THERE ARE NO NONTRIVIAL TWO-SIDED
MULTIPLICATIVE (GENERALIZED)-SKEW DERIVATIONS
IN PRIME RINGS

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Abstract: As originally defined by Mozumder and Dhara ([15]), multiplicative
(generalized)-skew derivations must satisfy two identities. In this short note we
show that, as a consequence of the simultaneous satisfaction of both identities, a
multiplicative (generalized)-skew derivation of a prime ring is either a multiplica-
tive (generalized) derivation (i.e., not skew), or a generalized skew derivation (i.e.,
additive). Therefore only one of the identities should be taken in the definition of
multiplicative (generalized)-skew derivations in order to get a new class of deriv-
ations in prime rings.

Keywords: prime rings, derivations, generalized derivations, skew derivations.

1. Introduction

The fundamental concept of derivation of an associative ring $R$, an additive
map $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$, has been progressively gen-
eralized in recent literature: by a twisting by an automorphism or a secondary
derivation of the ring, by dropping the additivity assumption, by combining
both previous ideas, and by repeating the process on the secondary deriv-
ation when present ([4],[2],[13],[5],[7],[15]). One of the main purposes of these
generalizations is to extend to more sophisticated maps the classic results on
derivations in the tradition of Herstein’s theory of rings ([16]), in which strong
knowledge is gained about the map or the ring through some special (and a
priori weaker) property of the map. The main focus is on prime and semiprime
rings, or on rings with well-behaved idempotents, which provide a context rich

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enough for the theory to be satisfactorily developed. For example one tries to extend to a more general setting Posner’s second theorem for derivations of prime rings ([17, Theorem 2]), which states that a prime ring $R$ is commutative when it has a derivation $d \neq 0$ such that $xd(x) - d(x)x$ is central for every $x \in R$. These efforts have generated literature in abundance (e.g. [1],[3],[6],[8],[9],[10],[12],[14],[18]).

**Definitions 1.1.** Let $R$ be a ring.

a) A skew derivation ([11, page 170]) is an additive map $d : R \to R$ together with an automorphism $\alpha : R \to R$ such that $d(xy) = d(x)y + \alpha(x)d(y)$.

b) A multiplicative derivation ([4]) is a map $d : R \to R$, not necessarily additive, such that $d(xy) = d(x)y + xd(y)$.

c) A multiplicative skew derivation is a not necessarily additive map $d : R \to R$ together with an automorphism $\alpha : R \to R$ such that $d(xy) = d(x)y + \alpha(x)d(y)$.

d) A generalized derivation ([2]) is an additive map $F : R \to R$ together with a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$.

e) A generalized skew derivation ([13]) is an additive map $F : R \to R$ together with an automorphism $\alpha : R \to R$ and a skew derivation $d : R \to R$ for $\alpha$ such that $F(xy) = F(x)y + \alpha(x)d(y)$.

f) A multiplicative generalized derivation ([5]) is a map $F : R \to R$, not necessarily additive, together with a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$.

g) A multiplicative (generalized) derivation ([7]) is a map $F : R \to R$, not necessarily additive, together with a map (not necessarily additive nor a derivation) $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$.

As defined in [15], a multiplicative (generalized)-skew derivation (M(G)S derivation) is a not necessarily additive map $F : R \to R$, together with a not necessarily additive map $d : R \to R$ and an automorphism $\alpha : R \to R$ such that

\[ F(xy) = F(x)\alpha(y) + xd(y) \quad \text{(Identity 1)} \]

\[ F(xy) = F(x)y + \alpha(x)d(y) \quad \text{(Identity 2)} \]

Since in this case we get two different identities in the definition, accordingly we will call these two-sided M(G)S derivations. We will say that a map is a type 1 M(G)S derivation (resp. type 2 M(G)S derivation) when it satisfies Identity 1 (resp. Identity 2).
2. Main theorem

In what follows we show that in prime rings there are no nontrivial two-sided M(G)S derivations, since either they are not skew or they actually are generalized skew derivations.

Lemma 2.1. If $R$ is a semiprime ring and $F$ is a M(G)S derivation of type 1 (resp. type 2) with map $d : R \rightarrow R$ and automorphism $\alpha : R \rightarrow R$ then $d$ is in fact a multiplicative skew derivation with $\alpha$ as automorphism (resp. satisfies $d(xy) = d(x)\alpha(y) + xd(y)$).

Proof: For type 1 this is [18, Lemma 2.1]. For type 2 the same proof works. ■

Theorem 2.2. Let $R$ be a prime ring and $F$ be a two-sided M(G)S derivation with map $d : R \rightarrow R$ and automorphism $\alpha : R \rightarrow R$. Then either

i) $\alpha = \text{id}_R$, so $F$ is a multiplicative (generalized) derivation, or

ii) $F$ and $d$ are additive, so $F$ is a generalized skew derivation.

Proof: From Identities 1 and 2, $F(x)y + \alpha(x)d(y) = F(xy) = F(x)\alpha(y) + xd(y)$ for every $x, y \in R$, so

$$F(x)(y - \alpha(y)) = (x - \alpha(x))d(y). \quad (1)$$

Linearizing in $x$ we get, for every $x, y, z \in R$,

$$F(x + y)(z - \alpha(z)) = (x + y - \alpha(x + y))d(z) = (x + y - \alpha(x) - \alpha(y))d(z) =$$

$$= (x - \alpha(x))d(z) + (y - \alpha(y))d(z) = F(x)(z - \alpha(z)) + F(y)(z - \alpha(z))$$

by (1). So $(F(x + y) - F(x) - F(y))(z - \alpha(z)) = 0$.

Put $G(x, y) := F(x + y) - F(x) - F(y)$. We have, for every $x, y, z \in R$,

$$G(x, y)z = G(x, y)\alpha(z). \quad (2)$$

Therefore, for every $w \in R$,

$$G(x, y)wz = G(x, y)\alpha(wz) = (G(x, y)\alpha(w))\alpha(z) = (G(x, y)w)\alpha(z)$$

by (2), hence $G(x, y)w(z - \alpha(z)) = 0$ for every $x, y, z, w \in R$. Since $R$ is prime, either $\alpha(z) = z$ for every $z \in R$ or $G(x, y) = 0$ for every $x, y \in R$.

In the first case $\alpha = \text{id}_R$ and $F$ is a multiplicative (generalized) derivation. In the second case we get $\alpha \neq \text{id}_R$ and $F(x + y) = F(x) + F(y)$ for every $x, y \in R$, so $F$ is additive. Now, by Lemma 2.1 above $d$ is another M(G)S derivation associated to $\alpha \neq \text{id}_R$, so analogously $d$ is additive, whence it is a skew derivation and $F$ is a generalized skew derivation. ■
References


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