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SPLIT COURANT ALGEBROIDS AS L_∞-STRUCTURES

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Abstract: We show that split Courant algebroids, i.e., those defined on a Whitney sum $A \oplus A^*$, are in a one-to-one correspondence with multiplicative curved L_∞-algebras. This one-to-one correspondence extends to Nijenhuis morphisms and behaves well under the operation of twisting by a bivector.

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1. Introduction

Courant algebroids were introduced by Liu, Weinstein and Xu [16] to interpret the bracket defined by Courant to study constraints on Dirac manifolds. In short, a Courant algebroid is a vector bundle $E \to M$ equipped with a symmetric nondegenerate bilinear form, together with a morphism of vector bundles $\rho : E \to TM$ and such that the space of sections $\Gamma(E)$ has the structure of a Leibniz algebra. All these data satisfy some compatibility conditions that we recall in Section 2. This is not the original definition introduced in [16], but an equivalent non-skew-symmetric version that uses the so-called Dorfman bracket instead of the Courant bracket.

There is an alternative way to define Courant algebroids, introduced by Roytenberg [18], which is the one that we consider in this paper. Courant algebroids can be described as degree 2 symplectic graded manifolds together with a degree 3 function Θ satisfying $\{\Theta, \Theta\} = 0$, where $\{\cdot, \cdot\}$ is the graded Poisson bracket corresponding to the graded symplectic structure. To the graded Poisson bracket we call *big bracket* [12]. The morphism ρ and the Dorfman bracket are recovered as derived brackets (see [18]).

When the Courant structure is defined on the Whitney sum $A \oplus A^*$ of a vector bundle A and its dual, we have what we call a *split* Courant algebroid. The Courant structure on $A \oplus A^*$ can be the double of a Lie bialgebroid structure on (A, A^*), the double of a quasi-Lie bialgebroid structure on (A, A^*) or, more generally, the double of a proto-Lie bialgebroid structure on (A, A^*)[19].

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Besides Courant algebroids, the other relevant structures in this paper are L_∞-algebras, also known as strongly homotopy Lie algebras. They were introduced by Lada and Stasheff [14] and consist of collections of n-ary brackets satisfying higher Jacobi identities. In the original definition of [14], the n-ary brackets are skew-symmetric, but in this paper we consider the equivalent definition where the brackets are graded symmetric. Roytenberg and Weinstein [20] showed that to each Courant algebroid one can associate a Lie 2-algebra and, recently, Lang, Sheng and Xu [15] proved a converse of this result.

In this paper we show that split Courant algebroids $A \oplus A^*$ are in a one-to-one correspondence with multiplicative curved L_∞-algebra structures on $\Gamma(\wedge^\bullet A)[2]$. This extends other previous results. In 2002, Roytenberg [19] mentions that each split Courant algebroid which is the double of a quasi-Lie bialgebroid has an associated L_∞-algebra defined on $\Gamma(\wedge^\bullet A)[2]$ and that the converse holds. No proof is given. In 2015, Frégier and Zambon [8] proved that each split Courant algebroid which is the double of a proto-Lie bialgebroid determines a curved L_∞-algebra structure on $\Gamma(\wedge^\bullet A^*)[2]$. The proof uses the higher derived brackets construction of Voronov [24]. We give an alternative and simpler proof that only uses the properties of the graded Poisson bracket, and we also prove the converse (Theorems 4.1 and 4.3).

Having established a one-to-one correspondence between split Courant algebroids and multiplicative curved L_∞-algebras, it seemed interesting to discuss the behavior of Nijenhuis operators under this correspondence. Nijenhuis morphisms on Courant algebroids were initially considered in [6] and then revisited in [10], under the graded manifold approach to Courant algebroids. Regarding Nijenhuis forms on L_∞-algebras, they were introduced in [4]. This notion also appears in [17], although with a simpler definition which turns out to be a particular case of the one in [4]. In this paper we consider the definition of [4]. Using the Lie 2-algebra associated to each Courant algebroid according to [20], some relations between Nijenhuis morphisms on Courant algebroids and Nijenhuis forms on Lie 2-algebras were already established in [4]. In the current paper the approach is different since split Courant algebroids $A \oplus A^*$ are seen as graded manifolds, which is not the case in [4], and the curved L_∞-algebra structure is defined on $\Gamma(\wedge^\bullet A)[2]$.

One of the advantages of viewing split Courant algebroids as graded manifolds, besides simpler and more efficient computations, is the relation with Lie algebroid structures on A. Indeed, we have that $(A \oplus A^*, \Theta = \mu)$ is a

Courant algebroid if and only if (A, μ) is a Lie algebroid. Having this is mind, we characterize some know structures on Lie algebroids as Nijenhuis forms on L_∞-algebra structures on $\Gamma(\wedge^\bullet A)[2]$.

Another type of operation that behave well under the one-to-one correspondence that we established, is the twisting on Courant algebroids and on L_∞-algebras. The twisting of a split Courant algebroid by a bivector was defined in [19], and the same operation can be done on L_∞-algebras. In [9] it is shown that the twisting of a L_∞-algebra by a degree zero element π is an L_∞-algebra provided that π is a Maurer-Cartan element. In the case of a curved L_∞-algebra, we show that π no longer needs to be a Maurer-Cartan element.

The paper is organized as follows. Section 2 contains a brief review of the main notions concerning (pre-)Courant algebroids as well as Nijenhuis morphisms on (pre-)Courant algebroids. In Section 3 we recall the definition of curved L_∞-algebras and of Nijenhuis forms on curved L_∞-algebras. Section 4 contains the main theorem, that establishes a one-to-one correspondence between split Courant algebroids and curved L_∞-algebras. In Section 5 we show that the one-to-one correspondence preserves deformations by Nijenhuis operators. In particular, some Nijenhuis morphisms on Courant algebroids are characterized as Nijenhuis forms on curved L_∞-algebras. Some well known structures on Lie algebroids are viewed as Nijenhuis form on L_∞-algebras. In Section 6 we discuss the twisting of a split Courant algebroid and of a curved L_∞-algebra by $\pi \in \Gamma(\wedge^2 A)$ and we show that the one-to-one correspondence preserve these twisting operations. In Section 7 we combine the one-to-one correspondence with the operations of twisting by π and deformation by a skew-symmetric vector-valued form on $\Gamma(\wedge^\bullet A)[2]$. The commutative diagrams included along Sections 4 to 7 can be combined to form a commutative cubic diagram, presented at the end of the paper.

2. Preliminaries on Courant algebroids and their Nijenhuis morphisms

In this section we recall the definition of Courant algebroid and how it can be seen as a Q-manifold, following the approach of [23, 18]. The notion of Nijenhuis morphism on a (pre)-Courant algebroid is also recalled.

Let $E \to M$ be a vector bundle equipped with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$.

Definition 2.1. [3] A *pre-Courant* structure on $(E, \langle\cdot,\cdot\rangle)$ is a pair $(\rho, [\cdot,\cdot])$, where $\rho : E \to TM$ is a morphism of vector bundles called the *anchor*, and $[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a $\mathbb{R}$-bilinear bracket, called the *Dorfman bracket*, satisfying the relations

$$\rho(u) \cdot \langle v, w\rangle = \langle [u, v], w\rangle + \langle v, [u, w]\rangle$$

and

$$\rho(u) \cdot \langle v, w\rangle = \langle u, [v, w] + [w, v]\rangle,$$

for all $u, v, w \in \Gamma(E)$. The quadruple $(E, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$ is a *pre-Courant algebroid.*

If a pre-Courant structure $(\rho, [\cdot,\cdot])$ satisfies the Jacobi identity,

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]],$$

for all $u, v, w \in \Gamma(E)$, then the pair $(\rho, [\cdot,\cdot])$ is called a *Courant* structure on $(E, \langle\cdot,\cdot\rangle)$ and $(E, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$ is a *Courant algebroid.*

Next, we recall the notion of Nijenhuis morphism on a (pre-)Courant algebroid $(E, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$. Given an endomorphism $\mathscr{I} : E \to E$, the transpose morphism $\mathscr{I}^* : E^* \simeq E \to E^* \simeq E$ is defined by $\langle \mathscr{I}^* u, v\rangle = \langle u, \mathscr{I} v\rangle$ for all $u, v \in E$. If $\mathscr{I} = -\mathscr{I}^*$, the morphism $\mathscr{I}$ is said to be *skew-symmetric.* For a skew-symmetric endomorphism $\mathscr{I} : E \to E$, we define a *deformed* pre-Courant algebroid structure $(\rho_{\mathscr{I}}, [\cdot,\cdot]_{\mathscr{I}})$ on $(E, \langle\cdot,\cdot\rangle)$ by setting

$$\begin{cases} \rho_{\mathscr{I}} = \rho \circ \mathscr{I} \\ [u, v]_{\mathscr{I}} = [\mathscr{I} u, v] + [u, \mathscr{I} v] - \mathscr{I}[u, v], \quad \forall u, v \in \Gamma(E). \end{cases} \tag{1}$$

A skew-symmetric endomorphism $\mathscr{I} : E \to E$ on a pre-Courant algebroid $(E, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$ is a *Nijenhuis morphism* if its Nijenhuis torsion $\mathcal{T}\mathscr{I}$ vanishes, where

$$\mathcal{T}\mathscr{I}(u, v) = \frac{1}{2}\left([\mathscr{I} u, \mathscr{I} v] - \mathscr{I}\left([u, v]_{\mathscr{I}}\right)\right),$$

for all $u, v \in \Gamma(E)$. If $\mathscr{I}$ is a Nijenhuis morphism, then $(E, \langle\cdot,\cdot\rangle, \rho_{\mathscr{I}}, [\cdot,\cdot]_{\mathscr{I}})$ is a Courant algebroid.

When the underlying vector bundle $E \to M$ of a (pre-)Courant algebroid is the Whitney sum $E = A \oplus A^*$ of a vector bundle $A \to M$ and its dual $A^* \to M$ we have a *split* (pre-)Courant algebroid. The graded manifold approach of split (pre-)Courant algebroids will be extensively used in this paper, and so we briefly recall it.

Given a vector bundle $A \to M$, we denote by $A[m]$ the graded manifold obtained by shifting the fibre degree by m. The graded manifold $T^*[2]A[1]$ is equipped with a canonical symplectic structure which induces a graded Poisson bracket on its algebra of functions $\mathscr{F} := C^\infty(T^*[2]A[1])$. This graded Poisson bracket is sometimes called the *big bracket*. (see [12]).

Let us describe locally this Poisson algebra (see [1] for more details). Fix local coordinates $x_i, p^i, \xi_a, \theta^a$, $i \in \{1, \dots, n\}, a \in \{1, \dots, d\}$, in $T^*[2]A[1]$, where x_i, ξ_a are local coordinates on $A[1]$ and p^i, θ^a are their associated moment coordinates. In these local coordinates, the Poisson bracket is given by

$$\{p^i, x_i\} = \{\theta^a, \xi_a\} = 1, \quad i = 1, \dots, n,\ a = 1, \dots, d,$$

while all the remaining brackets vanish.

