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CURVATURE ADAPTED SUBMANIFOLDS OF BI-INVARIANT LIE GROUPS

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Abstract: We study submanifolds of arbitrary codimension in a Lie group G equipped with a bi-invariant metric. In particular, we show that, if the normal bundle of $M \subset \mathsf{G}$ is abelian, then the normal Jacobi operator of M equals the square of its invariant shape operator. This allows us to obtain geometric conditions which are necessary and sufficient for the submanifold M to be curvature adapted to G.

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1. Introduction and main result

Let M be an m-dimensional submanifold of a Riemannian manifold $(Q, g) \equiv Q$, $N^1 M$ its unit normal bundle, and R the ambient curvature tensor. For $(p, \eta) \equiv \eta \in N^1_p M$, the *normal Jacobi operator*

$$\begin{aligned} K\colon T_pM &\to T_pQ \\ x &\mapsto R(\eta, x)\eta \end{aligned}$$

of M (with respect to η) measures the curvature of the ambient manifold along η. On the other hand, denoting by N a unit normal local extension of η along M, and by ∇ the Levi-Civita connection of Q, the *shape operator*

$$\begin{aligned} A\colon T_pM &\to T_pM \\ x &\mapsto \pi^\top \nabla_x N \end{aligned}$$

of M (with respect to η) describes the curvature of M as a submanifold of Q. Here $\pi^\top$ denotes orthogonal projection onto T_pM. One says that M is *curvature adapted* (*to* Q) if, for every $(p, \eta) \in N^1 M$,

(1) K leaves T_pM invariant, i.e., $K(T_pM) \subset T_pM$;
(2) K and A commute, i.e., $K \circ A = A \circ K$.

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Note that the first condition is always satisfied for hypersurfaces. Moreover, since both A and K are self-adjoint with respect to the Riemannian metric, fulfilment of both (1) and (2) is equivalent to the existence of a common (orthonormal) basis of eigenvectors.

It is easy to see that every hypersurface in a real space form is curvature adapted. Indeed, if $\mathit{\Pi}_x$ denotes the 2-plane determined by $x \in T_pM$ and $\eta \in N_pM$, then $\sec(\mathit{\Pi}_x) = -g(K(x), x)$. However, for other ambient spaces, the definition is restrictive. For example, if Q is an $(m+1)$-dimensional non-flat complex space form with complex structure J, then A and K commute precisely when $-J\eta$ is an eigenvector of A; Also, if Q is an $(m+1)$-dimensional non-flat quaternionic space form with quaternionic structure $\mathfrak{J}$, then A and K commute precisely when the maximal subspace of T_pM invariant under $\mathfrak{J}$ is also invariant under A; see [1] and [4, Sec. 9.8].

In a symmetric space of non-constant curvature, the situation is more involved, yet many interesting results have been obtained. Among others (see for example [6, 10, 7]), the most important is arguably Gray's Theorem [5, Th. 6.14], which states that any tubular hypersurface about a curvature adapted submanifold is itself curvature adapted.

Remarkably, Gray's Theorem has been further generalized to so-called $\mathfrak{B}$-spaces, namely Riemannian manifolds such that, for every geodesic γ, the Jacobi operator $R(\dot{\gamma}, \cdot)\dot{\gamma}$ is diagonalizable by a *parallel* orthonormal frame field along γ. It turns out that, in such spaces, the classification of the curvature-adapted submanifolds is fully determined by that of the curvature-adapted hypersurfaces; see [3, 2].

In this note, we shall examine the case where Q is a Lie group G equipped with a bi-invariant metric $\langle \cdot, \cdot \rangle$. In particular, we shall focus our attention to the class of submanifolds of G having abelian normal bundle. Recall that the normal bundle of $M \subset \mathsf{G}$ is called *abelian* if, for every $p \in M$, $\exp(N_pM)$ is contained in some totally geodesic, flat submanifold of G; see [14].

In order to explain our main result, we first set up some notation. Let $(e_1, \ldots, e_m)$ be an orthonormal basis of eigenvectors of K, that is, a basis of T_pM such that $K(e_j) = \lambda_j e_j$ for all $j = 1, \ldots, m$. Let $(E_1, \ldots, E_m)$ be the left-invariant extension of $(e_1, \ldots, e_m)$.

