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BOUNDS FOR THE ZEROS OF UNILATERAL OCTONIONIC POLYNOMIALS

ROGÉRIO SERÔDIO, P. D. BEITES AND JOSÉ VITÓRIA

ABSTRACT: In the present work it is proved that the zeros of an unilateral octonionic polynomial are the latent roots of an appropriate lambda-matrix. This allows the use of matricial norms, matrix norms in particular, to obtain upper and lower bounds for the zeros of unilateral octonionic polynomials.

KEYWORDS: Octonions, octonionic polynomials, matricial norms, bounds for the zeros.

MATH. SUBJECT CLASSIFICATION (2010): 11R52, 12D10, 15A60.

1. Introduction

One of the relevant aspects related to polynomials is their zeros, being polynomial root finding a classical subject. The theory of polynomials over commutative fields and, in general, over commutative integral domains is a well known one. For polynomials over noncommutative rings, even division rings, there are fewer results than in the commutative setting since familiar properties from this setting do not hold in the noncommutative context.

In [22] and references therein the characterization of the zeros of quaternionic polynomials was studied, based on Niven's algorithm. Other numerical methods based on Newton, Weierstrass and based on Sebastião e Silva's methods have been given in [13, 14, 26].

The difficulty for obtaining strong results on the zeros of a polynomial increases in a nonassociative setting, reason why less has been done for octonionic polynomials. The first mention to octonionic polynomials is in [12], where Eilenberg and Niven asserted that their proof of the Fundamental Theorem of Algebra for quaternionic polynomials could be extended to octonionic polynomials.

A proof of the Fundamental Theorem of Algebra for octonionic polynomials was presented in [17] by Jou, who used a topological method similar to that in [12]. In the former reference, an octonionic polynomial is a sum of a

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finite number of monomials where, due to nonassociativity, each of which is parenthesized. It was proved that an octonionic polynomial has at least one zero if there is a unique monomial of the highest degree.

A geometric description of the set of zeros of an octonionic polynomial was given by Datta and Nag in [6]. Moreover, they proved a generalization of the known De Moivre's formula for complex numbers, obtaining one for octonions which gives the zero set of the octonionic polynomial $x^m - \boldsymbol{o}$. Independently, Leite and Vitória arrived at the same result in [18].

More recently, Serôdio [23] proved that, despite belonging to the same set of conjugacy classes, the zeros of a non-monic octonionic polynomial do not coincide, in general, with the zeros of the corresponding monic polynomial. In [24] he searched for methods to construct octonionic polynomials with a prescribed set of zeros.

Although it can be hard to compute the zeros of a polynomial, localization and bounds for them are very often what matters most. As can be seen by [1] and references therein, where left and right eigenvalues of quaternionic matrices were considered, these two aspects have received a lot of attention in the quaternionic context.

In contrast to quaternionic polynomials, the literature on octonionic polynomials is scarce even on those aspects. Applying strategies whose main advantage is the use of real matrices, hence avoiding the octonionic eigenvalue problem, localization and bounds for the zeros of an unilateral octonionic polynomial are dealt within this paper.

For completeness, in Section 2, we recall some definitions and results which are needed in the following sections. In Section 3, upper bounds for norms of the zeros of an unilateral octonionic polynomial are obtained through upper bounds for norms of the latent roots of certain lambda-matrices. In particular, we generalize known results on complex polynomials, obtained by Cauchy [4], Dehmer [7], Deutsch [9], Marden [19], Melman [20], and Vitória [29], to unilateral octonionic polynomials. Moreover, lower bounds for norms of the zeros of an unilateral octonionic polynomial are also presented.

2. Preliminaries

In this section we recall definitions and results related to: the octonion algebra, unilateral octonionic polynomials, matricial norms and spectral radius, and matrix polynomials.

2.1. Octonion Algebra. Let

$$\mathbb{O} = \left\{ \sum_{\ell=0}^{7} o_\ell \boldsymbol{e}_\ell : o_\ell \in \mathbb{R},\ \ell = 0, \dots, 7 \right\}$$

be the octonion field, where the addition is accomplished by adding corresponding coefficients and the multiplication table can be summarized by the relations

$$\boldsymbol{e_i} \boldsymbol{e_j} = -\delta_{ij} \boldsymbol{e_0} + \varepsilon_{ijk} \boldsymbol{e_k},$$

where δ_{ij} is the Kronecker delta, ε_{ijk} is a Levi-Civita symbol, i.e., a completely antisymmetric tensor with a positive value $+1$ when $ijk = 123$, 145, 167, 246, 275, 374, 365 and $\boldsymbol{e_0}$ is the identity. This element will be omitted whenever it is clear from the context.

Given $\boldsymbol{o} \in \mathbb{O}$, it can be written as $\boldsymbol{o} = \mathrm{Re}(\boldsymbol{o}) + \mathrm{Im}(\boldsymbol{o})$, where $\mathrm{Re}(\boldsymbol{o}) = o_0$ and $\mathrm{Im}(\boldsymbol{o}) = \sum_{\ell=1}^{7} o_\ell \boldsymbol{e}_\ell$ are called the *real* part and the *imaginary* part, respectively. The conjugate of $\boldsymbol{o}$ is defined as $\overline{\boldsymbol{o}} = \mathrm{Re}(\boldsymbol{o}) - \mathrm{Im}(\boldsymbol{o})$. The *norm* of $\boldsymbol{o}$, denoted by $n_{\boldsymbol{o}}$, and the *trace* of $\boldsymbol{o}$, denoted by $t_{\boldsymbol{o}}$, are given, respectively, by $n_{\boldsymbol{o}} = \overline{\boldsymbol{o}}\boldsymbol{o} = \boldsymbol{o}\overline{\boldsymbol{o}} = \sum_{\ell=0}^{7} o_\ell^2$ and $t_{\boldsymbol{o}} = 2\mathrm{Re}(\boldsymbol{o}) = \boldsymbol{o} + \overline{\boldsymbol{o}}$. The inverse of a non-zero octonion $\boldsymbol{o}$ is $\boldsymbol{o}^{-1} = n_{\boldsymbol{o}}^{-1}\overline{\boldsymbol{o}}$.

The elements of the basis of $\mathbb{O}$ can also be written as

$$\boldsymbol{e_0} = 1, \quad \boldsymbol{e_1} = \boldsymbol{i}, \quad \boldsymbol{e_2} = \boldsymbol{j}, \quad \boldsymbol{e_3} = \boldsymbol{ij},$$
$$\boldsymbol{e_4} = \boldsymbol{k}, \quad \boldsymbol{e_5} = \boldsymbol{ik}, \quad \boldsymbol{e_6} = \boldsymbol{jk}, \quad \boldsymbol{e_7} = \boldsymbol{ijk}.$$

The octonions satisfy some properties. The most often used are presented in the following result.

Theorem 1 (Ward, [32])**.** *Let* $\boldsymbol{o}_1, \boldsymbol{o}_2, \boldsymbol{o}_3 \in \mathbb{O}$. *Then*

(i) $n_{\boldsymbol{o}_1} = 0$ *if and only if* $\boldsymbol{o}_1 = 0$,
(ii) $n_{\boldsymbol{o}_1\boldsymbol{o}_2} = n_{\boldsymbol{o}_2\boldsymbol{o}_1} = n_{\boldsymbol{o}_1} n_{\boldsymbol{o}_2}$,
(iii) $\boldsymbol{o}_1 = \sqrt{n_{\boldsymbol{o}_1}}\ \boldsymbol{u}$ *for some unit octonion* $\boldsymbol{u}$,
(iv) $\overline{\boldsymbol{o}_1\boldsymbol{o}_2} = \overline{\boldsymbol{o}_2}\ \overline{\boldsymbol{o}_1}$,
(v) $\overline{\boldsymbol{o}_1 + \boldsymbol{o}_2} = \overline{\boldsymbol{o}_1} + \overline{\boldsymbol{o}_2}$,
(vi) $\overline{\boldsymbol{o}_1} = \boldsymbol{o}_1$ *if and only if* $\boldsymbol{o}_1 \in \mathbb{R}$,
(vii) $\overline{\boldsymbol{o}_1^{-1}} = (\overline{\boldsymbol{o}_1})^{-1}$,
(viii) $(\boldsymbol{o}_1\boldsymbol{o}_2)^{-1} = \boldsymbol{o}_2^{-1}\boldsymbol{o}_1^{-1}$.

