neq 0$ and $\mathrm{Id}_R(hxh) \subseteq \mathrm{Id}_R(h)$, for each $x \in R$ there exists $\mu_x \in C(R)$ such that $hxh = \mu_x h$. Hence $0 \neq hRh \subset C(R)h$. Moreover, since $hx^*h = hxh$, $2hxh = hxh + hx^*h = (\mu_x + \mu_x^*)h \in H(C(R), *)h$, so $hRh \subseteq H(C(R), *)h$.

Let us suppose that $\mathrm{Id}_R(h)$ is essential in R and let us show that $\mathrm{Skew}(C(R), *)$ is zero: Take $\lambda \in \mathrm{Skew}(C(R), *)$ and $y \in R$. Then $(\lambda h)y(\lambda h) = \lambda h(y\lambda)h = \lambda\mu_{\lambda y}h \in K$ for some $\mu_{\lambda y} \in H(C(R), *)$. On the other hand $(\lambda h)y(\lambda h) =$

$\lambda^2 hyh = \lambda^2 \mu_y h \in H$ for some $\mu_y \in H(C(R),*)$. Therefore $(\lambda h)y(\lambda h) = 0$ for every $y \in R$, and by semiprimeness of R, $\lambda h = 0$, so $\lambda = 0$ because $\mathrm{Id}_R(h)$ is essential.

(3) Suppose first that R is prime, and let $h \in H := H(R,*)$ and $k \in K$ be elements such that $hKh = 0$ and $hKk = 0$. Since $R = H + K$ we only need to show that $hHk = 0$. Let $x \in H$ and $y \in R$. Then

$$0 = h(xky - (xky)^*)h = hxkyh + hy^*kxh = hxkyh + hykxh$$

since $h(y^* - y)k = 0$ for every $y \in R$. By Corollary 2.8, since $\mathrm{Id}_R(hxk) \subset \mathrm{Id}_R(h)$, for each $x \in R$ there exists $\mu_x \in C(R)$ such that $hxk = \mu_x h$. If $\mu_x = 0$ then $hxk = 0$ and we are done. Otherwise, $0 = hxkxk = \mu_x hxk = \mu_x^2 h$, hence $h = 0$ and we are also done.

Suppose now that R is semiprime. Then there exists a family of prime ideals $\{I_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} I_\alpha = 0$. In each prime quotient R/I_α we have $\bar{h}R/I_\alpha\bar{k} = \bar{0}$, so $hRk \subset I_\alpha$ for all α, hence $hRk = 0$. ■

Remark 5.2. Let R be a centrally closed ring with involution and free of 2-torsion. Recall that $R = H+K$, so every $x \in R$ can be expressed as $x = x_h + x_k$ with $x_h \in H$ and $x_k \in K$. If $a \in K$ is an ad-nilpotent element of K of index n, then for every $x \in R$

$$\begin{aligned}\mathrm{ad}_a^n(ax + xa) &= \mathrm{ad}_a^n(ax_k + x_k a) + \mathrm{ad}_a^n(ax_h + x_h a)\\ &= a\,\mathrm{ad}_a^n(x_k) + \mathrm{ad}_a^n(x_k)a + \mathrm{ad}_a^n(ax_h + x_h a) = 0,\end{aligned}$$

since $ax_h + x_h a \in K$. On the other hand, expanding this expression,

$$\begin{aligned}0 &= \mathrm{ad}_a^n(ax + xa) =\\ &= (-1)^n x a^{n+1} + \sum_{i=1}^{n} \left(\binom{n}{i} - \binom{n}{i-1} \right) (-1)^{n-i} a^i x a^{n+1-i} + a^{n+1}x.\end{aligned}$$

Observe that a nilpotent element in K is ad-nilpotent of both K and R, but its index of ad-nilpotence in R may be higher than the one found in K. In the following proposition we describe the ad-nilpotent elements of K of index n that are already nilpotent of certain index s. The description depends on the equivalence class of the index of ad-nilpotence modulo 4 and relates the index of nilpotence to the index of ad-nilpotence.

