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# NONLINEAR SYSTEMS OF PARABOLIC IBVP: A SUPER-SUPRACONVERGENT FULLY DISCRETE PIECEWISE LINEAR FEM

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Abstract: The main objective of this paper is the design and convergence analysis of discretizations of second order nonlinear parabolic initial boundary value problems with second order of convergence with respect to $H^1$-discrete norms. The results are established assuming that the solutions are in $H^2$. As the methods can be simultaneously seen as piecewise linear finite element methods and finite difference methods, the convergence results can be seen simultaneously as supraconvergence results and superconvergence results. Numerical results illustrating the sharpness of the smoothness assumptions are also included.

## 1. Introduction

The main goal of the present paper is to propose numerical discretizations of the following system of second order nonlinear parabolic equations

$$\frac{\partial U}{\partial t} + F(U, \nabla U) = A\Delta U + G \quad \text{in } \Omega \times (0, T], \tag{1}$$

that depends only on $x$ and $t$, and where, to simplify, $\Omega = (0, 1)$, $U = (u, v)$, $A$ is a diagonal matrix with entries $\alpha > 0$ and $\beta > 0$, $\nabla U = (\nabla u, \nabla v)$, $F = (f_1, f_2)$ and $G = (g_1, g_2)$. This system is complemented with homogeneous Dirichlet boundary conditions

$$U = 0 \quad \text{on } \partial\Omega \times (0, T], \tag{2}$$

with $\partial\Omega = \{0, 1\}$, and initial conditions

$$U(x, 0) = U_0(x), \ \ x \in \Omega, \tag{3}$$

having $U_0 = (u_0, v_0)$.

A huge number of physical, biological, and engineering science phenomena are described by the system of parabolic equations (1). Another application

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that we would like to focus, in a future work, is related with the numerical solutions of the 2D magnetohydrodynamic equations with dispersion. Although magnetohydrodynamic equations are of hyperbolic type, a common approach to solve numerically these equations is to add artificial diffusion terms and solve numerically, for small values of the diffusion coefficients, the corresponding parabolic problems. We observe that the system of equations (1) can be seen as the 1D magnetohydrodynamic equations, with velocity and magnetic dissipation (see [10]). In this case, $u$ denotes the velocity of the fluid, $v$ the magnetic field, and the pressure is assumed to be known.

To gain some insights in the design and convergence analysis of super-supraconvergent discretizations of second order nonlinear equations, we start by considering the elliptic boundary value problem

$$-u'' + f(u,u') = g \quad \text{in } \Omega, \tag{4}$$

with homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega. \tag{5}$$

We wish to extend the results obtained for the previous elliptic problem, to the following system of nonlinear elliptic equations

$$-\Delta U + F(U, \nabla U) = G \text{ in } \Omega, \tag{6}$$

where $U$ depends only on the spatial variable, with homogeneous Dirichlet boundary conditions

$$U = 0 \text{ on } \partial\Omega. \tag{7}$$

For linear elliptic boundary value problems, in [2, 5], numerical methods based on piecewise linear finite element methods (FEM), that are equivalent to finite difference methods (FDM), were proposed. In these papers, the authors presented a new approach to analyse the convergence properties of a fully discrete in space piecewise linear finite element method. In this new approach, the Bramble-Hilbert lemma [4] is the main tool in the convergence analysis that allows reducing the smoothness assumptions on the solutions of the differential problems usually required when Taylor expansion is used. This methodology has been largely used as can be seen, for instance, in [1, 3, 6, 7, 8, 9].

The aim of this paper is to propose numerical methods for the IBVP (1)-(3) and provide their convergence analysis using the numerical approach mentioned before. We propose fully discrete piecewise linear methods that can be seen as finite difference methods, defined in nonuniform grids. We show that these methods exhibit second convergence order with respect to a norm that can be seen as the discrete version of the usual $H^1$-norm. These results can be seen simultaneously as superconvergence results in the finite element framework, and supraconvergence results in the finite difference context. The main question here is the nonlinearity of the IBVP (1)-(3). We start by considering the stationary elliptic boundary value problems (4) and (6), to gain some insights in the convergence analysis for nonlinear problems. For the fully discrete piecewise linear finite element approximation for the solution of the nonlinear problem, we prove second order convergence for the gradient without any post-processing, a popular procedure followed in different contexts, that leads to an improvement of the accuracy of the gradient approximations (see for instance [11, 13]).

The paper is organized as follows. In Section 1 we propose and analyse superconvergence and supraconvergence results for (4) with homogeneous Dirichlet boundary conditions. The extension of these results to the boundary value problem (6)-(7) is the main goal of Section 2. Section 3 is devoted to the main goal of this paper: the construction of the numerical methods for the IBVP (1)-(3) and their convergence analysis. The convergence analysis is not based on the stability properties of the methods, but it will be established constructing the error equation. Numerical results illustrating the theoretical support developed in this paper are presented in Section 4. Finally, in Section 5 we present some conclusions.

## 2. A nonlinear elliptic equation

We start by introducing a non-uniform grid $\overline{\Omega}_h$, in $\overline{\Omega}$, of size $N$, induced by a vector $h = (h_1, ..., h_N)$, $h_i > 0, \forall i \in \{1, ..., N\}$, and $\sum_{i=1}^{N} h_i = x_N - x_0 = 1$. Let $\Lambda$ be a sequence of vectors $h$, and $h_{\max} = \max_{i=1,...,N} h_i \to 0$. Assuming $x_i = x_{i-1} + h_i$, the non-uniform grid considered is $\overline{\Omega}_h = \{x_i : i = 0, ..., N\}$. Now, we define the set of interior nodes of the grid by $\Omega_h = \Omega \cap \overline{\Omega}_h$, and the set of boundary points of the grid by $\partial\Omega_h = \partial\Omega \cap \overline{\Omega}_h$. By $W_h$ and $W_{h,0}$ we denote, respectively, the space of grid functions defined in $\overline{\Omega}_h$, and the space of grid functions defined in $\overline{\Omega}_h$ and null on $\partial\Omega_h$.

Let $u_h, w_h \in W_{h,0}$. We introduce the $L^2$-discrete inner product in $W_{h,0}$ defined by

$$(u_h, w_h)_h = \sum_{i=1}^{N-1} h_{i+1/2} u_h(x_i) w_h(x_i),$$

considering $h_{i+1/2} = \frac{h_i+h_{i+1}}{2}$. The norm induced by this inner product is denoted by $\|\cdot\|_h$.

Considering $u_h, w_h \in W_h$, we use introduce the inner product,

$$(u_h, w_h)_+ = \sum_{i=1}^{N} h_i u_h(x_i) w_h(x_i).$$

and its induced norm $\|\cdot\|_+$. In $W_{h,0}$ we also introduce the norm

$$\|v_h\|_{h,\infty} := \max_{i=1,\ldots,N-1} |v_h(x_i)|, \ \forall v_h \in W_{h,0}.$$

Let $D_{-x}u_h$ be the first-order backward finite difference operator, defined by,

$$D_{-x}u_h(x_i) = \frac{u_h(x_i) - u_h(x_{i-1})}{h_i}, \ i = 1, \ldots, N.$$

We recall some useful results regarding functions in $W_{h,0}$.

