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ON THE MAXIMUM OF A BIVARIATE INFINITE MA MODEL WITH INTEGER INNOVATIONS

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ABSTRACT: We study the limiting behaviour of the maximum of a bivariate moving average model, based on discrete random variables. We assume that the bivariate distribution of the innovations belong to the Anderson' class (Anderson, 1970). The innovations have an impact on the random variables of the MA model by binomial thinning. We show that the limiting distribution of the bivariate maximum is also of Anderson' class, and that the components of the bivariate maximum are asymptotically independent.

KEYWORDS: Bivariate maximum, MA model, interger random variables, limit distribution.

1. Introduction

Hall (2003) studied the limiting distribution of the maximum term $M_n = \max(X_1, \cdots, X_n)$ of stationary sequences defined by non-negative integer-valued moving sequences of the form

$$X_n = \sum_{i=-\infty}^{\infty} \alpha_i \circ V_{n-i},$$

where the innovation sequence $\{V_n\}$ is an iid sequence of non-negative integer-valued random variables (rv's) with exponential type tails of the form

$$1 - F_V(n) = n^{\xi} L(n)(1+\lambda)^{-n} , \quad n \in \mathbb{N}_0,\ \xi \in \mathbb{R}, \lambda > 0, \tag{1}$$

where $L(n)$ is slowly varying at $+\infty$, and $\circ$ denotes binomial thinning with probabilities $\alpha_i \in [0, 1]$. Hall (2003) proved that $\{X_n\}$ satisfies Leadbetter's

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conditions $D(x+b_n)$ and $D'(x+b_n)$, for a suitable real sequence b_n, and then

$$\begin{cases} \limsup_{n\to\infty} P(M_n \leq x + b_n) \leq \exp(-(1+\lambda/\alpha_{\max})^{-x}) \\ \liminf_{n\to\infty} P(M_n \leq x + b_n) \geq \exp(-(1+\lambda/\alpha_{\max})^{-(x-1)}) \end{cases}$$

for all real x and $\alpha_{\max} := \max\{\alpha_i,\, i \in \mathbb{Z}\}$. This is an extension of Theorem 2 of Anderson (1970), where it is proved that for sequences of iid rv's with an integer-valued distribution function (df) F with infinite right endpoint, the limit

$$\lim_{n\to+\infty} \frac{1-F(n-1)}{1-F(n)} = r > 1, \tag{2}$$

is equivalent to

$$\begin{cases} \limsup_{n\to\infty} F^n(x+b_n) \leq \exp(-r^{-x}) \\ \liminf_{n\to\infty} F^n(x+b_n) \geq \exp(-r^{-(x-1)}) \end{cases}$$

for all real x.

The class of df's satisfying (1), which is a particular case of (2) (see, e.g., Hall and Temido (2007)) is called Anderson's class.

In this paper we extend the result of Hall (2003) for the bivariate case of an integer-valued MA model. Concretely, we study the limiting distribution of the maximum term of stationary sequences $\{(X_n, Y_n)\}$ where the two marginals are defined by non-negative integer-valued moving sequences of the form

$$(X_n, Y_n) = \left(\sum_{i=-\infty}^{\infty} \alpha_i \circ V_{n-i}, \sum_{i=-\infty}^{\infty} \beta_i \circ W_{n-i} \right),$$

where X_n and Y_n are defined as above with respect to a two-dimensional innovation sequence $\{V_n, W_n\}$. The possible class of bivariate discrete distributions $F_{V,W}$ (see (4)) includes also the bivariate geometric models. .

We assume that $X = \alpha \circ V$ and $Y = \beta \circ W$ are conditionally independent given (V, W), i.e.

$$\begin{aligned} &P(X \in A, Y \in B | V, W) = P(X \in A | V, W)\, P(Y \in B | V, W) \\ &= P(X \in A | V)\, P(Y \in B | W) \end{aligned}$$

for all events A and B. We assume that $\alpha_i, \beta_i \in [0,1]$ and

$$\alpha_i, \beta_i = O\left(|i|^{-\delta}\right), |i| \to +\infty, \tag{3}$$

for some $\delta > 2$.

Following similar ideas of Hall (2003) for the univariate case, we

- Define a bivariate model $F_{V,W}$ which contains the bivariate geometric model
- Characterize the tail of $(\alpha \circ V, \beta \circ W)$ and the tail of (X_n, Y_n), defined in terms of the model $F_{V,W}$.
- Establish the limiting behaviour of the bivariate maximum $(M_n^{(1)}, M_n^{(2)})$ of the stationary sequence $\{(X_n, Y_n)\}$ which is defined componentwise.

2. Preliminaries results for bivariate innovations

Let (V, W) be a non-negative random vector (rv) with bivariate distribution function (df) $F_{V,W}$ satisfying

$$\begin{aligned} 1 - F_{V,W}(v, w) = (1 + \lambda_1)^{-[v]} [v]^{\xi_1} L_1(v) + (1 + \lambda_2)^{-[w]} [w]^{\xi_2} L_2(w) \\ -(1 + \lambda_1)^{-[v]}(1 + \lambda_2)^{-[w]}\theta^{\min([v],[w])} L_3(v)L_4(w)v^{\xi_3}w^{\xi_4}\ell(v, w) \end{aligned} \tag{4}$$

as $v, w \to +\infty$, for positive real constants $\lambda_i > 0$, $i = 1, 2$, $\theta > 0$ such that $\theta < \min\{1 + \lambda_1, 1 + \lambda_2\}$ and $\theta > 1 - \lambda_1\lambda_2$, some real constants ξ_i, and slowly varying functions L_i, $i = 1, 2, 3, 4$, and where $\ell(v, w)$ is a positive bounded (say by ϑ) function which converges to a positive constant, which may depend on $v < w$, $v = w$ or $w > w$.

By $[x]$ we denote the greatest integer not greater than x.

Remark 2.1. *The marginal distributions are of the form:*

$$1-F_V(v) \sim [v]^{\xi_1}(1+\lambda_1)^{-[v]}L_1(v) \text{ and } 1-F_W(w) \sim [w]^{\xi_2}(1+\lambda_2)^{-[w]}L_2(w) \tag{5}$$

for $v, w \to \infty$. Both marginal dfs belong to the Anderson class since

$$\lim_{v\to\infty} \frac{1 - F_V(v)}{1 - F_V(v + 1)} = 1 + \lambda_1 \quad \text{and} \quad \lim_{\mathrm{w}\to\infty} \frac{1 - \mathrm{F_W(w)}}{1 - \mathrm{F_W(w + 1)}} = 1 + \lambda_2.$$

From (4), we can derive the probability function (pf) of (V, W):

Proposition 2.1. *. The pf of the random vector (V, W) with df (4) is given by*

$$\begin{aligned} P(V = v, W = w) \ &\sim (1 + \lambda_1)^{-v}(1 + \lambda_2)^{-w}\theta^{\min([v],[w])-1} \times \\ &\times L_3(v)L_4(w)v^{\xi_3}w^{\xi_4}\ell(v, w)\ell^*(v, w) \end{aligned}$$

for v, w large integers, where

$$\lim_{v,w\to\infty} \ell^*(v, w) = \begin{cases} \lambda_2 (1 + \lambda_1 - \theta) \ , & v < w, \\ \lambda_1\lambda_2 + \theta - 1 \ , & w = v, \\ \lambda_1 (1 + \lambda_2 - \theta) \ , & w < v, \end{cases} \tag{6}$$

and $\ell(v,w)\ell^*(v,w)$ is bounded and converges to a positive constant.

Proof: Since for $v \le w-1$

$$\begin{aligned}
P(V=v, W=w) &= P(V \le v, W \le w) - P(V \le v-1, W \le w) \\
&\quad - P(V \le v, W \le w-1) + P(V \le v-1, W \le w-1)
\end{aligned}$$

$$\begin{aligned}
&= (1+\lambda_1)^{-v}\, v^{\xi_1} L_1(v) - (1+\lambda_1)^{-v+1}\,(v-1)^{\xi_1} L_1(v-1) \\
&- (1+\lambda_1)^{-v}\, v^{\xi_1} L_1(v) + (1+\lambda_1)^{-v+1}\,(v-1)^{\xi_1} L_1(v-1) \\
&+ (1+\lambda_2)^{-w}\, w^{\xi_2} L_2(w) - (1+\lambda_2)^{-w}\, w^{\xi_2} L_2(w) \\
&- (1+\lambda_2)^{-w+1}\,(w-1)^{\xi_2} L_2(w-1) + (1+\lambda_2)^{-w+1}\,(w-1)^{\xi_2} L_2(w-1) \\
&+ (1+\lambda_1)^{-v}\,(1+\lambda_2)^{-w}\,\theta^v v^{\xi_3} w^{\xi_4} L_3(v) L_4(w) \ell(v,w) \\
&- (1+\lambda_1)^{-v+1}\,(1+\lambda_2)^{-w}\,\theta^{v-1}(v-1)^{\xi_3} w^{\xi_4} L_3(v-1) L_4(w) \ell(v-1,w) \\
&- (1+\lambda_1)^{-v}\,(1+\lambda_2)^{-w+1}\,\theta^v v^{\xi_3}(w-1)^{\xi_4} L_3(v) L_4(w-1) \ell(v,w-1) \\
&+ (1+\lambda_1)^{-v+1}\,(1+\lambda_2)^{-w+1}\,\theta^{v-1}(v-1)^{\xi_3}(w-1)^{\xi_4} L_3(v-1) \times \\
&\times L_4(w-1)\ell(v-1,w-1)
\end{aligned}$$

then, we deduce for v and w large

$$\begin{aligned}
&= (1+\lambda_1)^{-v}\,(1+\lambda_2)^{-w}\,\theta^{v-1} v^{\xi_3} w^{\xi_4} L_3(v) L_4(w) \ell(v,w) \times \\
&\quad \Bigg[\theta - (1+\lambda_1)\left(1-\frac{1}{v}\right)^{\xi_3} \frac{L_3(v-1)}{L_3(v)} \frac{\ell(v-1,w)}{\ell(v,w)} \\
&\quad -(1+\lambda_2)\theta\left(1-\frac{1}{w}\right)^{\xi_4} \frac{L_4(w-1)}{L_4(w)} \frac{\ell(v,w-1)}{\ell(v,w)} \\
&\quad + (1+\lambda_1)(1+\lambda_2)\left(1-\frac{1}{v}\right)^{\xi_3}\left(1-\frac{1}{w}\right)^{\xi_4} \\
&\quad \times \frac{L_3(v-1)L_4(w-1)}{L_3(v)L_4(w)} \frac{\ell(v-1,w-1)}{\ell(v,w)}\Bigg] \\
&= (1+\lambda_1)^{-v}(1+\lambda_2)^{-w}\theta^{v-1} v^{\xi_3} w^{\xi_4} L_3(v) L_4(w) \ell(v,w) \ell^*(v,w)
\end{aligned}$$

where $\ell^*(v,w) \longrightarrow \lambda_2(1+\lambda_1-\theta)$, as $v,w \to +\infty$.