Proposition 5.3. *Let R be a centrally closed semiprime ring with involution $*$ and free of 2-torsion, and let $a \in K$ be a nilpotent element of index of nilpotence s. Then a is ad-nilpotent both of K and R. If the index of ad-nilpotence of a in K is n and R is free of $\binom{n}{t}$-torsion for $t := [\frac{n+1}{2}]$, then:*

(1) *If $n \equiv 0 \pmod 4$ then $s = t+1$ and $a^t K a^t = 0$.*
(2) *If $n \equiv 1 \pmod 4$ then $s = t$ and the index of ad-nilpotence of a in R is also n.*
(3) *The case $n \equiv 2 \pmod 4$ is not possible.*
(4) *If $n \equiv 3 \pmod 4$ then there exists an idempotent $e \in C(R)$ such that $ea^t = a^t$. Moreover, when we write $a = ea + (1-e)a$, we have:*
   (4.1) *If $ea \neq 0$ then ea is nilpotent of index $t+1$, $ea^t = a^t$ generates an essential ideal in eR and $(ea)^{t-1}k(ea)^t = (ea)^t k (ea)^{t-1}$ for every $k \in K$.*
   (4.2) *If $(1-e)a \neq 0$, then the index of ad-nilpotence of $(1-e)a$ in R is also n, and $(1-e)a^t = 0$.*

Furthermore, if a is a pure ad-nilpotent element of K then in (2) and in (4.2) we obtain pure ad-nilpotent elements of R of index n.

Proof: Let $a \in K$ be a nilpotent element of index of nilpotence s. Then a is ad-nilpotent of K of a certain index n. If we apply Lemma 4.1 to the second formula obtained in Remark 5.2 we get that all the monomials appearing in it are zero. We will now focus on certain monomials depending on the parity of n.

• If n is even, $n = 2t$. Let us see that $s = t+1$: on the one hand, for any $x \in R$ we know that

$$\left(\binom{n}{t} - \binom{n}{t-1}\right)(-1)^t a^t x a^{t+1} = 0$$

and, since

$$\binom{n}{t} - \binom{n}{t-1}$$

is a divisor of $2\binom{n}{t}$ and R is free of $2\binom{n}{t}$-torsion, we have that $a^t x a^{t+1} = 0$ for all x. Therefore $a^{t+1} = 0$ by semiprimeness, hence $s \leq t+1$. On the other hand, if $s = t$ then $a^t = 0$ and $\mathrm{ad}_a^{2t-1}(R) = 0$, a contradiction.

Let us see that $n \equiv 0 \pmod 4$: For any $k \in K$,

$$0 = \mathrm{ad}_a^{2t}(k) = \sum_{i=1}^{2t} \binom{2t}{i} (-1)^{2t-i} a^i k a^{2t-i} = \binom{2t}{t} (-1)^t a^t k a^t,$$

so $a^t k a^t = 0$ for every $k \in K$, which implies that t has to be even, since otherwise $a^t \in K$ and $a^t K a^t = 0$ imply $a^t = 0$ by Lemma 5.1(1), a contradiction. We have shown that, if n is even, $n \equiv 2 \pmod 4$ is not possible.

- If n is odd, $n = 2t - 1$, and for any $x \in R$,

$$\left(\binom{n}{t-1} - \binom{n}{t-2} \right) a^{t-1} x a^{t+1} = 0.$$

Since $\binom{n}{t-1} - \binom{n}{t-2}$ is a divisor of $2\binom{n}{t}$ and R is free of $2\binom{n}{t}$-torsion, we have that $a^{t-1} x a^{t+1} = 0$ for all x. Therefore $a^{t+1} = 0$ by semiprimeness, hence $s \leq t + 1$. On the other hand $s > t - 1$ since otherwise $\mathrm{ad}_a^{2t-2}(R) = 0$, a contradiction.

If $a^t = 0$ then a is already an ad-nilpotent element of R of index n. In this case $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ by Proposition 4.3(2). Furthermore, if a is pure in K then a is pure in R by Lemma 3.3.

Suppose from now on that $a^t \neq 0$. Let us show that $n \equiv 3 \pmod 4$. By Proposition 2.9 there exists an idempotent $e \in H(C(R), *)$ such that $ea^t = a^t$ and $\mathrm{Ann}_R(\mathrm{Id}_R(a^t)) = (1 - e)R$ (so $a^t = ea^t$ generates an essential ideal in eR). Notice that $ea \neq 0$ (otherwise $0 = (ea)^t = ea^t = a^t$, a contradiction). For every $k \in K$ we have

$$\begin{aligned} 0 = \mathrm{ad}_{ea}^n k &= \sum_{i=1}^{n} \binom{n}{i} (-1)^{n-i} e a^i k a^{n-i} = \\ &= \binom{n}{t-1} (-1)^t e a^{t-1} k a^t + \binom{n}{t} (-1)^{t-1} e a^t k a^{t-1} = \\ &= \binom{n}{t} (-1)^{t-1} (-e a^{t-1} k a^t + e a^t k a^{t-1}). \end{aligned}$$

Since R has no $\binom{n}{t}$-torsion, $ea^{t-1} k a^t = e a^t k a^{t-1}$ for every $k \in K$. Moreover, multiplying by a on the right we get $e a^t k a^t = a^t k a^t = 0$, so $a^t K a^t = 0$, which by Lemma 5.1(1) is only possible if $a^t \neq 0$ is symmetric, hence t is even and $n \equiv 3 \pmod 4$.

