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A SECOND ORDER CONVERGENT METHOD FOR A MULTIPHYSICS MODEL OF ENHANCED DRUG TRANSPORT

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Abstract: Ultrasound enhanced drug transport is a multiphysics problem that involves acoustic waves propagation, bioheat transfer and drug transport. The numerical modeling of this problem requires the solution of a coupled system of partial differential equations. A wave-type equation for acoustic pressure and two nonlinear parabolic-type equations: a diffusion-reaction equation for bioheat transfer and a convection-diffusion-reaction equation for drug transport.

In this paper we focus on the numerical analysis of such coupled system. We propose and derive convergence estimates for a piecewise linear finite element method (FEM) with quadrature. We prove that the FEM is second order convergent for concentration with respect to a discrete L^2-norm. Since concentration depends on the gradient of acoustic pressure, this result shows that the FEM is superconvergent. In fact, piecewise linear FEM have optimal order one in the H^1-norm then, the optimal convergence rate for concentration in a L^2-norm should be at most one. Numerical results backing the theoretical findings are included.

Keywords: Multiphysics, ultrasound, temperature, drug transport, coupled PDEs, piecewise linear FEM, superconvergence.

1. Introduction

In this paper we are concerned with the numerical analysis of the following system of partial differential equations

$$a\frac{\partial^2 p}{\partial t^2} + b\frac{\partial p}{\partial t} = \nabla \cdot (E\nabla p) + f_3, \tag{1}$$

$$\frac{\partial T}{\partial t} = \nabla \cdot (D_T(T)\nabla T) + kT + f_2(p), \tag{2}$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (v(p, \nabla p)c) - \nabla \cdot (D_c(p, T)\nabla c) = f_1, \tag{3}$$

defined in $\Omega \times (0, T_f]$, where $\Omega \subset \mathrm{I\!R}^2$ is a bounded domain with boundary $\partial\Omega$ and $T_f > 0$ is a time duration. The differential system (1)-(3) is

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complemented with the initial conditions

$$p(0) = p_0, \frac{\partial p}{\partial t}(0) = p_{v,0} \text{ in } \Omega, \tag{4}$$

$$T(0) = T_0 \text{ in } \Omega, \tag{5}$$

$$c(0) = c_0 \text{ in } \Omega, \tag{6}$$

and for simplicity with homogeneous Dirichlet boundary conditions

$$p(t) = 0 \text{ on } \partial\Omega \times (0, T_f], \tag{7}$$

$$T(t) = 0 \text{ on } \partial\Omega \times (0, T_f], \tag{8}$$

$$c(t) = 0 \text{ on } \partial\Omega \times (0, T_f]. \tag{9}$$

The coupled initial boundary value problem (IBVP) (1)-(9) arises, for instance, in the modeling of ultrasound enhanced drug transport. In this context, ultrasound is applied to improve the transport of therapeutic drug agents, and has been used, among other applications, in transdermal drug delivery (see, e.g., [1], [2], [3], [4], [5]). Basically, from the physical point of view, ultrasound enhanced drug transport involves the propagation of acoustic waves through a biological target tissue. The propagation of these waves generates heat that dissipates through the tissue. The drug transport is enhanced by both these factors, temperature rise and propagation of acoustic waves. The acoustic waves propagation can be modeled by the wave-type equation (1), while bioheat transfer and drug transport can be modeled by the parabolic-type equations (2) and (3), respectively.

The exact mechanisms induced by the propagation of acoustic waves through the target tissue are not completely elucidated, but it is accepted that involves thermal and mechanical processes or combination of both. In the first case, acoustic waves lead to a temperature increase due to the absorption of acoustic energy by the target tissue. This fact has been used to control drug release from temperature-sensitive nanocarriers loaded with drug, and/or to enhance drug transport due to the increase of blood flow and permeability in the target tissue (see, e.g., [6] and its references). On the other hand, the mechanical processes are related with the so-called cavitation phenomenon, which involves the expansion and compression of endogenous or exogenous gas microbubbles that oscillate in the medium. If violent changes in the acoustic wave amplitude occur, the microbubbles collapse (inertial cavitation) generating shock waves that can lead to pores formation in the cellular membranes (see [7] and [4] and its references). When the bubbles oscillate without collapsing, the cavitation is called stable. The increase of tissue permeability due to cavitation is not completely understood (see [5] and its references). Nevertheless, it is established that the micro-scale phenomena associated with cavitation induce a convective and diffusive transport at a macro-scale (see, e.g., [8], [9], [10], [11]). For instance, in [11], a convection enhanced transport is generated applying an external ultrasound pressure gradient that induces a convective flow through a soft tissue.

Mathematically, the influence of acoustic waves in drug transport can therefore be translated into a convective velocity that depends on the acoustic pressure and on its gradient. This is given by the term $v(p, \nabla p)$ in equation (3). Where p represents acoustic pressure. The increase in permeability can be translated into a diffusion coefficient that depends on the acoustic pressure. As previously mentioned, an increase in permeability is also expected due to the temperature rise originated by the absorption of acoustic energy by the tissue. These two mechanisms are translated into the diffusion term $D_c(p, T)$ in equation (3). Here T represents temperature. The connection between temperature and acoustic pressure is translated into the reaction heat source term $f_2(p)$ in equation (2).

Modeling and simulation of ultrasound enhanced drug transport has been subject of research in the last years. For instance, in [12], the authors developed a numerical framework for the numerical simulation of thermal ablation of brain tumors with ultrasound. Thermal ablation by ultrasound is a noninvasive procedure that consists in heating the tumor tissue above a cytotoxic temperature. For the acoustic simulation, the authors used the linear acoustic pressure wave equation

$$\frac{1}{v_s^2}\frac{\partial^2 p}{\partial t^2} + \frac{\tilde{a}}{v_s^2}\frac{\partial p}{\partial t} = \rho \nabla .(\frac{1}{\rho}\nabla p), \tag{10}$$

and also, for comparison, the nonlinear Westervelt-Lighthill equation

$$\frac{1}{v_s^2}\frac{\partial^2 p}{\partial t^2} - \frac{\delta}{v_s^4}\frac{\partial^3 p}{\partial t^3} - \frac{\beta}{2\rho v_s^4}\frac{\partial^2 p}{\partial t^2} = \rho \nabla .(\frac{1}{\rho}\nabla p), \tag{11}$$

Where p is the acoustic pressure, ρ is the tissue density, v_s is the sound speed and $\tilde{a} = a\sqrt{\frac{a^4 v_s^4}{4\pi^2 f^2} + v_s^2}$, being a the material attenuation coefficient and f the wave frequency. In (11), δ is the acoustic diffusivity in a thermoviscous fluid and β is a nonlinear coefficient of the medium. For thermal modeling, equation (10), or (11), was coupled with the Pennes's bioheat equation

$$\rho K \frac{\partial T}{\partial t} = \nabla.(\hat{K}\nabla T) + \rho Q + \rho S - \rho_b c_b \rho \omega (T - T_b), \tag{12}$$

where T is the temperature, K is the specific heat capacity, $\hat{K}$ is the thermal conductivity, Q is the metabolic heat generation rate, ω is the perfusion rate, ρ_b, c_b and T_b are the density, specific heat capacity and temperature of the blood, and $S = a\dfrac{p^2}{\rho c}$. This study does not consider drug transport.

A multiphysics approach to describe ultrasound enhanced drug transport is presented in [13]. It covers drug release from thermosensitive nanocarriers (liposomes), drug transport and drug absorption by a solid tumor. Pennes's bioheat equations, similar to (12), are used to model temperature rise in the tumor, normal tissue and blood. The model is able to simulate the evolution of drug concentration in all the domain (tumor, normal tissue, blood and liposomes). The transport of free drug in the intersticial fluid and the drug effect on the tumor cells dynamics is also considered. Some limitations, are the fact that the drug diffusion coefficients are constant, i.e., temperature and acoustic pressure independent, and that some simplifications are made in order to obtain an analytical solution for the acoustic pressure.