For $v \geq w+1$ the steps are similar, with $\ell^*(v,w)$

$$\begin{aligned}
\lim_{v,w} \ell^*(v,w) &= \theta - (1+\lambda_1)\lim_{v,w}\frac{\ell(v-1,w)}{\ell(v,w)} - (1+\lambda_2)\lim_{v,w}\frac{\ell(v,w-1)}{\ell(v,w)} + \\
&\quad + (1+\lambda_1)(1+\lambda_2)\lim_{v,w}\frac{\ell(v-1,w-1)}{\ell(v,w)} \\
&= \lambda_1(1+\lambda_2-\theta)
\end{aligned}$$

For $v = w \in \mathbb{N}$ large, we get with similar steps as above

$$\begin{aligned}
&P(V=v, W=v) = \\
&= (1+\lambda_1)^{-v}(1+\lambda_2)^{-v}\theta^{v-1}v^{\xi_3+\xi_4}L_3(v)L_4(v)\ell(v,v)\times \\
&\quad \Bigg[\theta - (1+\lambda_1)\left(1-\frac{1}{v}\right)^{\xi_3}\frac{L_3(v-1)}{L_3(v)}\frac{\ell(v-1,v)}{\ell(v,v)} - (1+\lambda_2)\left(1-\frac{1}{v}\right)^{\xi_4}\times \\
&\qquad\qquad \times\frac{L_4(v-1)}{L_4(v)}\frac{\ell(v,v-1)}{\ell(v,v)} \\
&\quad + (1+\lambda_1)(1+\lambda_2)\left(1-\frac{1}{v}\right)^{\xi_3+\xi_4}\times \\
&\qquad\qquad \times\frac{L_3(v-1)L_4(v-1)}{L_3(v)L_4(v)}\frac{\ell(v-1,v-1)}{\ell(v,v)}\Bigg] \\
&= (1+\lambda_1)^{-v}(1+\lambda_2)^{-v}\theta^{v-1}v^{\xi_3+\xi_4}L_3(v)L_4(v)\ell(v,v)\ell^*(v,v)
\end{aligned}$$

where $\ell^*(v,v) \longrightarrow \lambda_1\lambda_2 + \theta - 1 > 0$ as $v \to +\infty$, a positive constant by assumption. ∎

Example 2.1. *The Bivariate Geometric (BG) distribution is a particular case of the model (4) with margins (5). Indeed, using the construction of a BG distribution based on sequences of bivariate Bernoulli random variables (B_1, B_2) with success probabilities p_{+1} and p_{1+}, the pf and the df of a random vector (V,W) with BG distribution are given, respectively, by Mitov and Nadarajah (2005)*

$$f_{V,W}(v,w) = P(V=v, W=w) = \begin{cases} p_{00}^v p_{10} p_{+0}^{w-v-1} p_{+1}\ , & 0 \leq v < w, \\ p_{00}^v p_{11}\ , & v = w, \\ p_{00}^w p_{01} p_{0+}^{v-w-1} p_{1+}\ , & 0 \leq w < v, \end{cases} \qquad (7)$$

for $v, w \in \mathbb{N}_0$*, and*

$$\begin{aligned} F_{V,W}(v,w) &= P(V \le v, W \le w) \\ &= 1 - p_{0+}^{[v]+1} - p_{+0}^{[w]+1} + \begin{cases} p_{00}^{[v]+1} p_{+0}^{[w]-[v]}, & 0 \le v \le w, \\ p_{00}^{[w]+1} p_{0+}^{[v]-[w]}, & 0 \le w < v, \end{cases} \end{aligned} \tag{8}$$

for $v, w \in \mathbb{R}_0^+$*, assuming that* $0 < p_{0+}, p_{+0} < 1$*. Hence this distribution satisfies (4) with the constants* λ_1*,* λ_2 *given by* $1+\lambda_1 = \frac{1}{p_{0+}} > 1$ *and* $1+\lambda_2 = \frac{1}{p_{+0}} > 1$ *and the index* θ *associated to the dependence structure of the rv* (B_1, B_2) *is*

$$\theta = \frac{p_{00}}{p_{0+}p_{+0}}.$$

The slowly varying functions are constants and $\xi_i = 0$ *for* $i = 1,2,3,4$*. The independence case occurs when* $\theta = 1$*. For dependence cases, we can have* $0 < \theta < 1$ *or* $\theta > 1$*. Finally, we note that* $\ell(v,w)$ *is a constant. For instance, take* $L_1(v) = L_3(v) = 1/(1+\lambda_1)$*,* $L_2(v) = L_4(v) = 1/(1+\lambda_2)$*, we have* $\ell(v,w) = \theta$*) with* $\ell^*(v,w)$ *as in (6).*

The marginal df of V *and* W *are*

$$P(V \le v) = 1 - p_{0+}^{[v]+1} \quad \text{and} \quad P(W \le w) = 1 - p_{+0}^{[w]+1}, \quad \text{for } v, w \ge 0$$

which means V *and* W *follow a geometric distribution with parameter* p_{1+} *and* p_{+1}*, respectively.* □

In order to characterize the df of $(X,Y) = (\alpha \circ V, \beta \circ W)$ we start by establishing the relationship between the probability generating function (pgf) of (V,W) and (X,Y), defined e.g. for (V,W) as

$$P_{V,W}(s_1,s_2) := \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P(V=k_1, W=k_2) s_1^{k_1} s_2^{k_2}$$

which exists in the region $\mathcal{R}$ of convergence of the double series in $P_{V,W}$. Obviously, any $(s_1,s_2) \in \mathcal{R}$ with $|s_i| \le 1$.

Proposition 2.2. *The pgf of* $(X,Y) = (\alpha \circ V, \beta \circ W)$ *is given in terms of the pgf of* (V,W) *by* $P_{X,Y}(s_1,s_2) = P_{V,W}(\alpha s_1 + 1 - \alpha, \beta s_2 + 1 - \beta)$ *for all* $(s_1,s_2) \in \mathcal{R}$*.*

Proof: Denoting $B(n,p)$ a random variable following a binomial distribution with parameters n and p, we have:

$$P_{X,Y}(s_1,s_2) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} P(X=j_1, Y=j_2) s_1^{j_1} s_2^{j_2}$$

$$\begin{aligned}
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty} P(X=j_1, Y=j_2|V=k_1, W=k_2)\times \\
&\qquad \times P(V=k_1, W=k_2) s_1^{j_1} s_2^{j_2} \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty} P(X=j_1|V=k_1)P(Y=j_2|W=k_2)\times \\
&\qquad \times P(V=k_1, W=k_2) s_1^{j_1} s_2^{j_2} \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{k_1=j_1}^{\infty}\sum_{k_2=j_2}^{\infty} P(B(k_1,\alpha)=j_1)P(B(k_2,\beta)=j_2)\times \\
&\qquad \times P(V=k_1, W=k_2) s_1^{j_1} s_2^{j_2} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2} P(B(k_1,\alpha)=j_1)P(B(k_2,\beta)=j_2)\times \\
&\qquad \times P(V=k_1, W=k_2) s_1^{j_1} s_2^{j_2} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(\sum_{j_1=0}^{k_1} P(B(k_1,\alpha)=j_1)s_1^{j_1}\right)\times \\
&\quad \times\left(\sum_{j_2=0}^{k_2} P(B(k_2,\beta)=j_2)s_2^{j_2}\right) P(V=k_1, W=k_2) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} (\alpha s_1+1-\alpha)^{k_1}(\beta s_2+1-\beta)^{k_2} P(V=k_1, W=k_2) \\
&= P_{V,W}(\alpha s_1+1-\alpha, \beta s_2+1-\beta)
\end{aligned}$$

■

Taking into account Proposition 2.1, the series $P_{V,W}(\alpha s_1+1-\alpha, \beta s_2+1-\beta)$ converges obviously for any $s_i \leq 1$. Even for some $s_i > 1$ the series converges because of the assumption (4). By this assumption, we have $E(s_1^V) < \infty$ if

$s_1 < 1 + \lambda_1$ and $E(s_2^W) < \infty$ if $s_2 < 1 + \lambda_2$. For the joint distribution with $s_i > 1$, the series exists shown in the following lemma.