The Poisson algebra $(\mathscr{F}, \{\cdot,\cdot\})$ is endowed with an $(\mathbb{N}_0 \times \mathbb{N}_0)$-valued bidegree. We define this bidegree (locally but it is well defined globally, see [23, 18]) as follows: the coordinates on the base manifold M, x_i, $i \in \{1, \dots, n\}$, have bidegree $(0,0)$, while the coordinates on the fibres, ξ_a, $a \in \{1, \dots, d\}$, have bidegree $(0,1)$ and their associated moment coordinates, p^i and θ^a, have bidegree $(1,1)$ and $(1,0)$, respectively. We denote by $\mathscr{F}^{k,l}$ the space of functions of bidegree (k,l) and by $\mathscr{F}^t$ the space of functions of (total) degree t,

$$\mathscr{F}^t = \bigoplus_{k+l=t} \mathscr{F}^{k,l}.$$

Notice that $\mathscr{F}^0 = C^\infty(M)$, $\mathscr{F}^{0,1} = \Gamma(A)$ and $\mathscr{F}^{1,0} = \Gamma(A^*)$. The big bracket has bidegree $(-1,-1)$, i.e., $\{\mathscr{F}^{k_1,l_1}, \mathscr{F}^{k_2,l_2}\} \subset \mathscr{F}^{k_1+k_2-1,l_1+l_2-1}$ and, for all $f, g \in \mathscr{F}^0 = C^\infty(M)$ and $X+\alpha, Y+\beta \in \mathscr{F}^1 = \Gamma(A \oplus A^*)$, we have

$$\{f,g\} = 0, \ \ \{f, X+\alpha\} = 0 \ \text{ and } \ \{X+\alpha, Y+\beta\} = \langle X+\alpha, Y+\beta\rangle,$$

where $\langle\cdot,\cdot\rangle$ stands for the usual pairing between A and A^*,

$$\langle X+\alpha, Y+\beta\rangle := \alpha(Y) + \beta(X).$$

There is a one-to-one correspondence between pre-Courant structures $(\rho, [\cdot,\cdot])$ on $(A \oplus A^*, \langle\cdot,\cdot\rangle)$ and functions $\Theta \in \mathscr{F}^3$. In other words, a pre-Courant structure on $(A \oplus A^*, \langle\cdot,\cdot\rangle)$ corresponds to a hamiltonian vector field $X_\Theta = \{\Theta, \cdot\}$ on the graded manifold $T^*[2]A[1]$. The anchor and Dorfman bracket associated to a given $\Theta \in \mathscr{F}^3$ are defined, for all $X+\alpha, Y+\beta \in$

$\Gamma(A \oplus A^*)$ and $f \in C^\infty(M)$, by the derived bracket expressions

$$\rho(X+\alpha)\cdot f = \{\{X+\alpha, \Theta\}, f\} \quad \text{and} \quad [X+\alpha, Y+\beta] = \{\{X+\alpha, \Theta\}, Y+\beta\}. \tag{2}$$

In [18, 23] it is proved that there is a one-to-one correspondence between Courant structures on $(A \oplus A^*, \langle \cdot, \cdot \rangle)$ and functions $\Theta \in \mathscr{F}^3$ such that the hamiltonian vector field X_Θ on $T^*[2]A[1]$ is a homological vector field, i.e., $\{\Theta, \Theta\} = 0$. Thus, a Courant algebroid $(A \oplus A^*, \langle \cdot, \cdot \rangle, \Theta)$ corresponds to a Q-manifold $(T^*[2]A[1], X_\Theta)$.

In what follows, a split (pre-)Courant algebroid will be denoted simply by $(A \oplus A^*, \Theta)$.

A (pre-)Courant structure $\Theta \in \mathscr{F}^3$ can be decomposed using the bidegrees, as follows:

$$\Theta = \psi + \gamma + \mu + \phi, \tag{3}$$

with $\psi \in \mathscr{F}^{3,0} = \Gamma(\wedge^3 A), \gamma \in \mathscr{F}^{2,1}, \mu \in \mathscr{F}^{1,2}$ and $\phi \in \mathscr{F}^{0,3} = \Gamma(\wedge^3 A^*)$. We recall from [19] that, when $\psi = \gamma = \phi = 0$, Θ is a Courant structure on $A \oplus A^*$ *if and only if* (A, μ) is a Lie algebroid. When $\psi = \phi = 0$, Θ is a Courant structure on $A \oplus A^*$ *if and only if* $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid and when $\phi = 0$ (resp. $\psi = 0$), Θ is a Courant structure on $A \oplus A^*$ *if and only if* $((A, A^*), \mu, \gamma, \psi)$ (resp. $((A^*, A), \gamma, \mu, \phi)$) is a quasi-Lie bialgebroid. In the more general case, $\Theta = \psi + \gamma + \mu + \phi$ is a Courant structure *if and only if* $((A, A^*), \mu, \gamma, \psi, \phi)$ is a proto-Lie bialgebroid. In this general case,

$$\{\Theta, \Theta\} = 0 \;\Leftrightarrow\; \begin{cases} \{\gamma, \psi\} = 0 \\ \{\gamma, \gamma\} + 2\{\mu, \psi\} = 0 \\ \{\mu, \gamma\} + \{\psi, \phi\} = 0 \\ \{\mu, \mu\} + 2\{\gamma, \phi\} = 0 \\ \{\mu, \phi\} = 0. \end{cases} \tag{4}$$

Now, we shall see what is the function on $\mathscr{F}^3$ corresponding to the deformed (pre-)Courant structure (1) on $A \oplus A^*$. A skew-symmetric endomorphism on $A \oplus A^*$, $J : A \oplus A^* \to A \oplus A^*$, is of the type

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix}, \tag{5}$$

with $N : A \to A, \pi \in \Gamma(\wedge^2 A)$, $\omega \in \Gamma(\wedge^2 A^*)$ and where $N^* : A^* \to A^*$, $\pi^\sharp : A^* \to A$, $\omega^\flat : A \to A^*$ are defined by

$$\begin{cases} \langle N^*\alpha, X\rangle = \langle \alpha, NX\rangle \\ \langle \pi^\sharp(\alpha), \beta\rangle = \pi(\alpha,\beta) \\ \langle \omega^\flat(X), Y\rangle = \omega(X,Y), \end{cases}$$

for all $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$.*

We have

$$J(X+\alpha) = \{X+\alpha, \pi + N + \omega\},$$

so the morphism J corresponds to the function $\pi+N+\omega \in \Gamma(\wedge^2(A\oplus A^*)) \subset C^\infty(T^*[2]A[1])$, that we also denote by J.

The deformation of the (pre-)Courant structure Θ by J is the function $\Theta_J = \Theta_{\pi+N+\omega} := \{\pi + N + \omega, \Theta\} \in \mathscr{F}^3$, that corresponds to $(\rho_J, [\cdot,\cdot]_J)$ (via (2)).

When J satisfies $J^2 = \lambda\, \mathrm{id}_{A\oplus A^*}$, for some $\lambda \in \mathbb{R}$, the Nijenhuis torsion of J is given by [10, 1]

$$\mathcal{T}_\Theta J = \frac{1}{2}((\Theta_J)_J - \lambda\Theta), \tag{6}$$

where $(\Theta_J)_J$ denotes the deformation of Θ_J by J.

3. Review on L_∞-algebras and Nijenhuis forms

In this section we recall the definitions of curved (pre-)L_∞-algebra and Nijenhuis form on an L_∞-algebra, following [4]. For the definition of an L_∞-algebra we consider graded symmetric brackets, which is not the case in the original definition introduced in [14]. Both definitions are equivalent, and the equivalence is given by the so-called *décalage isomorphism* (see [24, 4] for more details).

In what follows, we consider graded vector spaces with all components of finite dimension.

Definition 3.1. A *curved pre-L_∞-algebra* $(\mathscr{L}, \ell)$ is a graded vector space $\mathscr{L} = \bigoplus_{i\in\mathbb{Z}} \mathscr{L}_i$ together with a family of symmetric vector-valued forms (brackets) $\ell_i : \otimes^i \mathscr{L} \to \mathscr{L}, i \geq 0$, of degree 1. For $i = 0$, $\ell_0 \in \mathscr{L}_1$. The term ℓ_0 is called the *curvature*. We write $\ell = \sum_{i\geq 0} \ell_i$.

*We use the same notation for the maps induced on the space of sections $\Gamma(A)$ and $\Gamma(A^*)$.

The pair $(\mathscr{L},\ell)$ is called a *curved L_∞-algebra* if the generalized Jacobi identity is satisfied:

$$\sum_{i+j=n+1}\ \sum_{\sigma\in Sh(i,j-i)} \epsilon(\sigma)\ell_j(\ell_i(X_{\sigma(1)},\cdots,X_{\sigma(i)}),\cdots,X_{\sigma(n)})=0 \tag{7}$$

for all $n\in\mathbb{N}_0$, where $Sh(i,j-i)$ stands for the set of $(i,j-i)$-unshuffles and $\epsilon(\sigma)$ is the (graded commutative) Koszul sign defined by

$$X_{\sigma(1)}\otimes\ldots\otimes X_{\sigma(n)}=\epsilon(\sigma)\,X_1\otimes\ldots\otimes X_n,$$

for all $X_1,\ldots,X_n\in\mathscr{L}$. When the curvature vanishes, i.e. $\ell_0=0$, $(\mathscr{L},\ell)$ is simply called an *L_∞-algebra.*

For $k\geq 0$, we denote by $S^k(\mathscr{L}^*)\otimes\mathscr{L}$ the space of symmetric vector-valued k-forms on the graded vector space $\mathscr{L}$, i.e., graded symmetric k-linear maps on $\mathscr{L}$, and we set

$$S^\bullet(\mathscr{L}^*)\otimes\mathscr{L}=\bigoplus_{k\geq 0}S^k(\mathscr{L}^*)\otimes\mathscr{L}.$$

For $k=0$, $S^0(\mathscr{L}^*)\otimes\mathscr{L}$ is isomorphic to $\mathscr{L}$.

The insertion operator of a symmetric vector-valued k-form K is an operator

$$\imath_K : S^\bullet(\mathscr{L}^*)\otimes\mathscr{L}\to S^\bullet(\mathscr{L}^*)\otimes\mathscr{L}$$

defined by

$$\imath_K H(X_1,\ldots,X_{k+h-1})=\sum_{\sigma\in Sh(k,h-1)}\epsilon(\sigma)H\left(K(X_{\sigma(1)},\ldots,X_{\sigma(k)}),\ldots,X_{\sigma(k+h-1)}\right),$$

for all $H\in S^h(\mathscr{L}^*)\otimes\mathscr{L}$ and $X_1,\ldots,X_{k+h-1}\in\mathscr{L}$. If $H\in S^0(\mathscr{L}^*)\otimes\mathscr{L}\simeq\mathscr{L}$, $\imath_K H=0$.

Given a symmetric vector-valued k-form $K\in S^k(\mathscr{L}^*)\otimes\mathscr{L}$ and a symmetric vector-valued h-form $H\in S^h(\mathscr{L}^*)\otimes\mathscr{L}$, the *Richardson-Nijenhuis bracket* of K and H is the symmetric vector-valued $(k+h-1)$-form $[K,H]$ on $\mathscr{L}$, given by

$$[K,H]=\imath_K H-(-1)^{\overline{K}\,\overline{H}}\imath_H K, \tag{8}$$

where $\overline{K}$ is the degree of K as a graded map, that is $K(X_1,\ldots,X_k)\in\mathscr{L}_{x_1+\ldots+x_k+\bar{K}}$, for all $X_i\in\mathscr{L}_{x_i}, i=1,\ldots,k$. The pair $(S^\bullet(\mathscr{L}^*)\otimes\mathscr{L},[\cdot,\cdot])$ is a graded skew-symmetric Lie algebra.

Curved L_∞-algebras can be characterized using the Richardson-Nijenhuis bracket.

Proposition 3.2. [4] *A curved pre-L_∞-algebra $(\mathscr{L}, \ell)$ is a curved L_∞-algebra if and only if $[\ell, \ell] = 0$.*

Assume that there exists an associative graded commutative algebra structure of degree zero on $\mathscr{L}$, denoted by $\wedge$. A vector-valued k-form $K \in S^k(\mathscr{L}^*) \otimes \mathscr{L}$ is said to be a *multiderivation symmetric vector-valued k-form* if

$$
\begin{aligned}
K(X_1, \cdots, X_{k-1}, Y \wedge Z) = K(X_1, \cdots, X_{k-1}, Y) \wedge Z \\
+ (-1)^{yz} K(X_1, \cdots, X_{k-1}, Z) \wedge Y,
\end{aligned}
$$

for all $X_1, \cdots X_{k-1} \in \mathscr{L}$, $Y \in \mathscr{L}_y$ and $Z \in \mathscr{L}_z$.

The space of all multiderivation symmetric vector-valued forms on $\mathscr{L}$ is a graded Lie subalgebra of $(S^\bullet(\mathscr{L}^*) \otimes \mathscr{L}, [\cdot, \cdot])$.

A curved L_∞- algebra $(\mathscr{L}, \ell)$ is called *multiplicative* if all the brackets are multiderivations. Multiplicative (curved) L_∞- algebras are also called (curved) P_∞- *algebras* [7]. They can be viewed as a symmetric version of G_∞-algebras.

Given a curved L_∞-structure ℓ and a symmetric vector-valued form of degree zero, n, on a graded vector space, we call $[n, \ell]$ the *deformation of ℓ by n* and denote the deformed structure by $\ell_n := [n, \ell]$.

Next we recall the definition of Nijenhuis vector-valued form on an L_∞-algebra, introduced in [4].

Definition 3.3. Let $(\mathscr{L}, \ell)$ be a curved pre-L_∞-algebra. A symmetric vector-valued form on $\mathscr{L}$, n, of degree zero, is called a *Nijenhuis form* on $(\mathscr{L}, \ell)$ if there exists a vector-valued form k of degree zero, such that

$$[n, [n, \ell]] = [k, \ell] \quad \text{and} \quad [n, k] = 0.$$

Such a vector-valued form k is called a *square* of n.