**Proposition 1.** *For $v_h \in W_{h,0}$ it holds*

$$\|v_h\|_h \leq \|D_{-x}v_h\|_+$$

*and*

$$\|v_h\|_{h,\infty} \leq \|D_{-x}v_h\|_+.$$

We now consider the second-order centered finite difference operator, $\Delta_h$, as the discrete version of the second derivative in space, defined by

$$\Delta_h u_h(x_i) = \frac{h_i u_h(x_{i+1}) - (h_i + h_{i+1}) u_h(x_i) + h_{i+1} u_h(x_{i-1})}{h_i h_{i+1} h_{i+1/2}}, \ i = 1, ..., N-1,$$

For this operator, the following result holds, which is an analogue of the integration by parts formula known in calculus.

**Proposition 2.** *For $u_h, v_h \in W_{h,0}$,*

$$(\Delta_h u_h, v_h)_h = -(D_{-x}u_h, D_{-x}v_h)_+.$$

We also introduce the discrete operator $\nabla_h$ defined as

$$\nabla_h u_h(x_i) = \frac{h_i}{h_i + h_{i+1}} D_{-x}u_h(x_{i+1}) + \frac{h_{i+1}}{h_i + h_{i+1}} D_{-x}u_h(x_i), \ i = 1, ...N-1,$$

with

$$\nabla_h u_h(x_0) = D_{-x}u_h(x_1) \ \text{ and } \ \nabla_h u_h(x_N) = D_{-x}u_h(x_N).$$

The source term $g$ is discretized by

$$g_h(x_i) = \frac{1}{h_{1+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} g(x)dx, \tag{8}$$

where $x_{i-1/2} = x_i - \frac{h_i}{2}$, and $x_{i+1/2} = x_i + \frac{h_{i+1}}{2}$.

We consider, for the elliptic equation (4), the finite differences approximation

$$-\Delta_h u_h + f(u_h, \nabla_h u_h) = g_h \quad \text{in } \Omega_h, \tag{9}$$

complemented with the boundary condition

$$u_h = 0 \quad \text{on } \partial\Omega_h. \tag{10}$$

We observe that the finite difference discretization (9) can be obtained from the weak problem

$$\begin{aligned} &\text{find } u \in H_0^1(\Omega) \text{ such that} \\ &\qquad (u', v') + (f(u, u'), v) = (g, v),\ \forall v \in H_0^1(\Omega). \end{aligned} \tag{11}$$

In fact, if $u_h \in W_{h,0}$ and $P_h u_h$ denotes the piecewise linear interpolator of $u_h$, the last variational problem is replaced by the following finite dimensional problem:

$$\begin{aligned} &\text{find } u_h \in W_{h,0} \text{ such that} \\ &((P_h u_h)', (P_h v_h)') + (f(P_h u_h, (P_h u_h)'), P_h v_h) = (g, P_h v_h),\ \forall v_h \in W_{h,0}, \end{aligned}$$

then, taking into account that

$$((P_h u_h)', (P_h v_h)') = (D_{-x} u_h, D_{-x} v_h)_+$$

and

$$(f(P_h u_h, (P_h u_h)'), P_h v_h) \simeq (f(u_h, \nabla_h u_h), v_h)_h, \quad (g, P_h v_h) \simeq (g_h, v_h)_h$$

we get the weak variational problem:

$$\begin{aligned} &\text{find } u_h \in W_{h,0} \text{ such that} \\ &(D_{-x} u_h, D_{-x} v_h)_+ + (f(u_h, \nabla_h u_h), v_h)_h = (g_h, v_h)_h,\ \forall v_h \in W_{h,0}. \end{aligned} \tag{12}$$

The discrete equality (12) can be easily obtained from (9), taking into account Proposition 2. In what follows we need to impose a condition on the grid to get a second-order estimate for the nonlinear term. We assume that there exists a positive constant $C_R$ such that, for all $h \in \Lambda$, we have

$$\frac{h_{max}}{h_{min}} \le C_r, \tag{13}$$

where $h_{min} = \min_{i=1,\dots,N} h_i$.

**Theorem 1.** *Let us suppose that the sequence of grids* $\Lambda$ *satisfies* (13)*, the solution* $u$ *of* (11) *belongs to* $H^3(\Omega) \cap H_0^1(\Omega)$*,* $f$ *is a Lipschitz function with Lipschitz constant* $C_L$ *such that* $1 - C_L(1+\sqrt{2C_r}) > 0$*, and* $f(u, u') \in H^2(\Omega)$*. Let* $u_h \in W_{h,0}$ *be a solution of* (12) *or* (9)-(10)*, and let* $E_u$ *be the discretization error,* $E_u = u_h - R_h u$*. Then there exists a positive constant* $C$ *such that*

$$\|D_{-x}E_u\|_+^2 \leq C \sum_{i=1}^{N} h_i^4 \left( \|u\|^2_{H^3(x_{i-1},x_{i+1})} + \|f(u,u')\|^2_{H^2(x_{i-1},x_i)} \right), \tag{14}$$

*where* $R_h : C(\overline{\Omega}) \to W_h$ *denotes the restriction operator.*

*Proof*: It can be shown that

$$(g_h, v_h)_h = (f(R_h u, \nabla_h R_h u), v_h)_h + (D_{-x}R_h u, D_{-x}v_h)_+ + \sum_{i=1}^{3} T_h^{(i)} \tag{15}$$

where

$$T_h^{(1)} = (\hat{R}_h u' - D_{-x}R_h u, D_{-x}v_h)_+,$$

$$T_h^{(2)} = ((f(u,u'))_h - R_h f(u,u'), v_h)_h,$$

and

$$T_h^{(3)} = (R_h f(u,u') - f(R_h u, \nabla_h R_h u), v_h)_h,$$

where $\hat{R}_h : C(\overline{\Omega}) \to W_h$ is defined by $\hat{R}_h v(x_i) = v(x_{i-1/2})$, $i = 1, \ldots, N$, $\hat{R}_h v(x_0) = v(x_0)$. To show (15) we observe that we have successively

$$\begin{aligned}
(g_h, v_h)_h &= \sum_{i=1}^{N-1} \int_{x_{i-1/2}}^{x_{i+1/2}} (-u'' + f(u,u'))\, dx\, v_h(x_i) \\
&= \sum_{i=1}^{N} h_i \left( u'(x_{i-1/2}) - D_{-x}u(x_i) \right) D_{-x}v_h(x_i) \\
&\quad + \sum_{i=1}^{N} h_i D_{-x}u(x_i) D_{-x}v_h(x_i) \\
&\quad + \sum_{i=1}^{N-1} \left( \int_{x_{i-1/2}}^{x_{i+1/2}} f(u,u')dx - h_{i+1/2} f(u(x_i), u'(x_i)) \right) v_h(x_i)
\end{aligned}$$

$$+\sum_{i=1}^{N-1} h_{i+1/2}\left(f(u(x_i),u'(x_i)) - f(u(x_i),\nabla_h u(x_i))\right) v_h(x_i)$$
$$+\sum_{i=1}^{N-1} h_{i+1/2} f(u(x_i),\nabla_h u(x_i)) v_h(x_i).$$

In what follows we estimate separately $T_h^{(i)},\ i=1,2,3.$

- Estimation of $T_h^{(1)}$

As in [2], there exists a positive constant $C_1$ such that

$$|T_h^{(1)}| \leqslant C_1 \sum_{i=1}^{N} h_i^2 \int_{x_{i-1}}^{x_i} |u^{(3)}(x)|\,dx\,|D_{-x}v_h(x_i)|.$$

This leads to

$$\begin{aligned}
|T_h^{(1)}| &\leqslant C_1 \Bigg(\sum_{i=1}^{N} h_i^4 \|u^{(3)}\|^2_{L^2(x_{i-1},x_i)}\Bigg)^{1/2} \|D_{-x}v_h\|_+ \\
&\leqslant C_1 \Bigg(\sum_{i=1}^{N} h_i^4 \|u\|^2_{H^3(x_{i-1},x_i)}\Bigg)^{1/2} \|D_{-x}v_h\|_+ \\
&\leqslant \frac{C_1}{4\epsilon_1^2} \sum_{i=1}^{N} h_i^4 \|u\|^2_{H^3(x_{i-1},x_i)} + \epsilon_1^2 \|D_{-x}v_h\|_+^2,
\end{aligned}$$

where $\epsilon_1$ is an arbitrary nonzero real constant.