Theorem 1. *Assume that the normal bundle of $M \subset \mathsf{G}$ is abelian. Then the following are equivalent:*

(i) *A and K commute;*

(ii) *For all* $j < h \in \{1, \dots, m\}$ *such that* $\lambda_j \neq \lambda_h$,

$$e_j(\langle N, E_h \rangle) = 0\,;$$

(iii) *The left-invariant extension of each eigenspace of* K *is orthogonal to* N *along all the others.*

Remark 2. For all $j \in \{1, \dots, m\}$, $\lambda_j = -\sec(e_j, \eta) \leq 0$; see Section 2.

Clearly, when M is a hypersurface, the condition on the normal bundle is automatically fulfilled. Specializing the theorem to that case, we obtain:

Corollary 3. *If* $\dim \mathsf{G} = m + 1$*, then the following are equivalent:*
(i) A *and* K *commute;*
(ii) *The left-invariant extension of each eigenspace of* K *is tangent to* M *along all the others.*

The basic fact which allows us to prove Theorem 1 is that the shape operator of $M \subset \mathsf{H}$ with respect to η, being H any Lie group with a left-invariant metric, decomposes as the sum of two terms [12]: an *invariant shape operator*, which depends only on η and H; plus a second term, here denoted by W, which is closely related to the Gauss map of M, see Section 3 for details. In particular, if the metric is bi-invariant, then the invariant shape operator commutes with K (Proposition 9), and so, by linearity, commutativity of A and K reduces to that of W and K.

In fact, if the metric is bi-invariant, then the non-zero eigenvalues of K have even multiplicities (Corollary 11), which allows us to establish the following result:

Proposition 4. *Assume the normal bundle of* $M \subset \mathsf{G}$ *is abelian. Then:*
(i) *If* $\dim M = 2$*, then* M *is curvature adapted;*
(ii) *If* $\dim M = 3$ *and* $K \neq 0$ *for all* $\eta \in N^1_p M$*, then the following are equivalent:*
  (a) M *is curvature adapted in a neighborhood* U *of* p*;*
  (b) *For all* $\eta \in N^1 U$*, the* 0*-eigenvector of* K *is an eigenvector of* A*.*

Eventually, combining the first part of Proposition 4 with Gray's Theorem, we get:

Corollary 5. *If* $M \subset \mathsf{G}$ *is two-dimensional and has abelian normal bundle, then every tubular hypersurface about* M *is curvature adapted.*

The paper is organized as follows. In the next section we briefly review some background material. In Section 3 we introduce the invariant shape operator and examine its properties. In Section 4 we then prove Theorem 1 and Proposition 4. In Section 5 we study curvature adapted product submanifolds; in so doing we obtain further examples of curvature adapted submanifolds of bi-invariant Lie groups. We finish off with Appendix A, where – for illustrative purposes – we give a direct proof that condition (ii) in Theorem 1 holds whenever p is an umbilical point of M.

Notation. Throughout the paper, the indices j, h, i satisfy $j, h \in \{1, \dots, m\}$ and $i \in \{1, \dots, m+n\}$. Note that we always use Einstein summation convention.

2. Preliminaries

Here we recall some basic results which are used throughout the paper; see e.g. [9] for further details about metric Lie groups and [8] about Riemannian geometry.

To begin with, let (Q, g) be a Riemannian manifold and ∇ its Levi-Civita connection. The curvature endomorphism $R\colon \mathfrak{X}(Q)^3 \to \mathfrak{X}(Q)$ of (Q, g) is the $(1,3)$-tensor field on Q defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,.$$

(Caution: Milnor in [9] defines the curvature endomorphism as the negative of ours.)

Note that, if x, y, z are vectors in T_pQ, then the value $R(x,y)z$ is independent of the extension of x, y, z and thus well-defined. If, in particular, x, y are orthonormal, then the sectional curvature $\sec(x,y)$ of the plane spanned by x and y may be computed by the formula:

$$\sec(x,y) = g(R(x,y)y, x)\,.$$

Next we consider products. Let $Q = Q_1 \times Q_2$, where Q_1 and Q_2 are Riemannian manifolds. The tangent space T_pQ at $p = (p_1, p_2)$ naturally splits as the orthogonal direct sum $T_pQ = T_{p_1}Q_1 \oplus T_{p_2}Q_2$. As a consequence, Q has a canonical Riemannian metric, called the *product metric*, defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle\,.$$

Here and in what follows, objects related to Q_1 (resp. Q_2) are denoted by adding a "1" (resp. "2") as a subscript – or as a superscript when the object in question already has one.