Lemma 2.1. *For $s_1, s_2 > 1$, $P_{V,W}(s_1, s_2) = E(s_1^V s_2^W)$ exists if $s_i \le 1 + \lambda_i, i = 1, 2$ in case $\theta \le 1$, and if $s_i\sqrt{\theta} \le 1 + \lambda_i, i = 1, 2$, or $s_1 \le 1 + \lambda_1$ and $s_2\theta \le 1 + \lambda_2$ in case of $\theta > 1$.*

Proof: We have

$$\begin{aligned}
E(s_1^V s_2^W) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} s_1^k s_2^\ell P(V = k, W = \ell) \\
&= \sum_{k=0}^{m}\sum_{\ell=0}^{n} s_1^k s_2^\ell P(V = k, W = \ell) + \sum_{k=0}^{m}\sum_{\ell=n+1}^{\infty} s_1^k s_2^\ell P(V = k, W = \ell) \\
+ &\sum_{k=m+1}^{\infty}\sum_{\ell=0}^{n} s_1^k s_2^\ell P(V = k, W = \ell) + \sum_{k=m+1}^{\infty}\sum_{\ell=n+1}^{\infty} s_1^k s_2^\ell P(V = k, W = \ell)
\end{aligned}$$

The first partial sum is finite. The second one is bounded by

$$\frac{(s_1^{m+1} - 1)}{s_1 - 1} \sum_{\ell=n+1}^{\infty} s_2^\ell P(W = \ell)$$

which is finite for $s_2 < 1 + \lambda_2$. Analogously, the third one is bounded by

$$\frac{(s_2^{n+1} - 1)}{s_2 - 1} \sum_{k=m+1}^{\infty} s_1^k P(V = k)$$

which is finite for $s_1 < 1 + \lambda_1$. Finally, for the last partial sum we use Proposition 2.1 for large k, ℓ if $\theta > 1$ with

$$\theta^{\min(k,\ell)} \le \theta^{(k+\ell)/2} \quad \text{or} \quad \theta^{\min(k,\ell)} \le \theta^\ell$$

for the second conditions. This sum is finite if $s_1\sqrt{\theta} \le 1 + \lambda_1$ and $s_2\sqrt{\theta} \le 1 + \lambda_2$ or if

$$s_1 \le 1 + \lambda_1 \quad \text{and} \quad s_2\theta \le 1 + \lambda_2.$$

If $\theta \le 1$, the last sum exists by the first mentioned conditions $s_i < 1 + \lambda_i$, $i = 1, 2$. ■

We want to derive an exact relationship of the two distributions $F_{V,W}(v, w)$ and $F_{X,Y}(x, y)$. First, we investigate the pgf of the two distributions and define the modified pgf

$$Q_{V,W}(s_1,s_2)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(1-F_{(V,W)}(k_1,k_2)\right)s_1^{k_1}s_2^{k_2},$$

and analogously for X,Y. Between $Q_{V,W}(v,w)$ and $P_{V,W}(v,w)$ we have the relationship

$$Q_{V,W}(s_1,s_2)=\frac{1-P_{V,W}(s_1,s_2)}{(1-s_1)(1-s_2)},\quad s_1,s_2\in\mathbb{R},$$

if the series converge.

Proposition 2.3. *The modified pgf of (X,Y) and (V,W) satisfy*

$$Q_{X,Y}(s_1,s_2)=\alpha\beta Q_{V,W}(\alpha s_1+1-\alpha,\beta s_2+1-\beta).$$

Proof: Write $a_1=\alpha s_1+1-\alpha$ and $a_2=\beta s_2+1-\beta$. By Proposition 2.2 we have

$$\begin{aligned}Q_{X,Y}(s_1,s_2)&=\frac{1-P_{X,Y}(s_1,s_2)}{(1-s_1)(1-s_2)}=\frac{1-P_{V,W}(\alpha s_1+1-\alpha,\beta s_2+1-\beta)}{(1-s_1)(1-s_2)}\\&=\frac{1-P_{V,W}(a_1,a_2)}{\frac{1-a_1}{\alpha}\frac{1-a_2}{\beta}}\\&=\alpha\beta\frac{1-P_{V,W}(a_1,a_2)}{(1-a_1)(1-a_2)}\\&=\alpha\beta Q_{V,W}(a_1,a_2).\end{aligned}$$

for all $(s_1,s_2)\in\mathcal{R}$.

■

From Propositions 2.2 and 2.3, we can derive now the tail $1-F_{X,Y}(x,y)$ in terms of $1-F_{V,W}(x,y)$.

Proposition 2.4. *The joint df $F_{X,Y}$ is given in terms of the joint df $F_{V,W}$, with $x,y\in\mathbb{Z}$, by*

$$1-F_{X,Y}(x,y)=$$

$$=\sum_{k_1=x}^{\infty}\sum_{k_2=y}^{\infty}\binom{k_1}{x}\binom{k_2}{y}(1-\alpha)^{k_1-x}(1-\beta)^{k_2-y}\alpha^{x+1}\beta^{y+1}\left(1-F_{V,W}(k_1,k_2)\right)$$

Proof: By Proposition 2.3, and the definition of $Q_{V,W}$ we have

$$\begin{aligned}
Q_{X,Y}(s_1,s_2) &= \alpha\beta Q_{V,W}(\alpha s_1+1-\alpha,\beta s_2+1-\beta)\\
&= \alpha\beta\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(1-F_{V,W}(k_1,k_2)\right)\left(\alpha s_1+1-\alpha\right)^{k_1}\left(\beta s_2+1-\beta\right)^{k_2}\\
&= \alpha\beta\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(1-F_{V,W}(k_1,k_2)\right)\sum_{i=0}^{k_1}\binom{k_1}{i}(1-\alpha)^{k_1-i}(s_1\alpha)^i\times\\
&\qquad\times\sum_{j=0}^{k_2}\binom{k_2}{j}(1-\beta)^{k_2-j}(s_2\beta)^j\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\{\sum_{k_1=i}^{\infty}\sum_{k_2=j}^{\infty}\binom{k_1}{i}\binom{k_2}{j}(1-\alpha)^{k_1-i}(1-\beta)^{k_2-j}\alpha^{i+1}\beta^{j+1}\right.\\
&\qquad\left.\times\left(1-F_{V,W}(k_1,k_2)\right)\right\}s_1^is_2^j\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(1-F_{X,Y}(i,j)\right)s_1^is_2^j
\end{aligned}$$

∎

Hence the tail of $F_{X,Y}$ can be estimated by the assumption (4).

Proposition 2.5. *If the joint df of* (V,W) *satisfies (4), then for integers* x *and* y

$$1-F_{X,Y}(x,y)=\left(1+\frac{\lambda_1}{\alpha}\right)^{-x}x^{\xi_1}L_1^*(x)+\left(1+\frac{\lambda_2}{\beta}\right)^{-y}y^{\xi_2}L_2^*(y)-H(x,y)$$

where

$$0\le H(x,y)\le\vartheta L_3^*(x)x^{\xi_3}\left(1+\frac{\lambda_1}{\alpha}\right)^{-x}L_4^*(w)y^{\xi_4}\left(1+\frac{\lambda_{2\theta}}{\beta}\right)^{-y}$$

where L_i^* *are slowly varying functions, being*

$$L_1^*(x)=\alpha\left(\frac{1+\lambda_1}{\lambda_1+\alpha}\right)^{\xi_1+1}L_1(x),\qquad L_2^*(y)=\beta\left(\frac{1+\lambda_2}{\lambda_2+\beta}\right)^{\xi_2+1}L_2(y)$$

$$L_3^*(x) = \alpha \left(\frac{1+\lambda_1}{\lambda_1+\alpha}\right)^{\xi_3+1} \cdot L_3(x) \qquad L_4^*(y) = \beta \left(\frac{1+\lambda_{2\theta}}{\lambda_{2\theta}+\beta}\right)^{\xi_4+1} \cdot L_4(y)$$

where ϑ bounds $\ell(v,w)$ and

$$\lambda_{2\theta} = \begin{cases} \lambda_2 \ , & \theta \leq 1 \\ \frac{1+\lambda_2}{\theta} - 1 \ , & \theta > 1 \end{cases} \tag{9}$$

For $\theta > 1$, we have $0 < \lambda_{2\theta} < \lambda_2$.

Proof: Due to the assumption (2) and Proposition 2.4, for x and y large, $1 - F_{XY}(x,y)$ is given by the sum of three terms. Each term, defined by double sums, can be determined or bounded by (unique) sums associated to univariate tail functions satisfying Theorem 4 of Hall (2003) or see also Hall and Temido (2007).

Hence, for $x, y \in \mathbb{Z}$, for the first sum,

$$\begin{aligned}
&\sum_{i=x}^{+\infty}\sum_{j=y}^{+\infty} \binom{i}{x}\binom{j}{y} (1-\alpha)^{i-x} (1-\beta)^{j-y} \alpha^{x+1}\beta^{y+1} (1+\lambda_1)^{-i} i^{\xi_1} L_1(i) \\
&= \sum_{i=x}^{+\infty} \binom{i}{x} (1-\alpha)^{i-x} (1+\lambda_1)^{-i} i^{\xi_1} L_1(i)\, \alpha^{x+1} \sum_{j=y}^{+\infty} \binom{j}{y} (1-\beta)^{j-y} \beta^{y+1} \\
&= \sum_{i=0}^{+\infty} \binom{i+x}{x} (1-\alpha)^{i} (1+\lambda_1)^{-i-x} (i+x)^{\xi_1} L_1(i+x)\, \alpha^{x+1} \\
&\sim \alpha \left(\frac{1+\lambda_1}{\lambda_1+\alpha}\right)^{\xi_1+1} x^{\xi_1} L_1(x) \left(1+\frac{\lambda_1}{\alpha}\right)^{-x} \\
&= \left(1+\frac{\lambda_1}{\alpha}\right)^{-x} x^{\xi_1} L_1^*(x)\,.
\end{aligned}$$

The second sum can be dealt with in the same way.