- Estimation of $T_h^{(2)}$

Taking into account the definition of $T_h^{(2)}$, for $w(x)=f(u(x),u'(x))$, we have

$$T_h^{(2)} = \sum_{i=1}^{N-1} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} w(x)dx - h_{i+1/2} w(x_i)\right) v_h(x_i),$$

and consequently, there exists a positive constant $C_2$ such that

$$\begin{aligned}
|T_h^{(2)}| &\leqslant C_2 \Bigg(\sum_{i=1}^{N} h_i^4 \|w\|^2_{H^2(x_{i-1},x_i)}\Bigg)^{1/2} \|D_{-x}v_h\|_+ \\
&\leqslant \frac{C_2}{4\epsilon_2^2} \sum_{i=1}^{N} h_i^4 \|f(u,u')\|^2_{H^2(x_{i-1},x_i)} + \epsilon_2^2 \|D_{-x}v_h\|_+^2,
\end{aligned}$$

where $\epsilon_2$ is an arbitrary nonzero real constant, provided that $f(u,u') \in H^2(\Omega)$.

- Estimation of $T_h^{(3)}$

  Taking into account that $f$ is a Lipschitz function, for $T_h^{(3)}$ we deduce

$$|T_h^{(3)}| \leqslant C_L \sum_{i=1}^{N-1} h_{i+1/2} |u'(x_i) - \nabla_h u(x_i)||v_h(x_i)|.$$

We remark that $u'(x_i) - \nabla_h u(x_i)$ admits the representation

$$u'(x_i) - \nabla_h u(x_i) = \frac{1}{h_{i+1} + h_i} \lambda(w),$$

where $\lambda : H^3(0,1) \longrightarrow \mathbb{R}$ is the functional defined as

$$\lambda(v) = v'(\rho) - \Big[\hat{\rho}\left(v(1) - v(\rho)\right) + \frac{1}{\hat{\rho}}\left(v(\rho) - v(0)\right)\Big],$$

and $w(\xi) = u(x_{i-1} + \xi(h_i + h_{i+1}))$, $\xi \in [0,1]$, with $\rho = \frac{h_i}{h_i + h_{i+1}}$, $\hat{\rho} = \frac{h_i}{h_{i+1}}$. As in [12], Bramble-Hilbert lemma leads to

$$\begin{aligned}
|T_h^{(3)}| &\leqslant C_3 \Bigg( \sum_{i=1}^{N} h_i^4 \|u^{(3)}\|^2_{L^2(x_{i-1},x_i)} \Bigg)^{1/2} \|D_{-x} v_h\|_+ \\
&\leqslant \frac{C_3}{4\epsilon_3^2} \sum_{i=1}^{N} h_i^4 \|u\|^2_{H^3(x_{i-1},x_i)} + \epsilon_3^2 \|D_{-x} v_h\|_+^2.
\end{aligned}$$

Fixing $v_h = E_u$ in (12) and taking into account representation (15) it follows

$$\begin{aligned}
\left(1 - \sum_{i=1}^{3} \epsilon_i^2\right) \|D_{-x} E_u\|_+^2 &\leq (f(R_h u, \nabla_h R_h u) - f(u_h, \nabla_h u_h), E_u)_h + T_h \\
&\leq C_L \left(\sqrt{2C_r} \|E_u\|_h \|D_{-x} E_u\|_+ + \|E_u\|_h^2\right) + T_h \\
&\leq C_L \left(1 + \sqrt{2C_r}\right) \|D_{-x} E_u\|_+^2 + T_h,
\end{aligned} \tag{16}$$

where the last inequality was established taking into account Proposition 1 and

$$T_h \leq \left(\sum_{j=1}^{3} \frac{C_j}{4\epsilon_j^2}\right) \sum_{i=1}^{N} h_i^4 \left(\|u\|^2_{H^3(x_{i-1},x_i)} + \|f(u,u')\|^2_{H^2(x_{i-1},x_i)}\right).$$

From (16) we establish

$$\left(1 - \sum_{i=1}^{3} \epsilon_i^2 - C_L(1+\sqrt{2C_r}\right) \|D_{-x}E_u\|_+^2 \leq T_h.$$

If $C_L\left(1+\sqrt{2C_r}\right) < 1$ then there exist positive values for $\epsilon_i,\ i = 1,2,3$, such that

$$1 - \sum_{i=1}^{3} \epsilon_i^2 - C_L\left(1+\sqrt{2C_r}\right) > 0,$$

which leads to the existence of a positive constant $C$ satisfying (14).

■

From Proposition 1 and Theorem 1 we conclude for the norm

$$\|v_h\|_{1,h} = \left(\|v_h\|_h^2 + \|D_{-x}v_h\|_+^2\right)^{1/2},\ v_h \in W_{h,0},$$

the following estimate.

**Corollary 1.** *Under the assumptions of Theorem 1, the following bound hold for the error $E_u = u_h - R_h u$,*

$$\|E_u\|_{1,h} \leq C h_{max}^2.$$

Corollary 1 states that the finite difference scheme (9)-(10) or equivalently, the fully discrete piecewise linear finite element method (12), is second-order convergent with respect to the norm $\|\cdot\|_{1,h}$ which is a discrete version of the usual $H^1$-norm.

Taking into account Theorem 1 and Proposition 1, we conclude the following corollary that establishes the uniform boundness of the solution of the finite difference scheme (9)-(10), or equivalently, the uniform boundness of the fully discrete piecewise linear finite element method (12).

**Corollary 2.** *If $u_h \in W_{h,0}$ is defined by the finite difference scheme* (9)-(10)*, or equivalently, by the fully discrete piecewise linear finite element method*

(12), *then under the assumptions of Theorem 1, there exists a positive constant* $C$, $h-$*independent, such that*

$$\|u_h\|_{h,\infty} \leq C,\ h \in \Lambda, \tag{17}$$

$$\|D_{-x}u_h\|_{h,\infty} \leq C,\ h \in \Lambda. \tag{18}$$

*Proof*: We have

$$u_h(x_i) = E_u(x_i) + R_h u(x_i),\ i = 1, \ldots, N-1,$$

and consequently

$$\begin{aligned}\|u_h\|_{h,\infty} &\leq \|E_u\|_{h,\infty} + \|u\|_{L^\infty}\\ &\leq \|D_{-x}E_u\|_+ + \|u\|_{L^\infty}\\ &\leq Ch_{max}^2 + \|u\|_{L^\infty},\ h \in \Lambda,\end{aligned}$$

that concludes the proof of (17).