Another study dealing with ultrasound enhanced drug transport is presented in [14]. In this study, the authors are concerned with the mathematical simulation of drug release from thermosensitive liposomes loaded with doxorubicin (a drug used to treat cancer). Using simulation, the authors concluded that controlled drug release by heating with ultrasound allows a significant increase in drug penetration into the tumor. For the simulation, the Penne's bioheat equation (12) is coupled with a nonlinear acoustic equation of the type (11). The drug concentration in the liposomes is governed by a convection-diffusion-reaction equation with a temperature dependent reaction term and a convective velocity given by Darcy's law. The released drug admits three states: free, cancer cell-bound and internalized in the tumor cells. The free drug concentration is described by a convection-diffusion-reaction equation with a convective velocity given again by Darcy's law. We note that the diffusion coefficients, as well as the convective velocity, are considered acoustic pressure independent. As it is often the case in simulation studies, a detailed convergence analysis of the numerical methods is not provided ([12], [13] and [14]).

The main contribution of this work is to provide convergence estimates for a piecewise linear FEM with quadrature that approximates the IBVP (1)-(9). Crucial points in the numerical discretization are the approximations of the temperature T, the acoustic pressure p and its gradient ∇p. The concentration equation (3) is linked to these quantities through the convective and diffusion terms. If such approximations are not properly handled the accuracy for concentration c can deteriorate. A piecewise linear FEM with quadrature for equations (1) and (3) was already analyzed by the authors in [15]. In that paper, we prove that the numerical approximation for c is second order convergent with respect to a discrete L^2-norm. The main motivation of the present paper is to extend that result for the IBVP (1)-(9).

The convergence analysis followed here is based on the approach introduced in [16] and [17] for one-dimensional and two-dimensional linear elliptic problems, respectively. That approach relies heavily on the Bramble-Hilbert Lemma ([18]), which allows the replacement of regularity assumption on the exact solution from the space of continuous functions $C^4(\overline{\Omega})$ to the less restrictive Sobolev space $H^3(\Omega)$. That approach was used, for instance, in [19] to study the numerical discretization of a coupled elliptic and integro-differential system, and in [20] for a coupled system of parabolic equations.

The rest of the paper is organized as follows. In Section 2, we give a weak solution to the IBVP (1)-(9). In Section 3, we present our piecewise linear FEM with quadrature. An equivalent formulation as a finite difference method (FDM) is also given. In Section 4, we study the convergence of our method and in Section 5 we present some numerical results that illustrate the theoretical findings. The sharpness of the convergence results, regarding the regularity assumptions on the solution of the IBVP (1)-(9), is also numerically investigated. At last, in Section 6, some conclusions are drawn.

2. Weak solution

In this section we introduce some notations and definitions and present a weak solution to the IBVP (1)-(9). In the following, without loss of generality we set $\Omega = [0,1]^2$. Also, if $w : \overline{\Omega} \times [0, T_f] \to \mathbb{R}$, then for $t \in (0, T_f]$, $w(t) : \overline{\Omega} \times [0, T_f] \to \mathbb{R}$ is given by $w(t)(x, y) = w(x, y, t), (x, y) \in \overline{\Omega}$.

In equation (1), E is a second order diagonal matrix with positive entries $e_i : \overline{\Omega} \to \mathbb{R}, i = 1, 2$, with a positive lower bound e_0 in $\overline{\Omega}$. For the constants a and b it holds that $a \geq a_0 > 0$ and $b \geq b_0 > 0$ in $\overline{\Omega}$. In equation (2), D_T is a second order diagonal matrix with positive entries $d_{T,i} : \mathbb{R} \to \mathbb{R}, i = 1, 2,$

with a positive lower bound β_0 in $\mathbb{R}$. In equation (3), $D_c(p,T)$ is a second order diagonal matrix with entries $d_{c,i} : \mathbb{R}^2 \to \mathbb{R}$ with a positive lower bound β_1 in $\mathbb{R}^2$. In equation (3), we assume that the convective velocity $v : \mathbb{R}^3 \to \mathbb{R}^2$ is such that its components $v_i : \mathbb{R}^2 \to \mathbb{R}$ are given by $v(x,y,z) = (v_1(x,y), v_2(x,z)), x, y, z \in \mathbb{R}$.

By $H^n(\Omega)$, $n \in \mathbb{N}_0$, we denote the classical Sobolev spaces equipped with the norm

$$\|w\|_{H^n(\Omega)} = \left(\sum_{|\alpha|\leq n} \|D^\alpha w\|^2\right)^{1/2}, w \in H^n(\Omega),$$

where $D^\alpha w = \dfrac{\partial^{|\alpha|} w}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, with $\alpha = (\alpha_1, \alpha_2)$. When $n = 0$ we set $H^0(\Omega) = L^2(\Omega)$ and we denote the usual inner product by $(.,.)$. In $[L^2(\Omega)]^2$, the usual inner product is denoted by $((.,.))$. We represent by $H^1_0(\Omega)$ the subspace of $H^1(\Omega)$ with null trace on $\partial\Omega$. We observe that this space can be given equivalently by the closure of $C_0^\infty(\Omega)$ with respect the norm $\|.\|_{H^1(\Omega)}$.

By $C^m([0,T_f], H^n(\Omega))$, we denote the space of functions $v : [0,T_f] \to H^n(\Omega)$ such that $v^{(j)} : [0,T_f] \to H^n(\Omega)$, $j = 0, \dots, m$, are continuous and

$$\|v\|_{C^m(H^n(\Omega))} = \|v\|_{C^m([0,T_f],H^n(\Omega))} = \max_{j=0,\dots,m} \left\|v^{(j)}(t)\right\|_{H^n(\Omega)} < +\infty.$$

We also consider the space $H^m(0,T_f,H^n(\Omega))$ of functions $v : (0,T_f) \to H^n(\Omega)$ with weak derivatives $v^{(j)} : (0,T_f) \to H^n(\Omega)$, $j = 0, \dots, m$, such that

$$\|v\|_{H^m(H^n(\Omega))} = \|v\|_{H^m((0,T_f),H^n(\Omega))} = \left(\sum_{j=0}^{m} \int_0^{T_f} \left\|v^{(j)}(t)\right\|^2_{H^n(\Omega)} dt\right)^{1/2} < +\infty.$$

Let us consider the following assumptions:

[A_1] f_2 is a L_{f_2}-Lipschitz function ,
[A_2] $|v_i(z_1,z_2)| \leq C_v\left(|z_1| + |z_2|\right), \forall z_1, z_2 \in \mathbb{R}, i = 1, 2,$
[A_3] $v_i, i = 1, 2,$ are L_v-Lipschitz functions,
[A_4] $d_{T,i}, i = 1, 2,$ are L_{D_T}-Lipschitz functions and $d_{T,i} \geq \beta_0 > 0$ in $\mathbb{R}$,
[A_5] $d_{c,i}, i = 1, 2,$ are L_{D_c} -Lipschitz functions and $d_{c,i} \geq \beta_1 > 0$ in $\mathbb{R}^2$.