For the third term observe that due to the fact $\ell(v,w)$ is a bounded function, with bound ϑ, we get for large integers x, y,

$$\begin{aligned}
&H(x,y) = P(X > x, Y > y) \\
&\leq \vartheta \sum_{k_1=x}^{\infty}\sum_{k_2=y}^{\infty}\binom{k_1}{x}\binom{k_2}{y}(1-\alpha)^{k_1-x}(1-\beta)^{k_2-y}\alpha^{x+1}\beta^{y+1}\left(1+\lambda_1\right)^{-k_1}\left(1+\lambda_2\right)^{-k_2} \\
&\qquad \times L_3(k_1)L_4(k_2)\max(1,\theta^{\min(k_1,k_2)})k_1^{\xi_3}k_2^{\xi_4} \\
&\qquad \leq \vartheta \sum_{k_1=x}^{+\infty}\binom{k_1}{x}(1-\alpha)^{k_1-x}\,\alpha^{x+1}\left(1+\lambda_1\right)^{-k_1}k_1^{\xi_3}L_3(k_1) \\
&\qquad\qquad \times \sum_{k_2=y}^{+\infty}\binom{k_2}{y}(1-\beta)^{k_2-y}\,\beta^{y+1}\left(1+\lambda_2\right)^{-k_2}\max(1,\theta^{k_2})\,k_2^{\xi_4}L_4(k_2) \\
&\qquad \sim L_3^*(x)\,x^{\xi_3}\left(1+\frac{\lambda_1}{\alpha}\right)^{-x}L_4^*(y)\,y^{\xi_4}\left(1+\frac{\lambda_{2\theta}}{\beta}\right)^{-y}. \qquad \blacksquare
\end{aligned}$$

3. The bivariate stationary sequence

We consider now the stationary bivariate sequence (X_n, Y_n). We establish first the tail behaviour of (X_n, Y_n). Dominating are the maximal values of α_i and β_i as in the univariate case. Therefore we write $\alpha_{\max} = \max\{\alpha_i : |i| \geq 0\}$ and $\beta_{\max} = \max\{\beta_i : |i| \geq 0\}$. It may happen in the bivariate case that $\alpha_{\max}$ and $\beta_{\max}$ occurs at the same index or at different ones, and also whether they are unique or not. We assume for simplicity that they are unique and the α_i's and β_i's are such that

$$\sum_{i=-\infty}^{+\infty}\alpha_i < \infty,\ \sum_{i=-\infty}^{+\infty}\beta_i < \infty. \tag{10}$$

Note that (10) holds because of (3).

Suppose first that $\alpha_{\max}$ and $\beta_{\max}$ are occuring at different indexes i_0 and i_1, respectively. We write

$$X_n = \alpha_{\max}\circ V_{n-i_0} + \alpha_{i_1}\circ V_{n-i_1} + \sum_{i\neq i_0,i_1}\alpha_i\circ V_{n-i}$$

and

$$Y_n = \beta_{\max} \circ W_{n-i_1} + \beta_{i_0} \circ W_{n-i_0} + \sum_{i \neq i_0, i_1} \beta_i \circ W_{n-i}\,.$$

Denote $S_1 = \alpha_{\max} \circ V_{n-i_0}$, $S_2 = \alpha_{i_1} \circ V_{n-i_1}$, $S_3 = \sum\limits_{i \neq i_0, i_1} \alpha_i \circ V_{n-i}$, $S = S_2 + S_3$, $T_1 = \beta_{\max} \circ W_{n-i_1}$, $T_2 = \beta_{i_0} \circ W_{n-i_0}$ and $T_3 = \sum\limits_{i \neq i_0, i_1} \beta_i \circ W_{n-i}$, $T = T_2 + T_3$. Hence, $X_n = S_1 + S_2 + S_3 = S_1 + S$ and $Y_n = T_1 + T_2 + T_3 = T_1 + T$.

For the proof of the main proposition of this section we need the following lemmas.

Lemma 3.1. *If Z belongs to the Anderson class, then*

$$E(1+h)^Z = 1 + hE(Z)(1 + o_h(1)), \quad \text{as } \; h \to 0^+.$$

Proof: We first note that Z has all moments finite. Applying the mean value theorem twice to the function $f(1+h) = (1+h)^k$, $h > 0$, we get, for some small positive values $h_2 < h_1 < h$,

$$\begin{array}{rcl} (1+h)^k & = & 1 + hk(1+h_1)^{k-1} \\ & = & 1 + hk\left(1 + h_1(k-1)(1+h_2)^{k-2}\right) \\ & < & 1 + h\left(k + k^2 h_1 (1+h_2)^k\right) \end{array}$$

Since $\sum_{k=0}^{+\infty} k^2 (1+h_2)^k P(Z=k)$ is finite for small h_2, we obtain

$$E(1+h)^Z < 1 + hE(Z)(1 + o_h(1)).$$

Due to the fact that $(1+h)^k > 1 + hk$ the proof is complete. ■

Lemma 3.2. *For any set I of integers, let $\alpha^* = \max\{\alpha_i, i \in I\}$. Consider the rv $Z = \sum_{i \in I} \alpha_i \circ V_{n-i}$. Then $E(1+h)^Z$ is finite for any $0 < h < \frac{\lambda_1}{\alpha^*}$.*

Proof: Let $\epsilon > 0$ and take M large such that $\alpha_i h < \epsilon$ for $|i| > M$. Then,

$$E(1+h)^Z = \prod_{i \in I} E(1+h)^{\alpha_i \circ V_{n-i}} = \prod_{i \in I} E(1 + \alpha_i h)^{V_{n-i}} =$$

$$= \prod_{i \in I, |i| \leq M} E(1 + \alpha_i h)^{V_{n-i}} \times \prod_{i \in I, |i| > M} E(1 + \alpha_i h)^{V_{n-i}}$$

Note that $E(1+\alpha_i h)^{V_{n-i}}$ is finite for all $h < \lambda_1/\alpha_i$ which holds because $h < \frac{\lambda_1}{\alpha^*}$. Using Lemma 3.1 we obtain

$$\begin{aligned}
\prod_{|i|>M}^{\infty} E(1+\alpha_i h)^{V_{n-i}} &= \prod_{|i|>M}^{\infty} (1+\alpha_i h E(V)(1+o_\epsilon(1))) \\
&\le \exp\left(\sum_{|i|>M}^{\infty} \ln(1+\alpha_i h E(V)(1+o_\epsilon(1)))\right) \\
&\le \exp\left((1+o_\epsilon(1))hE(V)\sum_{|i|>M}^{\infty}\alpha_i\right) < \infty
\end{aligned}$$

because $\sum_{i=1}^{\infty}\alpha_i$ is finite. The bound tends to 1 for $h \to 0+$. Note that the term $o_\epsilon(1)$ does not depend on i. ■

We derive now the limiting behaviour of the tail of (X_n, Y_n). Besides of the univariate tail distributions we derive only an appropriate bound $H^*(x,y)$ for the joint tail which is sufficient for the asymptotic limit distribution of the maxima. We will see that we get asymptotic independence of the components $(M_n^{(1)}, M_n^{(2)})$ of the bivariate maxima, since this normalized $H^*(x,y)$ is vanishing, not contributing to the limit.

For this derivation, we use $\psi, \rho < 1$ and $\lambda > 0$ such that $\frac{\lambda_1}{\alpha_{\max}} < \lambda < \frac{\lambda_1}{\alpha^*}$, with $\alpha^* = \max\{\alpha_i, i \neq i_0\}$,

$$1 + \frac{\lambda_1}{\alpha_{\max}} < (1+\lambda)^{\psi} < 1+\lambda < 1 + \frac{\lambda_1}{\alpha^*} \tag{11}$$

and

$$\rho < B = \log\left(1+\frac{\lambda_2}{\beta_{\max}}\right) / \log\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right). \tag{12}$$

Proposition 3.1. *If (V,W) satisfies (4) and $\alpha_{\max}$ and $\beta_{\max}$ are unique and taken at different indexes, then*

(i) *for the marginal distributions*

$$1 - F_{X_n}(x) \sim [x]^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-[x]} L_1^{**}(x)$$

and

$$1-F_{Y_n}(y)\sim [y]^{\xi_2}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-[y]}L_2^{**}(y),$$

(ii) *for the joint distribution with* ψ,ρ,λ *satisfying (11) and (12)*

$$\begin{aligned}1-F_{X_n,Y_n}(x,y)=&[x]^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-[x]}L_1^{**}(x)\left(1+o_x(1)\right)\\&+[y]^{\xi_2}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-[y]}L_2^{**}(y)(1+o_y(1))-H^*(x,y),\end{aligned}$$

where

$$L_1^{**}(x)=L_1^*(x)E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S,\quad L_2^{**}(y)=L_2^*(y)E\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^T \tag{13}$$

and

$$\begin{aligned}H^*(x,y)\leq &o_y(1)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y}x^{\xi_3}L_3^*(x)+Cx^{\xi_1+1}y^{\xi_2}L_1^*(x)L_2^*(y)\times\\&\times\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y+(\log y)^2}+O\left(P(S>\psi_x)\right)\end{aligned} \tag{14}$$

for some constant $C>0$ *with* $\psi_x=[\psi x]$.

We show also that $P(S>\psi_x)=o(P(S_1>x))$.

Proof: We deal with the three terms in (15), separately.

$$1-F_{(X_n,Y_n)}(x,y)=1-F_{X_n}(x)+1-F_{Y_n}(y)-P(X_n>x,Y_n>y) \tag{15}$$

(i) Since $\frac{\lambda_1}{\alpha_{\max}}<\frac{\lambda_1}{\alpha^*}$, taking the sum $S=Z$ in Lemma 3.2, we conclude that $E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S$ is finite. Similarly $E\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^T$ is also finite.