In the forthcoming sections we will often use the so-called *Euler map*. Given a graded vector space $\mathscr{L} = \bigoplus_{i \in \mathbb{Z}} \mathscr{L}_i$, the Euler map $\mathscr{E} : \mathscr{L} \to \mathscr{L}$ is a linear map of degree zero defined by

$$\mathscr{E}(P) = pP, \tag{9}$$

for all homogeneous elements $P \in \mathscr{L}_p$.

4. From Courant algebroids to L_∞-algebras and back

In this section we prove a theorem that generalizes a result initially established by Roytenberg [19] in the case of a split Courant algebroid which is the double of a quasi-Lie bialgebroid, and then extended by Frégier and Zambon [8] to the case where the Courant structure is the double of a proto-bialgebroid. A result similar to the one in [8] was obtained by Gualtieri, Matviichuk and Scott [11]. In all cases, given a split Courant algebroid structure, a (curved) L_∞-algebra is constructed. Our theorem includes the converse and the proof uses a technique different from the one in [8].

Let $(A \oplus A^*, \Theta)$ be a pre-Courant algebroid where $\Theta \in \mathscr{F}^3$ can be decomposed using the bidegrees as in (3):

$$\Theta = \psi + \gamma + \mu + \phi.$$

Set $L = \Gamma(\wedge^\bullet A)[2]$. Thus $L = \sum_{i=-2}^{\infty} L_i$ is a graded vector space where $L_{-2} = C^\infty(M), L_{-1} = \Gamma(A)$ and $L_i = \Gamma(\wedge^{i+2} A)$, $i \geq 0$.

Let us consider the map

$$\begin{array}{rccc} \mathscr{M}: & \mathscr{F}^3 & \longrightarrow & S^\bullet(L^*) \otimes L \\ & \Theta = \psi + \gamma + \mu + \phi & \longmapsto & l = l_0 + l_1 + l_2 + l_3 \end{array}$$

where

- $\mathscr{M}(\psi) = l_0 \in L_1 = \Gamma(\wedge^3 A)$ is defined by
$$l_0 = \psi; \tag{10}$$
- $\mathscr{M}(\gamma) = l_1 \in S^1(L^*) \otimes L$ is defined by
$$l_1(P) = \{\gamma, P\}; \tag{11}$$
- $\mathscr{M}(\mu) = l_2 \in S^2(L^*) \otimes L$ is defined by
$$l_2(P, Q) = \{\{\mu, P\}, Q\}; \tag{12}$$
- $\mathscr{M}(\phi) = l_3 \in S^3(L^*) \otimes L$ is defined by
$$l_3(P, Q, R) = \{\{\{\phi, P\}, Q\}, R\}, \tag{13}$$

for all $P, Q, R \in \Gamma(\wedge^\bullet A)$.

Theorem 4.1. *The map $\mathscr{M}$, defined by Equations (10)-(13), establishes a one-to-one correspondence between pre-Courant structures on $A \oplus A^*$ and multiplicative curved pre-L_∞-algebra structures $l = l_0 + l_1 + l_2 + l_3$ on $L = \Gamma(\wedge^\bullet A)[2]$.*

Before proving Theorem 4.1, let us recall Lemma 3.1.6 from [1].

Lemma 4.2. *Consider $F \in \mathscr{F}^{r,s}$, with $s > 0$. If F satisfies $\{F, X\} = 0$, for all $X \in \Gamma(A)$, then $F = 0$.*

Now let us prove Theorem 4.1

Proof of Theorem 4.1: Let $\Theta = \psi + \gamma + \mu + \phi$ be a pre-Courant structure on $A \oplus A^*$. First, let us prove that $l = \mathscr{M}(\Theta)$, defined by Equations (10)-(13), is a multiplicative curved pre-L_∞-algebra structure. The fact that l is multiplicative is a direct consequence of the definition of l and the Leibniz rule for the big bracket $\{\cdot, \cdot\}$. The remaining part of the statement claims that $l = \sum_{i=0}^{3} l_i$ is a graded symmetric linear map of degree 1. This is immediate due to the definition of $l = \mathscr{M}(\Theta)$ and to the properties of the big bracket in $C^\infty(T^*[2]A[1])$. For example, let us check explicitly that l_2 is a graded symmetric map $S^2(L) \to L$ of degree 1.

For all $P \in L_p = \Gamma(\wedge^{p+2} A)$ and $Q \in L_q = \Gamma(\wedge^{q+2} A)$, using the Jacobi identity of the big bracket, we have

$$\begin{aligned} l_2(Q, P) = \{\{\mu, Q\}, P\} = \{\mu, \{Q, P\}\} + (-1)^{(p+2)(q+2)} \{\{\mu, P\}, Q\} = \\ = (-1)^{pq} \{\{\mu, P\}, Q\} = (-1)^{pq} l_2(P, Q), \end{aligned}$$

which proves that l_2 is a graded symmetric map. Furthermore, in $C^\infty(T^*[2]A[1])$, the big bracket is a map of bidegree $(-1, -1)$ and the elements μ, P and Q have bidegrees $(1, 2)$, $(p+2, 0)$ and $(q+2, 0)$, respectively. Thus, $\{\{\mu, P\}, Q\}$ has bidegree

$$((1, 2) + (p + 2, 0) + (-1, -1)) + (q + 2, 0) + (-1, -1) = (p + q + 3, 0),$$

which means that

$$l_2(P, Q) = \{\{\mu, P\}, Q\} \in \Gamma(\wedge^{p+q+3} A) = L_{p+q+1}.$$

Then l_2 is a map of degree 1.

Conversely, given a multiplicative curved pre-L_∞-algebra structure $l = l_0 + l_1 + l_2 + l_3$ on $L = \Gamma(\wedge^\bullet A)[2]$, let us prove that there is an unique $\Theta_n \in \mathscr{F}^{3-n,n}$ such that $\mathscr{M}(\Theta_n) = l_n$, for each $n = 0, 1, 2, 3$.

- For $n = 0$, we have $\Theta_0 = l_0 \in \Gamma(\wedge^3 A) = \mathscr{F}^{3,0}$.

- For $n=1$, we need to define $\Theta_1 \in \mathscr{F}^{2,1}$, such that $\mathscr{M}(\Theta_1) = l_1$, i.e, such that,
$$\{\Theta_1, P\} = l_1(P), \quad \forall P \in \Gamma(\wedge^\bullet A). \tag{14}$$
We claim that Equation (14) defines explicitly an unique $\Theta_1 \in \mathscr{F}^{2,1}$. Indeed, locally, on coordinates $(x_i, p^i, \xi_a, \theta^a)$, such an element is written as $\Theta_1 = A_{ia}(x)p^i\theta^a + B^c_{ab}(x)\theta^a\theta^b\xi_c$ and, using Equation (14), its coefficients are determined (apart from signs that depend on conventions) by l_1, as follows:
$$\begin{cases} A_{ia} = \pm\{\{\Theta_1, x_i\}, \xi_a\} = \pm < l_1(x_i), \xi_a > \\ B^c_{ab} = \pm\frac{1}{2}\{\{\{\Theta_1, \theta^c\}, \xi_a\}, \xi_b\} = \pm\frac{1}{2} < l_1(\theta^c), \xi_a \wedge \xi_b > . \end{cases}$$
Thus, the existence of Θ_1 satisfying Equation (14) is guaranteed. Furthermore, we can not have two elements Θ_1 and Θ_1' satisfying Equation (14) because Lemma 4.2 would imply that $\Theta_1 - \Theta_1' = 0$.
- Analogously, for $n=2$, we need to define $\Theta_2 \in \mathscr{F}^{1,2}$, such that $\mathscr{M}(\Theta_2) = l_2$, i.e., such that
$$\{\{\Theta_2, P\}, Q\} = l_2(P, Q), \quad \forall P, Q \in \Gamma(\wedge^\bullet A). \tag{15}$$
Locally, $\Theta_2 = C^a_i(x)p^i\xi_a + D^{ab}_c(x)\xi_a\xi_b\theta^c$ and, using Equation (15), the coefficients are determined by l_2 as follows:
$$\begin{cases} C^a_i = \pm\{\{\Theta_2, x_i\}, \theta^a\} = \pm\, l_2(x_i, \theta^a) \\ D^{ab}_c = \pm\frac{1}{2}\left\{\left\{\{\Theta_2, \theta^a\}, \theta^b\right\}, \xi_c\right\} = \pm\frac{1}{2} < l_2(\theta^a, \theta^b), \xi_c > . \end{cases}$$
- Finally, for $n=3$, $\Theta_3 \in \mathscr{F}^{0,3} = \Gamma(\wedge^3 A^*)$ is a 3-form and condition $\mathscr{M}(\Theta_3) = l_3$ implies that
$$\{\{\{\Theta_3, X\}, Y\}, Z\} = l_3(X, Y, Z),$$
for all $X, Y, Z \in \Gamma(A)$, and this defines uniquely Θ_3.

■

Theorem 4.3. *Let $\Theta \in \mathscr{F}^3$ be a pre-Courant structure on $A \oplus A^*$ and $l = \mathscr{M}(\Theta)$ its corresponding multiplicative curved pre-L_∞-algebra structure on $L = \Gamma(\wedge^\bullet A)[2]$. Then, the following assertions are equivalent:*

i) $(A \oplus A^*, \Theta)$ is a Courant algebroid;

ii) (L, l) is a multiplicative curved L_∞-algebra.

Proof: The generalized Jacobi identity (7) satisfied by l corresponds exactly to the different conditions we obtained in (4), after splitting the condition $\{\Theta,\Theta\}=0$ using bidegree. Indeed, for $n=0$, we have

$$l_1(l_0)=0 \Leftrightarrow \{\gamma,\psi\}=0 \tag{16}$$

while for $n=1$ and for all $P\in L$,

$$\begin{aligned} l_2(l_0,P)+l_1(l_1(P))=0 &\Leftrightarrow \{\{\mu,\psi\},P\}+\{\gamma,\{\gamma,P\}\}=0\\ &\Leftrightarrow \left\{\{\mu,\psi\}+\frac{1}{2}\{\gamma,\gamma\},P\right\}=0\\ &\Leftrightarrow \{\mu,\psi\}+\frac{1}{2}\{\gamma,\gamma\}=0, \end{aligned} \tag{17}$$

where the last equivalence follows from Lemma 4.2. For $n=2$ we have, for all $P\in L_p$ and $Q\in L_q$,

$$\begin{aligned} & l_3(l_0,P,Q)+l_2(l_1(P),Q)+(-1)^{pq}l_2(l_1(Q),P)+l_1(l_2(P,Q))=0\\ \Leftrightarrow\quad & \{\{\{\phi,\psi\},P\},Q\}+\{\{\mu,\{\gamma,P\}\},Q\}+(-1)^{pq}\{\{\mu,\{\gamma,Q\}\},P\}\\ & +\{\gamma,\{\{\mu,P\},Q\}\}=0\\ \Leftrightarrow\quad & \{\{\{\phi,\psi\}+\{\mu,\gamma\},P\},Q\}=0\\ \Leftrightarrow\quad & \{\phi,\psi\}+\{\mu,\gamma\}=0. \end{aligned} \tag{18}$$

Equations (16), (17) and (18) are precisely the first, second and third equations on the right side of (4).

For $n=3,4$ and 5, since more terms are involved, computations are rather cumbersome but straightforward and only use the properties of the big bracket (essentially Jacobi identity). Computations for $n=3$ and 4 lead to the last two equations on the right side of (4). For $n=5$, we prove that for all $P,Q,R,S,T\in L$,

$$\circlearrowleft l_3(l_3(P,Q,R),S,T)=\frac{1}{2}\{\{\{\{\{\{\phi,\phi\},P\},Q\},R\},S\},T\},$$

where $\circlearrowleft$ stands for the sum of the ten terms corresponding to the graded $(3,2)$-unshuffled permutations of the set $\{P,Q,R,S,T\}$. This condition is trivially satisfied because $\{\phi,\phi\}=0$, for bidegree reasons, for any $\phi\in\Gamma(\wedge^3A^*)$. ■

Notice that the roles of the vector bundle A and its dual A^* can be reversed everywhere in this section, since $(A\oplus A^*,\Theta)$ is a Courant algebroid if and only if $(A^*\oplus A,\Theta)$ is a Courant algebroid [19]. As a consequence, in Theorem

4.3, instead of considering the graded vector space $L = \Gamma(\wedge^\bullet A)[2]$, we can take $\mathfrak{L} := \Gamma(\wedge^\bullet A^*)[2]$ and define the following graded symmetric brackets of degree 1:

$$\begin{cases} \lambda_0 = \phi \\ \lambda_1(\alpha) = \{\mu, \alpha\} \\ \lambda_2(\alpha, \beta) = \{\{\gamma, \alpha\}, \beta\} \\ \lambda_3(\alpha, \beta, \eta) = \{\{\{\psi, \alpha\}, \beta\}, \eta\} \end{cases} \tag{19}$$

for all $\alpha, \beta, \eta \in \Gamma(\wedge^\bullet A^*)$. Set $\lambda = \sum_{i=0}^{3} \lambda_i$.