In addition to these properties, it is well known that any two elements of $\mathbb{O}$ generate an associative algebra isomorphic to the quaternions.

For theoretical and computational reasons, we introduce a pseudo matrix representation of an octonion. For further representations of octonions see Tian [28].

Definition 1. *The* **left matrix representation** $\omega(\mathbf{o})$ *of the octonion* $\mathbf{o} = \sum_{\ell=0}^{7} o_\ell \boldsymbol{e_\ell}$ *is*

$$\omega(\mathbf{o}) = \begin{bmatrix} o_0 & -o_1 & -o_2 & -o_3 & -o_4 & -o_5 & -o_6 & -o_7 \\ o_1 & o_0 & -o_3 & o_2 & -o_5 & o_4 & -o_7 & o_6 \\ o_2 & o_3 & o_0 & -o_1 & -o_6 & o_7 & o_4 & -o_5 \\ o_3 & -o_2 & o_1 & o_0 & o_7 & o_6 & -o_5 & -o_4 \\ o_4 & o_5 & o_6 & -o_7 & o_0 & -o_1 & -o_2 & o_3 \\ o_5 & -o_4 & -o_7 & -o_6 & o_1 & o_0 & o_3 & o_2 \\ o_6 & o_7 & -o_4 & o_5 & o_2 & -o_3 & o_0 & -o_1 \\ o_7 & -o_6 & o_5 & o_4 & -o_3 & -o_2 & o_1 & o_0 \end{bmatrix}. \tag{1}$$

Associated with these matrices we have the usual subordinated matrix norms. Given $\boldsymbol{o} = \sum_{\ell=0}^{7} o_\ell \boldsymbol{e}_\ell$,

$$\|\boldsymbol{o}\|_2 \equiv \|\omega(\boldsymbol{o})\|_2 = \sqrt{n_{\boldsymbol{o}}} = \sqrt{\sum_{\ell=0}^{7} o_\ell^2} \tag{2}$$

$$\|\boldsymbol{o}\|_1 \equiv \|\omega(\boldsymbol{o})\|_1 = \sum_{\ell=0}^{7} |o_\ell| \tag{3}$$

$$\|\boldsymbol{o}\|_\infty \equiv \|\omega(\boldsymbol{o})\|_\infty = \sum_{\ell=0}^{7} |o_\ell| \tag{4}$$

Hence, $\|\boldsymbol{o}\|_1 = \|\boldsymbol{o}\|_\infty$.

Due to the non-associativity, the octonion algebra cannot be isomorphic to the real matrix algebra with the usual multiplication. With the purpose of introducing a convenient matrix multiplication, we show another way of representing the octonions by a matrix which is closely related to the one in (1).

Definition 2. *Let* $\mathbf{o} = \sum_{\ell=0}^{7} o_\ell \boldsymbol{e}_\ell \in \mathbb{O}$. *The* **column**, **vectorial** *or* **ket** *representation of* $\mathbf{o}$ *is* $|\mathbf{o}\rangle = [o_0 \;\; o_1 \;\; \cdots \;\; o_7]^T$.

With this notation, the following theorem enumerates some properties relating these two representations.

Theorem 2 (Eganova and Shirokov, [11]; Tian, [28]). *Let* $\boldsymbol{o_1}, \boldsymbol{o_2}, \boldsymbol{o_3} \in \mathbb{O}$ *and* $\lambda \in \mathbb{R}$. *Then*

(i) $\boldsymbol{o_1} = \boldsymbol{o_2}$ *if, and only if,* $\omega(\boldsymbol{o_1}) = \omega(\boldsymbol{o_2})$,
(ii) $\omega(\boldsymbol{o_1} + \boldsymbol{o_2}) = \omega(\boldsymbol{o_1}) + \omega(\boldsymbol{o_2})$,
(iii) $\omega(\lambda \boldsymbol{o_1}) = \lambda \omega(\boldsymbol{o_1})$,
(iv) $\omega(1) = I_8$,
(v) $\omega(\overline{\boldsymbol{o_1}}) = \omega(\boldsymbol{o_1})^T$,
(vi) $\omega(\boldsymbol{o_1}^2) = \omega(\boldsymbol{o_1})^2$,
(vii) $\omega\left((\boldsymbol{o_1}\boldsymbol{o_2})\boldsymbol{o_1}\right) = \omega(\boldsymbol{o_1})\omega(\boldsymbol{o_2})\omega(\boldsymbol{o_1})$,
(viii) $\omega(\boldsymbol{o_1}\boldsymbol{o_2}) + \omega(\boldsymbol{o_2}\boldsymbol{o_1}) = \omega(\boldsymbol{o_1})\omega(\boldsymbol{o_2}) + \omega(\boldsymbol{o_2})\omega(\boldsymbol{o_1})$,
(ix) $|\boldsymbol{o_1}\boldsymbol{o_2}\rangle = \omega(\boldsymbol{o_1})|\boldsymbol{o_2}\rangle$,
(x) $|\boldsymbol{o_1}(\boldsymbol{o_2}\boldsymbol{o_3})\rangle = \omega(\boldsymbol{o_1})\omega(\boldsymbol{o_2})|\boldsymbol{o_3}\rangle$.

An equivalence relation $\sim$ over $\mathbb{O}$ is also defined. For any two octonions $\boldsymbol{o}$ and $\boldsymbol{o}'$, $\boldsymbol{o} \sim \boldsymbol{o}'$ if there exists $\sigma \in \mathbb{O}$, $\sigma \neq 0$, such that $\boldsymbol{o}' = \sigma \boldsymbol{o} \sigma^{-1}$. In this case, $\boldsymbol{o}$ and $\boldsymbol{o}'$ are said to be *similar*. The *conjugacy* class of $\boldsymbol{o}$, denoted by $[\boldsymbol{o}]$, is the set $\{\boldsymbol{x} \in \mathbb{O} : \boldsymbol{x} \sim \boldsymbol{o}\}$.

This equivalence relation is a main concept in the theory of quaternionic and octonionic polynomials. The following result characterizes the conjugacy classes.

Theorem 3 (Tian, [28]). *Two octonions,* $\boldsymbol{o}$ *and* $\boldsymbol{o}'$, *are similar if and only if* $\mathrm{Re}(\boldsymbol{o}) = \mathrm{Re}(\boldsymbol{o}')$ *and* $n_{\boldsymbol{o}} = n_{\boldsymbol{o}'}$.

Theorem 4 (Serôdio, Beites, and Vitória, [25]). *Given an octonion* $\boldsymbol{o} = \sum_{\ell=0}^{7} o_\ell \boldsymbol{e}_\ell$, *the real matrix* $\omega(\boldsymbol{o})$ *has two complex eigenvalues,* $\lambda = o_0 \pm \boldsymbol{i}\sqrt{\sum_{\ell=1}^{7} o_\ell^2}$, *each with multiplicity 4.*

2.2. Unilateral Octonionic Polynomials. As the coefficients can be on the left, on the right or on both sides of a variable, there are several ways to define octonionic polynomials. The octonionic polynomials whose coefficients are on the left of the variable are called left unilateral octonionic polynomials. Right unilateral octonionic polynomials are defined in an analogous way. Moreover, all the results for left unilateral polynomials have corresponding results for right unilateral polynomials. For this reason, we restrict our attention to left unilateral octonionic polynomials, by referring to them simply as unilateral octonionic polynomials.