The tail function of X_n is given, with $\psi_x = [\psi x]$, by

$$
\begin{aligned}
1 - F_{X_n}(x) &= P\left(S_1 + S > x\right) = \sum_{k=0}^{\infty} P(S_1 > x-k)P(S=k) \\
&= P\left(S_1 > x\right) \sum_{k=0}^{\psi_x} \frac{P(S_1 > x-k)}{P(S_1 > x)} P(S=k) + \qquad (16) \\
&\quad + \sum_{k=\psi_x+1}^{\infty} P(S_1 > x-k)P(S=k).
\end{aligned}
$$

For the first sum of (16), we get by applying Proposition 2.5 for the marginal distributions

$$
\begin{aligned}
&\sum_{k=0}^{\psi_x} \frac{P(S_1 > x-k)}{P(S_1 > x)} P(S=k) = \sum_{k=0}^{\psi_x} \left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^k (1 + o_x(1))P(S=k) \\
&\to \sum_{k=0}^{\infty} \left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^k P(S=k) \\
&= E\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^S, \; x \to +\infty,
\end{aligned}
$$

by dominated convergence.
For the second sum in (16), we get for x large

$$
\begin{aligned}
&\sum_{k=\psi_x+1}^{\infty} P(S_1 > x-k)P(S=k) \le P(S > \psi x) \\
&= P\left((1+\lambda)^S \ge (1+\lambda)^{\psi_x}\right) \le \frac{E\left(1+\lambda\right)^S}{(1+\lambda)^{\psi_x}}
\end{aligned} \qquad (17)
$$

using the Markov inequality, since $E\left(1+\lambda\right)^S$ is finite for $\lambda < \lambda_1/\alpha^*$. Since $(1+\lambda)^{\psi} > 1 + \frac{\lambda_1}{\alpha_{\max}}$ we get by Theorem 4 of Hall (2003)

$$
\frac{(1+\lambda)^{-\psi_x}}{P(S_1 > x)} \to 0, \; x \to +\infty, \qquad (18)
$$

and thus together

$$
\begin{aligned}
1-F_{X_n}(x) &= P(S_1>x)E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S + O((1+\lambda)^{-\psi_x}) \\
&= P(S_1>x)\left[E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S + O\left(\frac{(1+\lambda)^{-\psi_x}}{P(S_1>x)}\right)\right] \\
&= P(S_1>x)E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S(1+o_x(1)).
\end{aligned}
$$

With the same arguments we characterize the tail $1-F_{Y_n}$. Hence, the statements on the marginal distributions are shown.

(ii) Now we deal with the third term in (15). Note that (S_1,T_2), (S_2,T_1) and (S_3,T_3) in the representation of X_n and Y_n are independent. For any $\psi\in(0,1)$ and $\lambda>0$ satisfying (11), we use that (17) and (18) imply

$$
P(S_2+S_3>\psi x) = P\left(S\ge \psi x\right) = O((1+\lambda)^{-\psi x}) \tag{19}
$$

with $S:=S_2+S_3$. The probability in the third term of (15) is split into four summands with $\psi<1$ satisfying (11), $\psi_x=[\psi x]$ and $\delta_y=[y-(\log y)^2]$. We get for x and y large,

$$
\begin{aligned}
&P(X_n>x,\ Y_n>y) = P(S_1+S_2+S_3>x,\ T_1+T_2+T_3>y) \\
&=\sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} P(S_1>x-k,T_2>y-\ell)P(S_2+S_3=k,T_1+T_3=\ell)+ \\
&\quad+\sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{\infty} P(S_1>x-k,T_2>y-\ell)P(S_2+S_3=k,T_1+T_3=\ell)+ \\
&\quad+\sum_{k=\psi_x+1}^{\infty}\sum_{\ell=0}^{\delta_y} P(S_1>x-k,T_2>y-\ell)P(S_2+S_3=k,T_1+T_3=\ell)+ \\
&\quad+\sum_{k=\psi_x+1}^{\infty}\sum_{\ell=\delta_y+1}^{\infty} P(S_1>x-k,T_2>y-\ell)P(S_2+S_3=k,T_1+T_3=\ell).
\end{aligned} \tag{20}
$$

The last sum is bounded by $P(T_1 + T_3 > \delta_y, S_2 + S_3 > \psi_x) \leq P(S_2 + S_3 > \psi_x) = O((1+\lambda)^{-\psi x})$ by (19).

For the first sum of (20), we obtain with $\rho < 1$ such that (12) holds, using Proposition 2.5,

$$\sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} P(S_1 > x-k, T_2 > y-\ell) P(S_2+S_3 = k, T_1+T_3 = \ell)$$

$$\leq \vartheta \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} \left\{ ([x]-k)^{\xi_3} ([y]-\ell)^{\xi_4} \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-([x]-k)} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-([y]-\ell)} \times \right.$$

$$\left. \times L_3^*([x]-k)\ L_4^*([y]-\ell) P\left(S_2+S_3 = k, T_1+T_3 = \ell\right) \right.$$

$$\leq \vartheta \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} \left\{ ([x]-k)^{\xi_3} ([y]-\ell)^{\xi_4} \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-([x]-k)} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-((1-\rho)+\rho)([y]-\ell)} \right.$$

$$\left. \times L_3^*([x]-k) L_4^*([y]-\ell) P\left(S_2+S_3 = k, T_1+T_3 = \ell\right) \right.$$

$$\leq o_y(1) x^{\xi_3} L_3^*(x) \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y} \times$$

$$\times \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{k} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{\rho\ell} P\left(S_2+S_3 = k, T_1+T_3 = \ell\right)$$

$$\leq o_y(1) x^{\xi_3} L_3^*(x) \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y} \times$$

$$\times E\left(\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{(S_2+S_3)} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{\rho(T_1+T_3)}\right)$$

$$\leq o_y(1) x^{\xi_3} L_3^*(x) \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x} \left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y}$$

since $(y-\ell) > (\log y)^2$ for $\ell \leq \delta_y$, and

$$(1+\frac{\lambda_{2\theta}}{\beta_{i_0}})^{-(1-\rho)([y]-\ell)}([y]-\ell)^{\xi_4} L_4^*([y]-\ell)$$

$$\leq (1+\frac{\lambda_{2\theta}}{\beta_{i_0}})^{-(1-\rho)([y]-\ell)}([y]-\ell)^{\xi_4+\epsilon} \to 0$$

as $y \to 0$ uniformly, for any small $\epsilon > 0$.

The last pgf exists due to Lemma 3.2 and (12). Let $I = \{i \neq i_0\}$,

$$
\begin{aligned}
&E\left(\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{(S_2+S_3)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{\rho(T_1+T_3)}\right)\\
&= E\left(\left(\prod_{i\in I}(1+\frac{\lambda_1}{\alpha_{\max}})^{\alpha_i\circ V_i}\left((1+\frac{\lambda_{2\theta}}{\beta_{i_0}})^{\rho}\right)^{\beta_i\circ W_i}\right)\right)\\
&= \prod_{i\in I} E\left((1+\frac{\lambda_1}{\alpha_{\max}})^{\alpha_i\circ V_i}\left((1+\frac{\lambda_{2\theta}}{\beta_{i_0}})^{\rho}\right)^{\beta_i\circ W_i}\right)\\
&= \prod_{i\in I} E\left((1+\frac{\alpha_i\lambda_1}{\alpha_{\max}})^{V_i}(1+\beta_i([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^{\rho}-1))^{W_i}\right).
\end{aligned}
$$

The expectations exist by assumption (4) since $(1+\frac{\alpha_i\lambda_1}{\alpha_{\max}}) < 1+\lambda_1$, and also $1+\beta_i([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^{\rho}-1) \leq 1+\beta_{\max}([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^{\rho}-1) < 1+\lambda_2$, for all i, by the choice of ρ in (12). For $|\alpha_i|, |\beta_i| < \epsilon$ for any small $\epsilon > 0$, we can approximate the expectations as in Lemma 3.1 by

$$
\left(1+\frac{\alpha_i\lambda_1}{\alpha_{\max}}E(V)\right)\left(1+\beta_i([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^{\rho}-1)E(W)\right)(1+o(\alpha_i+\beta_i)).
$$

Because of the summability of the α_i and β_i, the pgf exists.

We consider now the approximation of the second sum in (20). We have with some positive constant C

$$
\begin{aligned}
&\sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{\infty} P(S_1 > x-k, T_2 > y-\ell)P(S_2+S_3 = k, T_1+T_3=\ell)\\
&\leq \sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{\infty} P(S_1 > x-k)P(T_1+T_3=\ell)\\
&\leq Cx^{\xi_1+1}L_1^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x} P(T_1+T_3>\delta_y) \qquad (21)
\end{aligned}
$$

By the arguments used to approximate $P(X_n > x) = P(S_1 + S_2 + S_3 > x)$ in (i), we also obtain

$$P(T_1+T_3 > \delta_y) \sim Cy^{\xi_2}\, L_2^*(y) E\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{T_3}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-\delta_y}, \quad \text{as } y \to +\infty.$$

with some generic constant C. Hence, it implies together with (21)

$$\sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{\infty} P(S_1 > x-k, T_2 > y-\ell)P(S_2+S_3 = k, T_1+T_3 = \ell)$$

$$\leq Cx^{\xi_1+1}y^{\xi_2}L_1^*(x)\; L_2^*(y)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y+(\log y)^2}$$

as $y \to +\infty$.

For the third sum in (20), we get analogously to the derivation of the second sum

$$\sum_{k=\psi_x+1}^{\infty}\sum_{\ell=0}^{\delta_y} P(S_1 > x-k, T_2 > y-\ell)P(S_2+S_3 = k, T_1+T_3 = \ell)$$
$$\leq \delta_y P(T_2 > y-\delta_y)P(S_2+S_3 > \psi_x)$$
$$\leq y(\log y)^{2\xi_2}L_4^*((\log y)^2)(1+\frac{\lambda_2}{\beta_{i_0}})^{-(\log y)^2}P(S_2+S_3 > \psi_x)$$
$$= o_y(1)P(S_2+S_3 > \psi_x) = o_x(P(S > \psi x))$$

Combining now the four bounds, we get our statement. ∎

Suppose now the case that the unique $\alpha_{\max}$ and $\beta_{\max}$ are taken at the same index i_0, say. Write

$$X_n = \alpha_{\max} \circ V_{n-i_0} + \sum_{i \neq i_0} \alpha_i \circ V_{n-i}$$

and

$$Y_n = \beta_{\max} \circ W_{n-i_0} + \sum_{i \neq i_0} \beta_i \circ W_{n-i}$$

Denote $S_1 = \alpha_{\max} \circ V_{n-i_0}$, $S = \sum_{i \neq i_0} \alpha_i \circ V_{n-i}$, $T_1 = \beta_{\max} \circ W_{n-i_0}$, and $T = \sum_{i \neq i_0} \beta_i \circ W_{n-i}$, as used for Proposition 3.1. Observe that (S_1, T_1) and (S, T)

are independent. Then the corresponding statement of Proposition 3.1 holds for this case (letting $\beta_{i_0} = \beta_{\max}$) given in Proposition 3.2. We omit the proof since it is very similar to the given one with a few obvious changes.