On the other hand, since

$$D_{-x}u_h(x_i) = D_{-x}E_u(x_i) + D_{-x}R_h u(x_i),\ i = 1, \ldots, N,$$

and

$$\begin{aligned}|D_{-x}E_u(x_i)| &\leq \frac{1}{h_{min}}\sum_{j=1}^{N} h_j |D_{-x}E_u(x_j)|\\ &\leq \frac{1}{h_{min}}\|D_{-x}E_u\|_+,\end{aligned}$$

from (14)

$$\begin{aligned}|D_{-x}E_u(x_i)| &\leq C\frac{h_{max}^2}{h_{min}}\\ &\leq C\, C_r\, h_{max},\end{aligned}$$

we conclude the proof of (18).

■

## 3. A nonlinear system of elliptic equations

In this section we extend Theorem 1 for the solution of the following finite difference method

$$-\Delta_h U_h + F(U_h, \nabla_h U_h) = G_h \text{ in } \Omega_h, \tag{19}$$

where $U_h = (u_h, v_h)$, with boundary conditions

$$U_h = 0 \quad \text{on } \partial\Omega_h, \tag{20}$$

that leads to an approximation for the solution of the differential system (1)-(2). In (19), $G_h = (g_{1,h}, g_{2,h})$ with $g_{\ell,h}$ defined by (8) with $g$ replaced by $g_\ell,\ \ell = 1, 2$.

We remark that the finite difference method (19)-(20) is equivalent to the following fully discrete piecewise linear finite element method:

find $U_h \in [W_{h,0}]^2$ such that

$$(D_{-x}U_h, D_{-x}Q_h)_+ + (F(U_h, \nabla_h U_h), Q_h)_h = (G_h, Q_h)_h,\ \forall Q_h \in [W_{h,0}]^2 \tag{21}$$

where $D_{-x}U_h = (D_{-x}u_h, D_{-x}v_h)$, and the inner product $(\cdot,\cdot)_h$ in $[W_{h,0}]^2$ is defined in the usual way. In (21), if $Q_h = (q_h, p_h)$, then $(D_{-x}U_h, D_{-x}Q_h)_+ = (D_{-x}u_h, D_{-x}q_h)_+ + (D_{-x}v_h, D_{-x}p_h)_+$.

The fully discrete variational problem is a fully discrete version of the variational problem:

find $U \in [H_0^1(\Omega)]^2$ such that

$$(\nabla U, \nabla Q) + (F(U, \nabla U), Q) = (G, Q),\ \forall Q \in [H_0^1(\Omega)]^2. \tag{22}$$

For $V_h = (v_{1,h}, v_{2,h}) \in [W_{h,0}]^2$ we use the following notation $\|D_{-x}V_h\|_+^2 = \|D_{-x}v_{1,h}\|_+^2 + \|D_{-x}v_{2,h}\|_+^2$.

**Theorem 2.** *Let us suppose that the sequence of grids* $\Lambda$ *satisfies* (13)*, the solution* $U = (u, v)$ *of* (22) *belongs to* $[H^3(\Omega) \cap H_0^1(\Omega)]^2$*,* $F$ *is a Lipschitz function with Lipschitz constant* $C_L$*, such that* $1 - 2(1 + \sqrt{2C_r})C_L > 0$ *and* $F(U, \nabla U) \in [H^2(\Omega)]^2$*. Let* $U_h = (u_h, v_h) \in [W_{h,0}]^2$ *be solution of the FDM* (19)-(20) *or, equivalently, of the fully discrete piecewise FEM* (21)*, and let* $E_h = U_h - R_hU$*. Then there exists a positive constant* $C$*,* $h$*-independent such that*

$$\|D_{-x}E_h\|_+^2 \leq C \sum_{i=1}^{N} h_i^4 \left( \|U\|^2_{[H^3(x_{i-1},x_i)]^2} + \|F(U, \nabla U)\|^2_{[H^2(x_{i-1},x_i)]^2} \right), \tag{23}$$

*for* $h \in \Lambda$.

*Proof*: The proof of this result follows the proof of Theorem 1.

Let $E_h = (E_u, E_v)$. We observe that for

$$Q_1 := (f_1(R_hU, \nabla_h R_hU) - f_1(U_h, \nabla_h U_h), E_u)_h$$

we have successively the following

$$\begin{aligned}
Q_1 &\leq C_L \left( \left( \|E_u\|_h + \sqrt{2C_r} \|D_{-x}E_u\|_+ + \|E_v\|_h + \sqrt{2C_r} \|D_{-x}E_v\|_+ \right) \|E_u\|_h \right) \\
&\leq C_L \left( (1+\sqrt{2C_r}) \|D_{-x}E_u\|_+^2 + (1+\sqrt{2C_r}) \|D_{-x}E_v\|_+ \|D_{-x}E_u\|_+ \right) \\
&\leq C_L \left( \frac{3}{2}(1+\sqrt{2C_r}) \|D_{-x}E_u\|_+^2 + \frac{1}{2}(1+\sqrt{2C_r}) \|D_{-x}E_v\|_+^2 \right).
\end{aligned}$$

Analogously, for $Q_2 := (f_2(R_hU, R_h\nabla U) - f_2(U_h, \nabla_h U_h), E_v)_h$ we easily get

$$Q_2 \leq C_L \left( \frac{1}{2}(1+\sqrt{2C_r}) \|D_{-x}E_u\|_+^2 + \frac{3}{2}(1+\sqrt{2C_r}) \|D_{-x}E_v\|_+^2 \right).$$

Then for $Q_1 + Q_2$ we deduce

$$Q_1 + Q_2 \leq 2C_L \left( 1 + \sqrt{2C_r} \right) \|D_{-x}E_h\|_+^2.$$

Consequently, following the proof of Theorem 1 we obtain

$$\begin{aligned}
&\left( 1 - \sum_{i=1}^{3} \epsilon_i^2 - 2(1+\sqrt{2C_r})C_L \right) \|D_{-x}E_u\|_+^2 \\
&\qquad + \left( 1 - \sum_{i=1}^{3} \eta_i^2 - 2(1+\sqrt{2C_r})C_L \right) \|D_{-x}E_v\|_+^2 \\
&\qquad \leq C \sum_{i=1}^{N} h_i^4 \left( \|U\|_{[H^3(x_{i-1},x_i)]^2}^2 + \|F(U,\nabla U)\|_{[H^2(x_{i-1},x_i)]^2}^2 \right).
\end{aligned}$$

If $1 - 2\left(1+\sqrt{2C_r}\right) C_L > 0$ then there exists a positive constant $C$ such that (23) holds.

■

**Corollary 3.** *Under the assumptions of Theorem 2, there exists a positive constant $C$, $h$-independent, such that*

$$\|E_h\|_{1,h} \leq C\, h_{max}^2.$$

Under the assumptions of Theorem 2, following the proof of Corollary 2 we easily prove the next result.