Let $\mathbb{O}[X]$ denote the ring of unilateral polynomials in the variable x over $\mathbb{O}$. Every polynomial $p \in \mathbb{O}[X]$ can be written as $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$, for some nonnegative integer m, and $\boldsymbol{a}_r \in \mathbb{O}$, $r = 0, 1, \ldots, m$, where $\boldsymbol{a}_m \neq 0$. Under these conditions, m is called the degree of p, which will be denoted by $\deg(p) = m$. If $\boldsymbol{a}_m = 1$ the polynomial is said to be *monic*; otherwise it is said to be *non-monic*. For the particular case where the coefficients of the octonionic polynomial are real, the corresponding polynomial ring will be denoted by $\mathbb{R}[X]$.

The equality and the addition of two polynomials are defined in the usual way. Contrarily, the multiplication of two octonionic polynomials is defined in a standard although unexpected way, treating the variable x as real, i.e., commuting with the octonion coefficients. The multiplication of two octonionic polynomials, p and q, will be denoted by $p \star q$.

The evaluation of an octonionic polynomial p at an octonion $\boldsymbol{o}$ can only be done after expressing p with all the coefficients at the left of the variable. Then, the evaluation of p at $\boldsymbol{o}$ is the result of replacing $\boldsymbol{o}$ by x. The evaluation at $\boldsymbol{o}$ is not a ring homomorphism from $\mathbb{O}[X]$ to $\mathbb{O}$.

An octonion $\boldsymbol{o}$ is said to be a *zero* of p if $p(\boldsymbol{o}) = 0$. We will denote the set of all the zeros of p by $Zero(p)$.

Definition 3. *Let $p \in \mathbb{O}[X]$. The* **spectral radius** *of p is denoted by $\rho(p)$ and is defined by*

$$\rho(p) = \max \left\{ \|\boldsymbol{o}\|_2 \ : \ \boldsymbol{o} \in Zero(p) \right\},$$

i.e, is the largest of the 2-norm values of the zeros of p.

For any $\boldsymbol{o} \in \mathbb{O}$, the real polynomial $\Delta_{\boldsymbol{o}}(x) = (x - \overline{\boldsymbol{o}}) \star (x - \boldsymbol{o}) = x^2 - t_{\boldsymbol{o}} x + n_{\boldsymbol{o}}$ is called the *characteristic* polynomial of the octonion $\boldsymbol{o}$. Note that $\Delta_{\boldsymbol{o}}$ is an irreducible real quadratic polynomial if $\boldsymbol{o} \in \mathbb{O} \backslash \mathbb{R}$.

Lemma 1 (Serôdio, [23]). *Let $\boldsymbol{o} \in \mathbb{O}$. Then $\Delta_{\boldsymbol{o}}(\boldsymbol{o}) = 0$.*

From the definitions of conjugacy class and characteristic polynomial of an octonion, it follows that two octonions are similar if and only if they have the same characteristic polynomial.

Given $p(x) = \sum_{r=1}^{m} \boldsymbol{a}_r x^r \in \mathbb{O}[X]$, we define $\overline{p}(x) = \sum_{r=1}^{m} \overline{\boldsymbol{a}}_r x^r$ and the *normal* polynomial of p, denoted by n_p, as

$$n_p(x) \;=\; (p \star \overline{p})(x) = (\overline{p} \star p)(x) = \sum_{0 \le r,s \le m} \boldsymbol{a}_r \overline{\boldsymbol{a}}_s x^{r+s}. \tag{5}$$

Notice that $n_p \in \mathbb{R}[X]$ and $\deg(n_p) = 2\deg(p)$.

Remark 0.1. *We can write the octonionic polynomial p as*

$$p(x) = \sum_{i=0}^{7} \boldsymbol{e}_i p_i(x) = p_0(x) + \sum_{i=1}^{7} \boldsymbol{e}_i p_i(x).$$

Hence,

$$\overline{p}(x) = p_0(x) - \sum_{i=0}^{7} \boldsymbol{e}_i p_i(x).$$

Multiplying p by $\overline{p}$, it is easy to verify that

$$n_p(x) = (p \star \overline{p})(x) = \sum_{i=0}^{7} p_i^2(x). \qquad \square$$

Two key results on octonionic polynomials are enunciated in what follows. They relate the classes of conjugacy of the zeros of an octonionic polynomial to the zeros of the corresponding normal polynomial.

Theorem 5 (Serôdio, [23])**.** *Let $p \in \mathbb{O}[X]$ and $\boldsymbol{o} \in \mathbb{O}$. Then $\Delta_{\boldsymbol{o}}$ divides n_p if, and only if, there exists at least an $\boldsymbol{o}' \sim \boldsymbol{o}$ such that $p(\boldsymbol{o}') = 0$.*

Corollary 5.1. *The zeros of $p \in \mathbb{O}[X]$ belong to one of the classes $[\boldsymbol{o}']$, where $\boldsymbol{o}'$ is a zero of n_p.*

2.3. Matricial Norms and Spectral Radius. Consider the set of non-negative reals, $\mathbb{R}_+$. Let $\mathbb{R}^{m\times m}$ and $\mathbb{R}_+^{k\times k}$ denote, respectively, the algebra of real $m \times m$ matrices and the set of all $k \times k$ matrices with entries in $\mathbb{R}_+$.

Definition 4. *The* **spectral radius** *of $A \in \mathbb{R}^{m\times m}$ is defined by*

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|,$$

where $\sigma(A)$ denotes the set of distinct eigenvalues of A.

Definition 5. *A mapping $\mu : \mathbb{R}^{m\times m} \to \mathbb{R}_+^{k\times k}$ is a* **matricial norm** *if, for any $A, B \in \mathbb{R}^{m\times m}$ and $\alpha \in \mathbb{R}$,*

(i) $\mu(\alpha A) = |\alpha|\mu(A)$
(ii) $\mu(A + B) \leq \mu(A) + \mu(B)$
(iii) $\mu(AB) \leq \mu(A)\mu(B)$
(iv) $\mu(A) \neq 0$ *if* $A \neq 0$.

If $k = 1$ *then* μ *is a* **matrix norm**.

Particular classes of matrix norms can be obtained in the following ways:

$$\|A\|_1 = \max_{1\leq j\leq m} \sum_{i=1}^{m} |a_{ij}|, \quad \|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{m} |a_{ij}|, \quad \|A\|_2 = \sqrt{\rho(A^T A)}.$$

These matrix norms are vector-induced (or subordinate) and are usually known as 1-norm or column norm, ∞-norm or row norm, 2-norm or spectral norm, respectively, [16].

The subsequent result shows how one can generate a matricial norm on $\mathbb{R}^{m\times m}$ starting with a matrix norm on $\mathbb{R}^{m\times m}$.

Theorem 6 (Deutsch, [9]). *Let* $E_1, \ldots, E_k$ *be the projections associated with a direct-sum decomposition* $\mathbb{R}^n = X_1 \oplus \cdots \oplus X_k$ *of* $\mathbb{R}^n$, *and let* ψ *be a matrix norm on* $\mathbb{R}^{m\times m}$. *Then the mapping*

$$\mu : \mathbb{R}^{m\times m} \to \mathbb{R}_+^{k\times k}$$

$$\mu(A) = (\psi(E_i A E_j))_{i,j=1,\ldots,k}$$

is a matricial norm on $\mathbb{R}^{m\times m}$, *called the matricial norm induced by the direct-sum decomposition* $\mathbb{R}^n = X_1 \oplus \cdots \oplus X_k$ *and the matrix norm* ψ.

Remark 0.2. *In a concrete form, we generate a matricial norm in the following way: given a real* $m \times m$ *matrix, we partition it into a* $k \times k$ *block matrix such that the diagonal matrices are square not necessarily of the same order, and then consider a matrix norm of each block.*

We finish this subsection with properties involving matricial norms and spectral radii.