Proposition 3.2. *If (V, W) satisfies (4) and $\alpha_{\max}$ and $\beta_{\max}$ are unique, occurring at the same index, then*

$$\begin{aligned}1 - F_{X_n,Y_n}(x, y) \sim\ & [x]^{\xi_1}\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-[x]} L_1^{**}(x)\,(1 + o_x(1)) \\ & + [y]^{\xi_2}\left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^{-[y]} L_2^{**}(y)\,(1 + o_y(1)) \\ & - H^*(x, y)\end{aligned}$$

where

$$L_1^{**}(x) = L_1^*(x)E\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^S, \quad L_2^{**}(y) = L_2^*(y)E\left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^T$$

and

$$\begin{aligned}H^*(x, y) \le\ & o_y(1)x^{\xi_3}L_3^*(x)\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1 + \frac{\lambda_{2\theta}}{\beta_{\max}}\right)^{-y} + \\ & + Cy^{\xi_2}L_2^*(y)x^{\xi_1+1}L_1^*(x)\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x}\left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^{-y+(\log y)^2} \\ & + O(P(S > \psi x))\end{aligned}$$

for some constant $C > 0$ and $\psi \in (0, 1)$ satisfying (11).

Now we investigate the limiting behaviour for the bivariate maxima, in case of iid. (X_n, Y_n).

Theorem 3.1. *Let (V, W) be such that (4) holds and $\alpha_{\max}$ and $\beta_{\max}$ are unique, occurring either at the same or not the same index. Let*

$$d_1 = 1/\log(1 + \frac{\lambda_1}{\alpha_{\max}}), \quad d_2 = 1/\log(1 + \frac{\lambda_2}{\beta_{\max}})$$

Define the normalizations

$$u_n(x) = x + d_1[\log n + \xi_1 \log\log n + \log L_1^{**}(\log(n)) + \xi_1 \log d_1] \tag{22}$$

and

$$v_n(y) = y + d_2[\log n + \xi_2 \log\log n + \log L_2^{**}(\log(n))] + \xi_2 \log d_2] \tag{23}$$

Then for x, y real

$$\limsup\,(\liminf)\, n(1-F_{(X_n,Y_n)})(u_n(x),v_n(y)) \lesseqgtr$$
$$\lesseqgtr \left(1+\tfrac{\lambda_1}{\alpha_{\max}}\right)^{-x+1(0)} + \left(1+\tfrac{\lambda_2}{\beta_{\max}}\right)^{-y+1(0)}$$

Proof: The convergence for the marginal distributions holds by applying Proposition 3.1 or 3.2 with the chosen normalization sequences. Since $u_n(x)$ and $v_n(y)$ are similar in type, we only show the derivation of the first marginal. Because the normalization $u_n(x)$ is not always an integer, we have to consider lim sup and lim inf. Let us consider lim sup. The lim inf derivation follows similarly. Note that

$$\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1\log n} = \left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-d_2\log n} = \frac{1}{n}$$

and

$$\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1(\xi_1\log\log n+\log L_1^{**}(\log n)+\xi_1\log d_1)} = \frac{(d_1\log n)^{-\xi_1}}{L_1^{**}(\log n)}$$

For the normalization we get

$$[u_n(x)] \geq x-1+d_1(\log n+\xi_1\log\log n+\log L_1^{**}(\log n)+\xi_1\log d_1) \sim d_1\log n$$

So, with $b_n := d_1(\log n+\xi_1\log\log n+\log L_1^{**}(\log n)+\xi_1\log d_1)$, we have

$$\begin{aligned} & n\times[u_n(x)]^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-[u_n(x)]} L_1^{**}([u_n(x)]) \\ & \lesssim n\times(d_1\log n)^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x+1-b_n} L_1^{**}(\log n) \\ & = (1+\lambda_1/\alpha_{\max})^{-(x-1)} \end{aligned}$$

The derivation of the lim inf is similar taking into account that $[u_n(x)] \leq u_n(x)$.

Now for the joint distribution we use the bounds $H^*(x,y)$ of the two propositions. First we consider the case of Proposition 3.1. We have to derive the limits of three terms multiplied with n. The last term tends to 0 since from

(13) we get

$$
\begin{aligned}
nP(S_1 > u_n(x)) &= n \times (u_n(x))^{\xi_1} \left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n(x)} L_1^*(u_n(x)) \\
&\sim (1 + \lambda_1/\alpha_{\max})^{-x} L_1^*(\log n)/L_1^{**}(\log n) \\
&\sim (1 + \lambda_1/\alpha_{\max})^{-x} / E\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^S
\end{aligned}
$$

which is bounded.
The first of the three boundary terms is smaller than

$$
\begin{aligned}
& no_n(1)\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n(x)} \left(1 + \frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho v_n(y)} (u_n(x))^{\xi_3} L_3^*(u_n(x)) \\
&= no_n(1)\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1 \log n + o(\log n)} \left(1 + \frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho d_2 \log n + o(\log n)} \times \\
&\qquad \times (d_1 \log n)^{\xi_3} L_3^*(\log n) \\
&= o_n(1)(\log n)^{\xi_3} L_3^*(\log n) \exp\left(-\rho d_2 \log n \log\left(1 + \frac{\lambda_{2\theta}}{\beta_{i_0}}\right) + o(\log n)\right) \\
&= o_n(1)(\log n)^{\xi_3} L_3^*(\log n) \exp\left(-(\rho/B)(1 + o_n(1)) \log n\right\} \\
&= o_n(1),
\end{aligned}
$$

because $\rho/B > 0$ with B given by (12).
The second term is smaller than

$$
\begin{aligned}
& nC_1(d_1 \log n)^{\xi_1+1} (d_2 \log n)^{\xi_2} L_1^*(\log n) L_2^*(\log n) \\
&\qquad \times \left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)d_1 \log n + o(\log n)} \left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^{-d_2 \log n + (\log(d_2 \log n))^2} \\
&\leq \; C_1 (\log n)^{\xi_1+\xi_2+1} L_1^*(\log n) L_2^*(\log n) \times \\
&\qquad \times \exp\left(\log n - (1-\psi)\log n - \log n + o(\log n)\right) \\
&= \; o_n(1)
\end{aligned}
$$

since $1 - \psi > 0$ and where C_1 represents a generic positive constant.
Thus the limiting distribution is proved in case of Proposition 3.1.

Now let us consider the changes of the proof for the case of Proposition 3.2. Again we have three boundary terms in the bound where the last two are as in Proposition 3.1. In the first term we have similarly

$$
\begin{aligned}
&no_n(1)(u_n(x))^{\xi_3} L_3^*(u_n(x)) \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n(x)} \left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right)^{-v_n(y)} \\
&= o_n(1)(\log n)^{\xi_3} L_3^*(\log n) \exp\left(-d_2 \log n \log\left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right) + o(\log n)\right) \\
&= o_n(1).
\end{aligned}
$$

since $d_2 \log\left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right) > 0$.

Thus the statements are shown. ∎

4. Conditions $D(u_n, v_n)$ and $D'(u_n, v_n)$

We deal now with the mixing condition and the local dependence condition used in the bivariate extreme value theory of the limiting distribution.

We consider the conditions $D(u_n, v_n)$ and $D'(u_n, v_n)$ of Hüsler (1990) (see also Hsing (1989) and Falk et al. (1990)) since $\{(X_n, Y_n)\}$ is a stationary sequence.

Definition 4.1. *The sequence of rv's $\{(X_n, Y_n)\}$ satisfies condition $D(u_n, v_n)$ if for any integers $1 \le i_1 < ... < i_p < j_1 < ... < j_q \le n$, for which $j_1 - i_p > \ell_n$, we have*

$$
\begin{aligned}
&\Big|P\Big(\bigcap_{s=1}^{p}\{X_{i_s} \le u_n, Y_{i_s} \le v_n\}, \bigcap_{t=1}^{q}\{X_{j_t} \le u_n, Y_{j_t} \le v_n\}\Big) \\
&\quad - P\Big(\bigcap_{s=1}^{p}\{X_{i_s} \le u_n, Y_{i_s} \le v_n\}\Big) P\Big(\bigcap_{t=1}^{q}\{X_{j_t} \le u_n, Y_{j_t} \le v_n\}\Big)\Big| \le \alpha_{n,\ell_n}
\end{aligned}
$$

for some α_{n,ℓ_n} with $\lim_{n\to+\infty} \alpha_{n,\ell_n} = 0$, for some integer sequence $\ell_n = o(n)$.

Under this long range condition, extreme values occurring in largely separated intervals of positive integers are asymptotically independent. The local dependence condition $D'(u_n, v_n)$ excludes the occurrence of local clusters of extreme values in individual margins of $\{(X_n, Y_n)\}$ as well as together in the two components.

Definition 4.2. *Let $\{s_n\}$ and $\{\ell_n\}$ be sequences of positive integers satisfying*

$$\lim_{n\to\infty} s_n^{-1} = \lim_{n\to\infty} \frac{s_n \ell_n}{n} = \lim_{n\to\infty} s_n \alpha_{n,\ell_n} = 0. \tag{24}$$

The sequence of rv's $\{(X_n, Y_n)\}$ satisfies condition $D'(u_n, v_n)$ if

$$n \sum_{j=1}^{[n/s_n]} \Big\{ P\left(X_0 > u_n, X_j > u_n\right) + P\left(X_0 > u_n, Y_j > v_n\right)$$

$$+ P\left(Y_0 > v_n, Y_j > v_n\right) + P\left(Y_0 > v_n, X_j > u_n\right) \Big\} \to 0 \ , n \to +\infty.$$

Write $M_n^{(1)} = \max\{X_1, \cdots, X_n\}$ and $M_n^{(2)} = \max\{Y_1, \cdots, Y_n\}$. For the stationary sequence $\{(X_n, Y_n)\}$ satisfying $D(u_n, v_n)$ and $D'(u_n, v_n)$, the limiting behaviour of the bivariate maxima $\left(M_n^{(1)}, M_n^{(2)}\right)$, under linear normalization, is given in Theorem 3.1.