**Corollary 4.** *Let $U_h \in [W_{h,0}]^2$ be solution of the FDM* (19)-(20)*, or equivalently, of the fully discrete piecewise FEM* (21)*. Then, under the conditions of Theorem 2, there exists a positive constant $C$, $h-$independent, such that*

$$\|U_h\|_{h,\infty} \leq C,\ h \in \Lambda,$$

$$\|D_{-x}U_h\|_{h,\infty} \leq C,\ h \in \Lambda.$$

## 4. A nonlinear system of parabolic equations

To compute an approximation for the IBVP (1)-(3), we propose the following semi-discrete scheme

$$\frac{dU_h}{dt}(t) + F(U_h(t), \nabla_h U_h(t)) = A\Delta_h U_h(t) + G_h(t),\ t \in (0,T] \tag{24}$$

with boundary and initial conditions

$$U_h = 0 \ \text{ on } \ \partial\Omega_h \times (0,T],$$
$$U_h(0) = R_h U_0 \ \text{ in } \ \Omega_h. \tag{25}$$

We remark that the semi-discrete scheme is equivalent to the following fully discrete piecewise linear FEM:

find $U_h(t) \in [W_{h,0}]^2$ such that for all $t \in (0,T]$ and $Q_h \in [W_{h,0}]^2$

$$\left(\frac{dU_h}{dt}(t), Q_h\right)_h + (AD_{-x}U_h(t), D_{-x}Q_h)_+ + (F(U_h(t), \nabla_h U_h(t)), Q_h)_h = (G_h(t), Q_h)_h, \tag{26}$$

with

$$(U_h(0), Q_h)_h = (R_h U_0, Q_h)_h,\ \forall Q_h \in [W_{h,0}]^2. \tag{27}$$

This fully discrete method is obtained from the piecewise linear FEM:

find $U_h(t) \in [W_{h,0}]^2$ such that for all $t \in (0,T]$ and $Q_h \in [W_{h,0}]^2$

$$\left(\frac{dP_hU_h}{dt}(t), P_hQ_h\right) + (A\nabla P_hU_h(t), \nabla P_hQ_h) + (F(P_hU_h(t), \nabla P_hU_h), P_hQ_h) = (G(t), P_hQ_h),$$

with

$$(P_hU_h(0), P_hQ_h) = (U_0, P_hQ_h), \forall Q_h \in [W_{h,0}]^2$$

considering convenient quadrature rules.

**Theorem 3.** *Let us suppose that the sequence of grids* $\Lambda$ *satisfies* (13), *and* $U(t) \in [H^3(\Omega) \cap H_0^1(\Omega)]^2$, $U \in [C^1([0,T], C(\overline{\Omega}))]^2$, $F$ *is a Lipschitz function with Lipschitz constant* $C_L$, *and* $F(U(t), \nabla U(t)) \in [H^2(\Omega)]$. *Let* $U_h(t) \in [W_{h,0}]^2$ *be a solution of the initial value problem* (24)-(25) *or, equivalently, solution of the fully discrete piecewise linear FEM* (26)- (27) *that we suppose in* $[C^1([0,T], W_{h,0})]^2$. *Then there exists a positive constant* $C$, $h$ *and* $t$ *independent, such that for* $E_h = U_h - R_h U$ *we have*

$$\|E_h(t)\|_h^2 + D\int_0^t e^{S(t-s)}\|D_{-x}E_h(s)\|_+^2 ds \leq \int_0^t e^{S(t-s)} T_h(s) ds,\ t \in [0,T], \quad (28)$$

*where*

$$T_h(t) = C\sum_{i=1}^N h_i^4 \left( \left\| \frac{\partial U}{\partial t}(t) \right\|^2_{[H^2(x_{i-1},x_i)]^2} + \|U(t)\|^2_{[H^3(x_{i-1},x_i)]^2} \right.$$
$$\left. + \|F(U(t), \nabla U(t))\|^2_{[H^2(x_{i-1},x_i)]^2} \right),$$

*the coefficients* $D$ *and* $S$ *are given by*

$$D = 2\min\left\{ \alpha - \sum_{i=1}^5 \epsilon_i^2 - \eta_6^2, \beta - \sum_{i=1}^5 \eta_i^2 - \epsilon_6^2 \right\}, \quad (29)$$

*and*

$$S = C_L\left(4 + C_r \max\left\{ \frac{1}{\epsilon_5^2} + \frac{1}{\epsilon_6^2}, \frac{1}{\eta_5^2} + \frac{1}{\eta_6^2} \right\}\right), \quad (30)$$

*respectively, with* $\epsilon_i$ *and* $\eta_i$ *for* $i = 1, \ldots, 6$, *fixed such as*

$$\alpha - \sum_{i=1}^5 \epsilon_i^2 - \eta_6^2 > 0, \quad (31)$$

*and*

$$\beta - \sum_{i=1}^5 \eta_i^2 - \epsilon_6^2 > 0. \quad (32)$$

*Proof*: We start by remarking that it can be shown that

$$(g_{1,h}(t), E_u(t))_h = \left( R_h \frac{\partial u}{\partial t}(t), E_u(t) \right)_h + \left( \left( \frac{\partial u}{\partial t}(t) \right)_h - R_h \frac{\partial u}{\partial t}(t), E_u(t) \right)_h$$
$$+ \alpha(D_{-x}R_h u(t), D_{-x}E_u(t))_+$$

$$
\begin{aligned}
&+\alpha\left(\hat{R}_h\nabla u(t)-D_{-x}R_h u(t), D_{-x}E_u(t)\right)_+\\
&+(f_1(R_hU(t),\nabla_h R_hU(t)),E_u(t))_h\\
&+(f_{1,h}(t)-f_1(t),E_u(t))_h\\
&+(f_1(t)-(f_1(R_hU(t),\nabla_h R_hU(t)),E_u(t))_h,
\end{aligned}
$$

where, to simplify, the following notation was used

$$f_1(t)=f_1(R_hU(t),R_h\nabla U(t)),$$

$f_{1,h}(t)$ is defined by (8) with $g$ replaced by $f_1(t)$.

Taking this into account, from the first equation of (26) with $p_h=E_u(t)$ we easily get

$$
\begin{aligned}
&\left(\frac{dE_u}{dt}(t),E_u(t)\right)_h+\alpha(D_{-x}E_u(t),D_{-x}E_u(t))_+\\
&\quad=(f_1(R_hU(t),\nabla_h R_hU(t))-f_1(U_h(t),\nabla_h U_h(t)),E_u(t))_h+\sum_{i=1}^{4}T_h^{(i)}, \quad (33)
\end{aligned}
$$

where

$$
\begin{aligned}
T_h^{(1)}&=\left(\left(\frac{\partial u}{\partial t}\right)_h-R_h\frac{\partial u}{\partial t}(t),E_u(t)\right)_h,\\
T_h^{(2)}&=\alpha(\hat{R}_h\nabla u(t)-D_{-x}R_hu(t),D_{-x}E_u(t))_+,\\
T_h^{(3)}&=(f_{1,h}(t)-f_1(t),E_u(t))_h,\\
T_h^{(4)}&=(f_1(t)-f_1(R_hU(t),\nabla_h R_hU(t)),E_u(t))_h.
\end{aligned}
$$