Theorem 7 (Deutsch, [9]; Horn and Johnson, [16]). *Let* $\mu : \mathbb{R}^{m\times m} \to \mathbb{R}_+^{k\times k}$ *be a matricial norm and let* $\|\cdot\|$ *be a matrix norm. Then, for all* $A \in \mathbb{R}^{m\times m}$ *and for all positive integers* q,

$$\rho(A) \leq [\rho(\mu(A^q))]^{1/q} \text{ and } \rho(A) \leq \|A\|_i,\ i = 1, 2, \infty.$$

2.4. Matrix Polynomials. We finish Section 2 focusing on the concept of block companion matrix associated with a matrix polynomial. This subject has been given considerable attention in the literature and is well-known.

The following definitions and results can be found in [8].

Definition 6. *Let $A_0, A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$. The matrix function in the variable $X \in \mathbb{R}^{n\times n}$ given by*

$$M(X) = A_m X^m + A_{m-1} X^{m-1} + \cdots + A_1 X + A_0$$

is called a matrix polynomial. If $A_m = I_n$, then $M(X)$ is said to be monic and, for simplicity, it is usual to omit A_m.

Definition 7. *Let $M(X)$ be a matrix polynomial. If $X = \lambda I_n$, then*

$$M(\lambda I_n) = M(\lambda) = A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0,$$

is called a lambda-matrix.

Definition 8. *Let $M(\lambda)$ be a lambda-matrix. The solutions of* $\det(M(\lambda)) = 0$*, which are the eigenvalues of $M(\lambda)$, are called latent roots of $M(\lambda)$.*

Definition 9. *Let $M(X) = X^m + A_{m-1}X^{m-1} + \cdots + A_1 X + A_0$ be a monic matrix polynomial. The matrix*

$$C(M) = \left[\begin{array}{cccc:c} 0_n & 0_n & \cdots & 0_n & -A_0 \\ \hdashline I_n & 0_n & \cdots & 0_n & -A_1 \\ 0_n & I_n & \cdots & 0_n & -A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_n & 0_n & \cdots & I_n & -A_{m-1} \end{array}\right]$$

is called the block companion matrix associated with $M(X)$. For simplicity, we will write C instead of $C(M)$ when there is no doubt which polynomial we are referring to.

In the next section we will relate the subsequent result to the octonionic polynomials and take advantage of known results.

Theorem 8 (Barnett, [2]; Dennis, Traub and Weber, [8])**.** *Let C be the block companion matrix associated with a monic matrix polynomial $M(X)$. The eigenvalues of C are the latent roots of the associated lambda-matrix $M(\lambda)$.*

3. Main Results on Bounds

The present section is devoted to the localization of the zeros of unilateral octonionic polynomials. We will show how this localization problem is related to the localization of latent roots.

3.1. Octonionic Polynomials and Lambda-Matrices. We start by associating the octonionic polynomial p with a matrix polynomial P and see how the zeros of p are related to the latent roots of P.

Definition 10. *Let $p \in \mathbb{O}[X]$ be given by $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_0$ of degree m. The associated matrix polynomial P is given by*

$$P(X) = A_m X^m + A_{m-1} X^{m-1} + \cdots + A_0,$$

where $A_i = \omega(\boldsymbol{a}_i) \in \mathbb{R}^{8\times 8}$, $i = 0, \ldots, m$, with ω defined in (1).

Lemma 2. *Let $p \in \mathbb{O}[X]$ and P be its associated matrix polynomial. Then the associated matrix polynomial $\overline{P}$ of $\overline{p} \in \mathbb{O}[X]$ satisfies*

$$\overline{P}(\lambda) = (P(\lambda))^T .$$

Proof: Let $p \in \mathbb{O}[X]$. Writing p as $p(x) = \sum_{i=0}^{7} \boldsymbol{e}_i p_i(x)$ the result follows directly from property (v) of Theorem **??**. ■

Lemma 3. *Let $p \in \mathbb{O}[X]$ and P be its associated matrix polynomial. If $p(x) = \sum_{i=0}^{7} \boldsymbol{e}_i p_i(x)$, then P satisfies*

$$P(\lambda) \left(P(\lambda)\right)^T = \sum_{i=0}^{7} p_i^2(x) I_8.$$

Proof: Straightforward from the multiplication of the matrices $P(\lambda)$ and $\overline{P}(\lambda)$. ■

Theorem 9. *Let $p \in \mathbb{O}[X]$ and let P be its associated matrix polynomial. Then there exists an octonion $\boldsymbol{o} \in [\lambda]$ such that $p(\boldsymbol{o}) = 0$ if and only if* $\det\left(P(\lambda)\right) = 0$.

Proof: Let $p \in \mathbb{O}[X]$ be of degree m and the normal polynomial of p be given by

$$n_p(x) = (p \star \overline{p})(x) = b_{2m} x^{2m} + b_{2m-1} x^{2m-1} + \cdots + b_1 x + b_0,$$

where $b_i \in \mathbb{R}$, $i = 0, \ldots, 2m$.

Thus N_p, the associated matrix polynomial of n_p,

$$N_p(X) = B_{2m}X^{2m} + B_{2m-1}X^{2m-1} + \cdots + B_1X + B_0,$$

is a polynomial whose coefficients are scalar matrices. Hence the lambda-matrix $N_p(\lambda)$, corresponding to $N_p(X)$, is

$$\begin{aligned} N_p(\lambda) &= B_{2m}\lambda^{2m} + B_{2m-1}\lambda^{2m-1} + \cdots + B_0 \\ &= b_{2m}\lambda^{2m}I_8 + b_{2m-1}\lambda^{2m-1}I_8 + \cdots + b_0I_8 \\ &= n_p(\lambda)I_8. \end{aligned}$$

On the other hand, by Lemmas 2 and 3 and Remark 0.1, we obtain

$$N_p(\lambda) = P(\lambda)\overline{P}(\lambda).$$

We have successively,

$$\begin{aligned} \det(N_p(\lambda)) &= \det(P(\lambda)\overline{P}(\lambda)) \\ \det(n_p(\lambda)I) &= \det(P(\lambda)P(\lambda)^T) \\ (n_p(\lambda))^8 &= \det(P(\lambda))^2. \end{aligned}$$

Hence,

$$\det(P(\lambda)) = \pm\,(n_p(\lambda))^4\,.$$

Since the leading coefficients of $\det(P(\lambda))$ and $n_p(\lambda)$ are positive, we conclude that $\det(P(\lambda)) = (n_p(\lambda))^4$.

■

Corollary 9.1. *Let $p \in \mathbb{O}[X]$, P its associated matrix polynomial and $\boldsymbol{o} \in \mathbb{O}$ such that $\boldsymbol{o} \in Zero(p)$. The following statements are equivalent:*

(1) $\boldsymbol{o} \in [\lambda]$*;*
(2) $\det(P(\lambda)) = 0$*;*
(3) $\det(\lambda I - C(\widetilde{P})) = 0$*, where $\widetilde{P}$ is the corresponding monic matrix polynomial of P.*

Proof: A direct consequence of Theorem 8 and Theorem 9. ■

Corollary 9.2. *Let $p \in \mathbb{O}[X]$ and C be the companion matrix of $\widetilde{P} = A_m^{-1}P(X)$, where P is the associated matrix polynomial of p. Then, $\rho(p) = \rho\,(C)$.*

Proof: A direct consequence of the previous Corollary. ■

3.2. Localization and Upper Bounds for the Zeros of Unilateral Octonionic Polynomials. Taking into account Section 3.1, we can claim that locating the zeros of an octonionic polynomial p is equivalent to locate the latent roots of its associated lambda-matrix $P(\lambda)$. This implies that all research developed for the localization of latent roots of lambda-matrices can be applied to the localization of octonionic polynomials without restriction. Even more, this problem is less restrictive since the leading matrix coefficient is always invertible. It is worth to mention that this is also valid for quaternionic polynomials.