So it remains to show that the conditions $D(u_n, v_n)$ and $D'(u_n, v_n)$ hold with u_n and v_n given by (22) and (23), respectively. We begin with the first condition $D(u_n, v_n)$. Let $1 \le i_1 \le \cdots \le i_p < j_1 \le \cdots \le j_q \le n$ with $j_1 - i_p > 2\ell_n$, $\ell_n = n^{\phi}$, $\phi < 1$, and $s_n = n^{\zeta}$, $\zeta < 1$. We select ϕ and ζ later. Due to the fact that

$$\left\{ \sum_{k=-\ell_n+1}^{\infty} \alpha_k \circ V_{i-k}, \sum_{k=-\ell_n+1}^{\infty} \beta_k \circ W_{i-k} \, , i \le i_p \right\}$$

and

$$\left\{ \sum_{k=-\infty}^{\ell_n-1} \alpha_k \circ V_{j-k}, \sum_{k=-\infty}^{\ell_n-1} \beta_k \circ W_{j-k} \, , j \ge j_1 \right\}$$

are independent and writing

$$M_n^{(1,1)} = \max_{0\le j\le n} \sum_{k=-\infty}^{-\ell_n} \alpha_k \circ V_{j-k} \qquad M_n^{(1,2)} = \max_{0\le j\le n} \sum_{k=-\infty}^{-\ell_n} \beta_k \circ W_{j-k}$$

$$M_n^{(2,1)} = \max_{0\le j\le n} \sum_{k=\ell_n}^{+\infty} \alpha_k \circ V_{j-k} \qquad M_n^{(2,2)} = \max_{0\le j\le n} \sum_{k=\ell_n}^{+\infty} \beta_k \circ W_{j-k},$$

we have

$$
\begin{aligned}
&P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}, \bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\Big) \qquad (25)\\
&\le P\Big(\bigcap_{s=1}^{p}\{X_{i_s}-\sum_{k=-\infty}^{-\ell_n}\alpha_k\circ V_{i_s-k}\le u_n, Y_{i_s}-\sum_{k=-\infty}^{-\ell_n}\beta_k\circ W_{i_s-k}\le v_n\}\Big)\\
&\quad \times P\Big(\bigcap_{t=1}^{q}\{X_{i_t}-\sum_{k=\ell_n}^{\infty}\alpha_k\circ V_{j_t-k}\le u_n, Y_{j_t}-\sum_{k=\ell_n}^{\infty}\beta_k\circ W_{j_t-k}\le v_n\}\Big)\\
&\le P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n+M_n^{(1,1)}, Y_{i_s}\le v_n+M_n^{(1,2)}\}\Big)\times\\
&\quad \times P\Big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n+M_n^{(2,1)}, Y_{j_t}\le v_n+M_n^{(2,2)}\}\Big).
\end{aligned}
$$

We split furthermore this upper bound.

$$
\begin{aligned}
&P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}, \bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\Big)\\
&\le \Big[P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n+M_n^{(1,1)}, Y_{i_s}\le v_n+M_n^{(1,2)}\}, M_n^{(1,1)}=0, M_n^{(1,2)}=0\Big)\\
&\qquad + P\Big(M_n^{(1,1)}\ge 1\vee M_n^{(1,2)}\ge 1\Big)\Big]\\
&\qquad \times\Big[P\Big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n+M_n^{(2,1)}, Y_{j_t}\le v_n+M_n^{(2,2)}\}, M_n^{(2,1)}=0, M_n^{(2,2)}=0\Big)\\
&\qquad + P\Big(M_n^{(2,1)}\ge 1\vee M_n^{(2,2)}\ge 1\Big)\Big]\\
&\le \Big[P\big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}\big)+P\big(M_n^{(1,1)}\ge 1\vee M_n^{(1,2)}\ge 1\big)\Big]\\
&\qquad \times\Big[P\big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\big)+P\big(M_n^{(2,1)}\ge 1\vee M_n^{(2,2)}\ge 1\big)\Big]
\end{aligned}
$$

$$\leq P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\leq u_n, Y_{i_s}\leq v_n\}\Big)\times P\Big(\bigcap_{t=1}^{q}\{X_{j_t}\leq u_n, Y_{j_t}\leq v_n\}\Big)$$
$$+2P\big(M_n^{(1,1)}\geq 1\big)+2P\big(M_n^{(1,2)}\geq 1\big)+2P\big(M_n^{(2,1)}\geq 1\big)+2P\big(M_n^{(2,2)}\geq 1\big)$$

These last four terms tend to zero as it is proved in Hall (2003) depending on ℓ_n. We show it for one term.

$$\begin{aligned}P\left(M_n^{(1,1)}\geq 1\right) &\leq (n+1)P\left(\sum_{k=-\infty}^{-\ell_n}\alpha_k\circ V_{-k}\geq 1\right)\\ &\leq (n+1)\sum_{k=-\infty}^{-\ell_n}E(\alpha_k\circ V_{-k})=(n+1)\sum_{k=-\infty}^{-\ell_n}\alpha_k E(V_{-k})\\ &\leq Cn\sum_{k=\ell_n}^{\infty}\frac{1}{k^\delta}\leq Cn\ell_n^{1-\delta}\end{aligned}$$

for some generic constant C and δ satisfying (3). Selecting $\phi > 1/(\delta-1)$, this bound tends to 0.

In the same way we establish the lower bound of (25). In fact

$$\begin{aligned}&P\big(\bigcap_{s=1}^{p}\{X_{i_s}\leq u_n, Y_{i_s}\leq v_n\}\big)P\big(\bigcap_{t=1}^{q}\{X_{j_t}\leq u_n, Y_{j_t}\leq v_n\}\big)\\ &\quad\leq P\big(\bigcap_{s=1}^{p}\{\sum_{k=-\ell_n+1}^{\infty}\alpha_k\circ V_{i_s-k}\leq u_n,\sum_{k=-\ell_n+1}^{\infty}\beta_k\circ W_{i_s-k}\leq v_n\}\big)\\ &\qquad\times P\big(\bigcap_{t=1}^{q}\{\sum_{k=-\infty}^{\ell_n-1}\alpha_k\circ V_{j_t-k}\leq u_n,\sum_{k=-\infty}^{\ell_n-1}\beta_k\circ W_{j_t-k}\leq v_n\}\big)\\ &\quad\leq P\big(\bigcap_{s=1}^{p}\{\sum_{k=-\ell_n+1}^{\infty}\alpha_k\circ V_{i_s-k}\leq u_n,\sum_{k=-\ell_n+1}^{\infty}\beta_k\circ W_{i_s-k}\leq v_n\},\\ &\qquad\qquad\bigcap_{t=1}^{q}\{\sum_{k=-\infty}^{\ell_n-1}\alpha_k\circ V_{j_t-k}\leq u_n,\sum_{k=-\infty}^{\ell_n-1}\beta_k\circ W_{j_t-k}\leq v_n\}\big)\end{aligned}$$

$$\leq P\big(\textstyle\bigcap_{s=1}^{p}\{X_{i_s} \leq u_n + M_n^{(1,1)}, Y_{i_s} \leq v_n + M_n^{(1,2)}\},$$
$$\textstyle\bigcap_{t=1}^{q}\{X_{j_t} \leq u_n + M_n^{(2,1)}, Y_{j_t} \leq v_n + M_n^{(2,2)}\}$$

Since we need also that $s_n \alpha_{n,l_n} \to 0$, we select s_n (i.e. ζ) such that $s_n n \ell_n^{1-\delta} = n^{1+\zeta-\phi(\delta-1)} \to 0$, which holds for $1+\zeta < \phi(\delta-1)$. Hence condition $D(u_n, v_n)$ holds.

To prove the condition $D'(u_n, v_n)$ note first that

$$n \sum_{j=1}^{[n/s_n]} P\left(X_0 > u_n, X_j > v_n\right) \to 0, \quad n \to +\infty,$$

and

$$n \sum_{j=1}^{[n/s_n]} P\left(Y_0 > u_n, Y_j > v_n\right) \to 0, \quad n \to +\infty,$$

because $\{X_n\}$ and $\{Y_n\}$ satisfy conditions $D'(u_n)$ and $D'(v_n)$ which was shown in Hall (2003). Hence, it remains to consider the sums on the terms $P\left(X_0 > u_n, Y_j > v_n\right)$ and on the terms $P\left(Y_0 > u_n, X_j > v_n\right)$.

We show it for the sum of the first terms, since for the second one the proof follows in the same way. Let $\gamma_n = n^{\nu}$ with $\nu < 1-\zeta$. Then $\gamma_n = o(n/s_n) = o(n^{1-\zeta})$. For $j < \gamma_n$

$$(X_0, Y_j) = \left(\sum_{i=-\infty}^{\infty} \alpha_i \circ V_{-i}\,, \sum_{i=-\infty}^{\infty} \beta_{i+j} \circ W_{-i} \right)$$

Note that $\alpha_{i_0} = \alpha_{\max}$ for some i_0 and $\beta_{j_0} = \beta_{\max}$ for some j_0. For one j we have $i_0 + j = j_0$, i.e. $j = j_0 - i_0$. Hence the maximum terms occur at the same index for V_{-i_0} and W_{-i_0}. If $j_0 = i_0$ this case $j = 0$ does not occur in the sum. For all other j's the maxima is occurring at different indexes. We consider the bound established in Proposition 3.1 and 3.2 for H^*. For $j = j_0 - i_0$, we showed in the proof of Theorem 3.1 that $nH^*(u_n, v_n) \to 0$.