Assuming that Q is equipped with the product metric, it is well-known that its Levi-Civita connection decomposes by

$$\nabla_{(Y_1,Y_2)}(X_1,X_2) = \left(\nabla^1_{Y_1}X_1, \nabla^2_{Y_2}X_2\right),$$

and the curvature endomorphism by

$$R(X,Y)Z = (R_1(X_1,Y_1)Z_1, R_2(X_2,Y_2)Z_2), \tag{1}$$

where on the left hand side of the last equation we have used the notation $\mathfrak{X}(Q) \ni (V_1,V_2) = V$.

We now turn our attention to Lie groups. Let G be a Lie group equipped with a left- and right-invariant (i.e., bi-invariant) metric $\langle\cdot,\cdot\rangle$, and $\mathfrak{g}$ its Lie algebra, that is, the Lie algebra of left-invariant vector fields on G. As customary, we identify $\mathfrak{g}$ with the tangent space $T_e\mathsf{G}$ of G at the identity e.

Let $X,Y,Z \in \mathfrak{g}$. Then the Levi-Civita connection is given by

$$\nabla_X Y = -\nabla_Y X = \frac{1}{2}[X,Y], \tag{2}$$

while the curvature endomorphism by

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]].$$

In addition, the following equality holds:

$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle. \tag{3}$$

Let x,y be two orthonormal vectors in $T_p\mathsf{G}$, and X,Y their left-invariant extensions. The sectional curvature of the the two-plane spanned by x and y may be computed by

$$\sec(x,y) = \frac{1}{4}\langle [X,Y],[X,Y]\rangle.$$

Note that $\sec(x,y) \geq 0$, with equality if and only if $[X,Y]=0$.

3. The invariant shape operator

In this section we shall consider the general case of an orientable submanifold M of a Lie group H equipped with a left-invariant metric $\langle\cdot,\cdot\rangle$. Given a unit normal vector η of M at p, we denote by H its left-invariant extension. The *invariant shape operator* of M (with respect to η) is the map

$$\begin{aligned} \alpha\colon T_pM &\to T_pM \\ x &\mapsto \pi^\top \nabla_x H, \end{aligned}$$

where, as usual, $\pi^\top$ is the orthogonal projection onto T_pM.

The significance of the invariant shape operator lies in the fact that it represents the deviation of the ordinary shape operator from the differential of the Gauss map of M, in the sense described below.

Let N^1M be the unit normal bundle of M. Let S_e^{m+n-1} be the unit sphere inside the Lie algebra $\mathfrak{h}$ of H. The *Gauss map* of M is the map

$$\begin{aligned}\gamma\colon N^1M &\to S_e^{m+n-1}\\ (p,\eta) &\mapsto d(L_{p^{-1}})(\eta)\,.\end{aligned}$$

Let N be a unit normal vector field along M such that $N_p = \eta$. Consider the map $\bar\gamma \coloneqq \gamma \circ N$. Its differential at p is a linear map $T_pM \to \gamma(p,\eta)^\perp$. Thus, since $d(L_{p^{-1}})$ maps η to $\gamma(p,\eta)$ and is an isometry, it follows that

$$W \coloneqq \pi^\top \circ d(L_p) \circ d\bar\gamma$$

is an endomorphism of T_pM.

Clearly, if $\mathsf{G} = \mathbb{R}^{m+1}$, then γ is the classical Gauss map of M, whereas W its shape operator. In our setting, we shall prove that

Proposition 6 (cf. [13, p. 769])**.**

$$\forall x \in T_pM\colon A(x) = \alpha(x) + W(x)\,. \tag{4}$$

Proof: Let $(b_1,\dots,b_{m+n})$ be an orthonormal basis of $T_p\mathsf{G}$, such that $b_1,\dots,b_m \in T_pM$ and $b_{m+n} = \eta$. For each i, let B_i be the left-invariant extension of b_i, so that $(B_1,\dots,B_{m+n} = H)$ is an orthonormal frame for G (and a basis of $T_e\mathsf{G}$).