The weak solution for the IBVP (1)-(9) is given by the triplet $(p(t), T(t), c(t)) \in [H^1_0(\Omega)]^3$ satisfying the following:

i) $p(t) \in H_0^1(\Omega)$, $p^{(j)}(t) \in L^2(\Omega), j = 1, 2$, $t \in (0, T_f]$, and

$$(ap''(t), w) + (bp'(t), w) = -((E\nabla p(t), \nabla w)) + (f_3(t), w),$$

for $w \in H_0^1(\Omega)$, $t \in (0, T_f]$, and

$$\begin{cases} (p'(0), w) = (p_{v,0}, w), \ \forall w \in L^2(\Omega) \\ (p(0), q) = (p_0, q), \ \forall q \in L^2(\Omega); \end{cases}$$

ii) $T(t) \in H_0^1(\Omega)$, $T'(t) \in L^2(\Omega)$, $t \in (0, T_f]$, and

$$(T'(t), w)) = -((D_T(T(t))\nabla T(t), \nabla w)) + k(T(t), w) + (f_2(p(t)), w), \quad (13)$$

for $w \in H_0^1(\Omega)$, $t \in (0, T_f]$, and

$$(T(0), q) = (T_0, q), \ \forall q \in L^2(\Omega); \quad (14)$$

iii) $c(t) \in H_0^1(\Omega)$, $c'(t) \in L^2(\Omega)$, $t \in (0, T_f]$, and

$$\begin{aligned} (c'(t), w) - ((c(t)v(p(t), \nabla p(t)), \nabla w)) \ &= -((D_c(p(t), T(t))\nabla c(t), \nabla w))) \\ &\quad +(f_1(t), w), \end{aligned} \quad (15)$$

for $w \in H_0^1(\Omega)$, $t \in (0, T_f]$, and

$$(c(0), q) = (c_0, q), \ \forall q \in L^2(\Omega). \quad (16)$$

3. Fully discrete in space piecewise linear FEM

In this section, we present the fully discrete in space piecewise linear FEM to approximate the solution $(p(t), T(t), c(t))$ of the IBVP (1)-(9). The FEM is define over a special family of triangulations. Such triangulations are associated with non-uniform rectangular partitions of Ω and, consequently, they are not required to be quasi-uniform. First, we present some notations and definitions.

3.1. Notations and basic definitions. In $\overline{\Omega}$ we introduce a non-uniform rectangular grid defined by $H = (h, k)$ with $h = (h_1, \dots, h_N)$, $h_i > 0$, $i = 1, \dots, N$, $\sum_{i=1}^{N} h_i = 1$, and $k = (k_1, \dots, k_M)$, $k_j > 0$, $j = 1, \dots, M$, $\sum_{j=1}^{M} k_j = 1$. Let $\{x_i\}$ and $\{y_j\}$ be the non-uniform grids induced by h and k in $[0, 1]$ with $x_i - x_{i-1} = h_i$ and $y_j - y_{j-1} = k_j$. We represent by $\overline{\Omega}_H$ the rectangular grid introduced in $\overline{\Omega}$ that depends on H and let Ω_H and $\partial\Omega_H$ be defined by $\Omega_H = \Omega \cap \overline{\Omega}_H$ and $\partial\Omega_H = \partial\Omega \cap \overline{\Omega}_H$, respectively.

Let $H_{max} = \max\{h_i, k_j;\ i = 1, \cdots N;\ j = 1, \cdots, M\}$. We denote by Λ a sequence of vectors $H = (h, k)$ such that $H_{max} \to 0$. Let W_H be the space of grid functions defined in $\overline{\Omega}_H$ with $W_{H,0} = \{w_H \in W_H : w_H = 0 \text{ on } \partial\Omega_H\}$. Let $\mathcal{T}_H$ be a triangulation of $\overline{\Omega}$ defined using the set $\overline{\Omega}_H$ as vertices. We denote by diamΔ the diameter of the triangle $\Delta \in \mathcal{T}_H$. We represent by $P_H v_H$ the continuous piecewise linear interpolant of $v_H \in W_h$ with respect to the partition $\mathcal{T}_H$.

Next, we define fully discrete inner products and the corresponding norms. In $W_{H,0}$ we define the inner product

$$(u_H, w_H)_H = \sum_{(x_i,y_j)\in\overline{\Omega}_H} |\square_{i,j}| u_H(x_i, y_j) w_H(x_i, y_j),\ u_H, w_H \in W_{H,0},$$

where $\square_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}) \cap \Omega$, $|\square_{i,j}|$ denotes the area of $\square_{i,j}$, and $x_{i+1/2} = x_i + \dfrac{h_{i+1}}{2}$, $x_{i-1/2} = x_i - \dfrac{h_i}{2}$, $h_{i+1/2} = x_{i+1/2} - x_{i-1/2}$ being $y_{j\pm 1/2}$ and $k_{j+1/2}$ defined analogously. Let $\|.\|_H$ be the corresponding norm.

For $u_H = (u_{1,H}, u_{2,H})$, $w_H = (w_{1,H}, w_{2,H})$, and $u_{\ell,H}, w_{\ell,H} \in W_H$, for $\ell = 1, 2$, we use the notation

$$((u_H, w_H))_H = (u_{1,H}, w_{1,H})_{H,x} + (u_{2,H}, w_{2,H})_{H,y},$$

where

$$(u_{1,H}, w_{1,H})_{H,x} = \sum_{i=1}^{N} \sum_{j=1}^{M-1} h_i k_{j+1/2} u_{1,H}(x_i, y_j) w_{1,H}(x_i, y_j),$$

being $(u_{2,H}, w_{2,H})_{H,y}$ defined analogously.

Let D_{-x} and D_{-y} be the first-order backward finite difference operators with respect to the variables x and y, respectively, and let ∇_H be the discrete version of the gradient operator ∇ defined by $\nabla_H u_H = (D_{-x}u_H, D_{-y}u_H)$. We use the following notation

$$\begin{aligned}\|\nabla_H u_H\|_H &= \Big((D_{-x}u_H, D_{-x}u_H)_{H,x} + (D_{-y}u_H, D_{-y}u_H)_{H,y}\Big)^{1/2}\\ &= \Big(\|D_{-x}u_H\|_H^2 + \|D_{-y}u_H\|_H^2\Big)^{1/2},\ \ u_H \in W_H.\end{aligned}$$

Moreover, a straightforward calculation shows that the following Poincaré-Friedrichs inequality holds: there exists a positive constant C, independent of H, such that

$$\|u_H\|_H \le C\|\nabla_H u_H\|_H,\ \forall u_H \in W_{H,0}. \tag{17}$$

3.2. Fully discrete in space FEM. We are now in position to present our numerical scheme. We split the presentation into three steps, one for each equation of the IBVP (1)-(9).

The piecewise linear FEM for the wave IBVP (1), (4), (7) is defined as follows: find $p_H(t) \in W_{H,0}$ such that $P_H p_H(t)$ satisfies

$$\begin{aligned}(aP_H p''_H(t), P_H w_H) + (bP_H p'_H(t), P_H w_H) \ &= -((E\nabla P_H p_H(t), \nabla P_H w_H)) \\ &\quad +(f_3(t), P_H w_H),\end{aligned} \tag{18}$$

for $w_H \in W_{H,0}$, $t \in (0, T_f]$, and

$$\begin{cases} (P_H p'_H(0), P_H w_H) = (P_H R_H p_{v,0}, P_H w_H),\ \forall w_H \in W_{H,0}, \\ (P_H p_H(0), P_H q_H) = (P_H R_H p_0, P_H q_H),\ \forall q_H \in W_{H,0}. \end{cases} \tag{19}$$

We denote by $R_H : C(\overline{\Omega}) \to W_H$ the restriction operator from the space of continuous functions in $\overline{\Omega}$ to the space of grid functions W_H.