Next corollary summarizes what we have proved so far.

Corollary 4.4. *The following assertions are equivalent:*

i) $(A \oplus A^*, \Theta)$ *is a Courant algebroid;*
ii) $(A^* \oplus A, \Theta)$ *is a Courant algebroid;*
iii) (L, l) *is a curved* L_∞*-algebra;*
iv) $(\mathfrak{L}, \lambda)$ *is a curved* L_∞*-algebra.*

Remark 4.5. In [8] it is proved that (i) (or (ii)) implies (iv). The technique used to obtain the curved L_∞-algebra structure is the Voronov's higher derived brackets [24]. See also [11] for the case of exact Courant algebroids.

Having established Corollary 4.4, we can proceed over the next sections either with the curved L_∞-algebra $(L = \Gamma(\wedge^\bullet A)[2], l)$ or with the curved L_∞-algebra $(\mathfrak{L} = \Gamma(\wedge^\bullet A^*)[2], \lambda)$. We will continue with (L, l), but one should have in mind that all the forthcoming results have their *dual version* if we would consider $(\mathfrak{L}, \lambda)$ instead of (L, l).

5. Nijenhuis on Courant algebroids and on L_∞-algebras

In this section, to each skew-symmetric endomorphism on $A \oplus A^*$ we associate a vector-valued form of degree zero on $\Gamma(\wedge^\bullet A)[2]$ and we analyse how the induced deformations on pre-Courant algebroids and curved pre-L_∞-algebras are related under the map $\mathscr{M}$. This leads to a relationship between Nijenhuis operators and also enable us to see some structures on Lie algebroids as Nijenhuis forms on L_∞-algebras.

Consider a skew-symmetric endomorphism $J : A \oplus A^* \to A \oplus A^*$ given as in (5):

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix}.$$

Recall that J is identified with $\pi+N+\omega \in \Gamma(\wedge^2(A\oplus A^*))$ and that $J(X+\alpha) = \{X+\alpha, \pi+N+\omega\}$.

Let us define the *extensions* $\underline{N}$ and $\underline{\omega}$ of the tensors N and ω, respectively, by setting, for all functions $f \in C^\infty(M)$ and homogeneous elements $P = P_1 \wedge \ldots \wedge P_p \in \Gamma(\wedge^p A)$ and $Q = Q_1 \wedge \ldots \wedge Q_q \in \Gamma(\wedge^q A)$,

$$\begin{cases} \underline{N}(f) = 0 \\ \underline{N}(P) = \sum_{i=1}^{p} (-1)^{i-1} N(P_i) \wedge \widehat{P_i}, \end{cases}$$

and

$$\begin{cases} \underline{\omega}(P,f) = \underline{\omega}(f,P) = 0 \\ \underline{\omega}(P,Q) = \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{p+i+j-1} \omega(P_i,Q_j) \widehat{P_i} \wedge \widehat{Q_j}, \end{cases}$$

where $\widehat{P_i} = P_1 \wedge \ldots \wedge P_{i-1} \wedge P_{i+1} \wedge \ldots \wedge P_p$ and $\widehat{Q_j} = Q_1 \wedge \ldots \wedge Q_{j-1} \wedge Q_{j+1} \wedge \ldots \wedge Q_q$.

Lemma 5.1. *The extensions $\underline{N}$ and $\underline{\omega}$ are multiderivation symmetric vector-valued 1-form and 2-form, respectively, i.e.,*

i) $\underline{N}(P \wedge Q) = \underline{N}(P) \wedge Q + (-1)^{pq}\, \underline{N}(Q) \wedge P$,
ii) $\underline{\omega}(P,Q) = (-1)^{pq}\, \underline{\omega}(Q,P)$,
iii) $\underline{\omega}(P, Q\wedge R) = \underline{\omega}(P,Q) \wedge R + (-1)^{qr}\, \underline{\omega}(P,R) \wedge Q$,

for all $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$ and $R \in \Gamma(\wedge^\bullet A)$.

Proof: Let us consider homogeneous elements $P = P_1 \wedge \ldots \wedge P_p \in \Gamma(\wedge^p A)$, $Q = Q_1 \wedge \ldots \wedge Q_q \in \Gamma(\wedge^q A)$ and $R = R_1 \wedge \ldots \wedge R_r \in \Gamma(\wedge^r A)$. Then,

i)

$$\begin{aligned}
\underline{N}(P \wedge Q) &= \sum_{i=1}^{p} (-1)^{i-1} N(P_i) \wedge \widehat{P_i} \wedge Q + \sum_{j=1}^{q} (-1)^{p+j-1} N(Q_j) \wedge P \wedge \widehat{Q_j} \\
&= \underline{N}(P) \wedge Q + \left(\sum_{j=1}^{q} (-1)^{pq+j-1} N(Q_j) \wedge \widehat{Q_j} \right) \wedge P \\
&= \underline{N}(P) \wedge Q + (-1)^{pq}\, \underline{N}(Q) \wedge P.
\end{aligned}$$

ii)

$$
\begin{aligned}
\underline{\omega}(P,Q) &= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{p+i+j-1}\omega(P_i,Q_j)\widehat{P_i}\wedge\widehat{Q_j}\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{p+i+j+(p-1)(q-1)}\omega(Q_j,P_i)\,\widehat{Q_j}\wedge\widehat{P_i}\\
&= (-1)^{pq}\sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{q+i+j-1}\omega(Q_j,P_i)\,\widehat{Q_j}\wedge\widehat{P_i}\\
&= (-1)^{pq}\,\underline{\omega}(Q,P).
\end{aligned}
$$

iii)

$$
\begin{aligned}
\underline{\omega}(P,Q\wedge R) = \sum_{i=1}^{p}\Bigg(&\sum_{j=1}^{q}(-1)^{p+i+j-1}\omega(P_i,Q_j)\,\widehat{P_i}\wedge\widehat{Q_j}\wedge R\\
&+\sum_{k=1}^{r}(-1)^{p+i+q+k-1}\omega(P_i,R_k)\,\widehat{P_i}\wedge Q\wedge\widehat{R_k}\Bigg)\\
= \Bigg(&\sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{p+i+j-1}\omega(P_i,Q_j)\,\widehat{P_i}\wedge\widehat{Q_j}\Bigg)\wedge R\\
&+\sum_{i=1}^{p}\sum_{k=1}^{r}(-1)^{p+i+q+k-1+q(r-1)}\omega(P_i,R_k)\,\widehat{P_i}\wedge\widehat{R_k}\wedge Q\\
= {}&\underline{\omega}(P,Q)\wedge R+(-1)^{qr}\,\underline{\omega}(P,R)\wedge Q.
\end{aligned}
$$

■

If one uses the graded Poisson bracket (big bracket) on the graded symplectic manifold $T^*[2]A[1]$, the evaluation of $\underline{N}$ and $\underline{\omega}$ on sections of $\wedge^\bullet A$ is much simpler to handle, as it is shown in the next lemma.

Lemma 5.2. *For all $P,Q\in\Gamma(\wedge^\bullet A)$, we have:*

i) $\underline{N}(P)=\{-N,P\}$*;*
ii) $\underline{\omega}(P,Q)=\{\{-\omega,P\},Q\}$.

Proof: All the operators are derivations on each entry, so we only have to check the identities for sections of $\Gamma(A)$. But in this case the identities are obvious because the extensions $\underline{N}$ and $\underline{\omega}$ coincide with the original tensors N and ω. ■

Remark 5.3. In general, a tensor $\mathscr{V} \in \Gamma(\wedge^k A^* \otimes \wedge^l A)$ can be extended as a graded symmetric multiderivation $\underline{\mathscr{V}} \in S^\bullet L^* \otimes L$ of degree $k+l-2$, by setting

$$\underline{\mathscr{V}}(P^1, \ldots, P^k) = (-1)^{\frac{k(k+1)}{2}+k(n_1+\ldots+n_k)-\sum_{j=1}^{k-1} j\, n_j} \sum_{a_1=1}^{n_1} \cdots \sum_{a_k=1}^{n_k} (-1)^{a_1+\ldots+a_k} \mathscr{V}(P^1_{a_1}, P^2_{a_2}, \ldots, P^k_{a_k}) \wedge \widehat{P^1_{a_1}} \wedge \ldots \wedge \widehat{P^k_{a_k}},$$

for all homogeneous elements $P^i = P^i_1 \wedge \ldots \wedge P^i_{n_i} \in \Gamma(\wedge^{n_i} A)$ and where we used the notation $\widehat{P^i_{a_i}} = P^i_1 \wedge \ldots \wedge P^i_{a_i-1} \wedge P^i_{a_i+1} \wedge \ldots \wedge P^i_{n_i}$, $i = 1, \ldots, k$. Using derived brackets, the extension $\underline{\mathscr{V}}$ is simply defined by

$$\underline{\mathscr{V}}(P^1, \ldots, P^k) = (-1)^{kl-\frac{k(k-1)}{2}} \left\{ \ldots \left\{ \left\{ \mathscr{V}, P^1 \right\}, P^2 \right\}, \ldots, P^k \right\}. \tag{20}$$

Let us now consider the map

$$\begin{array}{rcl} \Upsilon : \ \Gamma(\wedge^2(A \oplus A^*)) & \longrightarrow & S^\bullet(L^*) \otimes L \\ J = \pi + N + \omega & \longmapsto & \mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_2 \end{array}$$

where

- $\Upsilon(\pi) = \mathcal{J}_0 \in L_0 \subset S^0(L^*) \otimes L$ is defined by
$$\mathcal{J}_0 = -\pi; \tag{21}$$
- $\Upsilon(N) = \mathcal{J}_1 \in S^1(L^*) \otimes L$ is defined by
$$\mathcal{J}_1(P) = \underline{N}(P); \tag{22}$$
- $\Upsilon(\omega) = \mathcal{J}_2 \in S^2(L^*) \otimes L$ is defined by
$$\mathcal{J}_2(P, Q) = \underline{\omega}(P, Q), \tag{23}$$

for all $P, Q \in \Gamma(\wedge^\bullet A)$.

By Lemma 5.2, we can rewrite the expressions defining the map Υ using the big bracket as follows:

$$\begin{cases} \mathscr{j}_0 = -\pi \\ \mathscr{j}_1(P) = \{-N, P\} \\ \mathscr{j}_2(P, Q) = \{\{-\omega, P\}, Q\}. \end{cases}$$

Lemma 5.4. *The map $\Upsilon(J) = \mathscr{j} : S^\bullet(L) \to L$ is a graded symmetric linear map of degree zero.*

Proof: The map $\mathscr{j}$ is of degree zero because $\mathscr{j}_0 \in L_0$ and $\mathscr{j}_1$ and $\mathscr{j}_2$ are both maps of degree zero. To check this we use the same procedure as in the proof of Theorem 4.1. For example, in the case of $\mathscr{j}_2$, if $P \in L_p$ and $Q \in L_q$, then $\mathscr{j}_2(P, Q) = \{\{-\omega, P\}, Q\}$ has bidegree

$$((0, 2) + (p + 2, 0) + (-1, -1)) + (q + 2, 0) + (-1, -1) = (p + q + 2, 0),$$

which means that $\mathscr{j}_2(P, Q) \in L_{p+q}$ and so $\mathscr{j}_2$ is a map of degree zero. ■

Having shown in Theorem 4.1 that the map $\mathscr{M}$ is invertible, it seems natural to ask if Υ also admits an inverse. The answer is yes. Indeed, given three maps $\mathscr{j}_i : S^i(L) \to L$, $i = 0, 1, 2$, of degree zero and such that $\imath_f \mathscr{j}_i = 0$, for all $f \in L_{-2} = C^\infty(M)$, we can define $J \in \Gamma(\wedge^2(A \oplus A^*))$ such that $\Upsilon(J) = \mathscr{j}_0 + \mathscr{j}_1 + \mathscr{j}_2$. The proof is analogous to the proof of converse part of Theorem 4.1.

Recall that $\Theta \in \mathscr{F}^3$ can be deformed by a skew-symmetric endomorphism J of $A \oplus A^*$, yielding Θ_J (see Section 2). Also, a curved pre-L_∞-algebra can be deformed by a degree zero symmetric vector-valued form $\mathscr{n}$ yielding the curved pre-L_∞-algebra $\ell_{\mathscr{n}} = [\mathscr{n}, \ell]$ (see Section 3).