Bounds for the latent roots of lambda-matrices have been widely studied and many localization theorems have been published [15, 20, 29, 30]. Some results for locating the latent roots are extended from well known results for complex polynomials [4, 21]. Nowadays some results for complex polynomials are still being obtained [5, 7, 10, 19, 27, 31].

In the following subsections we present some results for octonionic polynomials extended from matrix polynomials, which in turn have been extended from complex polynomials. For this reason, and taking into account previous sections, the proofs of the results undermentioned follow the proofs for the matricial case. Therefore, almost all theorems will be presented without proof.

The choice of these results is only indicative of their applicability, and they were chosen to compare the bounds.

3.2.1. *Cauchy-like and Pellet-like results.* The well known Cauchy's Theorem in [4] can now be extended to octonionic polynomials. We present a proof just to illustrate the above idea.

Theorem 10. *Let* $p \in \mathbb{O}[X]$*, of degree* m*, be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$*. Then* $Zero(p) \subset \{\boldsymbol{o} \in \mathbb{O} : \ \|\boldsymbol{o}\|_2 \leq R_i\}$*, where* R_i *is the unique positive solution of the real equation*

$$x^m - \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} x^{m-1} - \cdots - \frac{\|\boldsymbol{a}_1\|_i}{\|\boldsymbol{a}_m\|_i} x - \frac{\|\boldsymbol{a}_0\|_i}{\|\boldsymbol{a}_m\|_i} = 0, \quad i = 1, 2. \tag{6}$$

Proof: Given $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0 \in \mathbb{O}[X]$, we construct the respective monic associated matrix polynomial

$$\widetilde{P}(X) \ = \ X^m + \widetilde{A}_{m-1} X^{m-1} + \cdots + \widetilde{A}_0.$$

By Corollary 9.2, the spectral radius of p is equal to the spectral radius of C, where C is the companion matrix of $\widetilde{P}$, i.e, $\rho(p) = \rho(C)$.

In addition, by Theorem 7, $\rho(C) \leq \rho\left(\mu(C)\right)$, where $\mu(\cdot)$ is a matricial norm. Considering the matricial norm that applies the matrix norm $\|\cdot\|_i$, for $i = 1, 2$, to each 8×8 block, and taking into consideration that $\|A_m^{-1} A_\ell\|_i = \frac{\|A_\ell\|_i}{\|A_m\|_i}$, for $\ell = 0, \ldots, m-1$, then

$$\mu(C) \;=\; \begin{bmatrix} 0 & \cdots & 0 & \|a_0\|_i/\|a_m\|_i \\ 1 & \cdots & 0 & \|a_1\|_i/\|a_m\|_i \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \|a_{m-1}\|_i/\|a_m\|_i \end{bmatrix}.$$

This matrix is the companion matrix of the real polynomial in the first member of (6) which has a unique positive zero R_i. Hence, $\rho(p) = \rho(C) \leq \rho(\mu(C)) = R_i$. ■

The very nice Pellet's result in [20], which determines regions of exclusion, can also be extended to octonionic polynomials. Unlike Cauchy's result, this theorem does not always bear fruit.

Theorem 11. *Let $p \in \mathbb{O}[X]$ be given by*

$$p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0,$$

with $\boldsymbol{a}_m \boldsymbol{a}_0 \boldsymbol{a}_k \neq 0$ and $m \geq 3$. Let $1 \leq k \leq m-1$, and let $p_k \in \mathbb{R}[X]$ be given by

$$\begin{aligned} p_k(x) \;&=\; \|\boldsymbol{a}_m\|_2 x^m + \|\boldsymbol{a}_{m-1}\|_2 x^{m-1} + \cdots + \|\boldsymbol{a}_{k+1}\|_2 x^{k+1} - \\ &\quad -\|\boldsymbol{a}_k\|_2 x^k + \|\boldsymbol{a}_{k-1}\|_2 x^{k-1} + \cdots + \|\boldsymbol{a}_0\|_2 \end{aligned}$$

If p_k has two distinct positive zeros x_1 and x_2 with $x_1 < x_2$, then p has exactly k zeros in or on the circle $|x| = x_1$ and no zeros in the annular ring $x_1 < |x| < x_2$.

The next theorem is an extension to octonionic polynomials of a theorem that can be found in Marden [19].

Theorem 12. *Let $p \in \mathbb{O}[X]$ be given by $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. If $\|\boldsymbol{a}_k\|_2 < \|\boldsymbol{a}_m\|_2$, $k = 0, 1, \ldots, m-1$, then $Zero(p) \subset \{\boldsymbol{o} \in \mathbb{O} : \|\boldsymbol{o}\|_2 \leq 2\}$.*

3.2.2. *Other Bounds.* In this subsection, generalizations to octonionic polynomials of complex and matrix polynomials results are presented. The results for complex and matrix polynomials can be found in the work of Deutsch on applications of matricial norms to complex polynomials and in the work of

Vitória on applications of matricial norms to lambda matrices, respectively. When not mentioned, the octonion norm can be the norm 1 or the norm 2.

These first two theorems are extensions to octonionic polynomials of results that can be found in Deutsch [10].

Theorem 13. *Let* $p \in \mathbb{O}[X]$ *be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. *If* $k_0, k_1, \ldots, k_{m-2}$ *are arbitrary positive numbers, then*

$$\rho(p) \leq \frac{1}{2}\left[\beta + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \sqrt{\left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} - \beta\right)^2 + \frac{4\gamma k_{m-2}}{\|\boldsymbol{a}_m\|_i}}\right], \quad (7)$$

$$\rho(p) \leq \max\left\{\beta + k_{m-2}, \frac{\gamma + \|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}\right\}, \quad (8)$$

$$\rho(p) \leq \sqrt{\beta^2 + k_{m-2}^2 + \frac{\gamma^2 + \|\boldsymbol{a}_{m-1}\|_i^2}{\|\boldsymbol{a}_m\|_i^2}}, \quad (9)$$

where $\beta = \max\left\{\frac{k_0}{k_1}, \frac{k_1}{k_2}, \ldots, \frac{k_{m-3}}{k_{m-2}}\right\}$, $\gamma = \max\left\{\frac{\|\boldsymbol{a}_0\|_i}{k_0}, \frac{\|\boldsymbol{a}_1\|_i}{k_1}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i}{k_{m-2}}\right\}$, *and* $i \in \{1, 2\}$.

If we take $k_0 = \cdots = k_{m-2} = 1$ in Theorem 13 we obtain the following.

Corollary 13.1. *Let* $p \in \mathbb{O}[X]$ *be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. *Then*

$$\rho(p) \leq \frac{1}{2}\left[1 + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \sqrt{\left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} - 1\right)^2 + \frac{4M}{\|\boldsymbol{a}_m\|_i}}\right], \quad (10)$$

$$\rho(p) \leq \max\left\{2, \frac{\|\boldsymbol{a}_0\|_i + \|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i + \|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}\right\}, \quad (11)$$

$$\rho(p) \leq \sqrt{2 + \frac{M^2 + \|\boldsymbol{a}_{m-1}\|_i^2}{\|\boldsymbol{a}_m\|_i^2}}, \quad (12)$$

where $M = \max\{\|\boldsymbol{a}_0\|_i, \|\boldsymbol{a}_1\|_i, \ldots, \|\boldsymbol{a}_{m-2}\|_i\}$ *and* $i \in \{1, 2\}$.

If we take $k_\ell = \frac{\|\boldsymbol{a}_{\ell+1}\|_i}{\|\boldsymbol{a}_m\|_i}$, for $\ell = 0, \ldots, m-3$ in Theorem 13 we obtain the following.