Considering adequate quadrature rules to approximate the integrals in (18), we replace (18), (19) by the following fully discrete in space FEM: find $p_H(t) \in W_{H,0}$ such that

$$\begin{aligned}(a_H p''_H(t), w_H)_H + (b_H p'_H(t), w_H)_H \ &= -((E_H\nabla_H p_H(t), \nabla_H w_H))_H \\ &\quad +(f_{3,H}(t), w_H)_H,\end{aligned} \tag{20}$$

for $w_H \in W_{H,0}$, $t \in (0, T_f]$, and

$$\begin{cases} (p'_H(0), w_H)_H = (R_H p_{v,0}, w_H)_H,\ \forall w_H \in W_{H,0}, \\ (p_H(0), q_H)_H = (R_H p_0, q_H)_H,\ \forall q_H \in W_{H,0}. \end{cases} \tag{21}$$

In (20), E_H represents the diagonal matrix with entries $e_{1,H}(x_i, y_j) = e_1(x_{i-1/2}, y_j)$ and $e_{2,H}(x_i, y_j) = e_2(x_i, y_{j-1/2})$, a_H and b_H are defined by $a_H = R_H a$ and $b_H = R_H b$, and

$$f_{3,H}(t)(x_i, y_j) = \frac{1}{|\square_{i,j}|}\int_{\square_{i,j}} f_3(x, y, t)dxdy. \tag{22}$$

It can be shown that the FEM (20), (21) is equivalent to the FDM

$$a_H p''_H(t) + b_H p'_H(t) = \nabla^*_H \cdot (E_H\nabla_H p_H(t)) + f_{3,H}(t) \text{ in } \Omega_H,\ t \in (0, T_f], \tag{23}$$

with the initial conditions

$$\begin{cases} p'_H(0) = R_H p_{v,0}, \\ p_H(0) = R_H p_0, \end{cases} \tag{24}$$

and boundary condition

$$p_H(t) = 0 \text{ on } \partial\Omega_H \times (0, T_f]. \tag{25}$$

In (23), ∇_H^* represents the finite difference operator $\left(D_x^-, D_y^-\right)$, with

$$D_x^- v_H(x_i, y_j) = \frac{v_H(x_{i+1}, y_j) - v_H(x_i, y_j)}{h_{i+1/2}},$$

and D_y^- is defined analogously (see [15]).

The piecewise linear FEM for the parabolic IBVP (13), (14) is defined as follows: find $T_H(t) \in W_{H,0}$ such that

$$\begin{aligned} (P_H T_H'(t), P_H w_H) \; &= -((D_T(P_H T_H(t))\nabla P_H T_H(t), \nabla P_H w_H)) \\ &+ k(P_H T_H(t), P_H w_H) + (f_2(P_H p_H(t)), P_H w_H), \end{aligned} \tag{26}$$

for $w_H \in W_{H,0}$, $t \in (0, T_f]$, and

$$(P_H T_H(0), P_H q_H) = (P_H R_H T_0, P_H q_H), \; \forall q_H \in W_{H,0}. \tag{27}$$

Let us set $D_{T_H}(t)$ as $D_T(M_H T_H(t))$, where M_H represents the average operator $M_H(w_1, w_2) = (M_h w_1, M_k w_2)$, for $(w_1, w_2) \in [W_{H,0}]^2$, with

$$M_h(w_1(x_i, y_j)) = \frac{1}{2}\left(w_1(x_{i-1}, y_j) + w_1(x_i, y_j)\right), \tag{28}$$

being M_k defined analogously. Then, we replace (26), (27) by the following fully discrete in space FEM: find $T_H(t) \in W_{H,0}$ such that

$$\begin{aligned} (T_H'(t), w_H)_H \; &= -((D_{T,H}(t)\nabla_H T_H(t), \nabla_H w_H))_H \\ &+ k(T_H(t), w_H)_H + (f_2(p_H(t)), w_H)_H, \end{aligned} \tag{29}$$

for $w_H \in W_{H,0}$, $t \in (0, T_f]$, and

$$(T_H(0), q_H)_H = (R_H T_0, q_H)_H, \; \forall q_H \in W_{H,0}. \tag{30}$$

In (29), the diffusion tensor $D_{T,H}(t)$ is a 2-diagonal matrix with entries $d_{T,1}(M_h T_H(t))$ and $d_{T,2}(M_k T_H(t))$.

The FEM (29), (30) can also be seen as the following FDM

$$T_H'(t) = \nabla_H^* \cdot (D_{T,H}(t)\nabla_H T_H(t)) + kT_H(t) + f_2(p_H(t)) \text{ in } \Omega_H, t \in (0, T_f], \tag{31}$$

with the initial condition

$$T_H(0) = R_H T_0, \tag{32}$$

and the boundary condition

$$T_H(t) = 0 \text{ on } \partial\Omega_H. \tag{33}$$

At last, the piecewise linear FEM for the parabolic IBVP (15), (16) is defined as follows: find $c_H(t) \in W_{H,0}$ such that

$$\begin{aligned}(P_H c_H'(t), P_H w_H) \ & -((P_H c_H(t) v_H(t), \nabla P_H w_H)) \\ & = -((D_c(P_H p_H(t), P_H T_H(t))\nabla P_H c_H(t), \nabla P_H w_H)) \\ & \quad +(f_1(t), P_H w_H),\end{aligned} \tag{34}$$

for $w_H \in W_{H,0}$, $t \in (0, T_f]$, and

$$(P_H c_H(0), P_H q_H) = (P_H R_H c_0, P_H q_H),\ \forall q_H \in W_{H,0}. \tag{35}$$

Here, we get the following fully discrete in space FEM: find $c_H(t) \in W_{H,0}$ such that

$$\begin{aligned}(c_H'(t), w_H)_H \ & -((M_H(c_H(t) v_H(t)), \nabla_H w_H))_H \\ & = -((D_{c,H}(t)\nabla_H c_H(t), \nabla_H w_H))_H + (f_{1,H}(t), w_H)_H,\end{aligned} \tag{36}$$

for $w_H \in W_{H,0}$, $t \in (0, T_f]$, and

$$(c_H(0), q_H)_H = (R_H c_0, q_H)_H,\ \forall q_H \in W_{H,0}. \tag{37}$$

In (36), the convective velocity $v_H(t)$ is defined by

$$v_H(t) = (v_1(p_H(t), D_h^* p_H(t)), v_2(p_H(t), D_k^* p_H(t)))$$

with

$$\begin{aligned} D_h^* w_H(x_i, y_j) &= \frac{h_i D_{-x} w_H(x_{i+1}, y_j) + h_{i+1} D_{-x} w_H(x_i, y_j)}{h_i + h_{i+1}}, i = 1, \dots, N-1, \\ D_h^* w_H(x_N, y_j) &= D_{-x} w_H(x_N, y_j), \\ D_h^* w_H(x_0, y_j) &= D_{-x} w_H(x_1, y_j), \end{aligned}$$

for $j = 1, \dots, M-1$, and with D_k^* defined analogously. The diffusion tensor $D_{c,H}(t)$ is a 2-diagonal matrix with entries $d_{c,1}(M_h p_H(t), M_h T_H(t))$ and $d_{c,2}(M_k p_H(t), M_k T_H(t))$.

The FEM (36), (37) is equivalent to the FDM

$$c_H'(t) + \nabla_{c,H} \cdot (c_H(t) v_H(t)) = \nabla_H^* \cdot (D_{c,H}(t) \nabla_H c_H(t)) + f_{1,H}(t) \text{ in } \Omega_H, t \in (0, T_f], \tag{38}$$

with

$$c_H(0) = R_H c_0 \text{ in } \Omega_H, \tag{39}$$

$$c_H(t) = 0 \text{ on } \partial\Omega_H. \tag{40}$$

In (38), the finite difference operator $\nabla_{c,H}$ is defined by $\nabla_{c,H} \cdot (w_1, w_2) = D_{c,x} w_1 + D_{c,y} w_2$, with

$$D_{c,x} w_1(x_i, y_j) = \frac{w_1(x_{i+1}, y_j) - w_1(x_{i-1}, y_j)}{h_i + h_{i+1}},$$

for $(w_1, w_2) \in [W_{H,0}]^2$. $D_{c,y}$ is defined analogously.

4. Convergence analysis

This is main section of the paper, and we will discuss the convergence properties of our numerical method. We start by highlighting the relevance of the theoretical findings.

Our fully discrete piecewise FEM for the IBVP (1)-(9), is obtained coupling the IBVPs:

- (20), (21) to approximate the acoustic pressure;
- (29), (30) to approximate the temperature;
- (36), (37) to approximate the concentration.