If $(1-e)a \neq 0$ then $\mathrm{ad}^{2t-1}_{(1-e)a}(R) = 0$ and $(1-e)a$ is an ad-nilpotent element of R of index $n = 2t-1$. If a is pure in K then $(1-e)a$ is pure in R by Lemma 3.3. ■

Remark 5.4. Let $a \in K$ be a nilpotent element of index s. If we denote its index of ad-nilpotence in K by n, we obtain from Proposition 5.3 that, under the right torsion hypothesis,

$$2s - 3 \leq n \leq 2s - 1 \quad \text{and} \quad \frac{n+1}{2} \leq s \leq \frac{n+3}{2}.$$

Proposition 5.5. *Let R be a centrally closed semiprime ring with involution $*$ and free of 2-torsion, and let $a \in K$ be an ad-nilpotent element of K of index $n > 1$. Then:*

(1) *There exists an idempotent $e \in H(C(R), *)$ such that $(1-e)a$ is an ad-nilpotent element of R of index $\leq n$ and ea is nilpotent with $\mathrm{ad}^n_{\mu ea}(R) \neq 0$ for every $\mu \in C(R)$ such that $\mu ea \neq 0$.*

(2) *Moreover, if a is pure ad-nilpotent in K and R is free of $\binom{n}{t}$-torsion and t-torsion for $t := [\frac{n+1}{2}]$, when we write $a = ea + (1-e)a$ we have:*

 (2.1) *If $ea \neq 0$ then ea is nilpotent of index $t+1$.*

 (2.2) *If $(1-e)a \neq 0$ then $(1-e)a$ is pure ad-nilpotent in R of index n. In this case n is odd and there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $((1-e)a - \lambda)^t = 0$.*

Proof: Notice that $n \geq 3$ since $\mathrm{ad}^2_a(K) = 0$ implies $a \in Z(R)$ by [9, Corollary 4.8] and so $\mathrm{ad}_a(K) = 0$, which is not possible because $n > 1$ by hypothesis.
(1) Let us suppose first that R is a prime ring. Either $\mathrm{ad}^n_a(R) = 0$ or $\mathrm{ad}^n_a(R) \neq 0$. Suppose from now on that $\mathrm{ad}^n_a(R) \neq 0$; in particular there are no nonzero skew elements λ in $C(R)$, since otherwise by 2.3 $R = K + \lambda K$ would imply $\mathrm{ad}^n_a(R) = 0$.

Since $\mathrm{ad}^n_a(K) = 0$, by the second formula of Remark 5.2 and Corollary 2.8, a is an algebraic element of R over the field $\mathbb{F} := C(R)$. Let us consider the minimal polynomial $p(X) \in \mathbb{F}(X)$ of a. Let $\overline{\mathbb{F}}$ be the algebraic closure of $C(R)$ and let $\mu_1, \ldots, \mu_t \in \overline{\mathbb{F}}$ such that

$$p(X) = (X - \mu_1)^{k_1} \cdots (X - \mu_t)^{k_t}.$$

Let

$$q_1(X) := p(X)/(X - \mu_1),$$

so $q_1(a)a = \mu_1 q_1(a)$. Now, for any $x \in R \otimes \overline{\mathbb{F}}$,

$$\begin{aligned}
0 &= \mathrm{ad}_a^n(ax + xa)q_1(a) \\
&= a\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x a^{n-i} q_1(a) + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x a^{n-i} a q_1(a) \\
&= a\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x \mu_1^{n-i} q_1(a) + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x \mu_1^{n-i} \mu_1 q_1(a) \\
&= a\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i \mu_1^{n-i} x q_1(a) + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i \mu_1^{n-i} \mu_1 x q_1(a) \\
&= a(a-\mu_1)^n x q_1(a) + (a-\mu_1)^n \mu_1 x q_1(a) = (a-\mu_1)^n(a+\mu_1)xq_1(a)
\end{aligned}$$

and therefore, since $R \otimes \overline{\mathbb{F}}$ is a prime ring, $(a-\mu_1)^n(a+\mu_1) = 0$. If $\mu_1 = 0$ then a is nilpotent of index at most $n+1$. If $\mu_1 \neq 0$, since the involution is of the first kind on R, it extends to $R \otimes \overline{\mathbb{F}}$ via $(r \otimes \lambda)^* := r^* \otimes \lambda$, hence $0 = ((a-\mu_1)^n)^*(a+\mu_1)^* = (a^* - \mu_1)^n(a^* + \mu_1) = (-a-\mu_1)^n(-a+\mu_1)$ implies $(a+\mu_1)^n(a-\mu_1) = 0$. From the conditions $(a-\mu_1)^n(a+\mu_1) = 0$ and $(a+\mu_1)^n(a-\mu_1) = 0$ we obtain $p(X) = (X-\mu_1)(X+\mu_1)$. Thus $a^2 = \mu_1^2$, but then $\mathrm{ad}_a^3(k) = 4\mu_1^2[a,k]$ for every $k \in K$, a contradiction with $n \geq 3$.