Corollary 13.2. *Let $p \in \mathbb{O}[X]$ be given by $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. If $\boldsymbol{a}_j \neq 0$, $j = 0, 1, \ldots, m-1$, then*

$$\rho(p) \leq \frac{1}{2}\left[\beta' + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \sqrt{\left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} - \beta'\right)^2 + 4\gamma'\|\boldsymbol{a}_{m-1}\|_i}\,\right], \quad (13)$$

$$\rho(p) \leq \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \gamma', \quad (14)$$

$$\rho(p) \leq \sqrt{2\frac{\|\boldsymbol{a}_{m-1}\|_i^2}{\|\boldsymbol{a}_m\|_i^2} + \beta'^2 + \gamma'^2}, \quad (15)$$

where $\beta' = \max\left\{\frac{\|\boldsymbol{a}_1\|_i}{\|\boldsymbol{a}_2\|_i}, \frac{\|\boldsymbol{a}_2\|_i}{\|\boldsymbol{a}_3\|_i}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i}{\|\boldsymbol{a}_{m-1}\|_i}\right\}$, $\gamma' = \max\left\{\frac{\|\boldsymbol{a}_0\|_i}{\|\boldsymbol{a}_1\|_i}, \frac{\|\boldsymbol{a}_1\|_i}{\|\boldsymbol{a}_2\|_i}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i}{\|\boldsymbol{a}_{m-1}\|_i}\right\}$, *and* $i \in \{1, 2\}$.

The following result uses only one parameter t.

Theorem 14. *Let $p \in \mathbb{O}[X]$ be given by $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. If t is an arbitrary positive number, then*

$$\rho(p) \leq \frac{1}{2}\left[t + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \sqrt{\left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} - t\right)^2 + \frac{4\delta t}{\|\boldsymbol{a}_m\|_i}}\,\right], \quad (16)$$

$$\rho(p) \leq \max\left\{\frac{\|\boldsymbol{a}_0\|_i}{\|\boldsymbol{a}_m\|_i t^{m-1}} + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i}{\|\boldsymbol{a}_m\|_i t} + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}, 2t\right\}, \quad (17)$$

$$\rho(p) \leq \sqrt{2t^2 + \frac{\delta^2 + \|\boldsymbol{a}_{m-1}\|_i^2}{\|\boldsymbol{a}_m\|_i^2}}, \quad (18)$$

where $\delta = \max\left\{\frac{\|\boldsymbol{a}_0\|_i}{t^{m-1}}, \frac{\|\boldsymbol{a}_1\|_i}{t^{m-2}}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i}{t}\right\}$ *and* $i \in \{1, 2\}$.

If we take $t = \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}$ in Theorem 14 we obtain the following.

Corollary 14.1. *Let* $p \in \mathbb{O}[X]$ *be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. *If* $\boldsymbol{a}_{m-1} \neq 0$, *then*

$$\rho(p) \leq \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \sqrt{\delta'}, \tag{19}$$

$$\rho(p) \leq \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \max\left\{\delta' \frac{\|\boldsymbol{a}_m\|_i}{\|\boldsymbol{a}_{m-1}\|_i}, \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}\right\}, \tag{20}$$

$$\rho(p) \leq \sqrt{3\left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}\right)^2 + \left(\frac{\delta'\|\boldsymbol{a}_m\|_i}{\|\boldsymbol{a}_{m-1}\|_i}\right)^2}, \tag{21}$$

where $\delta' = \max\left\{\frac{\|\boldsymbol{a}_m\|_i^{m-3}\|\boldsymbol{a}_0\|_i}{\|\boldsymbol{a}_{m-1}\|_i^{m-2}}, \frac{\|\boldsymbol{a}_m\|_i^{m-4}\|\boldsymbol{a}_1\|_i}{\|\boldsymbol{a}_{m-1}\|_i^{m-3}}, \ldots, \frac{\|\boldsymbol{a}_{m-2}\|_i}{\|\boldsymbol{a}_m\|_i}\right\}$ *and* $i \in \{1, 2\}$.

If we take $t = N$ in Theorem 14, where N is defined by

$$N = \max\left\{\left(\frac{\|\boldsymbol{a}_0\|_i}{\|\boldsymbol{a}_m\|_i}\right)^{1/m}, \left(\frac{\|\boldsymbol{a}_1\|_i}{\|\boldsymbol{a}_m\|_i}\right)^{1/(m-1)}, \ldots, \left(\frac{\|\boldsymbol{a}_{m-2}\|_i}{\|\boldsymbol{a}_m\|_i}\right)^{1/2}\right\} \tag{22}$$

we obtain the following.

Corollary 14.2. *Let* $p \in \mathbb{O}[X]$ *be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$. *Then*

$$\rho(p) \leq \frac{1}{2}\left[N + \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} + \sqrt{\left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i} - N\right)^2 + 4N^2}\right], \tag{23}$$

$$\rho(p) \leq N + \max\left\{N, \frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}\right\}, \tag{24}$$

$$\rho(p) \leq \sqrt{3N^2 + \left(\frac{\|\boldsymbol{a}_{m-1}\|_i}{\|\boldsymbol{a}_m\|_i}\right)^2}, \tag{25}$$

where N *is given by (22) and* $i \in \{1, 2\}$.

The next theorem is an extension to octonionic polynomials of a generalization of a result that can be found in Vitória [29].

Theorem 15. *Let* $p \in \mathbb{O}[X]$ *be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$, *with* $\boldsymbol{a}_0 \neq 0$ *and* $m \geq 3$. *Let* C *be the companion matrix of*

$\widetilde{P} = A_m^{-1}P(X)$, *where* P *is the associated matrix polynomial of* p. *Then*

$$\rho(p) \leq \sqrt{\frac{\delta_{\kappa,i} + \beta_{\kappa,i} + \sqrt{(\delta_{\kappa,i} - \beta_{\kappa,i})^2 + 4\alpha_{\kappa,i}\gamma_{\kappa,i}}}{2}} \tag{26}$$

where

$$C^2 = \begin{bmatrix} \delta_{\kappa,i} & \alpha_{\kappa,i} \\ \gamma_{\kappa,i} & \beta_{\kappa,i} \end{bmatrix}$$

is partitioned into square diagonal blocks through column $\kappa \in \{8, 16, \ldots, 8(m-2)\}$ *and taking the norm* $\|\cdot\|_i$, $i \in \{1, 2, \infty\}$, *of these blocks.*

Proof: Let $\widetilde{P} = A_m^{-1}P(X)$, where P is the associated matrix polynomial of p. By Corollary 9.2, we know that $\rho(p) = \rho(C)$. Using Theorem 7 with $q = 2$, we obtain

$$\rho(p) \ \leq \ \sqrt{\rho\left(\mu\left(C^2\right)\right)},$$

where μ is a matricial norm. Taking μ in such a way that it partitions the matrix C^2 into a 2×2 matrix preserving the octonion blocks, i.e., taking $\kappa \in \{8, 16, \ldots, 8(m-2)\}$, we obtain the partitioned matrix

$$C^2 = \begin{bmatrix} \delta_{\kappa,i} & \alpha_{\kappa,i} \\ \gamma_{\kappa,i} & \beta_{\kappa,i} \end{bmatrix},$$

whose greatest eigenvalue is $\dfrac{\delta_{\kappa,i} + \beta_{\kappa,i} + \sqrt{(\delta_{\kappa,i} - \beta_{\kappa,i})^2 + 4\alpha_{\kappa,i}\gamma_{\kappa,i}}}{2}$. From here we obtain (26). ■