For the remaining terms $P(X_0 > u_n, Y_j > v_n)$ with $j \neq j_0 - i_0$, we have $\beta_{i_0+j} < \beta_{\max}$ and deduce from Proposition 3.1 the following upper bound for

$H^*(u_n, v_n)$ given by

$$o_n(1)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right)^{-\rho v_n} u_n^{\xi_3}L_3^*(u_n)+Cu_n^{\xi_1+1}v_n^{\xi_2}L_1^*(u_n)L_2^*(v_n)\times \\ \times\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)u_n}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-v_n+(\log v_n)^2} + O\left(P(S>\psi u_n)\right) \tag{26}$$

with $\rho, \psi \in (0,1)$ defined in (11) and (12). Note that $\rho = \rho(j)$ should be such that $\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right)^{\rho(j)} < \left(1+\frac{\lambda_2}{\beta_{\max}}\right)$ for all $j \neq j_0$ that (11) is satisfied. It implies that for any $\epsilon > 0$ we can select $\rho(j)$ for every $j \neq j_0$ such that

$$\log\left(1+\frac{\lambda_2}{\beta_{\max}}\right) > \rho(j)\log\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right) > (1-\epsilon)\log\left(1+\frac{\lambda_2}{\beta_{\max}}\right).$$

Then the sum on $\{j \leq 2\gamma_n, j \neq j_0\}$, with the first term in the bound of H^* in (26) multiplied by n, is bounded by

$$\begin{aligned}
&o_n(1)n^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1\log n+o(\log n)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_o+j}}\right)^{-\rho(j)d_2\log n+o(\log n)}\times \\
&\qquad \times (d_1\log n)^{\xi_3}L_3^*(\log n) \\
&= o_n(1)\exp\left\{\log n\cdot\left(1+\nu-d_1\log\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)-\right.\right. \\
&\qquad \left.\left.-d_2(1-\epsilon)\log\left(1+\frac{\lambda_2}{\beta_{\max}}\right)\right)+o(\log n)\right\} \\
&= o_n(1)\exp\{-\log n\cdot(1-\epsilon-\nu+o_n(1))\}\to 0, n\to\infty,
\end{aligned}$$

if also ν such that $\nu < 1-\epsilon$.
Let us consider the sum on $\{j \leq 2\gamma_n, j \neq j_0\}$ with the second terms in (26) multiplied with n. We have

$$\begin{aligned}
&n^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)u_n}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-v_n+(\log v_n)^2} \\
&= n^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)d_1\log n+o(\log n)}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-d_2\log n+o(\log n)} \\
&= \exp\{\log n\cdot[1+\nu-(1-\psi)-1]+o(\log n)\} \\
&= \exp\{\log n\cdot[\nu+\psi-1+o_n(1)]\}\to 0, n\to\infty,
\end{aligned}$$

if also $\nu < 1-\psi$. Hence we choose $\nu < \min(1-\epsilon, 1-\zeta, 1-\psi)$.

It remains to deal with the sum of the third terms in (26). We showed that $P(S > \psi_x) = O((1+\lambda)^{-\psi_x})$ in (19) with $(1+\lambda)^{\psi} > 1 + \frac{\lambda_1}{\alpha_{\max}}$ in (11). Let $\tilde{\psi} > 1$ such that $(1+\lambda)^{\psi/\tilde{\psi}} = (1 + \frac{\lambda_1}{\alpha_{\max}})$. Then with $\varepsilon > 0$ such that $\tilde{\psi} - \varepsilon > 1$, this sum on $\{j \le 2\gamma_n, j \ne j_0\}$ multiplied with n is bounded by

$$Cn^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(\tilde{\psi}-\varepsilon)u_n} = C\exp\left\{\log n \cdot \left[1+\nu-(\tilde{\psi}-\varepsilon)+o_n(1)\right]\right\} \to 0$$

if also $\nu < \tilde{\psi} - 1 - \varepsilon$.

Thus combining these three bounds we have that

$$n\sum_{j\le 2\gamma_n} P\left(X_0 > u_n, Y_j > v_n\right) \to 0$$

if $\nu < \min(1-\epsilon, 1-\zeta, 1-\psi, \tilde{\psi}-1-\varepsilon)$.

We consider now the sum on j with $2\gamma_n < j \le n/s_n$ and write

$$X_0' = \sum_{i=-\gamma_n}^{\infty} \alpha_i \circ V_{-i}\ , \quad X_0'' = \sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}$$

and

$$Y_j' = \sum_{i=-\infty}^{\gamma_n} \beta_i \circ W_{j-i}\ , \quad Y_j'' = \sum_{i=\gamma_n+1}^{\infty} \beta_i \circ W_{j-i}$$

Note that X_0' and Y_j' are independent. We have, for $j > 2\gamma_n$ and some $k > 1$ (chosen later),

$$\begin{aligned}
&P\left(X_0 > u_n, Y_j > v_n\right) = P\left(X_0' + X_0'' > u_n, Y_j' + Y_j'' > v_n\right)\\
\le\ & P\left(X_0' > u_n - X_0'', Y_j' > v_n - Y_j'', X_0'' < k, Y_j'' < k\right)\\
& + P\left(X_0'' \ge k\right) + P\left(Y_j'' \ge k\right)\\
\le\ & P\left(X_0' > u_n - k, Y_j' > v_n - k\right) + P\left(X_0'' \ge k\right) + P\left(Y_j'' \ge k\right)\\
\le\ & P\left(X_0 > u_n - k\right) P\left(Y_j > v_n - k\right) + P\left(X_0'' \ge k\right) + P\left(Y_j'' \ge k\right)\\
=\ & O\left(\frac{1}{n}\right) O\left(\frac{1}{n}\right) + P\left(X_0'' \ge k\right) + P\left(Y_j'' \ge k\right)
\end{aligned}$$

Similar to Hall (2003), the last two probabilities are sufficiently fast tending to 0. We have

$$\begin{aligned} P\left(X_0'' \geq k\right) &= P\left(\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i} \geq k\right) \\ &= P\left((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}} > (1+h_n)^k\right) \\ &\leq \frac{E\left((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}}\right)}{(1+h_n)^k} \end{aligned}$$

We select h_n such that $h_n \gamma_n^{1-\delta} = C > 0$, for some constant C, then we have for $i \leq -\gamma_n - 1$ and $\delta > 2$

$$0 < \alpha_i h_n \leq C|i|^{-\delta} h_n \leq C(\gamma_n + 1)^{-\delta} h_n = o(1/\gamma_n) \to 0, n \to \infty,$$

by the assumption (3) on the sequence $\{\alpha_i\}$. It implies

$$E\left((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}}\right) = E\left(\prod_{i=-\infty}^{-\gamma_n-1} (1+h_n)^{\alpha_i \circ V_{-i}}\right) = \prod_{i=-\infty}^{-\gamma_n-1} E\left((1+\alpha_i h_n)^{V_{-i}}\right)$$

and, due to Lemma 3.1,

$$\begin{aligned} &E\left((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}}\right) \\ &\leq \prod_{i=-\infty}^{-\gamma_n-1} \left(1 + \alpha_i h_n E(V_0)(1 + o_n(1))\right) \\ &= \exp\left(\sum_{i=-\infty}^{-\gamma_n-1} \ln\left(1 + \alpha_i h_n E(V_0)\right)(1 + o_n(1))\right) \\ &= \exp\left(E(V_0) \sum_{i=-\infty}^{-\gamma_n-1} \alpha_i h_n (1 + o_n(1))\right) \\ &= \exp\left(E(V_0) h_n O(1) \sum_{i=-\infty}^{-\gamma_n-1} |i|^{-\delta}\right) \\ &= \exp\left(O(1) h_n \gamma_n^{1-\delta}\right) = O(1),\ n \to +\infty, \end{aligned}$$

by the choice of h_n. Note that $h_n = C\gamma_n^{\delta-1} = Cn^{\nu(\delta-1)} \to \infty$. Now, select k depending on δ, ν and ζ such that

$$n^2/((1+h_n)^k s_n) \sim n^2/(C^k n^{k\nu(\delta-1)} n^\zeta) = o(1), n \to +\infty,$$

which holds for $k > (2-\zeta)/(\nu(\delta-1))$. This choice implies that

$$(n^2/s_n) P\left(X_0'' \geq k\right) \to 0, n \to +\infty.$$

In the same way we can show that also

$$n \sum_{j \leq n/s_n} P\left(Y_j'' \geq k\right) \to 0, n \to +\infty,$$

for such a k, since also $\beta_i \leq C\,|i|^{-\delta}$ for $|i| \geq \gamma_n$ and some constant $C > 0$. Hence condition $D'(u_n, v_n)$ holds.

Therefore, upper and lower bounds of the limiting distribution of the maximum term of non-negative integer-valued moving average sequences are found leading to a "quasi max-stable" limiting behavior of the bivariate maximum in the sense of Anderson type.

Theorem 4.1. *Consider the stationary sequences $\{(X_n, Y_n)\}$ defined by*

$$(X_n, Y_n) = \left(\sum_{i=-\infty}^{\infty} \alpha_i \circ V_{n-i}, \sum_{i=-\infty}^{\infty} \beta_i \circ W_{n-i} \right).$$

Suppose that the innovation sequence $\{V_n, W_n\}$ is an iid sequence of non-negative integer-valued random vectors with df of the form (4), the sequences of $\{\alpha_i\}$ and $\{\beta_i\}$ satisfies (3) and $\alpha_{\max}$ and $\beta_{\max}$ are unique. Then,

$$\limsup\,(\liminf)\, P\left(M_n^{(1)} \leq u_n(x), M_n^{(2)} \leq v_n(x)\right) \lesseqgtr$$
$$\lesseqgtr \exp\left(-\left(1+\tfrac{\lambda_1}{\alpha_{\max}}\right)^{-(x-0(1))} - \left(1+\tfrac{\lambda_2}{\beta_{\max}}\right)^{-(y-0(1))} \right)$$

for all real x and y and where $u_n(x)$ and $v_n(x)$ are defined by (22) and (23).

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