Let us study the semiprime case: If a is already ad-nilpotent in R of index n, take $e = 0$ and the claim holds. Suppose from now on that $\mathrm{ad}_a^n(R) \neq 0$. By Proposition 2.9 let $e \in H(C(R),*)$ be an idempotent such that $e\,\mathrm{ad}_a^n(x) = \mathrm{ad}_a^n(x)$ for every $x \in R$, $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = (1-e)R$ and $\mathrm{Ann}_{C(R)}(\mathrm{ad}_a^n(R)) = (1-e)C(R)$. Then $\mathrm{ad}_{(1-e)a}^n(R) = (1-e)\,\mathrm{ad}_a^n(R) = 0$.
Let us study the element ea: First notice that $\mathrm{ad}_{\mu ea}^n R \neq 0$ for every μ such that $\mu ea \neq 0$, since otherwise $\mu e\,\mathrm{ad}_a^n(R) = \mathrm{ad}_{\mu ea}^n R = 0$ implies $\mu e \in \mathrm{Ann}_{C(R)}(\mathrm{ad}_a^n(R)) = (1-e)C(R)$ and hence $\mu e = 0$, a contradiction. Let us see that ea is nilpotent. Since R is semiprime, the intersection of all $*$-prime ideals of R is zero. Consider the essential $*$-ideal $S := \mathrm{Id}_R(\mathrm{ad}_a^n(R)) \oplus \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = \mathrm{Id}_R(\mathrm{ad}_a^n(R)) \oplus (1-e)R$. Let us consider the families

$$\Delta_1 := \{I \triangleleft^* R \mid R/I \text{ is } *\text{-prime and } S \not\subset I\}$$

and

$$\Delta_2 := \{I \triangleleft^* R \mid R/I \text{ is } *\text{-prime and } S \subset I\}.$$

Since $S \subset \bigcap_{I\in\Delta_2} I$ and S is essential, $\bigcap_{I\in\Delta_1} I = 0$ and R is a subdirect product of R/I with $I \in \Delta_1$. Let us see that in any $*$-prime quotient ea is nilpotent of index not greater than $t+1$. Take $I \in \Delta_1$ and consider $\bar{R} := R/I$. We may have two cases:

- If $\overline{e} = \overline{0}$ then $\overline{ea} = \overline{0}$.
- If $\overline{e} \neq \overline{0}$ then $\overline{e} = \overline{1} \in R/I$ and $\overline{1-e} = \overline{0}$, so $(1-e)R \subset I$. Moreover, $\mathrm{ad}^n_{\overline{ea}}(R/I) \neq \overline{0}$ since otherwise $\mathrm{ad}^n_{\overline{ea}}(R/I) = 0$ would imply $S \subset I$, a contradiction. Let us see that R/I is prime: if R/I is $*$-prime and not prime there would exist a nonzero skew element λ in $C(R/I)$, which implies that $R/I = \mathrm{Skew}(R/I,*) \oplus \lambda\,\mathrm{Skew}(R/I,*)$ (see 2.3), so $\mathrm{ad}^n_{\overline{ea}}(R/I) = \mathrm{ad}^n_{\overline{ea}}(\mathrm{Skew}(R/I,*) \oplus \lambda\,\mathrm{Skew}(R/I,*)) = 0$, a contradiction. So R/I is a prime ring with involution and $\mathrm{ad}^n_{\overline{ea}}(R/I)) \neq \overline{0}$ which implies that $\overline{ea}$ is nilpotent of index not greater than $n+1$.