It is well known that continuous piecewise linear FEM lead to second order approximations, with respect to the L^2-norm, and to first order approximations with respect to the H^1-norm. We recall that in our IBVP (1)-(9) the convective velocity in the concentration equation depends on ∇p. Then, it should be expected that a continuous piecewise linear FEM for (1)-(9), would lead at most to a first order approximation for the concentration with respect to the L^2-norm. Naturally, this result should stand for the proposed fully discrete piecewise linear FEM.

However, we will show that the concentration approximation $c_H(t)$ is second order convergent for $c(t)$ with respect to a discrete L^2-norm, assuming that $p(t)$, $T(t)$ and $c(t)$ belong to $H^3(\Omega) \cap H^1_0(\Omega)$. To prove this result, we will rely on the fact the acoustic pressure approximation $p_H(t)$ is second order convergent for $p(t)$ with respect to a discrete H^1-norm. This means that its discrete gradient is a second order approximation for $\nabla p(t)$ with respect to a discrete L^2-norm. We will use the fact that the temperature approximation $T_H(t)$ is second order convergent for $T(t)$ with respect to a discrete L^2-norm. This higher than optimal convergence rate is known as superconvergence in the finite element context.

Since our FEM can be seen as a FDM, let us analyze this superconvergence phenomena from the finite difference perspective. The equivalent FDM for the IBVP (1)-(9), is obtained coupling the IBVPs (23)-(25), (31)-(33) and

(38)-(40). The FDM is defined on a non uniform rectangular grid $\overline{\Omega}_H$. Assuming that $p(t)$, $T(t)$ and $c(t)$ belong to $C^3(\overline{\Omega})$, it can be shown that the truncation error is first order for all variables with respect to the maximum norm $\|.\|_\infty$. Then, based on stability and consistency, we can expect the global error in space for $c_H(t)$ to be at most of first order. Nevertheless, taking into account the equivalence between FEM and FDM, we will be able to conclude that the finite difference approximation for $c_H(t)$ is second order with respect to a discrete L^2-norm, provided that $p(t), T(t), c(t) \in H^3(\Omega)$. This phenomena, where the global error is higher than the truncation error, is known as supraconvergence in the context of finite difference.

The convergence estimates are derived next. For ease of presentation, the analysis is divide into three subsections, namely, error analysis for the acoustic pressure, temperature and concentration equations, respectively.

4.1. Acoustic pressure. Let us define $e_{H,p}(t) = R_H p(t) - p_H(t)$ as the spatial discretization error associated with the FEM (20), (21) for the acoustic pressure. It holds the following upper bound for $e_{H,p}$.

Theorem 1 (Theorem 2 of [21]). *If the solution $p(t)$ of the IBVP (1), (4), (7) belongs to $H^3(0, T_f, H^2(\Omega)) \cap H^1(0, T_f, H^3(\Omega) \cap H^1_0(\Omega))$, and $a, b, e_i, i = 1, 2, \in W^{2,\infty}(\Omega)$, then, there exist positive constants $C_i, i = 1, 2,$ independent of $p(t)$, H and t such that, for $H \in \Lambda$, we have*

$$\begin{aligned}\|e'_{H,p}(t)\|^2_H &+ \int_0^t \|e'_{H,p}(s)\|^2_H ds + \|\nabla_H e_{H,p}(t)\|^2_H \\ &\le C_1 e^{C_2 t} \sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^4 \Big(\|p\|^2_{H^1(H^3)} + \|p\|^2_{H^3(H^2)}\Big),\ t \in [0, T_f].\end{aligned}$$

Using Theorem 1 and (17), we conclude that there exists a constant C such that

$$\|e_{H,p}(t)\|_H + \|\nabla_H e_{H,p}(t)\|_H \le C H^2_{max}. \tag{41}$$

The boundness of the sequences $(\|p_H(t)\|_\infty)_{H\in\Lambda}$ and $(\|\nabla_H p_H(t)\|_\infty)_{H\in\Lambda}$, where

$$\|p_H(t)\|_\infty = \max_{(x,y)\in\overline{\Omega}_H} |p_H(x, y, t)|, \tag{42}$$

$$\begin{aligned}\|\nabla_H p_H(t)\|_\infty &= \max_{i=1,\ldots,N, j=1,\ldots,M-1} |D_{-x} p_H(x_i, y_j, t)| \\ &\quad + \max_{i=1,\ldots,N-1, j=1,\ldots,M} |D_{-y} p_H(x_i, y_j, t)|,\end{aligned} \tag{43}$$

will also play an important role in the error analysis of the concentration equation. In what follows we prove such boundness under weaker assumptions than those considered in [15]. For the spatial grids $\overline{\Omega}_H, H \in \Lambda$, we impose that, for H_{max} small enough, there exists a positive constant C_m such that

$$\frac{H_{\max}}{H_{\min}} \leq C_m, H \in \Lambda. \tag{44}$$

We start by nothing that

$$\|p_H(t)\|_\infty^2 \leq 2\frac{1}{H_{min}^2}\|e_{H,p}(t)\|_H^2 + 2\|R_H p(t)\|_\infty^2.$$

Then, from (41), we get

$$\|p_H(t)\|_\infty^2 \leq C\frac{H_{max}^4}{H_{min}^2} + 2\|p(t)\|_\infty^2, H \in \Lambda,$$

and we derive the boundness of (42) by the fact that $p(t) \in C(\overline{\Omega})$ when $p(t) \in H^3(\Omega) \cap H_0^1(\Omega)$.

To prove the boundness of (43), we observe that we have

$$\begin{aligned}\|\nabla_H p_H(t)\|_\infty^2 &\leq 2\frac{1}{H_{min}^2}\|\nabla_H e_{H,p}(t)\|_H^2 + 2\|\nabla_H R_H p(t)\|_\infty^2 \\ &\leq C\frac{H_{max}^4}{H_{min}^2} + 2\|\nabla_H R_H p(t)\|_\infty^2 \\ &\leq C\frac{H_{max}^4}{H_{min}^2} + 2\|\nabla p(t)\|_\infty^2,\end{aligned}$$

where C denotes a positive constant, H, $p(t)$ and t independent. An upper bound to (43) follows from the fact that $p(t) \in C^1(\overline{\Omega})$ when $p(t) \in H^3(\Omega)$.

Corollary 1. *Under the assumptions of Theorem 1, if the sequence of stepsizes Λ satisfies (44) then, there exists a positive constant C such that*

$$\|p_H(t)\|_\infty \leq C \quad \textit{and} \quad \|\nabla_H p_H(t)\|_\infty \leq C,$$

for $H \in \Lambda$ with H_{max} small enough.

4.2. Temperature. Let us define $e_{H,T}(t) = R_H T(t) - T_H(t)$ as the spatial error associated with the FEM (29), (32) for the temperature. To derive an upper bound for $e_{H,T}$, we do not follow the classical error analysis method introduced by Wheeler in [22] for parabolic equations. Our approach, based on the direct analysis of the error equation for $e_{H,T}(t)$, allows us to reduce the regularity assumptions on the solution $T(t)$.

First, we note that the following identity holds

$$\begin{aligned}(e'_{H,T}(t), w_H)_H &= -((D_{T,H}(t)\nabla_H e_{H,T}(t), \nabla_H w_H))_H \\ &+ (((D_{T,H}(t) - D^*_{T,H}(t))\nabla_H R_H T(t), \nabla_H w_H))_H \\ &+ (R_H f_2(p(t)) - f_2(p_H(t)), w_H)_H \\ &+ (kR_H T(t) - kT_H(t), w_H)_H \\ &+ \tau_{D_T}(w_H) + \tau_k(w_H) + \tau_{f_2}(w_H) + \tau_d(w_H),\end{aligned} \tag{45}$$

where $w_H \in W_{H,0}$ and $D^*_{T,H}(t)$ is defined as $D_{T,H}(t)$ with $T_H(t)$ replaced by $R_H T(t)$, and

$$\begin{aligned}\tau_{D_T}(w_H) &= ((D^*_{T,H}(t)\nabla_H R_H T(t), \nabla_H w_H))_H \\ &\quad + ((\nabla \cdot (D_T(T(t))\nabla T(t)))_H, w_H)_H, &(46)\\ \tau_k(w_H) &= ((kT(t))_H, w_H)_H - (kR_H T(t), w_H)_H, &(47)\\ \tau_{f_2}(w_H) &= ((f_2(p(t)))_H, w_H)_H - (f_2(R_H p(t)), w_H)_H, &(48)\\ \tau_d(w_H) &= (R_H T'(t) - (T'(t))_H, w_H)_H, &(49)\end{aligned}$$

with $(\nabla\cdot(D_T(T(t))\nabla T(t))_H$, $(kT(t))_H$, $(f_2(p(t)))_H$ and $(T'(t))_H$ given by (22) with f_3 replace by $\nabla \cdot (D_T(T(t))\nabla T(t))$, $kT(t)$, $f_2(p(t))$, $T'(t)$, respectively.