A way to confirm that the maps $\mathscr{M}$ and Υ are a natural way to embed skew-symmetric endomorphisms of split pre-Courant algebroids into vector-valued forms on curved pre-L_∞-structures is by checking that the following diagram is commutative:

$$\begin{tikzcd} \Theta \arrow[r, "\mathscr{M}"] \arrow[d, "\text{deformation}"', "\text{by } J"'] & l = \mathscr{M}(\Theta) \arrow[d, "\text{deformation}", "\text{by } \Upsilon(J) = \mathscr{j}"] \\ \Theta_J \arrow[r, "\mathscr{M}"] \arrow[ur, phantom, "\circlearrowright"] & l_{\mathscr{j}} \end{tikzcd} \tag{24}$$

This is the purpose of the next theorem.

Theorem 5.5. *Let $(A\oplus A^*,\Theta)$ be a pre-Courant algebroid and $J : A\oplus A^* \to A\oplus A^*$ a skew-symmetric endomorphism. The diagram (24) is commutative, which means that*

$$\mathscr{M}(\Theta_J) = (\mathscr{M}(\Theta))_{\Upsilon(J)},$$

where $\mathscr{M}$ and Υ are defined by Equations (10)-(13) and (21)-(23), respectively.

Proof: Let us take $\Theta = \psi+\gamma+\mu+\phi$ and $J = \pi+N+\omega$ (see equations (3) and (5)). Then,

$$\Theta_J = \{J,\Theta\} = \{\pi+N+\omega, \psi+\gamma+\mu+\phi\}$$

can be decomposed as follows:

$$\begin{cases} (\Theta_J)_{(3,0)} = \{\pi,\gamma\}+\{N,\psi\} \\ (\Theta_J)_{(2,1)} = \{\pi,\mu\}+\{N,\gamma\}+\{\omega,\psi\} \\ (\Theta_J)_{(1,2)} = \{\pi,\phi\}+\{N,\mu\}+\{\omega,\gamma\} \\ (\Theta_J)_{(0,3)} = \{N,\phi\}+\{\omega,\mu\}\,. \end{cases}$$

Now, Equations (10)-(13) define explicitly the brackets forming $\mathscr{M}(\Theta_J)$:

$$\begin{cases} (\mathscr{M}(\Theta_J))_0 = \{\pi,\gamma\}+\{N,\psi\} \\ (\mathscr{M}(\Theta_J))_1(P) = \{\{\pi,\mu\}+\{N,\gamma\}+\{\omega,\psi\},P\} \\ (\mathscr{M}(\Theta_J))_2(P,Q) = \{\{\{\pi,\phi\}+\{N,\mu\}+\{\omega,\gamma\},P\},Q\} \\ (\mathscr{M}(\Theta_J))_3(P,Q,R) = \{\{\{\{N,\phi\}+\{\omega,\mu\},P\},Q\},R\} \end{cases} \tag{25}$$

for all $P,Q,R\in\Gamma(\wedge^\bullet A)$.

On the other hand, $\mathscr{M}(\Theta) = l = l_0+l_1+l_2+l_3$ is defined by Equations (10)-(13) while $\Upsilon(J) = \jmath = \jmath_0+\jmath_1+\jmath_2$ is defined by Equations (21)-(23). Thus, the curved pre-L_∞-structure

$$(\mathscr{M}(\Theta))_{\Upsilon(J)} = [\jmath, l] = [\jmath_0+\jmath_1+\jmath_2, l_0+l_1+l_2+l_3]$$

can be decomposed in four terms $[\jmath,l]_i\in S^i(L^*)\otimes L$, $i=0,1,2,3$, as follows:

$$\begin{cases} [\jmath,l]_0 = [\jmath_0,l_1]+[\jmath_1,l_0] \\ [\jmath,l]_1 = [\jmath_0,l_2]+[\jmath_1,l_1]+[\jmath_2,l_0] \\ [\jmath,l]_2 = [\jmath_0,l_3]+[\jmath_1,l_2]+[\jmath_2,l_1] \\ [\jmath,l]_3 = [\jmath_1,l_3]+[\jmath_2,l_2]. \end{cases} \tag{26}$$

We need to prove that the curved pre-L_∞-structures defined by Equations (25) and (26) coincide. For the 0-brackets, we have

$$\begin{aligned}[\mathcal{J}_0, l_1] + [\mathcal{J}_1, l_0] &= l_1(\mathcal{J}_0) - \mathcal{J}_1(l_0)\\ &= \{\gamma, -\pi\} - \{-N, \psi\}\\ &= \{\pi, \gamma\} + \{N, \psi\},\end{aligned}$$

where we used Equations (8), (10), (11), (21) and (22). The 1-brackets require more computations, but still straightforward. Besides the equations used for the 0-brackets we also use Equations (12) and (23) to carry out the computations, for any $P \in L$:

$$\begin{aligned}([\mathcal{J}_0, l_2] + [\mathcal{J}_1, l_1] + [\mathcal{J}_2, l_0])(P) &= l_2(\mathcal{J}_0, P) + l_1(\mathcal{J}_1(P)) - \mathcal{J}_1(l_1(P))\\ &\quad - \mathcal{J}_2(l_0, P)\\ &= l_2(-\pi, P) + l_1(\{-N, P\}) - \mathcal{J}_1(\{\gamma, P\})\\ &\quad - \mathcal{J}_2(\psi, P)\\ &= \{\{\mu, -\pi\}, P\} + \{\gamma, \{-N, P\}\}\\ &\quad - \{-N, \{\gamma, P\}\} - \{\{-\omega, \psi\}, P\}\\ &= \{\{\pi, \mu\}, P\} - \{\{\gamma, N\}, P\} + \{\{\omega, \psi\}, P\}\\ &= \{\{\pi, \mu\} + \{N, \gamma\} + \{\omega, \psi\}, P\}.\end{aligned}$$

For 2-brackets and 3-brackets, computations are more laborious (because they implicate more terms) but are similar. ■

In the next theorem we shall use Theorem 5.5 in order to relate some classes of Nijenhuis endomorphisms on $(A \oplus A^*, \Theta)$ with Nijenhuis vector-valued forms on (L, l).

Theorem 5.6. *Let $(A \oplus A^*, \Theta)$ be a pre-Courant algebroid and J be a skew-symmetric endomorphism of $A \oplus A^*$ such that $J^2 = \lambda\, \mathrm{id}_{A\oplus A^*}$, for some $\lambda \in \mathbb{R}$.*

Then, J is a Nijenhuis morphism on $(A \oplus A^*, \Theta)$ iff $\mathcal{J} = \Upsilon(J)$ is a Nijenhuis vector-valued form on the curved pre-L_∞-structure $l = \mathscr{M}(\Theta)$ with square $\mathcal{k} = -\lambda\mathscr{E}$, where $\mathscr{E}$ is the Euler map defined by (9).

Let us prove a calculatory lemma before proving Theorem 5.6

Lemma 5.7. *Let $\mathcal{k}$ be the vector-valued form on L, of degree zero, given by $\mathcal{k} = -\lambda\mathscr{E}$, for some $\lambda \in \mathbb{R}$.*

i) If $l_i, i = 0,1,2,3,$ are graded symmetric brackets of degree 1 *on L, then*

$$[\mathcal{K}, l_i] = \lambda\, l_i, \quad i = 0,1,2,3.$$

ii) If $\mathcal{J}_i, i = 0,1,2,$ are graded symmetric vector-valued forms of degree zero on L, then

$$[\mathcal{J}_i, \mathcal{K}] = 0, \quad i = 0,1,2.$$

Proof: i) The proof is done directly. We present here, as an example, the computations for $i = 2$. For all $P \in L_p$ and $Q \in L_q$, we have

$$\begin{aligned}
[\mathcal{K}, l_2](P,Q) &= (\imath_{\mathcal{K}} l_2 - \imath_{l_2}\mathcal{K})(P,Q) \\
&= l_2(\mathcal{K}(P), Q) + (-1)^{pq}\, l_2(\mathcal{K}(Q), P) - \mathcal{K}(l_2(P,Q)) \\
&= -\lambda\, p\, l_2(P,Q) - \lambda\, q\, (-1)^{pq}\, l_2(Q,P) + \lambda(p+q+1)\, l_2(P,Q) \\
&= \lambda\, l_2(P,Q).
\end{aligned}$$

ii) Analogous to i).

■

Let us now prove Theorem 5.6

Proof of Theorem 5.6: The statement of Lemma 5.7 ii) ensures that $\mathcal{J}$ is a Nijenhuis vector-valued form on $l = \mathscr{M}(\Theta)$ with square $\mathcal{K}$ if and only if

$$[\mathcal{J}, [\mathcal{J}, l]] = [\mathcal{K}, l]. \tag{27}$$

Using Theorem 5.5 twice, for the l.h.s. of Equation (27), we have :

$$[\mathcal{J}, [\mathcal{J}, l]] = \left(l_{\mathcal{J}}\right)_{\mathcal{J}} = \mathscr{M}\left((\Theta_J)_J\right).$$

Furthermore, by Lemma 5.7 i) we know that

$$[\mathcal{K}, l] = \lambda\, l = \lambda\, \mathscr{M}(\Theta) = \mathscr{M}(\lambda\, \Theta).$$

Thus, Equation (27) is equivalent to

$$\mathscr{M}((\Theta_J)_J) = \mathscr{M}(\lambda\, \Theta),$$

which is equivalent to $(\Theta_J)_J = \lambda\, \Theta$, because the map $\mathscr{M}$ is injective (this is part of the proof of Theorem 4.1). But, using Equation (6), this is equivalent to J being a Nijenhuis endomorphism on $(A \oplus A^*, \Theta)$. ■

Remark 5.8. In Theorem 5.6, if Θ is a Courant algebroid structure then Θ_J is also a Courant algebroid structure and both l and $l_{\mathcal{J}}$ are curved L_∞-structures.

As it was mentioned in Section 2, $(A \oplus A^*, \mu)$ is a Courant algebroid if and only if (A, μ) is a Lie algebroid. Moreover, if $\Theta = \mu + \phi$, from (4) we have that $(A \oplus A^*, \mu + \phi)$ is a Courant algebroid if and only if (A, μ) is a Lie algebroid and $\{\mu, \phi\} = 0$. The condition $\{\mu, \phi\} = 0$ means that ϕ is a closed 3-form on the Lie algebroid (A, μ), and we write $\mathrm{d}\phi = 0$.

Poisson quasi-Nijenhuis structures with background were intoduced in [1] as quadruples (π, N, φ, H) formed by a bivector π, a $(1,1)$-tensor N and two closed 3-forms φ and H on a Lie algebroid (A, μ), satisfying some conditions. In the case where φ is exact, $\varphi = \mathrm{d}\omega$, we have an exact Poisson quasi-Nijenhuis structure with background [2]. Denoting by $C(\pi, N)$ the Magri-Morosi concomitant of π and N and by $\mathcal{T}N$ the Nijenhuis torsion of N on (A, μ), the definition goes as follows:

Definition 5.9. [2] An *exact Poisson quasi-Nijenhuis structure with background* on a Lie algebroid (A, μ) is a quadruple (π, N, ω, H), where π is a bivector, ω is a 2-form, N is a $(1,1)$-tensor and H is a closed 3-form such that $N \circ \pi^\# = \pi^\# \circ N^*$, $\omega^\flat \circ N = N^* \circ \omega^\flat$ and

(i) π is Poisson,
(ii) $C(\pi, N)(\alpha, \beta) = 2H(\pi^\#(\alpha), \pi^\#(\beta), .)$, for all $\alpha, \beta \in \Gamma(A^*)$,
(iii) $\mathcal{T}N(X, Y) = \pi^\#(H(NX, Y, .) + H(X, NY, .) + \mathrm{d}\omega(X, Y, .))$, for all $X, Y \in \Gamma(A)$,
(iv) $i_N \mathrm{d}\omega - \mathrm{d}\omega_N - \mathscr{H} + \lambda H = 0$, for some $\lambda \in \mathbb{R}$,

with $\omega_N(X, Y) := \omega(NX, Y)$ and $\mathscr{H}(X, Y, Z) := \circlearrowleft_{X,Y,Z} H(NX, NY, Z)$, for all $X, Y, Z \in \Gamma(A)$, where $\circlearrowleft_{X,Y,Z}$ means sum after circular permutation on X, Y and Z.