In conclusion, for any $I \in \Delta_1$ we have $ea^{n+1} \in I$ and therefore $ea^{n+1} = 0$.
(2) Suppose now that a is a pure element of K of index n and R is free of $2\binom{n}{t}$-torsion and free of t-torsion for $t = [\frac{n+1}{2}]$. If a is already ad-nilpotent of R of index n then a is pure in R by Lemma 3.3 and we can use Theorem 4.4 to find that n is odd and there exists $\lambda \in \mathrm{Skew}(C(R),*)$ such that $(a-\lambda)^t = 0$. Otherwise write $a = ea + (1-e)a$ as before. Since ea is nilpotent and ad-nilpotent of K of index n (because we are assuming that a is pure in K), ea is nilpotent of index $t+1$ (it has index t or $t+1$ by Proposition 5.3, but $\mathrm{ad}^n_{ea}(R) \neq 0$). Moreover, $(1-e)a$ is a pure ad-nilpotent element of R of index n (if it is nonzero, its index of ad-nilpotence cannot be lower than n since $(1-e)a$ is ad-nilpotent in K of index n), and we can apply Theorem 4.4 to get that n is odd and there exists $\lambda \in \mathrm{Skew}(C(R),*)$ such that $((1-e)a-\lambda)^t = 0$. ∎

Theorem 5.6. *Let R be a centrally closed semiprime ring with involution $*$ and free of 2-torsion, and let $a \in K$ be a pure ad-nilpotent element of K of index $n > 1$. If R is free of $\binom{n}{t}$-torsion and t-torsion for $t := [\frac{n+1}{2}]$ then:*

1. *(1) If $n \equiv 0\,(mod\ 4)$ then $a^{t+1} = 0$, $a^t \neq 0$ and $a^tKa^t = 0$. Moreover, there exists an idempotent $e \in H(C(R),*)$ such that $ea = a$ and the ideal generated by a^t is essential in eR. In addition eR satisfies the GPI $a^txa^tya^t = a^tya^txa^t$ for every $x, y \in eR$.*
2. *(2) If $n \equiv 1\,(mod\ 4)$ then there exists $\lambda \in \mathrm{Skew}(C(R),*)$ such that $(a-\lambda)^t = 0$ (a is an ad-nilpotent element of R of index n).*
3. *(3) It is not possible that $n \equiv 2\,(mod\ 4)$.*
4. *(4) If $n \equiv 3\,(mod\ 4)$ then there exists an idempotent $e \in H(C(R),*)$ making $a = ea + (1-e)a$ such that:*

(4.1) If $ea \neq 0$ then $ea^{t+1} = 0$, $ea^t \neq 0$ and $ea^t kea^{t-1} = ea^{t-1} kea^t$ for every $k \in K$. The ideal generated by ea^t is essential in eR and eR satisfies the GPI $a^t xa^t ya^t = a^t ya^t xa^t$ for every $x, y \in eR$.

(4.2) If $(1-e)a \neq 0$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $((1-e)a - \lambda)^t = 0$ ($(1-e)a$ is a pure ad-nilpotent element of R of index n).

In particular, for all $n > 1$ there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(a - \lambda)^{t+1} = 0$, $(a-\lambda)^{t-1} \neq 0$.

Proof: By Proposition 5.5 there exists an idempotent $e \in H(C(R), *)$ such that $e \,\mathrm{ad}_a^n x = \mathrm{ad}_a^n x$ for every $x \in R$ and $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = (1-e)R$, and moreover:

- If $ea \neq 0$, it is nilpotent of index $t+1$ and ad-nilpotent of K of index n. By Proposition 5.3 this may happen if either $n \equiv 0 \,(\mathrm{mod}\ 4)$, in which case $a^{t+1} = 0$, $a^t \neq 0$, $a^t K a^t = 0$ and $(1-e)a = 0$ (because $(1-e)a$ is ad-nilpotent of R and its index cannot be even), or $n \equiv 3 \,(\mathrm{mod}\ 4)$. The case $n \equiv 1 \,(\mathrm{mod}\ 4)$ is not possible because $ea^t \neq 0$.
- If $(1-e)a \neq 0$ then $(1-e)a$ is a pure ad-nilpotent element of R, n is odd and there exists $\lambda \in \mathrm{Skew}(R, *)$ with $((1-e)a - \lambda)^t = 0$. By Proposition 5.3 this may happen if either $n \equiv 1 \,(\mathrm{mod}\ 4)$ (in this case $ea = 0$) or $n \equiv 3 \,(\mathrm{mod}\ 4)$. The decomposition $(1-e)a - \lambda = a_1 + a_2$ given by Proposition 5.3(4) occurs with $a_1 = 0$ since otherwise the index $t+1$ of a_1 would contradict $((1-e)a - \lambda)^t = 0$.