Corollary 15.1. *Let* $p \in \mathbb{O}[X]$ *be given by* $p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1}x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0$, *with* $m \geq 3$, *and* $P(X) = A_m X^m + \cdots + A_0$ *is the associated matrix polynomial of* p. *Then*

$$\rho(p) \leq \left(\frac{\beta_i + \sqrt{\beta_i^2 + 4\alpha_i}}{2}\right)^{1/2} \quad (i = 1, 2, \infty), \tag{27}$$

where

$$\alpha_i = \left\| -\widetilde{A}_0 \quad \widetilde{A}_0\widetilde{A}_2 \right\|_i, \beta_i = \left\| \begin{matrix} -\widetilde{A}_1 & -\widetilde{A}_0 + \widetilde{A}_1\widetilde{A}_2 \\ -\widetilde{A}_2 & -\widetilde{A}_1 + \widetilde{A}_2^2 \end{matrix} \right\|_i,$$

if $m = 3$, *and*

$$\alpha_i = \left\| \begin{array}{c:cc} 0_{2,m-4} & -\widetilde{A}_0 & \widetilde{A}_0\widetilde{A}_{m-1} \\ & -\widetilde{A}_1 & -\widetilde{A}_0 + \widetilde{A}_1\widetilde{A}_{m-1} \end{array} \right\|_i,$$

$$\beta_i = \left\| \begin{array}{c:cc} 0_{2,m-4} & -\widetilde{A}_2 & -\widetilde{A}_1 + \widetilde{A}_2\widetilde{A}_{m-1} \\ & -\widetilde{A}_3 & -\widetilde{A}_2 + \widetilde{A}_3\widetilde{A}_{m-1} \\ & \cdots & \cdots \\ I_{m-4} & -\widetilde{A}_{m-1} & -\widetilde{A}_{m-2} + \widetilde{A}_{m-1}\widetilde{A}_{m-1} \end{array} \right\|_i ,$$

if $m > 3$*, with* $\widetilde{A}_\ell = A_m^{-1} A_\ell$*,* $\ell \in \{0, \ldots, m-1\}$*.*

Proof: Take $\kappa = 8$ and $\kappa = 16$ in Theorem 15 for $m = 3$ and $m > 3$, respectively, where $\delta_{\kappa,i} = 0$ and $\gamma_{\kappa,i} = 1$ with $i \in \{1, 2, \infty\}$. ■

3.3. Lower Bounds for the Zeros of Unilateral Octonionic Polynomials. As a consequence of the results deduced in the previous subsection, lower bounds and exclusion regions for the zeros of unilateral octonionic polynomials are obtained in the present subsection. Concretely, the connection of the zeros of a reciprocal polynomial with those of its original polynomial is applied.

Let $p \in \mathbb{O}[X]$, of degree m, be given by

$$p(x) = \boldsymbol{a}_m x^m + \boldsymbol{a}_{m-1} x^{m-1} + \cdots + \boldsymbol{a}_1 x + \boldsymbol{a}_0, \text{ where } \boldsymbol{a}_0 \neq 0.$$

Consider the reciprocal polynomial of p, $p_r \in \mathbb{O}[X]$ defined by $x^m \star p(x^{-1})$, that is,

$$p_r(x) = \boldsymbol{a}_0 x^m + \boldsymbol{a}_1 x^{m-1} + \cdots + \boldsymbol{a}_{m-1} x + \boldsymbol{a}_m.$$

The zeros of p_r are the reciprocals of the zeros of p. To obtain an upper bound for the zeros of p_r is equivalent to obtaining a lower bound for the zeros of p.

3.4. Numerical Experiments. We finish with some illustrative examples of the (inclusion and exclusion) regions and the (lower and upper) bounds presented in Subsections 3.2 and 3.3. We will consider three examples. Almost all formulas are comparable, and a running example is given, as we can see, in Example 3.1. Indeed, the only one that is not comparable is Pellet-like result. Example 3.2 illustrates this result. Finally, in Example 3.3, we simulate over a set of polynomials applying Vitória-like results with various partitions.

Example 3.1. *For this example we could have constructed an octonionic polynomial with a prescribed set of zeros (see* [24] *for how to construct an octonionic polynomial with prescribed conditions). But this would probably* *imply a messy set of coefficients. Since we are not interested in the zeros but in a bound for them, we prevailed the coefficients choosing them with integer imaginary parts.*

The octonionic polynomial $p_1(x) = \boldsymbol{a}_3 x^3 + \boldsymbol{a}_2 x^2 + \boldsymbol{a}_1 x + \boldsymbol{a}_0$*, with coefficients*

$$\begin{aligned} \boldsymbol{a}_3 &= 1 + 2\boldsymbol{j} - 5\boldsymbol{ik} + \boldsymbol{jk} - 4\boldsymbol{ijk} \\ \boldsymbol{a}_2 &= -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} - \boldsymbol{ik} - \boldsymbol{jk} - \boldsymbol{ijk} \\ \boldsymbol{a}_1 &= 6 + 3\boldsymbol{k} \\ \boldsymbol{a}_0 &= 5\boldsymbol{i} - 4\boldsymbol{k} + 2\boldsymbol{ijk}, \end{aligned}$$

has exactly three zeros. They belong to the following conjugacy classes

$$\begin{gathered} [0.723477 + 0.480511i] \\ [-0.0121304 + 0.965987i] \\ [-0.924139 + 0.711349i]. \end{gathered}$$

This means that the greatest 2-norm value equals 1.166212 and the smallest equals 0.868510. We chose the coefficients so that Theorem 12 could be applied, and indeed the spectral radius 1.166212 is less than 2, as expected.

The upper bounds given by (7) – (27) and the corresponding lower bounds are presented in Table 2, where the best bounds are highlighted for each of the two norms used for the octonions. The values of k_0 *and* k_1 *for (7), (8), and (9) and those of* t *for (16), (17) and (18) were found using a computer. We chose an initial value for* t *and an initial step and went in the direction which decreased the bound value. When the neighbour values were both greater, the step was reduced. This process was maintained until we obtained a reasonable value for* t*. The same was done for the* k*'s, but this time the search was performed in two dimensions. These values are presented in Table 1, although they may not correspond to the best values.*

			1-norm		2-norm	
Result	Ref.	Bound	k_0	k_1	k_0	k_1
Theorem 13	(7)	Lower	10.000000	10.000000	10.091000	10.036100
		Upper	9.197100	11.001000	9.290100	10.901000
	(8)	Lower	1.000000	1.000000	1.000000	0.900000
		Upper	0.698977	0.836049	0.700000	0.800000
	(9)	Lower	1.100000	1.000000	1.014600	1.007290
		Upper	0.900000	1.000000	0.985600	0.992790
			t		t	
Theorem 14	(16)	Lower	1.000000		1.005432	
		Upper	0.836044		0.852234	
	(17)	Lower	1.000000		1.005458	
		Upper	0.836049		0.852244	
	(18)	Lower	1.057266		1.007278	
		Upper	0.945831		0.992767	

TABLE 1. Values of k_0, k_1, and t for formulas (7)–(9) and (16)–(18).

Result	Ref.	Lower bound		Upper bound	
		1-norm	2-norm	1-norm	2-norm
Theorem 10	(6)	**0.620488**	0.618505	**1.393802**	**1.472784**
Theorem 13	(7)	0.500000	0.497286	1.672098	1.704487
	(8)	0.500000	0.494565	1.672098	1.755141
	(9)	0.497906	0.497285	1.704975	1.756271
Corollary 13.1	(10)	0.500000	0.497282	1.689226	1.718724
	(11)	0.500000	0.494565	2.000000	2.000000
	(12)	0.495918	0.497245	1.711430	1.756447
Corollary 13.2	(13)	0.193838	0.198918	4.025371	4.398545
	(14)	0.335025	0.263240	1.961539	3.095908
	(15)	0.392906	0.316756	2.219467	3.905806
Theorem 14	(16)	0.500000	0.497286	1.672098	1.704487
	(17)	0.500000	0.497286	1.672098	1.704488
	(18)	0.498578	0.497285	1.702011	1.756271
Corollary 14.1	(19)	0.495042	0.497282	1.815545	2.012170
	(20)	0.387055	0.494565	4.433761	8.022150
	(21)	0.441725	0.497245	4.051864	7.689797
Corollary 14.2	(23)	0.499569	0.497285	1.680030	1.717423
	(24)	0.472919	0.496389	1.891675	1.985557
	(25)	0.498578	0.497285	1.702011	1.756271
Corollary 15.1	(27)	0.437152	**0.644557**	2.370628	1.517975

TABLE 2. Comparison of bounds given by formulas (6)–(27).