If $q \in M$ – writing N^i as a shorthand for $\langle N, B_i\rangle$ – then

$$\bar\gamma(q) = d(L_{q^{-1}})(N_q) = d(L_{q^{-1}})\big(N^i(q)B_i|_q\big) = N^i(q)B_i\,.$$

Thus, if $x \in T_pM$, then

$$d\bar\gamma(x) = dN^i(x)B_i = x(N^i)B_i\,.$$

Since

$$d(L_p)(d\bar\gamma(x)) = x(N^i)\,b_i\,,$$

it follows that

$$\pi^\top d(L_p)(d\bar\gamma(x)) = x(N^j)\,b_j\,, \tag{5}$$

On the other hand,

$$\begin{aligned} A(x) &= \pi^\top \nabla_x N^i B_i \\ &= \pi^\top \big(N^i(p)\, \nabla_x B_i + x(N^i)\, b_i \big). \end{aligned}$$

Since, by construction, $N^1(p) = \cdots = N^{m+n-1}(p) = 0$ and $N^{m+n}(p) = 1$, we have

$$A(x) = \alpha(x) + x(N^j)\, b_j\,, \tag{6}$$

which, together with (5), gives (4). ■

Remark 7. Proposition 6 shows that W does not depend on the particular choice of normal vector field N but only on its value at p.

Remark 8. Using equation (6), it is not difficult to see that statement (ii) in Theorem 1 is nothing but the coordinate expression, with respect to the frame $(E_1, \ldots, E_m)$, of the condition

$$\text{For all } j < h \in \{1, \ldots, m\} \text{ such that } \lambda_j \neq \lambda_h \colon \pi_j \operatorname{Im} W|_{\Lambda_h} = 0\,, \tag{7}$$

where Λ_j is the eigenspace of K corresponding to the eigenvalue λ_j, and π_j the orthogonal projection onto Λ_j. Note that (7) holds if and only if W leaves the eigenspaces of K invariant.

A useful property of the invariant shape operator, which is crucial in proving Theorem 1, is contained in the following

Proposition 9. *If $\langle\cdot,\cdot\rangle$ is bi-invariant and the normal bundle of M is abelian, then*

(1) *$K = \alpha \circ \alpha$, and so α and K commute;*
(2) *K leaves T_pM invariant.*

The proof will be based on a lemma:

Lemma 10. *Under the hypotheses of Proposition 9, $\alpha(x) = \nabla_x H$.*

Proof: Let ξ be a unit normal vector at p. Let X and $\varXi$ be the left-invariant extensions of $x \in T_pM$ and ξ, respectively. Note that, for the normal bundle of M is abelian, $[H, \varXi] = 0$. Indeed, since $H_p = \eta$ and ξ are tangent to a totally geodesic, flat submanifold, we infer from the Gauss equation that $\langle R(\eta, \xi)\xi, \eta\rangle = 0$, which is equivalent to $\sec(\eta, \xi) = 0$ when η and ξ are linearly independent.

Hence, by bi-invariance of the metric, it follows that

$$\langle [X, H], \Xi\rangle = \langle X, [H, \Xi]\rangle = 0\,,$$

which implies $[X, H]_p = \nabla_x H \in T_pM$, and so $\alpha(x) = \pi^\top \nabla_x H = \nabla_x H$. ∎

Proof of Proposition 9: Clearly, being the second assertion in the proposition a direct consequence of the first, we only need to prove the latter.

Let $x \in T_pM$. Since K is tensorial, the value $K(x)$ may be computed in terms of the left-invariant extensions X and H of x and η:

$$K(x) = \big(\nabla_H \nabla_X H - \nabla_X \nabla_H H - \nabla_{[H,X]} H\big)_p\,.$$

Assume that the metric is bi-invariant. Then, using (2), we have

$$\begin{aligned} K(x) &= (\nabla_H \nabla_X H)_p - 2(\nabla_{\nabla_H X} H)_p \\ &= -(\nabla_{\nabla_X H} H)_p + 2(\nabla_{\nabla_X H} H)_p \\ &= \nabla_{\nabla_x H} H\,. \end{aligned}$$