In the particular case of $n \equiv 3 \,(\mathrm{mod}\ 4)$ with $ea \neq 0$, the idempotent e_1 produced in Proposition 5.3(4) for the nilpotent element ea satisfies $e_1 ea^t = ea^t$, so $(1-e_1)e \in \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = (1-e)R$, thus $e_1 e = e$ and $ea^t = e_1 ea^t$ generates an essential ideal in eR. On the other hand, we know from Proposition 5.5 that $(ea)^{t-1} k (ea)^t = (ea)^t k (ea)^{t-1}$ for every $k \in K$; in particular $(ea)^t K (ea)^t = 0$. Therefore, by Lemma 5.1(2) the identity

$$a^t xa^t ya^t = a^t ya^t xa^t$$

holds in eR.

In the particular case of $n \equiv 0 \,(\mathrm{mod}\ 4)$ the idempotent e produced in Proposition 5.5 satisfies $ea^t xa^t = ea^t$ for every $x \in R$ and $\mathrm{Ann}_R \,\mathrm{Id}_R(a^t R a^t) = (1-e)R$. On the other hand, $(1-e)a$ must be zero because $\mathrm{ad}^n_{(1-e)a}(R) = 0$ and a is a pure ad-nilpotent element (so $a = ea$). Therefore, the ideal generated by a^t in eR is essential in eR and the identity $a^t x a^t y a^t = a^t y a^t x a^t$ holds in eR by Lemma 5.1(2). ■

In the next corollary we recover Lee's main result by taking into account that every ad-nilpotent element can be expressed as a sum of pure ad-nilpotent elements of decreasing indices.

Corollary 5.7. ([16, Theorem 1.5]) *Let R be a centrally closed semiprime ring with involution $*$ and free of $n!$-torsion, and let $a \in K$ be an ad-nilpotent element of K of index n. Then there exist $\lambda \in \mathrm{Skew}(C(R), *)$ and an idempotent $e \in H(C(R), *)$ such that $(ea - \lambda)^{t+1} = 0$ and $(ea - \lambda)^{t-1} \neq 0$ for $t := [\frac{n+1}{2}]$, and $(1 - e)R$ is a PI-algebra satisfying the standard identity S_4.*

Proof: By Proposition 3.4 there exists a family of orthogonal symmetric idempotents $\{e_i\}_{i=1}^k$ of the extended centroid such that $a = \sum_{i=1}^k e_i a$, with $e_i a$ a pure ad-nilpotent element of index n_i ($n = n_1 > n_2 > \dots$) of $e_i R$. If $n_k = 1$ then $e_k a$ can be decomposed as $e_k a = e_{k1} a + (1 - e_{k1}) a$, where $e_{k1} a \in Z(R)$ and $(1 - e_{k1})R$ is a PI-algebra satisfying the standard identity S_4 by [3, Theorem 4.2(i),(ii) and (*)]. The claim follows now from Theorem 5.6. ■

Let us extend this last result to Lie algebras of the form $K/(K \cap Z(R))$ and $[K, K]/([K, K] \cap Z(R))$.

Corollary 5.8. *Let R be a centrally closed semiprime ring with involution free of $n!$-torsion and consider the Lie algebra $L := K/(K \cap Z(R))$. If $\bar{a}$ is an ad-nilpotent element of L of index n then there exist $\lambda \in \mathrm{Skew}(C(R), *)$ and an idempotent $e \in H(C(R), *)$ such that $(ea - \lambda)^{t+1} = 0$ and $(ea - \lambda)^{t-1} \neq 0$ for $t := [\frac{n+1}{2}]$, and $(1 - e)R$ is a PI-algebra that satisfying the standard identity S_4.*

Proof: Notice that $\mathrm{ad}_a^n(K) \subset Z(R)$ implies $\mathrm{ad}_a^n(K) = 0$: if not, there would exist $0 \neq \lambda \in \mathrm{ad}_a^n(K) \cap Z(R)$, so $R = K + \lambda K$ by 2.3 and hence $\mathrm{ad}_a^n(R) \subset Z(R)$, which implies by Lemma 4.6 that $\mathrm{ad}_a^n(R) = 0$, a contradiction. The claim follows now from Corollary 5.7. ■

Now we turn to Lie algebras of the form $[K, K]/([K, K] \cap Z(R))$. We first need a technical lemma.