Example 3.2. *Theorem 11 is not applicable to all octonionic polynomials, but it is not difficult to find one that does. Consider the octonionic polynomial*

$$p_2(x) = \boldsymbol{i}x^6 + \frac{1}{2}x^5 + (3 - 4\boldsymbol{ij})x^4 + \frac{\boldsymbol{ik}}{3}x^3 + (6\boldsymbol{i} + 8\boldsymbol{jk})x^2 + \boldsymbol{ijk}x + 1.$$

The zeros belong to the following six conjugacy classes

$$[1.691188 + 1.861327\boldsymbol{i}], \quad [-1.690392 + 1.580116\boldsymbol{i}], \quad [-0.941026 + 0.977546\boldsymbol{i}],$$
$$[0.940187 + 0.900354\boldsymbol{i}], \quad [-0.215089 + 0.226091\boldsymbol{i}], \quad [\,0.215131 + 0.225644\boldsymbol{i}],$$

with 2-norm values equal to 2.5148867, 2.313913, 1.356881, 1.301764, 0.312058 and 0.311764, respectively.

Applying Theorem 10 we obtain an upper bound 2.802674 and a lower bound 0.265132.

By constructing Newton's polygon for this polynomial, we observe that some information from Theorem 11 can be obtained only if $k = 2$ or $k = 4$. Indeed, only $k = 2$ gives two positive real roots, 0.392328 and 1.082217, which means that p has two zeros with norm less than 0.392328 and four with norm greater than 1.082217. Between this gap there are no roots..

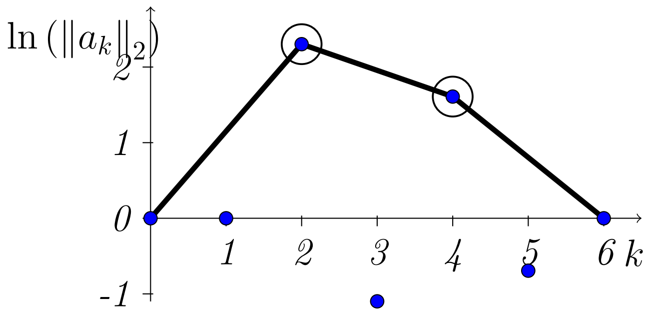

FIGURE 1. Newton's polygon for polynomial p_2 indicating the values of k for which it is worth to try to apply Theorem 11.

Example 3.3. *In this last example we have generated polynomials of degree $m = 3, \ldots, 14$ and for each polynomial we applied Theorem 15 with $\kappa = 1, \ldots, m-1$. We did this with real polynomials and octonionic polynomials.*

In the real case, for each degree m we randomly generated m coefficients, between 10^{-3} and 10^3 and random sign, of the monic real polynomial, constructed the respective companion matrix and applied Theorem 15 for each κ, from 1 to $m - 1$. The value of κ for the greatest upper bound was retained. This process was repeated 10000 times for each degree. These calculations were performed in R and the seed 12345 was used so that the experience could *be reproducible. The result of this experience is presented in Table 3 where the values are given in percentage. This table suggests that, in general, the best partition is the penultimate, and the second best is the last one. Furthermore, as the degree increases the penultimate becomes even more predominant.*

In the octonionic case, for each degree m we generated m coefficients for the monic octonionic polynomial. For each coefficient nine random numbers between 10^{-2} *and* 10^{2} *were generated, eight for the elements of the octonion and one as a factor so that the coefficients had different sizes. The elements of the octonion where also randomly assigned. As in the real case, all values of* κ *were used and the one that gave the best result was retained. This process was repeated 10000 times for each degree. The result is presented in Table 4. In this case, the predominance between the penultimate and the last partition is even greater, having the same tendency as the real case when the polynomials degree increases. The 12345 seed was also used in this case.*

	Polynomial's Degree											
κ	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
1	**53.39**	*5.94*	*10.91*	*8.59*	*7.66*	*6.99*	*6.37*	*5.97*	*5.38*	*5.21*	*4.95*	*4.22*
2	*46.61*	**64.73**	*15.19*	*9.70*	*7.43*	*6.41*	*5.21*	*4.37*	*4.06*	*3.63*	*3.03*	*2.84*
3	-	*29.33*	**46.64**	*3.43*	*1.18*	*0.92*	*0.93*	*0.83*	*0.69*	*0.81*	*0.60*	*0.52*
4	-	-	*27.26*	**54.70**	*3.17*	*0.78*	*0.61*	*0.56*	*0.54*	*0.56*	*0.50*	*0.66*
5	-	-	-	*23.58*	**59.41**	*2.62*	*0.48*	*0.47*	*0.34*	*0.43*	*0.38*	*0.28*
6	-	-	-	-	*21.15*	**62.89**	*2.44*	*0.40*	*0.32*	*0.38*	*0.27*	*0.23*
7	-	-	-	-	-	*19.39*	**66.66**	*2.18*	*0.28*	*0.19*	*0.17*	*0.10*
8	-	-	-	-	-	-	*17.30*	**68.30**	*2.00*	*0.23*	*0.16*	*0.13*
9	-	-	-	-	-	-	-	*16.92*	**69.60**	*1.88*	*0.10*	*0.15*
10	-	-	-	-	-	-	-	-	*16.79*	**71.48**	*1.90*	*0.13*
11	-	-	-	-	-	-	-	-	-	*15.20*	**73.41**	*1.97*
12	-	-	-	-	-	-	-	-	-	-	*14.53*	**73.71**
13	-	-	-	-	-	-	-	-	-	-	-	*15.06*

TABLE 3. Percentage of cases where the best partition is κ for each degree in the real case.

	Polynomial'sDegree											
κ	3	4	5	6	7	8	9	10	11	12	13	14
8	27.41	0.05	0	0	0	0	0	0	0	0	0	0
16	**72.59**	43.22	0.45	0.06	0	0	0	0	0	0	0	0
24	–	**56.73**	**50.34**	0	0.01	0	0	0	0	0	0	0
33	–	–	49.21	**55.19**	0	0	0	0	0	0	0	0
40	–	–	–	44.75	**58.10**	0	0	0	0	0	0	0
48	–	–	–	–	41.89	**60.89**	0	0	0	0	0	0
56	–	–	–	–	–	39.11	**63.40**	0	0	0	0	0
64	–	–	–	–	–	–	36.60	**65.12**	0	0	0	0
72	–	–	–	–	–	–	–	34.88	**64.62**	0	0	0
80	–	–	–	–	–	–	–	–	35.38	**67.19**	0	0
88	–	–	–	–	–	–	–	–	–	32.81	**67.57**	0
96	–	–	–	–	–	–	–	–	–	–	32.43	**68.51**
104	–	–	–	–	–	–	–	–	–	–	–	31.49

Table 4. Percentage of cases where the best partition is κ for each degree in the octonionic case.

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Rogério Serôdio
CMA and Departamento de Matemática, Universidade da Beira Interior, Portugal
E-mail address: rserodio@ubi.pt

P. D. Beites
CMA and Departamento de Matemática, Universidade da Beira Interior, Portugal
E-mail address: pbeites@ubi.pt

José Vitória
Departamento de Matemática, Universidade de Coimbra, Portugal
E-mail address: jvitoria@mat.uc.pt