In [2] we proved that, given a closed 3-form $H \in \Gamma(\wedge^3 A^*)$ on a Lie algebroid (A, μ) and a skew-symmetric endomorphism J of $A \oplus A^*$,

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix},$$

such that $J^2 = \lambda\, \mathrm{id}_{A \oplus A^*}$, for some $\lambda \in \mathbb{R}$, then *J is a Nijenhuis morphism on the Courant algebroid $(A \oplus A^*, \mu + H)$ if and only if the quadruple (π, N, ω, H) is an exact Poisson quasi-Nijenhuis structure with background on (A, μ).*

From Theorem 5.6, we get that each exact Poisson quasi-Nijenhuis structure with background on a Lie algebroid (A, μ) can be seen as a Nijenhuis vector-valued form with respect to a curved L_∞-algebra structure on $\Gamma(\wedge^\bullet A)[2]$. More precisely, we have:

Corollary 5.10. *Let (A,μ) be a Lie algebroid, π a bivector, ω a 2-form, N a $(1,1)-$tensor and ϕ a closed 3-form on (A,μ). Assume that $N\circ\pi^\# = \pi^\#\circ N^*$, $\omega^\flat \circ N = N^* \circ \omega^\flat$ and $N^2 = \lambda\,\mathrm{id}_A$, for some $\lambda \in \mathbb{R}$. Then, the quadruple (π, N, ω, ϕ) is an exact Poisson quasi-Nijenhuis structure with background on (A,μ) if and only if $\mathcal{J} = -\pi + \underline{N} + \underline{\omega}$ is a Nijenhuis vector-valued form on the L_∞-algebra $(\Gamma(\wedge^\bullet A)[2], l_2 + l_3)$, with square $\mathcal{k} = -\lambda\mathscr{E}$.*

In [5] a one-to-one correspondence between an exact Poisson quasi-Nijenhuis structure with background and a co-boundary Nijenhuis[†] vector-valued form is established. Indeed, Theorem 4.4 in [5] establishes that *$(\pi, N, -\omega, \phi)$ is an exact Poisson quasi-Nijenhuis structure with background on a Lie algebroid (A,μ) if and only if $\mathcal{N} = \pi + \underline{N} + \underline{\omega}$ is a co-boundary Nijenhuis vector-valued form on the L_∞-algebra* $(\Gamma(\wedge^\bullet A)[2], l_2 + \underline{\phi} = l_2 - l_3)$[‡], with square $\underline{N^2} + [\underline{\omega}, \pi]$.

The approach in [5] is different from the one considered in the current paper since, contrary to what happens in Corollary 5.10, $\Upsilon(\pi + N - \omega) = \pi + \underline{N} - \underline{\omega} \neq \mathcal{N}$.

When, in Definition 5.9, $\omega = 0$ and $H = 0$, the pair (π, N) is a *Poisson-Nijenhuis structure* on the Lie algebroid (A,μ). In the case where $N^2 = \lambda\,\mathrm{id}_A$, for some $\lambda \in \mathbb{R}$, Theorem 5.6 gives the following characterization of these Poisson-Nijenhuis structures in the setting of L_∞-algebras (see [2]).

Corollary 5.11. *Let (A,μ) be a Lie algebroid, π a bivector and N a $(1,1)$-tensor such that $N\circ\pi^\# = \pi^\#\circ N^*$ and $N^2 = \lambda\,\mathrm{id}_A$, for some $\lambda \in \mathbb{R}$. Then, the pair (π, N) is a Poisson-Nijenhuis structure on (A,μ) if and only if $\mathcal{J} = -\pi + \underline{N}$ is a Nijenhuis vector-valued form with respect to the L_∞-algebra $(\Gamma(\wedge^\bullet A)[2], l_2)$, with square $\mathcal{k} = -\lambda\mathscr{E}$.*

Recall that an *ΩN structure* on a Lie algebroid (A,μ) is a pair (ω, N), where N is a Nijenhuis tensor, ω is a closed 2-form such that $\omega^\flat\circ N = N^*\circ\omega^\flat$ and the 2-form $\omega_N(\cdot,\cdot) = \omega(N\cdot,\cdot)$ is closed.

In [2] we proved that, given a closed 2-form ω on a Lie algebroid (A,μ) and a skew-symmetric endomorphism $J_{\omega,N}$ of $A\oplus A^*$,

$$J_{\omega,N} = \begin{pmatrix} N & 0 \\ \omega^\flat & -N^* \end{pmatrix},$$

[†]If we remove condition $[\mathcal{n}, \mathcal{k}] = 0$ in Definition 3.3, $\mathcal{n}$ is called a co-boundary Nijenhuis form.

[‡]The extension $\underline{\phi}$ of the 3-form ϕ is given by (20). More precisely, for all $P, Q, R \in \Gamma(\wedge^\bullet A)$, $\underline{\phi}(P,Q,R) = \{\{\{-\phi, P\}, Q\}, R\} = -l_3(P,Q,R)$ and Lemma 4.2 yields $\underline{\phi} = -l_3$.

such that $J^2_{\omega,N} = \lambda\, \mathrm{id}_{A\oplus A^*}$, for some $\lambda \in \mathbb{R}$, then $J_{\omega,N}$ *is a Nijenhuis morphism on the Courant algebroid* $(A \oplus A^*, \mu)$ *if and only if* (ω, N) *is an* ΩN *structure on* (A, μ). So, from Theorem 5.6, we get the following characterization of these ΩN structures in the setting of L_∞-algebras.

Corollary 5.12. *Let* (A, μ) *be a Lie algebroid,* ω *a* 2*-form and* N *a* $(1, 1)$*-tensor such that* $\omega^\flat \circ N = N^* \circ \omega^\flat$ *and* $N^2 = \lambda\, \mathrm{id}_A$*, for some* $\lambda \in \mathbb{R}$*. Then, the pair* (ω, N) *is an* ΩN *structure on* (A, μ) *if and only if* $\mathcal{J} = \underline{N} + \underline{\omega}$ *is a Nijenhuis vector-valued form with respect to the* L_∞*-algebra* $(\Gamma(\wedge^\bullet A)[2], l_2)$*, with square* $\mathcal{k} = -\lambda \mathcal{E}$*.*

A $P\Omega$ *structure* on a Lie algebroid (A, μ) is a pair (π, ω), where π is a Poisson bivector and the 2-forms ω and ω_N are closed, with $N = \pi^\sharp \circ \omega^\flat$.

Using Theorem 5.6, we may establish a relation between a class of $P\Omega$ structures and L_∞-algebras. For that purpose we need to recall the next two lemmas.

Lemma 5.13. [1] *If* (π, ω) *is a* $P\Omega$ *structure on* (A, μ)*, then* $N = \pi^\sharp \circ \omega^\flat$ *is a Nijenhuis tensor on* (A, μ)*.*

Lemma 5.14. [13] *Let* N *be a* $(1, 1)$*-tensor on a Lie algebroid* (A, μ) *such that* $N^2 = \lambda\, \mathrm{id}_A$*, for some* $\lambda \in \mathbb{R}$*. Then* N *is a Nijenhuis tensor on* (A, μ) *if and only if* $J_N : A \oplus A^* \to A \oplus A^*$ *given by* $J_N = \begin{pmatrix} N & 0 \\ 0 & -N^* \end{pmatrix}$ *is a Nijenhuis morphism on the Courant algebroid* $(A \oplus A^*, \mu)$*.*

Corollary 5.15. *Let* (A, μ) *be a Lie algebroid,* π *a Poisson bivector and* ω *a closed* 2*-form such that* $N := \pi^\sharp \circ \omega^\flat$ *satisfies* $N^2 = \lambda\, \mathrm{id}_A$*, for some* $\lambda \in \mathbb{R}$*. If the pair* (π, ω) *is a* $P\Omega$ *structure on* (A, μ) *then* $\mathcal{J} = \underline{N}$ *is a Nijenhuis vector-valued form with respect to the* L_∞*-algebra* $(\Gamma(\wedge^\bullet A)[2], l_2)$*, with square* $\mathcal{k} = -\lambda \mathcal{E}$*.*

Proof: It is a direct consequence of Theorem 5.6, by application of Lemmas 5.13 and 5.14. ∎

6. Twisting by a bivector

The purpose of this section is to discuss the twisting of a Courant algebroid and of a curved L_∞-algebra by a bivector.

Given a pre-Courant structure Θ on $A \oplus A^*$, the notion of *twisting* Θ by a bivector $\pi \in \Gamma(\wedge^2 A)$ was introduced in [19] as the canonical transformation given by the flow of the Hamiltonian vector field $X_\pi := \{\pi, \cdot\}$ associated to π:

$$e^\pi := 1 + \{\pi, \cdot\} + \frac{1}{2!}\{\pi, \{\pi, \cdot\}\} + \frac{1}{3!}\{\pi, \{\pi, \{\pi, \cdot\}\}\} + \dots$$

When applied to $\Theta = \psi + \gamma + \mu + \phi$ yields

$$\begin{aligned} e^\pi \Theta = &\psi + \{\pi, \gamma\} + \frac{1}{2}\{\pi, \{\pi, \mu\}\} + \frac{1}{6}\{\pi, \{\pi, \{\pi, \phi\}\}\} \\ &+ \gamma + \{\pi, \mu\} + \frac{1}{2}\{\pi, \{\pi, \phi\}\} + \mu + \{\pi, \phi\} + \phi. \end{aligned}$$

Since $X_\pi := \{\pi, \cdot\}$ is of degree zero, $e^\pi\Theta$ has degree 3, that is to say $e^\pi\Theta \in \mathscr{F}^3$ is a pre-Courant structure on $A \oplus A^*$. Moreover, we have the following:

Proposition 6.1. [19] *If Θ is a Courant structure on $A \oplus A^*$ so is $e^\pi\Theta$.*

Replacing $\pi \in \Gamma(\wedge^2 A)$ by $\omega \in \Gamma(\wedge^2 A^*)$ one has the twisting of Θ by ω, $e^\omega\Theta$ [19]. Proposition 6.1 also holds for $e^\omega\Theta$.

Remark 6.2. The next Definition 6.3 and Propositions 6.4 and 6.5 also hold, without any change, for curved pre-L_∞-algebras but, for the sake of better reading, we shall not address them in the more general setting.

Recall that a Maurer-Cartan element of a curved L_∞-algebra $(\mathscr{L}, \sum_{i=0}^3 \ell_i)$ is a degree zero element $\pi \in \mathscr{L}_0$ such that

$$\ell_0 - \ell_1(\pi) + \frac{1}{2}\ell_2(\pi, \pi) - \frac{1}{6}\ell_3(\pi, \pi, \pi) = 0. \tag{28}$$

Let $(\mathscr{L}, \sum_{i\geq 0} \ell_i)$ be a curved L_∞-algebra and $\pi \in \mathscr{L}_0$, a degree zero element of $\mathscr{L}$. Let us define the operator

$$\varepsilon^\pi := 1 - [\pi, \cdot] + \frac{(-1)^2}{2!}[\pi, [\pi, \cdot]] + \frac{(-1)^3}{3!}[\pi, [\pi, [\pi, \cdot]]] + \dots$$

that, applied to $\ell := \sum_{i=0}^{3} \ell_i$, yields:

$$\begin{aligned} \varepsilon^\pi \ell = & \underbrace{\ell_0 - \ell_1(\pi) + \frac{1}{2}\ell_2(\pi,\pi) - \frac{1}{6}\ell_3(\pi,\pi,\pi)}_{(\varepsilon^\pi\ell)_0} \\ & + \underbrace{\ell_1 - \ell_2(\pi,\cdot) + \frac{1}{2}\ell_3(\pi,\pi,\cdot)}_{(\varepsilon^\pi\ell)_1} + \underbrace{\ell_2 - \ell_3(\pi,\cdot)}_{(\varepsilon^\pi\ell)_2} + \underbrace{\ell_3}_{(\varepsilon^\pi\ell)_3}. \end{aligned} \tag{29}$$

Definition 6.3. The pair $(\mathscr{L}, \varepsilon^\pi \ell)$ is called the *twisting* by π of the curved L_∞-algebra $(\mathscr{L}, \ell)$.

When the curvature vanishes ($\ell_0 = 0$) and so $(\mathscr{L}, \ell)$ is an L_∞-algebra, in general, the term $(\varepsilon^\pi \ell)_0 \in \mathscr{L}_0$ need not to vanish. The vanishing of $(\varepsilon^\pi \ell)_0$, which is equivalent to

$$\ell_1(\pi) - \frac{1}{2}\ell_2(\pi,\pi) + \frac{1}{6}\ell_3(\pi,\pi,\pi) = 0,$$

means that π is a Maurer-Cartan element of the L_∞-algebra $(\mathscr{L}, \sum_{i=1}^{3} \ell_i)$ (see (28)). So, we recover a result from [9]:

Proposition 6.4. *The twisting by π of the L_∞-algebra $(\mathscr{L}, \ell)$ is an L_∞-algebra provided that π is a Maurer-Cartan element of $(\mathscr{L}, \ell)$.*

Let us see that when dealing with curved L_∞-algebras, the condition of π being a Maurer-Cartan element can be removed. Next proposition holds for any curved L_∞-algebra, but we only consider the case where $\ell_i = 0$, for $i \geq 4$, which is the one we are interested in.