Lemma 5.9. *Let R be a centrally closed semiprime ring with involution $*$ and free of 2-torsion. Let $a \in K$ be such that $\operatorname{ad}_a^n([K,K]) \subset Z(R)$. If R is free of $\binom{n+1}{t}$-torsion for $t := [\frac{n+2}{2}]$ then $\operatorname{ad}_a^n(K) = 0$.*

Proof: Let us first suppose that R is a $*$-prime ring. If $\operatorname{Skew}(C(R),*) \neq 0$ then $R = K + \lambda K$ for any $0 \neq \lambda \in \operatorname{Skew}(C(R),*)$ (see 2.3); thus $\operatorname{ad}_a^n([R,R]) \subset Z(R)$, and by Lemma 4.7 a is an ad-nilpotent element of R of index n. Otherwise $\operatorname{Skew}(C(R),*) = 0$, in which case R must be prime and $K \cap Z(R) = 0$, so $\operatorname{ad}_a^n([K,K]) = 0$. From $\operatorname{ad}_a^{n+1} K \subset \operatorname{ad}_a^n([K,K]) = 0$ and $\operatorname{Skew}(C(R),*) = 0$ we get from Proposition 5.5 that a is a nilpotent element of R. Let s be its index of nilpotence. If $\operatorname{ad}_a^n K = 0$ we are done; suppose it is not and let us compare the index of ad-nilpotence of a in K with its index of nilpotence s (see Proposition 5.3) to get a contradiction:
(a) If $n+1 \equiv 0 \pmod 4$ then $s = \frac{n+3}{2}$ and $a^{s-1}Ka^{s-1} = 0$. From $\binom{n}{s-2} = \binom{n}{s-1}$ we get, for every $x \in R$, that $\operatorname{ad}_a^n x = (-1)^{s-1}\binom{n}{s-2}\left(a^{s-2}xa^{s-1} - a^{s-1}xa^{s-2}\right)$. Then, for every $k, k' \in K$,

$$
\begin{aligned}
&2(\operatorname{ad}_a^n k)k'(\operatorname{ad}_a^n k) = \\
&= 2\binom{n}{s-2}\binom{n}{s-2}\left(a^{s-2}ka^{s-1}k'a^{s-2}ka^{s-1} + a^{s-1}ka^{s-2}k'a^{s-1}ka^{s-2}\right) \\
&= 2\binom{n}{s-2}\binom{n}{s-2}a^{s-2}k(a^{s-1}k'a^{s-2} - a^{s-2}k'a^{s-1})ka^{s-1} + \\
&+ 2\binom{n}{s-2}\binom{n}{s-2}a^{s-1}k(a^{s-2}k'a^{s-1} - a^{s-1}k'a^{s-2})ka^{s-2} = \\
&= 2(-1)^{s-2}\binom{n}{s-2}(a^{s-2}k(\operatorname{ad}_a^n k')ka^{s-1} - a^{s-1}k(\operatorname{ad}_a^n k')ka^{s-2}) = \\
&= (-1)^{s-2}\binom{n}{s-2}(a^{s-2}\operatorname{ad}_k^2(\operatorname{ad}_a^n k')a^{s-1} - a^{s-1}\operatorname{ad}_k^2(\operatorname{ad}_a^n k')a^{s-2}) = \\
&= \operatorname{ad}_a^n(\operatorname{ad}_k^2(\operatorname{ad}_a^n k')) \in \operatorname{ad}_a^n([K,K]) = 0
\end{aligned}
$$

because $a\operatorname{ad}_a^n k = 0 = (\operatorname{ad}_a^n k)a$, $a^{s-1}Ka^{s-1} = 0$ and $s \geq 3$ implies $a^{s-1}a^{s-2} = 0$. Therefore $(\operatorname{ad}_a^n k)K(\operatorname{ad}_a^n k) = 0$ and hence $\operatorname{ad}_a^n k = 0$ for every $k \in K$ by Lemma 5.1(1).
(b) If $n+1 \equiv 1 \pmod 4$ then $s = \frac{n}{2} + 1$. For every $x \in R$, $\operatorname{ad}_a^n x =$

$(-1)^{s-1}\binom{n}{s-1}a^{s-1}xa^{s-1}$. Then, for every $k, k' \in K$,

$$\begin{aligned} 2(\operatorname{ad}_a^n k)k'(\operatorname{ad}_a^n k) &= 2\binom{n}{s-1}\binom{n}{s-1}a^{s-1}ka^{s-1}k'a^{s-1}ka^{s-1} = \\ &= \binom{n}{s-1}\binom{n}{s-1}a^{s-1}\operatorname{ad}_k^2(a^{s-1}k'a^{s-1})a^{s-1} = \\ &= \operatorname{ad}_a^n(\operatorname{ad}_k^2(\operatorname{ad}_a^n k')) \in \operatorname{ad}_a^n([K,K]) = 0 \end{aligned}$$

because $a^{s-1}a^{s-1} = 0$. Therefore $(\operatorname{ad}_a^n k)K(\operatorname{ad}_a^n k) = 0$ and hence $\operatorname{ad}_a^n k = 0$ for every $k \in K$ by Lemma 5.1(1).
(c) The case $n + 1 \equiv 2 \pmod 4$ is not possible.
(d) If $n + 1 \equiv 3 \pmod 4$ we are in case (a) or (b) by the primeness of R.
In any case $\operatorname{ad}_a^n(K) = 0$. Finally, the semiprime case follows because R is a subdirect product of $*$-prime rings. ■