We remark that an estimate for $(((D_{T,H}(t)-D^*_{T,H}(t))\nabla_H R_H T(t), \nabla_H w_H))_H$ is easily obtained assuming that $d_{T,i}, i = 1, 2$, are L_{D_T}-Lipschitz functions. In fact, it can be shown that

$$\begin{aligned}|(((D_{T,H}(t) \; &- D^*_{T,H}(t))\nabla_H R_H T(t), \nabla_H w_H))_H| \\ &\leq \sqrt{2} L_{D_T} \|\nabla_H R_H T(t)\|_\infty \|e_{H,T}(t)\|_H \|\nabla_H w_H\|_H,\end{aligned}$$

for $w_H \in W_{H,0}$, $H \in \Lambda$.

In the next propositions we estimates the quantities $\tau_{D_T}(w_H)$, $\tau_k(w_H)$, $\tau_{f_2}(w_H)$ and $\tau_d(w_H)$. Propositions 1, 2 and 3 follow directly from Lemma 5.7 of [17].

Proposition 1. *If $T(t) \in H^2(\Omega)$, for the functional $\tau_k : W_{H,0} \to \mathbb{R}$, $H \in \Lambda$, defined by (47) holds the following*

$$|\tau_k(w_H)| \leq C \left(\sum_{\Delta \in \tau_H} (diam\Delta)^4 \|T(t)\|^2_{H^2(\Delta)} \right)^{1/2} \|\nabla_H w_H\|_H .$$

Proposition 2. *If $T'(t) \in H^2(\Omega)$, for the functional $\tau_d : W_{H,0} \to \mathbb{R}$, $H \in \Lambda$, defined by (49) holds the following*

$$|\tau_d(w_H)| \leq C \left(\sum_{\Delta \in \tau_H} (diam \Delta)^4 \, \|T'(t)\|^2_{H^2(\Delta)} \right)^{1/2} \|\nabla_H w_H\|_H \, .$$

Proposition 3. *If f_2 is such that $f_2(p(t)) \in H^2(\Omega)$, then for the functional $\tau_{f_2} : W_{H,0} \to \mathbb{R}$, $H \in \Lambda$, defined by (48), there exists a positive constant C, H and t independent, such that*

$$|\tau_{f_2}(w_H)| \leq C \left(\sum_{\Delta \in \mathcal{T}_H} (diam \Delta)^4 \|f_2(p(t))\|^2_{H^2(\Delta)} \right)^{1/2} \|\nabla_H w_H\|_H .$$

Proposition 4. *If D_T satisfies the assumption A_4, $T(t) \in H^3(\Omega)$, $d_{T,i}(T(t)) \in L^\infty(\Omega), i = 1, 2$, and $D_T(t)\nabla T(t) \in [H^2(\Omega)]^2$, then for the functional $\tau_{D_T}(w_H) : W_{H,0} \to \mathbb{R}$, defined by (46) we have*

$$\begin{aligned}
|\tau_{D_T}(w_H)| \leq & C \Bigg(\sum_{\Delta \in \mathcal{T}_H} (diam \Delta)^4 \Big(L^2_{D_T} \|T(t)\|^2_{C^1(\Delta)} \|T(t)\|^2_{H^2(\Delta)} \\
& + \|D_T(T(t))\|^2_{\infty, L^\infty(\Delta)} \|T(t)\|^2_{H^3(\Delta)} \\
& + \|D_T(t) \nabla T(t)\|^2_{[H^2(\Delta)]^2} \Big) \Bigg)^{1/2} \|\nabla_H w_H\|_H \, ,
\end{aligned}$$

where $\|D_T(T(t))\|_{\infty, L^\infty(\Delta)} = \max_{i=1,2} \|d_{T,i}(T(t))\|_{L^\infty(\Delta)}$.

Proof: We start by noting that τ_{D_T} admits the representation

$$\begin{aligned}
\tau_{D_T}(w_H) = & (d_{T,1}(M_h R_H T(t)) D_{-x} R_H T(t), D_{-x} w_H)_{H,x} \\
& + ((\frac{\partial}{\partial x}(d_{T,1}(t) \frac{\partial T}{\partial x}(t)))_H, w_H)_H \\
& + (d_{T,2}(M_k R_H T(t)) D_{-y} R_H T(t), D_{-y} w_H)_{H,y} \\
& + ((\frac{\partial}{\partial y}(d_{T,2}(t) \frac{\partial T}{\partial y}(t)))_H, w_H t)_H \\
& := \tau_x(t) + \tau_y(t).
\end{aligned}$$

We split $\tau_x(t)$ as $\tau_x(t) = \tau_1(t) + \tau_2(t)$ with

$$\begin{aligned}\tau_1(t) = {} & (d_{T,1}(M_h R_H T(t)) D_{-x} R_H T(t), D_{-x} w_H)_{H,x} \\ & - (d_{T,1}(T(M_h(t))) D_{-x} R_H T(t), D_{-x} w_H)_{H,x}, \\ \tau_2(t) = {} & (d_{T,1}(T(M_h(t))) D_{-x} R_H T(t), D_{-x} w_H)_{H,x} + ((\frac{\partial}{\partial x}(d_{T,1}(t)\frac{\partial T}{\partial x}(t)))_H, w_H)_H.\end{aligned}$$

First we estimate $\tau_1(t)$. Let $\sigma(x_i, y_j, t)$ be defined by

$$\sigma(x_i, y_j, t) = T(x_{i-1/2}, y_j, t) - \frac{1}{2}\left(T(x_{i-1}, y_j, t) + T(x_i, y_j, t)\right),$$

which satisfies

$$k_{j+1/2}|\sigma(x_i, y_j, t)| \le \int_{y_{j-1/2}}^{y_{j+1/2}} |\sigma(x_i, y, t)|\, dy + k_{j+1/2}\int_{y_{j-1/2}}^{y_{j+1/2}} |\frac{\partial \sigma}{\partial y}(x_i, y, t)|\, dy.$$

We rewritten σ as

$$\sigma(x_i, y, t) = w\left(\frac{1}{2}\right) - \frac{1}{2}\left(w(1) + w(0)\right) = \lambda(w),$$

with $w(\xi) := T(x_i + \xi h_i, y, t)$ for $\xi \in [0,1]$, where $\lambda : W^{2,1}(0,1) \to \mathbb{R}$ is the functional

$$\lambda(g) := g\left(\frac{1}{2}\right) - \frac{1}{2}\left(g(1) + g(0)\right).$$

The functional λ is bounded in $W^{2,1}(0,1)$ and vanishes for $g = 1, \xi$. Bramble-Hilbert Lemma [18] guarantees the existence of a positive constant C such that

$$|\lambda(g)| \le C\,\|g''\|_{L^1(0,1)}, \; g \in W^{2,1}(0,1)$$

and consequently

$$|\sigma(x_i, y, t)| = |\lambda(w)| \le C h_i \int_{x_{i-1}}^{x_i} \left|\frac{\partial^2 T}{\partial x^2}(x, y, t)\right|\, dx. \tag{50}$$

Bramble-Hilbert Lemma also leads to

$$\left|\frac{\partial \sigma}{\partial y}(x_i, y, t)\right| \le C \int_{x_{i-1}}^{x_i} \left|\frac{\partial^2 T}{\partial x \partial y}(x, y, t)\right|\, dx, \tag{51}$$

for a positive constant C.