From here the statement follows directly from Lemma 10. ∎

Corollary 11. *Under the hypotheses of Proposition 9, the non-zero eigenvalues of K are negative and have even multiplicities.*

Proof: We deduce from equation (3) that the invariant shape operator α of G is skew-adjoint with respect to the Riemannian inner product. On the other hand, K is self-adjoint. Thence, if the hypotheses of Proposition 9 are fulfilled, then $K = \alpha \circ \alpha$, and the statement follows from [11, Theorem 2]. ∎

4. Proof of the main result

We are now ready to prove Theorem 1 and Proposition 4 in the Introduction.

Proof of Theorem 1: The equivalence of statements (ii) and (iii) is easily seen, so we prove (i) $\Leftrightarrow$ (ii).

Assume that the normal bundle of M is abelian. It follows from equation (6) that

$$A(e_j) = \alpha(e_j) + e_j(\langle N, E_h\rangle)e_h\,.$$

Hence, by linearity of K, we have:

$$K(A(e_j)) = K(\alpha(e_j)) + e_j(\langle N, E_h\rangle)K(e_h)\,,$$

whereas

$$A(K(e_j)) = \alpha(K(e_j)) + K(e_j)(\langle N, E_h\rangle)e_h\,.$$

Since G is equipped with a bi-invariant metric, K and α commute. It follows that $K(A(e_j)) = A(K(e_j))$ if and only if

$$e_j(\langle N, E_h\rangle)K(e_h) = K(e_j)(\langle N, E_h\rangle)e_h\,.$$

Therefore, being $(e_1, \dots, e_m)$ a basis of eigenvectors of K, we conclude that A and K commute if and only if $e_j(\langle N, E_h\rangle) = 0$ for all j *and* h such that $\lambda_j \neq \lambda_h$.

It only remains to show that $e_j(\langle N, E_h\rangle) = e_h(\langle N, E_j\rangle)$ when $\lambda_j \neq \lambda_h$. To this end, identify α, A, and K with their matrices in the basis $(e_j)_{j=1}^m$. The first is a skew-symmetric matrix, by equation (3), whereas A is symmetric and K diagonal. Since α and K commute, the (j, h)-entries of αK and $K\alpha$ are equal, and so we must have $\alpha_{jh}\lambda_h = \lambda_j\alpha_{jh}$.

Assume $\lambda_j \neq \lambda_h$. Then $\alpha_{jh} = -\alpha_{hj} = 0$ and so, by equation (4),

$$\begin{aligned} A_{jh} &= \langle W(e_j), e_h\rangle\,, \\ A_{hj} &= \langle W(e_h), e_j\rangle\,, \end{aligned}$$

from which we conclude that $\langle W(e_j), e_h\rangle = \langle W(e_h), e_j\rangle$ by symmetry of A. ■

Proof of Proposition 4: The first part is an immediate consequence of Corollary 11, so we shall prove the second.

Suppose that $\dim M = 3$ and $\alpha \neq 0$. It follows by Corollary 11 that K has one zero eigenvalue, while the remaining two are equal. Without loss of generality, we may assume that $\lambda_3 = 0$. Since $K \neq 0$, it is clear that $\lambda_1 = \lambda_2 \neq 0$.

Extend η to a unit normal vector field N along M. Then, by continuity, the multiplicity of λ_3 is locally constant, i.e., there exists a neighborhood $U = U(N)$ of p in M such that the extension of K has two negative definite eigenvalues in U.

Assume that A and K commute, i.e., they share a common basis of eigenvectors. Since the 0-eigenspace of K is one-dimensional, it follows that e_3 is an eigenvector of A. Conversely, if e_3 is an eigenvector of A, then its other two eigenvectors lie in the λ_1-eigenspace of K, from which we conclude that A and K commute. ■

5. Product manifolds

We have seen that any surface in a bi-invariant three-dimensional Lie group is curvature adapted. Further examples may be constructed via the next result, which states, in full generality, that the property of being curvature adapted behaves well under Cartesian products:

Proposition 12. *Let Q_1, Q_2 be Riemannian manifolds. If $Q_1 \times Q_2$ is equipped with the product metric, then the following are equivalent:*

(1) *$M_1 \subset Q_1$ and $M_2 \subset Q_2$ are curvature adapted submanifolds;*
(2) *$M_1 \times M_2$ is curvature adapted to $Q_1 \times Q_2$.*

Proof: For $k = 1, 2$, let $(p_k, \eta_k) \in N^1 M_k$. By definition of the product metric, it is clear that $\eta = 2^{-1/2}(\eta_1, \eta_2)$ is a unit normal vector of $M = M_1 \times M_2$ at (p_1, p_2). Conversely, every unit normal vector of M may be expressed in this form.