From this lemma and Corollary 5.7 we get:

Corollary 5.10. *Let R be a centrally closed semiprime ring with involution $*$ and free of $(n+1)!$-torsion, and consider the Lie algebra $L := [K,K]/(Z(R)\cap [K,K])$. If $\bar{a}$ is an ad-nilpotent element of L of index n then there exist $\lambda \in \operatorname{Skew}(C(R),*)$ and an idempotent $e \in H(C(R),*)$ such that $(ea-\lambda)^{t+1} = 0$ and $(ea-\lambda)^{t-1} \neq 0$ for $t := [\frac{n+1}{2}]$, and $(1-e)R$ is a PI-algebra satisfying the standard identity S_4.*

References

[1] Willard E. Baxter and Wallace S. Martindale, III. The extended centroid in $*$-prime rings. Comm. Algebra, 10(8):847–874, 1982.
[2] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev. Rings with generalized identities, volume 196 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1996.
[3] Jose Brox, Esther García, and Miguel Gómez Lozano. Jordan algebras at Jordan elements of semiprime rings with involution. J. Algebra, 468:155–181, 2016.
[4] Chen-Lian Chuang and Tsiu-Kwen Lee. Nilpotent derivations. J. Algebra, 287(2):381–401, 2005.
[5] L. O. Chung. Nil derivations. J. Algebra, 95(1):20–30, 1985.
[6] L. O. Chung and Jiang Luh. Nilpotency of derivations. Canad. Math. Bull., 26(3):341–346, 1983.
[7] L. O. Chung and Jiang Luh. Corrigendum ti the paper: “nilpotency of derivations". Canad. Math. Bull., 29(3):383–384, 1986.
[8] Lung O. Chung, Yuji Kobayashi, and Jiang Luh. Remark on nilpotency of derivations. Proc. Japan Acad. Ser. A Math. Sci., 60(9):329–330, 1984.

[9] E. García and M. Gómez Lozano. A characterization of the Kostrikin radical of a Lie algebra. J. Algebra, 346(1):266–283, 2011.
[10] Esther García and Miguel Gómez Lozano. A note on a result of Kostrikin. Comm. Algebra, 37(7):2405–2409, 2009.
[11] P. Grzeszczuk. On nilpotent derivations of semiprime rings. J. Algebra, 149(2):313–321, 1992.
[12] I. N. Herstein. Sui commutatori degli anelli semplici. Rend. Sem. Mat. Fis. Milano, 33:80–86, 1963.
[13] I. N. Herstein. Topics in ring theory. The University of Chicago Press, Chicago, Ill.-London, 1969.
[14] V. K. Kharchenko. Differential identities of prime rings. Algebra and Logic, 17(2):155–168, 1978.
[15] V. K. Kharchenko. Differential identities of semiprime rings. Algebra and Logic, 18(1):58–80, 1979.
[16] Tsiu-Kwen Lee. Ad-nilpotent elements of semiprime rings with involution. Canad. Math. Bull., 61(2):318–327, 2018.
[17] W. S. Martindale, III and C. Robert Miers. On the iterates of derivations of prime rings. Pacific J. Math., 104(1):179–190, 1983.
[18] W. S. Martindale, III and C. Robert Miers. Nilpotent inner derivations of the skew elements of prime rings with involution. Canad. J. Math., 43(5):1045–1054, 1991.
[19] Wallace S. Martindale, III. Prime rings satisfying a generalized polynomial identity. J. Algebra, 12:576–584, 1969.

Jose Brox
CMUC, Centre for Mathematics of the University of Coimbra, 3001-501 Coimbra, Portugal

E-mail address: josebrox@mat.uc.pt

Esther García
Departamento de Matemática Aplicada, Ciencia e Ingeniería de los Materiales y Tecnología Electrónica, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain

E-mail address: esther.garcia@urjc.es

Miguel Gómez Lozano
Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain

E-mail address: magomez@agt.cie.uma.es

Rubén Muñoz Alcázar
Departamento de Matemática Aplicada, Ciencia e Ingeniería de los Materiales y Tecnología Electrónica, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain

E-mail address: rubenj.alcazar@urjc.es

Guillermo Vera de Salas
Departamento de Matemática Aplicada, Ciencia e Ingeniería de los Materiales y Tecnología Electrónica, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain

E-mail address: guillermo.vera@urjc.es