Since

$$
\begin{aligned}
&k_{j+1/2}\left|d_{T,1}\left(M_h(R_H T(x_i,y_j,t))\right)-d_{T,1}\left(T(M_h(x_i,y_j,t))\right)\right|\\
&\qquad\le L_{d_T}\left(\int_{y_{j-1/2}}^{y_{j+1/2}}|\sigma(x_i,y,t)|\ dy+k_{j+1/2}\int_{y_{j-1/2}}^{y_{j+1/2}}\left|\frac{\partial\sigma}{\partial y}(x_i,y,t)\right|\ dy\right),
\end{aligned}
$$

and taking into account the estimates (50) and (51), we have

$$
\begin{aligned}
|\tau_1(t)|\le & L_{D_T}\sum_{i=1}^{N}\sum_{j=1}^{M-1}\left(h_i\left(\int_{y_{j-1/2}}^{y_{j+1/2}}|\sigma(x_i,y,t)|\ dy+k_{j+1/2}\int_{y_{j-1/2}}^{y_{j+1/2}}|\frac{\partial\sigma}{\partial y}(x_i,y,t)|\ dy\right)\right.\\
&\left.\times|D_{-x}T(x_i,y_j,t)||D_{-x}w_H(x_i,y_j)|\right)\\
&\le CL_{D_T}\left(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\|T(t)\|^2_{C^1(\Delta)}\|T(t)\|^2_{H^2(\Delta)}\right)^{1/2}\|D_{-x}w_H\|_H.
\end{aligned}
$$

To estimate $\tau_2(t)$ we apply directly Lemma 5.1 of [17], leading to

$$
\begin{aligned}
|\tau_2(t)|\le C\Big(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\Big(&\|d_{T,1}(t)\|^2_{L^\infty(\Delta)}\|T(t)\|^2_{H^3(\Delta)}\\
&+\|d_{T,1}(t)\frac{\partial T}{\partial x}\|^2_{H^2(\Delta)}\Big)\Big)^{1/2}\|D_{-x}w_H\|_{H,x},
\end{aligned}
$$

and therefore to

$$
\begin{aligned}
|\tau_x(t)|\le C\left(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\left(L^2_{D_T}\|T(t)\|^2_{C^1(\Delta)}\|T(t)\|^2_{H^2(\Delta)}\right.\right.&\\
+\|d_{T,1}(t)\|^2_{L^\infty(\Delta)}\|T(t)\|^2_{H^3(\Delta)}&\\
\left.\left.+\|d_{T,1}(t)\frac{\partial T}{\partial x}(t)\|^2_{H^2(\Delta)}\right)\right)^{1/2}\|D_{-x}w_H\|_H.&
\end{aligned}
$$

Proceeding analogously, we can obtain a similar result for $\tau_y(t)$, and finally get the inequality

$$
\begin{aligned}
|\tau_{D_T}(w_H)| \leq\ & C\Bigg(\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\Big(L_{D_T}^2\|T(t)\|_{C^1(\Delta)}^2\|T(t)\|_{H^2(\Delta)}^2 \\
& +\max_{i=1,2}\|d_{T,i}(t)\|_{L^\infty(\Delta)}^2\|T(t)\|_{H^3(\Delta)}^2 \\
& +\|D_T(t)\nabla T(t)\|_{[H^2(\Delta)]^2}^2\Big)\Bigg)^{1/2}\|\nabla_H w_H\|_H.
\end{aligned}
$$

■

With Propositions 1-4, we can now state the main result of this subsection, namely, an upper bound for $\|e_{H,T}(t)\|_H$. In what follows, by $C^m([0,T_f],W_{H,0})$ we represent the space of functions $v:[0,T_f]\to W_{H,0}$ such that $v^{(j)}:[0,T_f]\to W_{H,0}$, $j=0,\dots,m$, are continuous and

$$\|v\|_{C^m([0,T_f],W_{H,0})}=\max_{j=0,\dots,m}\left\|v^{(j)}(t)\right\|_H<+\infty.$$

Theorem 2. *Let us assume that the solution $T(t)$ of the IBVP (2), (5), (8) satisfies $R_HT(t)\in C^1([0,T_f],W_{H,0})$ and $T(t)\in L^2(0,T_f,H^3(\Omega)\cap H_0^1(\Omega))$ $\cap\ H^1(0,T_f,H^2(\Omega))$; the solution T_H of the initial value problem (29), (30) belongs to $C^1([0,T_f],W_{H,0})$; f_2 satisfies A_1 with $f_2(p(t))\in H^2(\Omega)$, where $p(t)$ is solution of the IBVP (1), (4), (7) and the assumptions of Proposition 4 hold. Then, there exists a positive constant C, H and t independent, such that, for the spatial error $e_{H,T}(t)=R_HT(t)-T_H(t)$ holds the following*

$$
\begin{aligned}
\|e_{H,T}(t)\|_H^2 & +2(\beta_0-5\epsilon^2)\int_0^t e^{\int_s^t g_H(T(\mu))d\mu}\|\nabla_H e_{H,T}(s)\|_H^2ds \\
& \leq e^{\int_0^t g_H(T(s))ds}\|e_{H,T}(0)\|_H^2+L_{f_2}^2\int_0^t e^{\int_s^t g_H(T(\mu))d\mu}\|e_{H,p}(s)\|_H^2ds \\
& +2\int_0^t e^{\int_s^t g_H(T(\mu))d\mu}\tau_1(s)ds,
\end{aligned}
$$

for $t\in[0,T_f]$, where

$$g_H(T(t))=\frac{1}{\epsilon^2}L_{d_T}^2\|\nabla_H R_H T(t)\|_\infty^2+1+2k,$$

$$\tau_1(t) = \frac{C}{4\epsilon^2}\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\Big(\big(L^2_{D_T}\|T(t)\|^2_{C^1(\Delta)} + \|D_T(T(t))\|^2_{\infty,L^\infty(\Delta)} + 1\big)\|T(t)\|^2_{H^3(\Delta)}$$
$$+ \|T'(t)\|^2_{H^2(\Delta)} + \|D_T(t)\nabla T(t)\|^2_{[H^2(\Delta)]^2} + \|f_2(p(t))\|^2_{H^2(\Delta)}\Big),$$

and $\epsilon \neq 0$ is an arbitrary constant.

Proof: From (45), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_{H,T}(t)\|^2_H + \beta_0\|\nabla_H e_{H,T}(t)\|^2_H \\ &\leq \sqrt{2}L_{d_T}\|\nabla_H R_H T(t)\|_\infty\|e_{H,T}(t)\|_H\|\nabla_H e_{H,T}(t)\|_H \\ &+\frac{1}{2}L^2_{f_2}\|e_{H,p}(t)\|^2_H + (\frac{1}{2}+k)\|e_{H,T}(t)\|^2_H \\ &+4\epsilon^2\|\nabla_H e_{H,T}(t)\|^2_H + \tau_1(t),\end{aligned} \tag{52}$$

for $\epsilon \neq 0$. Combining (52) with the inequality

$$\begin{aligned}&\sqrt{2}L_{d_T}\|\nabla_H R_H T(t)\|_\infty\|e_{H,T}(t)\|_H\|\nabla_H e_{H,T}(t)\|_H \\ &\qquad\leq \epsilon^2\|\nabla_H e_{H,T}(t)\|^2_H + \frac{1}{2\epsilon^2}L^2_{d_T}\|\nabla_H R_H T(t)\|^2_\infty\|e_{H,T}(t)\|^2_H,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\|e_{H,T}(t)\|^2_H &+2(\beta_0 - 5\epsilon^2)\|\nabla_H e_{H,T}(t)\|^2_H \leq L^2_{f_2}\|e_{H,p}(t)\|^2_H \\ &+g_H(T(t))\|e_{H,T}(t)\|^2_H + 2\tau_1(t),\end{aligned} \tag{53}$$

Inequality (53) leads to

$$\begin{aligned}\frac{d}{dt}\Big(&e^{-\int_0^t g_H(T(s))ds}\|e_{H,T}(t)\|^2_H \\ &+2(\beta_0 - 5\epsilon^2)\int_0^t e^{-\int_0^s g_H(T(\mu))d\mu}\|\nabla_H e_{H,T}(s)\|^2_H ds \\ &-L^2_{f_2}\int_0^t e^{-\int_0^s g_H(T(\mu))d\mu}\|e_{H,p}(s)\|^2_H ds \\ &-2\int_0^t e^{-\int_0^s g_H(T(\mu))d\mu}\tau_1(s)ds\Big) \leq 0,\end{aligned}$$

which allows us to conclude the proof.