It can be shown that there exist positive constants $C_i, i=1,\dots,4$, $h$ and $t$ independent, such that

$$
\begin{aligned}
|T_h^{(1)}|&\le C_1\left(\sum_{i=1}^{N}h_i^4\left\|\frac{\partial u}{\partial t}(t)\right\|^2_{H^2(x_{i-1},x_i)}\right)^{1/2}\|D_{-x}E_u(t)\|_+,\\
&\le\frac{C_1}{4\epsilon_1^2}\sum_{i=1}^{N}h_i^4\left\|\frac{\partial u}{\partial t}(t)\right\|^2_{H^2(x_{i-1},x_i)}+\epsilon_1^2\|D_{-x}E_u(t)\|_+^2
\end{aligned}
$$

$$
\begin{aligned}
|T_h^{(2)}| &\le C_2 \left( \sum_{i=1}^{N} h_i^4 \|u(t)\|^2_{H^3(x_{i-1},x_i)} \right)^{1/2} \|D_{-x} E_u(t)\|_+ \\
&\le \frac{C_2}{4\epsilon_2^2} \sum_{i=1}^{N} h_i^4 \|u(t)\|^2_{H^3(x_{i-1},x_i)} + \epsilon_2^2 \|D_{-x} E_u(t)\|_+^2,
\end{aligned}
$$

$$
\begin{aligned}
|T_h^{(3)}| &\le C_3 \left( \sum_{i=1}^{N} h_i^4 \|f_1(U(t), \nabla U(t))\|^2_{H^2(x_{i-1},x_i)} \right)^{1/2} \|D_{-x} E_u(t)\|_+ \\
&\le \frac{C_3}{4\epsilon_3^2} \sum_{i=1}^{N} h_i^4 \|f_1(U(t), \nabla U(t))\|^2_{H^2(x_{i-1},x_i)} + \epsilon_3^2 \|D_{-x} E_u(t)\|_+^2,
\end{aligned}
$$

$$
\begin{aligned}
|T_h^{(4)}| &\le C_4 \left( \sum_{i=1}^{N} h_i^4 \|U(t)\|^2_{[H^3(x_{i-1},x_i)]^2} \right)^{1/2} \|D_{-x} E_u(t)\|_+ \\
&\le \frac{C_4}{4\epsilon_4^2} \sum_{i=1}^{N} h_i^4 \|U(t)\|^2_{[H^3(x_{i-1},x_i)]^2} + \epsilon_4^2 \|D_{-x} E_u(t)\|_+^2
\end{aligned}
$$

and

$$
\begin{aligned}
&|(f_1(R_h U(t), \nabla_h R_h U(t)) - f_1(U_h(t), \nabla_h U_h(t)), E_u(t))_h| \\
&\qquad \le C_L \Big( \|E_u(t)\|_h + \|E_v(t)\|_h + \sqrt{2C_r} \, (\|D_{-x} E_u(t)\|_+ \\
&\qquad\quad + \|D_{-x} E_v(t)\|_+) \Big) \|E_u(t)\|_h \\
&\qquad \le C_L \left( \left( \frac{3}{2} + \frac{C_r}{2\epsilon_5^2} + \frac{C_r}{2\epsilon_6^2} \right) \|E_u(t)\|_h^2 + \frac{1}{2} \|E_v(t)\|_h^2 \right. \\
&\qquad\quad \left. + \left( \epsilon_5^2 \|D_{-x} E_u(t)\|_+^2 + \epsilon_6^2 \|D_{-x} E_v(t)\|_+^2 \right) \right)
\end{aligned}
$$

where $\epsilon_i \neq 0, i = 1, \ldots, 6$, are arbitrary positive constants.

Considering the previous estimates in (33) we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|E_u(t)\|_h^2 + \left(\alpha - \sum_{i=1}^{5}\epsilon_i^2\right)\|D_{-x}E_u(t)\|_+^2 \\
&\le C_L\left(\left(\frac{3}{2} + \frac{C_r}{2\epsilon_5^2} + \frac{C_r}{2\epsilon_6^2}\right)\|E_u(t)\|_h^2 + \frac{1}{2}\|E_v(t)\|_h^2 + \epsilon_6^2\|D_{-x}E_v(t)\|_+^2\right) \\
&\quad + T_{1,h}(t),
\end{aligned} \tag{34}
$$

where $T_{1,h}(t)$ is estimated as follows

$$
\begin{aligned}
T_{1,h}(t) \le \sum_{j=1}^{4}\frac{\tilde{C}_1}{4\epsilon_j^2}&\left(\sum_{i=1}^{N} h_i^4\left(\left\|\frac{\partial u}{\partial t}(t)\right\|_{H^2(x_{i-1},x_i)}^2 + \|U(t)\|_{[H^3(x_{i-1},x_i)]^2}^2\right.\right. \\
&\left.\left. + \|f_1(U(t),\nabla U(t))\|_{H^2(x_{i-1},x_i)}^2\right)\right),
\end{aligned}
$$

for some positive constant $\tilde{C}_1$, $h$ and $t$ independent.

Analogously, from the second equation of (26) with $q_h = E_v(t)$ we can establish

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|E_v(t)\|_h^2 + \left(\beta - \sum_{i=1}^{5}\eta_i^2\right)\|D_{-x}E_v(t)\|_+^2 \\
&\le C_L\left(\left(\frac{3}{2} + \frac{C_r}{2\eta_5^2} + \frac{C_r}{2\eta_6^2}\right)\|E_v(t)\|_h^2 + \frac{1}{2}\|E_u(t)\|_h^2 + \eta_6^2\|D_{-x}E_u(t)\|_+^2\right) \\
&\quad + T_{2,h}(t),
\end{aligned} \tag{35}
$$

with

$$
\begin{aligned}
T_{2,h}(t) \le \sum_{j=1}^{4}\frac{\tilde{C}_2}{4\eta_j^2}&\left(\sum_{i=1}^{N} h_i^4\left(\left\|\frac{\partial v}{\partial t}(t)\right\|_{H^2(x_{i-1},x_i)}^2 + \|U(t)\|_{[H^3(x_{i-1},x_i)]^2}^2\right.\right. \\
&\left.\left. + \|f_2(U(t),\nabla U(t))\|_{H^2(x_{i-1},x_i)}^2\right)\right),
\end{aligned}
$$

for some positive constant $\tilde{C}_2$, $h$ and $t$ independent.

Combining (34) and (35) we get the following differential inequality

$$\frac{d}{dt}\|E_h(t)\|_h^2 + 2\left(\alpha - \sum_{i=1}^{5}\epsilon_i^2 - 2\eta_6^2\right)\|D_{-x}E_u(t)\|_+^2$$
$$+ 2\left(\beta - \sum_{i=1}^{5}\eta_i^2 - 2\epsilon_6^2\right)\|D_{-x}E_u(t)\|_+^2$$
$$\leq C_L\left(4 + \frac{C_r}{\epsilon_5^2} + \frac{C_r}{\epsilon_6^2}\right)\|E_u(t)\|_h^2 + C_L\left(4 + \frac{C_r}{\eta_5^2} + \frac{C_r}{\eta_6^2}\right)\|E_v(t)\|_h^2$$
$$+ T_{1,h}(t) + T_{2,h}(t).$$

Fixing $\epsilon_i$ and $\eta_i, i = 1, \ldots, 6$, by (31) and (32), from (35) we get

$$\frac{d}{dt}\|E_h(t)\|_h^2 + D\|D_{-x}E_h(t)\|_+^2 \leq S\|E_h(t)\|_h^2 + T_{1,h}(t) + T_{2,h}(t),$$

where $D$ and $S$ are defined by (29) and (30). Taking into account the smoothness of $u(t)$ and $v(t)$ and of the semi-discrete approximations $u_h(t)$ and $v_h(t)$ we conclude (28).