For $x_k \in T_{p_k} M_k$, we compute:

$$\begin{aligned} 2^{1/2} A(x_1, x_2) &= \pi^\top \nabla_{(x_1, x_2)}(N_1, N_2) \\ &= \left(\pi_1^\top \nabla^1_{x_1} N_1, \pi_2^\top \nabla^2_{x_2} N_2\right) \\ &= \left(A_1(x_1), A_2(x_2)\right), \end{aligned}$$

where, as usual, objects related to Q_k are denoted by a sub- or super-scripted "k". It follows that (x_1, x_2) is an eigenvector of A if and only if x_k is an eigenvector of A_k for $k = 1, 2$.

Likewise, from equation (1), we deduce that

$$2K(x_1, x_2) = \left(K_1(x_1), K_2(x_2)\right),$$

and so an analogous statement holds for K. Hence, we conclude that A and K share a common eigenbasis precisely when A_k and K_k commute for each $k = 1, 2$. ∎

Corollary 13. *Under the assumptions of Proposition 12, the following are equivalent:*

(1) *M_1 is curvature adapted to Q_1;*
(2) *M_1 is curvature adapted to $Q_1 \times Q_2$;*
(3) *$M_1 \times Q_2$ is curvature adapted to $Q_1 \times Q_2$.*

Proof: Equivalence of the first and second statements follows by specializing Proposition 12 to the case where M_2 is a single point – which is obviously curvature adapted to Q_2. On the other hand, equivalence of the first and third statements follows by applying Proposition 12 to the case where $M_2 = Q_2$. ∎

In particular, Corollary 13, together with Gray's Theorem, implies that any surface in $\mathsf{SO}(3)$ (or $\mathsf{SU}(2)$) gives rise to a curvature adapted tubular hypersurface in $\mathsf{SO}(3) \times \mathsf{G}_2$, where the second factor is a bi-invariant Lie group of arbitrary dimension.

Appendix A.

Here we present a direct proof of the following obvious corollary of Theorem 1:

Corollary 14. *Assume that the normal bundle of M is abelian. If $A = \mu\,\mathrm{id}$ for some $\mu \in \mathbb{R}$, then $e_j(\langle N, E_h\rangle) = 0$ for all $j, h \in \{1, \dots, m\}$ such that $\lambda_j \neq \lambda_h$.*

Proof: Suppose the j-th eigenvalue λ_j of K has multiplicity r, meaning that there exists a multi-index $(j_1, \dots, j_r)$ of length $r \leq m$ such that $j_1, \dots, j_r \in \{1, \dots, m\}$ and $\lambda_j = \lambda_{j_1} = \cdots = \lambda_{j_r}$.

Assume that NM is abelian. Since α commutes with K, the j-th eigenspace $\Lambda_j = \mathrm{span}(e_{j_1}, \dots, e_{j_r})$ is invariant under α. Indeed, from $\alpha(K(e_j)) = \lambda_j \alpha(e_j) = K(\alpha(e_j))$, we observe that $\alpha(e_j)$ is an eigenvector of K corresponding to the eigenvalue λ_j.

Assume that $A = \mu\,\mathrm{id}$ for some $\mu \in \mathbb{R}$. Then, for every $j \neq h$,

$$\langle A(e_j), e_h\rangle = 0\,,$$

which, by equation (6), is equivalent to

$$\langle \alpha(e_j), e_h\rangle = -e_j(\langle N, E_h\rangle)\,.$$

Now, if $e_h \in \Lambda_j$, then $\lambda_h = \lambda_j$. Else, if $e_h \notin \Lambda_j$, then $e_j(\langle N, E_h\rangle) = 0$ because $\alpha(e_j) \in \Lambda_j$. ∎

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