Proposition 6.5. *Let $(\mathscr{L}, \sum_{i=0}^{3} \ell_i)$ be a curved L_∞-algebra and π a degree zero element of $\mathscr{L}$. Then, $(\mathscr{L}, \varepsilon^\pi \ell)$ is a curved L_∞-algebra.*

Proof: We have to prove that, for all homogeneous $X, X_1, \cdots, X_5 \in \mathscr{L}$, and using the notation of (29):

$$\begin{aligned} & i) (\varepsilon^\pi \ell)_1((\varepsilon^\pi \ell)_0) = 0; \\ & ii) (\varepsilon^\pi \ell)_2((\varepsilon^\pi \ell)_0, X) + (\varepsilon^\pi \ell)_1((\varepsilon^\pi \ell)_1(X)) = 0; \end{aligned}$$

$$iii)(\varepsilon^\pi\ell)_3((\varepsilon^\pi\ell)_0, X_1, X_2) + (\varepsilon^\pi\ell)_1((\varepsilon^\pi\ell)_2(X_1, X_2)) + (\varepsilon^\pi\ell)_2((\varepsilon^\pi\ell)_1(X_1), X_2) + (-1)^{x_1x_2}(\varepsilon^\pi\ell)_2((\varepsilon^\pi\ell)_1(X_2), X_1) = 0;$$

$$iv) \sum_{\sigma\in Sh(1,2)} \epsilon(\sigma)(\varepsilon^\pi\ell)_3((\varepsilon^\pi\ell)_1(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) + \sum_{\sigma\in Sh(2,1)} \epsilon(\sigma)(\varepsilon^\pi\ell)_2((\varepsilon^\pi\ell)_2(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)})) + (\varepsilon^\pi\ell)_1((\varepsilon^\pi\ell)_3(X_1, X_2, X_3)) = 0;$$

$$v) \sum_{\sigma\in Sh(3,1)} \epsilon(\sigma)(\varepsilon^\pi\ell)_2((\varepsilon^\pi\ell)_3(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}), X_{\sigma(4)}) + \sum_{\sigma\in Sh(2,2)} \epsilon(\sigma)(\varepsilon^\pi\ell)_3((\varepsilon^\pi\ell)_2(X_{\sigma(1)}, X_{\sigma(2)}), X_{\sigma(3)}, X_{\sigma(4)}) = 0;$$

$$vi) \sum_{\sigma\in Sh(3,2)} \epsilon(\sigma)(\varepsilon^\pi\ell)_3((\varepsilon^\pi\ell)_3(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}), X_{\sigma(4)}, X_{\sigma(5)}) = 0,$$

where x_i stands for the degree of X_i. For $i)$, we compute

$$\begin{aligned}
(\varepsilon^\pi\ell)_1((\varepsilon^\pi\ell)_0) &= (\varepsilon^\pi\ell)_1(\ell_0 - \ell_1(\pi) + \frac{1}{2}\ell_2(\pi,\pi) - \frac{1}{6}\ell_3(\pi,\pi,\pi)) \\
&= \Big[\ell_1(\ell_0)\Big] - \Big[\ell_1(\ell_1(\pi)) + \ell_2(\pi,\ell_0)\Big] \\
&+ \Big[\frac{1}{2}\ell_1(\ell_2(\pi,\pi)) + \ell_2(\pi,\ell_1(\pi)) + \frac{1}{2}\ell_3(\pi,\pi,\ell_0)\Big] \\
&- \Big[\frac{1}{6}\ell_1(\ell_3(\pi,\pi,\pi)) + \frac{1}{2}\ell_2(\pi, l_2(\pi,\pi)) + \frac{1}{2}\ell_3(\pi,\pi,\ell_1(\pi))\Big] \\
&+ \Big[\frac{1}{6}\ell_2(\pi,\ell_3(\pi,\pi,\pi)) + \frac{1}{4}\ell_3(\pi,\pi,\ell_2(\pi,\pi))\Big] \\
&- \Big[\frac{1}{12}\ell_3(\pi,\pi,\ell_3(\pi,\pi,\pi))\Big].
\end{aligned}$$

Since $(\mathscr{L}, \sum_{i=0}^3 \ell_i)$ is a curved L_∞-algebra, each expression inside the brackets $[\cdots]$ is zero as a consequence of the generalized Jacobi identity (7) for $n = 0, 1, \ldots, 5$, with $X_i = \pi$, $i = 1, \ldots, 5$. Thus, $i)$ is proved.

For $ii)$, we have

$$
\begin{aligned}
&(\varepsilon^\pi \ell)_2((\varepsilon^\pi \ell)_0, X) + (\varepsilon^\pi \ell)_1((\varepsilon^\pi \ell)_1(X)) = \Big[\ell_2(\ell_0, X) + \ell_1(\ell_1(X))\Big] \\
&- \Big[\ell_2(\ell_1(\pi), X) + \ell_3(\pi, \ell_0, X) + \ell_1(\ell_2(\pi, X)) + \ell_2(\ell_1(X), \pi)\Big] \\
&+ \Big[\frac{1}{2}\ell_2(\ell_2(\pi,\pi), X) + \ell_3(\ell_1(\pi), \pi, X) + \frac{1}{2}\ell_1(\ell_3(\pi,\pi,X)) + \ell_2(\ell_2(\pi, X), \pi) \\
&+ \frac{1}{2}\ell_3(\ell_1(X), \pi, \pi)\Big] - \Big[\frac{1}{6}\ell_2(\ell_3(\pi,\pi,\pi), X) + \frac{1}{2}\ell_3(\ell_2(\pi,\pi), \pi, X) \\
&+ \frac{1}{2}\ell_2(\ell_3(\pi,\pi,X), \pi) + \frac{1}{2}\ell_3(\ell_2(\pi, X), \pi, \pi)\Big] \\
&+ \Big[\frac{1}{6}\ell_3(\ell_3(\pi,\pi,\pi), \pi, X) + \frac{1}{4}\ell_3(\ell_3(\pi,\pi,X), \pi, \pi)\Big].
\end{aligned}
$$

Again, using (7) we get that each expression inside the brackets $[\cdots]$ is zero, and $ii)$ is proved. The proofs of $iii)$, $iv)$, $v)$ and $vi)$ are similar. ∎

Next we shall see that the map $\mathscr{M}$, given by Equations (10)-(13), commutes with the operations of twisting by π.

Let $(A \oplus A^*, \Theta = \psi + \gamma + \mu + \phi)$ be a pre-Courant algebroid and $(L = \Gamma(\wedge^\bullet A)[2], l = \mathscr{M}(\Theta))$ the corresponding curved pre-L_∞-algebra constructed in Section 4. Let π be a degree zero element of L, i.e., $\pi \in \Gamma(\wedge^2 A) = L_0$. Since Θ is a pre-Courant structure on $A \oplus A^*$, so it is $e^\pi \Theta$ and, according to Theorem 4.1, $\mathscr{M}(e^\pi \Theta)$ is a curved pre-L_∞-algebra.

Proposition 6.6. *We have,*

$$\mathscr{M}(e^\pi \Theta) = \varepsilon^\pi(\mathscr{M}(\Theta)) = \varepsilon^\pi l.$$

Equivalently, the next diagram is commutative:

$$
\begin{tikzcd}
\Theta \arrow[r, "\mathscr{M}"] \arrow[d, "e^\pi"'] \arrow[dr, phantom, "\circlearrowright"] & l \arrow[d, "\varepsilon^\pi"] \\
e^\pi \Theta \arrow[r, "\mathscr{M}"] & \varepsilon^\pi l
\end{tikzcd}
$$

Proof: Applying $\mathscr{M}$ to

$$e^\pi\Theta = \underbrace{\psi + \{\pi,\gamma\} + \frac{1}{2}\{\pi,\{\pi,\mu\}\} + \frac{1}{6}\{\pi,\{\pi,\{\pi,\phi\}\}\}}_{\text{bidegree }(3,0)}$$
$$+ \underbrace{\gamma + \{\pi,\mu\} + \frac{1}{2}\{\pi,\{\pi,\phi\}\}}_{\text{bidegree }(2,1)} + \underbrace{\mu + \{\pi,\phi\}}_{\text{bidegree }(1,2)} + \underbrace{\phi}_{\text{bidegree }(0,3)}$$

and using (10)-(13), yields $\mathscr{M}(e^\pi\Theta) = \sum_{i=0}^{3}(\mathscr{M}(e^\pi\Theta))_i$ with

$$\begin{cases} (\mathscr{M}(e^\pi\Theta))_0 = \psi + \{\pi,\gamma\} + \frac{1}{2}\{\pi,\{\pi,\mu\}\} + \frac{1}{6}\{\pi,\{\pi,\{\pi,\phi\}\}\} \\ (\mathscr{M}(e^\pi\Theta))_1(P) = \{\gamma,P\} + \{\{\pi,\mu\},P\} + \frac{1}{2}\{\{\pi,\{\pi,\phi\}\},P\} \\ (\mathscr{M}(e^\pi\Theta))_2(P,Q) = \{\{\mu,P\},Q\} + \{\{\{\pi,\phi\},P\},Q\} \\ (\mathscr{M}(e^\pi\Theta))_3(P,Q,R) = \{\{\{\phi,P\},Q\},R\}, \end{cases}$$

for all $P,Q,R \in \Gamma(\wedge^\bullet A)[2]$. Now, the twisting of $l = \mathscr{M}(\Theta)$ by π is, according to (29) and (10)-(13), given by $\varepsilon^\pi(\mathscr{M}(\Theta)) = \varepsilon^\pi l = \sum_{i=0}^{3}(\varepsilon^\pi l)_i$ with

$$\begin{cases} (\varepsilon^\pi l)_0 = \psi - \{\gamma,\pi\} + \frac{1}{2}\{\{\mu,\pi\},\pi\} - \frac{1}{6}\{\{\{\phi,\pi\},\pi\},\pi\} \\ (\varepsilon^\pi l)_1(P) = \{\gamma,P\} - \{\{\mu,\pi\},P\} + \frac{1}{2}\{\{\{\phi,\pi\},\pi\},P\} \\ (\varepsilon^\pi l)_2(P,Q) = \{\{\mu,P\},Q\} - \{\{\{\phi,\pi\},P\},Q\} \\ (\varepsilon^\pi l)_3(P,Q,R) = \{\{\{\phi,P\},Q\},R\}, \end{cases}$$

for all $P,Q,R \in \Gamma(\wedge^\bullet A)[2]$. Thus,

$$(\varepsilon^\pi l)_i = (\mathscr{M}(e^\pi\Theta))_i,$$

for $i = 0,\cdots,3$, which concludes the proof. ■

The next corollary is a consequence of the previous results.

Corollary 6.7. *The following assertions are equivalent:*

i) $(A \oplus A^*, e^\pi\Theta)$ *is a Courant algebroid;*

ii) $(L, \varepsilon^\pi l)$ *is a multiplicative curved* L_∞*-algebra.*

Next, we show that the L_∞-algebra attached to a bivector introduced in [21, 22] can be deduced from Corollary 4.4 as a particular case. This way, one can avoid the long direct proof presented in [21].

Let (A,μ) be a Lie algebroid over M and take a bivector $\pi \in \Gamma(\wedge^2 A)$. Next lemma appears in [19] for the case $A = TM$.

Lemma 6.8. *(A,μ) is a Lie algebroid if and only if $(A\oplus A^*, e^\pi\mu)$ is a Courant algebroid.*

Proof: Note that (A,μ) is a Lie algebroid if and only if $(A \oplus A^*,\mu)$ is a Courant algebroid. Applying Proposition 6.1 with $\Theta = \mu$, we have that if (A,μ) is a Lie algebroid then $(A\oplus A^*, e^\pi\mu)$ is a Courant algebroid. Conversely, the twisting of μ by π is

$$e^\pi\mu = \mu + \{\pi,\mu\} + \frac{1}{2}\{\pi,\{\pi,\mu\}\}$$

and, by bidegree reasons, we have that

$$\{e^\pi\mu, e^\pi\mu\} = 0 \;\Rightarrow\; \{\mu,\mu\} = 0.$$

So, if $(A\oplus A^*, e^\pi\mu)$ is a Courant algebroid, then (A,μ) is a Lie algebroid. ■

The twisting of μ by π can be written as

$$e^\pi\mu = \mu + \{\pi,\mu\} - \frac{1}{2}[\pi,\pi]_{_{SN}},$$

where $[\cdot,\cdot]_{_{SN}}$ is the Schouten-Nijenhuis bracket on the space $\Gamma(\wedge^\bullet A)$ of multivectors of A. The bivector π defines a *twisted-Poisson* structure on the Lie algebroid A and $(\mu, \{\pi,\mu\}, -\frac{1}{2}[\pi,\pi]_{_{SN}})$ is a quasi-Lie bialgebroid structure on (A^*, A) [19].