■

**Corollary 5.** *Under the assumptions of Theorem 3, there exists a positive constant $C$ such that holds the following*

$$\|E_h\|_h^2 + \int_0^t \|D_{-x}E_h(s)\|_+^2 ds \leq Ch_{max}^4.$$

## 5. Numerical results

In what follows, we present some numerical examples to illustrate the main convergence results, Theorems 1, 2 and 3. These numerical experiments also allow to show the sharpness of the smoothness assumptions imposed in these results. For each experiment, we consider $\Omega = (0, 1)$, and random nonuniform grids for the spatial discretization.

**5.1. Elliptic equations.** As a first example we consider

$$f(x_1, x_2) = \cos(x_1) + \sin(x_2),\ x_1, x_2 \in \mathbb{R}$$

and $g$ such that the boundary value problem (4)-(5) has the solution

$$u(x) = (e^x - 1)(x - 1),\ x \in \overline{\Omega}.$$

In this case we have $u \in H^3(\Omega)$, and $f$ satisfying the assumptions of the Theorem 1.

For the second example we consider we take

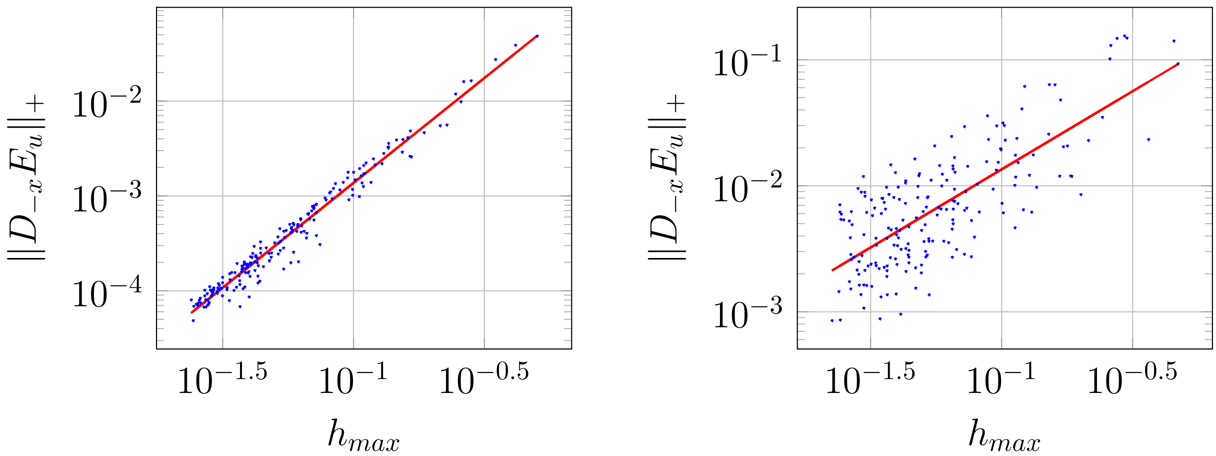
$$f(x_1, x_2) = \cos(x_1) + \sin(x_2),\ x_1, x_2 \in \mathbb{R}$$

and $g$ is such that

$$u(x) = |2x - 1|^{1.6} - 1,\ x \in \overline{\Omega},$$

is solution of the boundary value problem (4)-(5). In this case $u \in H^2(\Omega)$.

We report the discrete errors in these two situations in Figure 1. The slope




(a) $u \in H^3(\Omega)$ and the line has a slope of 2.21. (b) $u \in H^2(\Omega)$ and the line has a slope of 1.24.

FIGURE 1. Log-log plots of $\|D_{-x}E_u\|_+$ *versus* $h_{max}$ for the elliptic equation. The solid lines represent least-squares fittings.

of the line in the left-hand side of Figure 1 is 2.21, which confirms that if we assume the smoothness assumption specified in Theorem 1, i. e. $u \in H^3(\Omega)$, then we conclude a second-order convergence rate. The slope of the line in the plot in the right-hand side is 1.24, which shows that considering a weaker assumption of smoothness of the solution ($u \in H^2(\Omega)$), the convergence rate decreases.

**5.2. System of elliptic equations.** In the numerical experiments concerning the scheme (19)-(20), we replicate the previous procedure by exploring two different examples. In the first example we take $F = (f_1, f_2)$ defined by

$$f_1(x_1, x_2, x_3, x_4) = \cos(x_1) + \sin(x_2) + \cos(x_3) + \sin(x_4),\ x_1, x_2, x_3, x_4 \in \mathbb{R}$$

$$f_2(x_1, x_2, x_3, x_4) = x_1 + \sin(x_2) + x_3 + \sin(x_4),\ x_1, x_2, x_3, x_4 \in \mathbb{R}.$$

We take the function $G$ such that $U = (u, v)$ with

$$u(x) = (e^x - 1)(x - 1),\ x \in \overline{\Omega}, \tag{36}$$

$$v(x) = |2x - 1|^4 - 1,\ x \in \overline{\Omega},$$

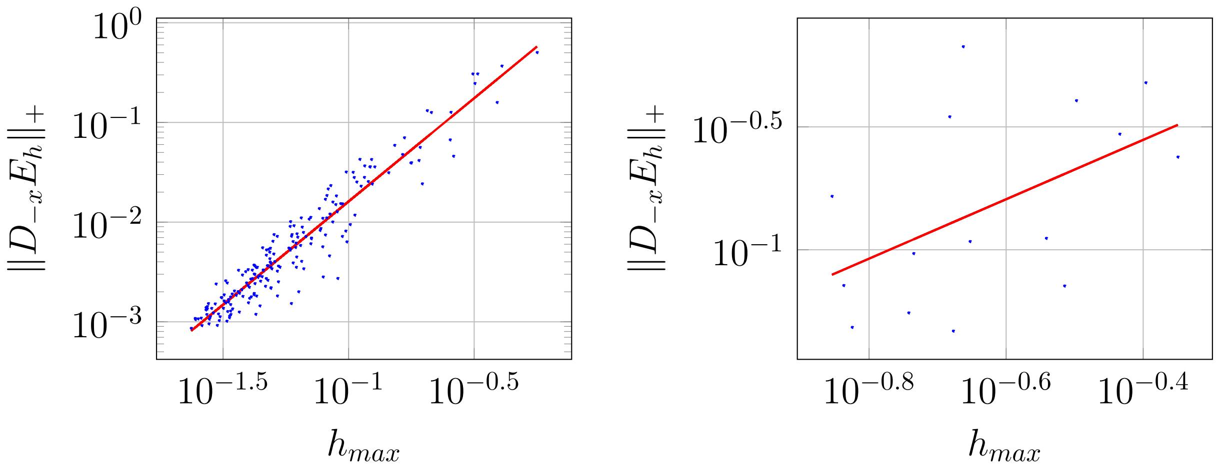
is solution of the boundary value problem (6)-(7). We observe that $U \in [H^3(\Omega)]^2$ and $F$ satisfies the assumptions of Theorem 2.

In the second example we take $F$ as before, $U = (u, v)$ with $u$ given by (36) and $v$ defined by

$$v(x) = |2x - 1|^{1.52} - 1,\ x \in \overline{\Omega}.$$

In this case we have $U \in [H^2(\Omega)]^2$.