The (curved) L_∞-algebra on $\Gamma(\wedge^\bullet A^*)[2]$ that corresponds to the Courant algebroid $(A^*\oplus A, \mu + \{\pi,\mu\} - \frac{1}{2}[\pi,\pi]_{_{SN}})$ is given by (19), with $\phi = 0$, $\gamma = \{\pi,\mu\}$ and $\psi = -\frac{1}{2}[\pi,\pi]_{_{SN}}$. More precisely, and denoting by $[\cdot,\cdot]_\pi$ the bracket on the space $\Gamma(\wedge^\bullet A^*)$ of forms on A, that is usually called the Koszul bracket, (19) gives:

$$\begin{cases} \lambda_1(\alpha) = \{\mu,\alpha\} \\ \lambda_2(\alpha,\beta) = \{\{\{\pi,\mu\},\alpha\},\beta\} = (-1)^{|\alpha|}[\alpha,\beta]_\pi \\ \lambda_3(\alpha,\beta,\eta) = \left\{\left\{\left\{-\dfrac{1}{2}[\pi,\pi]_{_{SN}},\alpha\right\},\beta\right\},\eta\right\}, \end{cases}$$

where $|\alpha|$ denotes the degree of α on the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A^*)[1], \wedge, [\cdot,\cdot]_\pi)$. For $A = TM$, this is the L_∞-algebra $(\Omega(M)[2], \lambda_1 + \lambda_2 + \lambda_3)$ introduced in [21].

7. Twisting and deformation

In this section we combine the operations of twisting and deformation on both (pre-)Courant algebroids and curved (pre-)L_∞-algebras.

Let $\pi \in \Gamma(\wedge^2 A)$ be a bivector. Take $N \in \Gamma(A \otimes A^*)$ such that $N \circ \pi^\# = \pi^\# \circ N^*$ and consider the bivector $\pi_N \in \Gamma(\wedge^2 A)$ defined, for all $\alpha, \beta \in \Gamma(A^*)$, by $\pi_N(\alpha, \beta) = \pi(N^*\alpha, \beta)$ or, using the big bracket, by $\pi_N = \frac{1}{2}\{\pi, N\}$. Miming the twisting of a pre-Courant structure by π, we may define the twisting of N by π, and set $e^\pi N := N + \{\pi, N\}$. We denote by J_N and $J_{\pi N}$ the skew-symmetric endomorphisms of $A \oplus A^*$ given, respectively, by

$$J_N = \begin{pmatrix} N & 0 \\ 0 & -N^* \end{pmatrix} \quad \text{and by} \quad J_{\pi_N} = \begin{pmatrix} N & 2\pi_N^\# \\ 0 & -N^* \end{pmatrix}.$$

J_N and J_{π_N} are identified with N and $e^\pi N$, respectively, since

$$J_N(X + \alpha) = \{X + \alpha, N\} \quad \text{and} \quad J_{\pi_N}(X + \alpha) = \{X + \alpha, e^\pi N\}.$$

The deformation by J_N of the (pre-)Courant structure $\Theta = \psi + \gamma + \mu + \phi$ is the pre-Courant structure $\Theta_N = \{N, \Theta\}$, while the deformation by J_{π_N} of the (pre-)Courant structure $e^\pi\Theta$ is the pre-Courant[§] structure $(e^\pi\Theta)_{e^\pi N} = \{e^\pi N, e^\pi \Theta\}$. On the other hand, the twisting of Θ_N by π is the pre-Courant structure $e^\pi(\Theta_N)$. The relation between these functions on $\mathscr{F}^3$ is given in the next proposition.

Proposition 7.1. *We have,*

$$e^\pi(\Theta_N) = (e^\pi\Theta)_{e^\pi N}.$$

Equivalently, the next diagram is commutative:

$$\begin{tikzcd} \Theta \arrow[r, "e^\pi"] \arrow[d, "N"'] \arrow[dr, phantom, "\circlearrowright"] & e^\pi\Theta \arrow[d, "e^\pi N"] \\ \Theta_N \arrow[r, "e^\pi"] & (e^\pi\Theta)_{e^\pi N} \end{tikzcd}$$

Proof: We compute,

[§]Note that in general $\{N, \Theta\}$ is a pre-Courant structure, even if Θ is Courant. When Θ is Courant, $\{N, \Theta\}$ is Courant if and only if N is a weak-Nijenhuis morphism [10].

$$\begin{aligned} e^{\pi}(\Theta_N) = e^{\pi}\{N,\Theta\} &= \{N,\mu\} + \{\pi,\{N,\phi\}\} \\ &+ \{N,\gamma\} + \{\pi,\{N,\mu\}\} + \frac{1}{2}\{\pi,\{\pi,\{N,\phi\}\}\} + \{N,\phi\} \\ &+ \{N,\psi\} + \{\pi,\{N,\gamma\}\} + \frac{1}{2}\{\pi,\{\pi,\{N,\mu\}\}\} + \frac{1}{6}\{\pi,\{\pi,\{\pi,\{N,\phi\}\}\}\}. \end{aligned}$$

Applying the Jacobi identity of the big bracket, we get

$$\begin{aligned} e^{\pi}\{N,\Theta\} = \{N, e^{\pi}\Theta\} &+ \{\{\pi,N\},\mu + \{\pi,\mu\}\} + \{\{\pi,N\},\gamma\} \\ &+ \{\{\pi,N\},\phi + \{\pi,\phi\} + \frac{1}{2}\{\pi,\{\pi,\phi\}\}\}. \end{aligned}$$

By bidegree reasons,

$$\{\{\pi,N\},\frac{1}{2}\{\pi,\{\pi,\mu\}\}\} = 0, \quad \{\{\pi,N\},\{\pi,\gamma\}\} = 0, \{\pi,\{\pi,\gamma\}\} = 0$$

and

$$\{\{\pi,N\},\{\pi,\{\pi,\{\pi,\phi\}\}\}\} = 0,$$

and so we may write

$$\begin{aligned} e^{\pi}\{N,\Theta\} =& \{N, e^{\pi}\Theta\} + \{\{\pi,N\}, e^{\pi}\Theta\} \\ =& (e^{\pi}\Theta)_{e^{\pi}N}. \end{aligned}$$

■

Corollary 7.2. *If J_N is a Nijenhuis morphism on the Courant algebroid $(A\oplus A^*,\Theta)$, then $e^{\pi}(\Theta_N) = (e^{\pi}\Theta)_{e^{\pi}N}$ is a Courant structure on $A\oplus A^*$.*

Proof: If J_N is a Nijenhuis morphism on the Courant algebroid $(A\oplus A^*,\Theta)$, then Θ_N is a Courant structure on $A\oplus A^*$ [1]. By Proposition 6.1, $e^{\pi}(\Theta_N)$ is a Courant structure on $A\oplus A^*$. ■

Next we shall see how Proposition 7.1 and Corollary 7.2 translate into curved (pre-)L_∞-algebras.

Proposition 7.3. *Let $(A\oplus A^*,\Theta)$ be a pre-Courant algebroid, $(L, l = \mathscr{M}(\Theta))$ the curved pre-L_∞-algebra determined by $\mathscr{M}$ and $(\Gamma(\wedge^\bullet A)[2], \varepsilon^{\pi} l)$ its twisting by $\pi\in\Gamma(\wedge^2 A)$. Consider $\jmath' = \jmath_1 + \underline{N}(\jmath_0)$, with $\jmath_0$ and $\jmath_1$ given by (21) and (22). Then,*

$$\varepsilon^{\pi}(l_{\jmath_1}) = (\varepsilon^{\pi} l)_{\jmath'}. \tag{30}$$

Equivalently, the next diagram is commutative:

$$\begin{tikzcd}
l \arrow[r, "\varepsilon^\pi"] \arrow[d, "\mathcal{J}_1"'] \arrow[dr, phantom, "\circlearrowright"] & \varepsilon^\pi l \arrow[d, "\mathcal{J}'"] \\
l_{\mathcal{J}_1} = [\mathcal{J}_1, l] \arrow[r, "\varepsilon^\pi"] & {[\mathcal{J}', \varepsilon^\pi l] = (\varepsilon^\pi l)_{\mathcal{J}'}}
\end{tikzcd}$$

Proof: We start by applying Proposition 7.1 to the pre-Courant structure Θ_{J_N}, to get

$$\mathscr{M}(e^\pi(\Theta_N)) = \mathscr{M}((e^\pi\Theta)_{e^\pi N}). \tag{31}$$

Then, applying Theorem 5.5 on the right-hand side of (31), for the pre-Courant structure $e^\pi\Theta$ and the endomorphism J_{π_N}, gives

$$\mathscr{M}((e^\pi\Theta)_{e^\pi N}) = (\mathscr{M}(e^\pi\Theta))_{\Upsilon(e^\pi N)} = (\mathscr{M}(e^\pi\Theta))_{\mathcal{J}'},$$

since by (21) and (22),

$$\Upsilon(e^\pi N) = \Upsilon(\{\pi, N\} + N) = -\{\pi, N\} + \mathcal{J}_1 = \underline{N}(\mathcal{J}_0) + \mathcal{J}_1.$$

Using Proposition 6.6, we have

$$(\mathscr{M}(e^\pi\Theta))_{\mathcal{J}'} = (\varepsilon^\pi l)_{\mathcal{J}'}.$$

Now, we take the left-hand side of (31) and apply Proposition 6.6 to it, to get

$$\mathscr{M}(e^\pi\Theta_N) = \varepsilon^\pi\mathscr{M}(\Theta_N).$$

Then, Theorem 5.5 for Θ and the endomorphism J_N, gives

$$\varepsilon^\pi\mathscr{M}(\Theta_N) = \varepsilon^\pi(l_{\Upsilon(N)}) = \varepsilon^\pi(l_{\mathcal{J}_1}),$$

which completes the proof. ∎

Remark 7.4. If $(A \oplus A^*, \Theta)$ is a Courant algebroid, then $\varepsilon^\pi\, l_{\mathcal{J}_1} = (\varepsilon^\pi l)_{\mathcal{J}'}$ is a curved L_∞-algebra structure on $(\Gamma(\wedge^\bullet A)[2]$.

Corollary 7.5. *If $\mathcal{J}_1 = \Upsilon(N)$ is a Nijenhuis form on the curved L_∞-algebra $(\Gamma(\wedge^\bullet A)[2], l)$, then $l_{\mathcal{J}_1}$ and $\varepsilon^\pi(l_{\mathcal{J}_1}) = (\varepsilon^\pi l)_{\mathcal{J}'}$ are curved L_∞-algebra structures on $\Gamma(\wedge^\bullet A)[2]$.*

Proof: If $\mathcal{J}_1 = \Upsilon(N)$ is Nijenhuis for l, then $l_{\mathcal{J}_1} = [\mathcal{J}_1, l] = [\underline{N}, l]$ is a curved L_∞-algebra structure on $\Gamma(\wedge^\bullet A)[2]$ [4]. By Proposition 6.5, $\varepsilon^\pi(l_{\mathcal{J}_1})$ is a curved L_∞-algebra on $\Gamma(\wedge^\bullet A)[2]$. ∎

The results of Theorem 5.5 and Propositions 6.6, 7.1 and 7.3 can be combined to form the following commutative cubic diagram:

$$
\begin{tikzcd}
& \varepsilon^{\pi} l \arrow[rr, "\mathcal{J}'"] & & (\varepsilon^{\pi} l)_{\mathcal{J}'} \\
l \arrow[ur, "\varepsilon^{\pi}"] \arrow[rr, "\mathcal{J}_1"] & & l_{\mathcal{J}_1} \arrow[ur, "\varepsilon^{\pi}"] & \\
& e^{\pi}\Theta \arrow[uu, "\mathscr{M}"'] \arrow[rr, "e^{\pi}N"] & & (e^{\pi}\Theta)_{e^{\pi}N} \arrow[uu, "\mathscr{M}"'] \\
\Theta \arrow[uu, "\mathscr{M}"] \arrow[ur, "e^{\pi}"] \arrow[rr, "N"] & & \Theta_N \arrow[uu, "\mathscr{M}"] \arrow[ur, "e^{\pi}"'] &
\end{tikzcd}
$$

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