Figure 2 illustrates the behaviour of the numerical method (19)-(20), in the last two scenarios: the results in the plot in the left-hand side were obtained with $U \in [H^3(\Omega)]^2$, and the estimated convergence rate is 2.07, while in the plot in the right-hand side we take $U \in [H^2(\Omega)]^2$, and the estimated convergence rate is 1.07. These results illustrate the sharpness of Theorem 2 smoothness assumptions on the solutions. In fact, we lose the second convergence rate when the solution is in $[H^2(\Omega)]^2$.




(a) $U \in [H^3(\Omega)]^2$ and the line has a slope of 2.07.

(b) $U \in [H^2(\Omega)]^2$ and the line has a slope of 1.07.

FIGURE 2. Plots of $\|D_{-x}E_h\|_+$ *versus* $h_{max}$ for the system of elliptic equations. The solid lines represent least-squares fittings.

**5.3. Systems of parabolic equations.** We now turn our attention to the numerical method (24)-(25). In the time domain $[0,T]$, we introduce the uniform grid $\{t_n = n\Delta t,\ n = 0,\dots,N_t\}$ with $t_{N_t} = T$ and $\Delta t$ denoting the step size.

We apply a first-order IMEX approach for the time discretization. The linear part is implicitly discretized while the nonlinear part is explicitly discretized. We denote by $u_h(t_n)$ and $v_h(t_n)$ the numerical approximations for $u(t_n)$ and $v(t_n)$, respectively. The fully discrete numerical scheme reads as

$$\frac{U_h^{n+1} - U_h^n}{\Delta t} - A\Delta_h U_h^{n+1} + F(U_h^n, \nabla_h U_h^n) = G_h^{n+1},\ n = 0,1,\dots,N_t - 1,$$

with initial conditions

$$U_h^0 = R_h U_0 \text{ in } \Omega_h,$$

and boundary conditions

$$U_h^n = 0 \text{ on } \partial\Omega_h,\ n = 0,...,N_t.$$

In order to estimate the convergence rate numerically, we define the error

$$\|E_h\|_\square = \sqrt{\|E_u\|_\square^2 + \|E_v\|_\square^2}$$

where

$$\|E_u\|_\square^2 = \max_{n=1,\dots,N_t} \left\{ \|E_u^n\|_h^2 + \Delta t \sum_{i=1}^{n} \|D_{-x} E_u^i\|_+^2 \right\}$$

and $\|E_v\|_\square$ is defined analogously.

We now set $\alpha = \beta = 0.1$ and define $F$ as

$$f_1(x_1,x_2,x_3,x_4) = \cos(x_1) + \sin(x_2) + \cos(x_3) + \sin(x_4),\ x_1,x_2,x_3,x_4 \in \mathbb{R}$$

and

$$f_2(x_1,x_2,x_3,x_4) = x_1 + \sin(x_2) + x_3 + \sin(x_4),\ x_1,x_2,x_3,x_4 \in \mathbb{R}.$$

Finally, we define the time step $\Delta t = 10^{-6}$, which is small enough so the first order error from the time discretization does not pollute the convergence rate wrt the space variable.

In the first example we consider $U = (u,v)$ defined by

$$u(x,t) = -e^{-t}(e^x - 1)(x - 1),$$

and

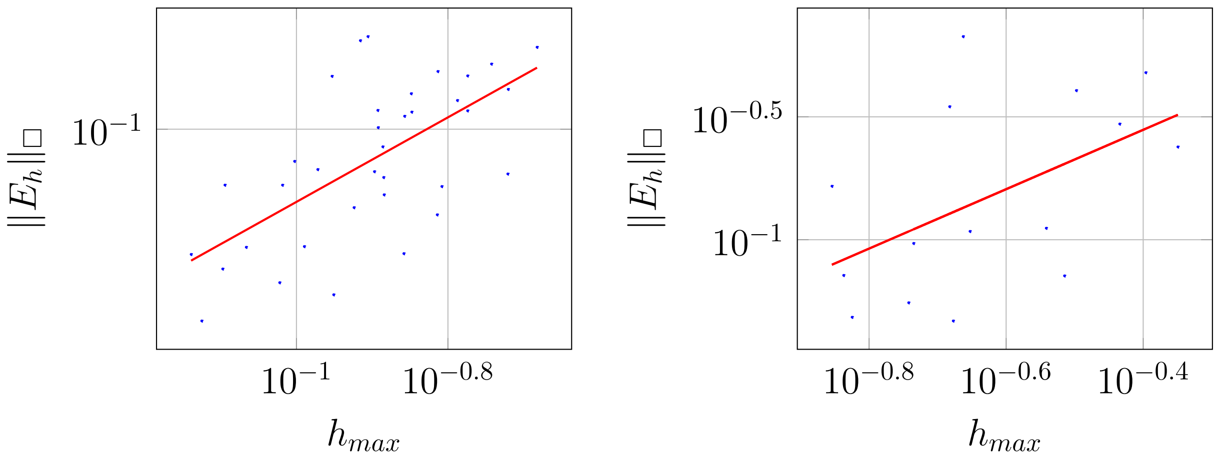
$$v(x,t) = e^{-t}((2x-1)^4 - 1),$$

for $x \in \overline{\Omega}$, $t \in [0,1]$ and determine $G$ such that $U$ is solution of the IBVP (1)-(3). In this case $U(t) \in [H^3(\Omega)]^2$ for $t \in [0,1]$ and $F$ satisfies the assumptions of Theorem 3.

In the second example we take $U$ with $u$ as before and $v$ given by

$$v(x,t) = e^{-t}|2x-1|^{1.52} - 1.$$

Figure 3 illustrates the error estimate in Theorem 3. The results included in the left figure were obtained with $U(t) \in [H^3(\Omega)]^2$. The slope of the linear regression, in this case, is 1.87 but it reduces to 1.2 when we take $v \in H^2(\Omega)$.




(a) $U(t) \in [H^3(\Omega)]^2$ and the line has a slope of 1.87.

(b) $U(t) \in [H^2(\Omega)]^2$ and the line has a slope of 1.21.

FIGURE 3. Log-log plots of $\|E_h\|_\square$ *versus* $h_{max}$ for the system of parabolic equations. The solid lines represent the least-squares fitting.

## 6. Conclusions

The main objective of the present paper is to propose numerical methods for nonlinear systems of parabolic equations (1)-(3) and to provide their convergence analysis. The proposed methods (24)-(25) can be seen as fully discrete piecewise linear finite element methods as well as finite difference methods. In the main result of this paper - Theorem 3 - we establish that if the solution $U(t)$ of (1)-(3) is in $[H^3(\Omega)]^2$, then the discrete $L^2$-norm of the error of the numerical solution defined by (24)-(25), and of its numerical gradient are second order convergent.

In the proof of the main result - Theorem 3 - we do not follow the usual approach, introduced by Mary Wheeler in [14], in the convergence analysis of finite element methods, and largely used in the literature. Basically, we use the error equation and the convergence analysis proposed in [2, 5]. Analogous results for systems of nonlinear elliptic equations (19)-(20), are also included in this work - Theorem 2.

We highlight that the authors wanted to extend some of their previously obtained results for numerical methods for elliptic linear equations (see [2, 5]), for an elliptic equation coupled with a parabolic equation in the scope of diffusion processes in porous media (see [1]), or for a parabolic equation coupled with another parabolic equation (see [8]).

Numerical results illustrating the convergence results are also included. These results show the sharpness of the smoothness assumptions that means that if the smoothness assumptions for the solutions of the differential problems are not satisfied, then the convergence rate is lost.

# References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