■

Corollary 2. *Under the assumptions of Theorems 1 and 2, there exists a positive constant C, H and t independent, such that*

$$\|e_{H,T}(t)\|_H^2 + \int_0^t \|\nabla_H e_{H,T}(s)\|_H^2 ds \le C H_{max}^4, t \in [0, T_f], H \in \Lambda.$$

The next corollary follows immediately from Corollary 2 and condition (44).

Corollary 3. *Under the conditions of Theorems 1 and 2, if the sequence of step-sizes Λ satisfies (44), there exists a positive constant C, H and t independent, such that*

$$\|T_H(t)\|_\infty \le C, \int_0^t \|\nabla_H T_H(s)\|_\infty ds \le C,\ t \in [0, T_f], H \in \Lambda,$$

with H_{max} small enough.

4.3. Concentration. In this subsection we establish the main result of this paper, Theorem 3. It proves the superconvergence of the FEM for the concentration $c(t)$. Naturally, the supraconvergence of the equivalent FDM, also follows from Theorem 3. The proof of this theorem relies heavily on Theorem 1, Corollary 1 and Corollary 2.

Let us define $e_{H,c}(t) = R_H c(T) - c_H(t)$ as the spatial discretization error associated with the FEM (36), (37) for the concentration. From (36), we obtain

$$\begin{aligned}(e'_{H,c}(t), w_H)_H = & -((D_{c,H}(t)\nabla_H e_{H,c}(t), \nabla_H w_H))_H \\ & + (((D_{c,H}(t) - D^*_{c,H}(t))\nabla_H R_H c(t), \nabla_H w_H))_H \\ & + ((M_H(v_H(t) e_{H,c}(t)), \nabla_H w_H))_H \\ & - ((M_H((v_H(t) - v^*_H(t)) R_H c(t)), \nabla_H w_H))_H \\ & + \tau_{D_c}(w_H) + \tau_v(w_H) + \tau_c(w_H),\end{aligned} \tag{54}$$

where $D^*_{c,H}(t)$ is defined as $D_{c,H}(t)$ with p_H and T_H replaced by $R_H p$ and $R_H T$, respectively, and $v^*_H(t)$ is defined as $v_H(t)$ with p_H replaced by $R_H p$. In (54), $\tau_{D_c}(w_H)$, $\tau_v(w_H)$ and $\tau_c(w_H)$ are defined by

$$\begin{aligned}\tau_{D_c}(w_H) \ & = ((D^*_{c,H}(t)\nabla_H R_H c(t), \nabla_H w_H))_H \\ & + ((\nabla \cdot (D_c(p(t), T(t))\nabla c(t)))_H, w_H)_H,\end{aligned} \tag{55}$$

$$\begin{aligned}\tau_v(w_H) \ & = -((M_H(v^*_H(t) R_H c(t)), \nabla_H w_H))_H \\ & - ((\nabla \cdot (v(p(t), \nabla p(t)) c(t)))_H, w_H)_H,\end{aligned} \tag{56}$$

and

$$\tau_c(w_H) = (R_H c'(t), w_H)_H - ((c'(t))_H, w_H)_H. \tag{57}$$

We observe that if $d_{c,i}, i = 1, 2$, are L_{D_c}-Lipschitz functions, then

$$|(((D_{c,H}(t) - D^*_{c,H}(t))\nabla_H R_H c(t), \nabla_H w_H))_H| \\ \leq 2L_{D_c}\big(\|e_{H,p}(t)\|_H + \|e_{H,T}(t)\|_H\big)\|\nabla_H R_H c(t)\|_\infty \|\nabla_H w_H\|_H \tag{58}$$

Assuming that v satisfies the assumption A_2 we easily conclude that

$$|((M_H(v_H(t)e_{H,c}(t)), \nabla_H w_H))_H| \\ \leq \sqrt{2}C_v\big(\|p_H(t)\|_\infty + \|\nabla_H p_H(t)\|_\infty\big)\|e_{H,c}(t)\|_H \|\nabla_H w_H\|_H. \tag{59}$$

An estimate for $| - ((M_H((v_H(t) - v^*_H(t))R_H c(t)), \nabla_H w_H))_H|$ can be obtained using condition (44) for the sequence of grids $\overline{\Omega}_H, H \in \Lambda$. In fact, it holds that

$$| - ((M_H((v_H(t) - v^*_H(t))R_H c(t)), \nabla_H w_H))_H| \\ \leq 2L_v\Big(\|e_{H,p}(t)\|_H + \sqrt{2C_m}\|\nabla_H e_{H,p}(t)\|_H\Big)\|R_H c(t)\|_\infty \|\nabla_H w_H(t)\|_H, \tag{60}$$

for $w_H \in W_{H,0}$, $H \in \Lambda$ with H_{max} small enough.

We derive now estimates for $\tau_{D_c}(w_H)$, $\tau_v(w_H)$ and $\tau_c(w_H)$, with $w_H \in W_{H,0}$.

Proposition 5. *If D_c satisfies A_5, $p(t) \in H^2(\Omega)$, $c(t) \in H^3(\Omega)$, $d_{c,i}(p(t), T(t)) \in L^\infty(\Omega), i = 1, 2,$ and $D_c(t)\nabla c(t) \in [H^2(\Delta)]^2$ then, for the functional $\tau_{D_c} : W_{H,0} \to \mathbb{R}$ defined by (55), there exists a positive constant C, H, t, p, T and c independent, such that*

$$|\tau_{D_c}(w_H)| \leq C\Big(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^4\big(L^2_{D_c}\big(\|T(t)\|^2_{H^2(\Delta)} + \|p(t)\|^2_{H^2(\Delta)}\big)\|c(t)\|^2_{C^1(\Delta)} \\ + \|D_c(p(t), T(t))\|^2_{\infty, L^\infty(\Delta)}\|c(t)\|^2_{H^3(\Delta)} \\ + \|D_c(p(t), T(t))\nabla c(t)\|^2_{[H^2(\Delta)]^2}\Big)^{1/2}\|\nabla_H w_H\|_H, \tag{61}$$

for $w_H \in W_{H,0}$, $H \in \Lambda$.

Proof: The functional (55) admits the representation

$$\tau_{D_c}(w_H) = \sum_{i=1}^{2} \tau_i(w_H), w_H \in W_{H,0}, \tag{62}$$

with

$$\tau_1(w_H) = (((D^*_{c,H}(t) - \tilde{D}_{c,H}(t))\nabla_H R_H c(t), \nabla_H w_H))_H,$$

where the diagonal entries of $\tilde{D}_{c,H}(t)$ at (x_i, y_j) are given by

$$d_{c,1}(p(x_{i-1/2}, y_j, t)), T(x_{i-1/2}, y_j, t))) \text{ and } d_{c,2}(p(x_i, y_{j-1/2}, t), T(x_i, y_{j-1/2}, t)),$$

and

$$\tau_2(w_H) = ((\tilde{D}_{c,H}(t)\nabla_H R_H c(t), \nabla_H w_H))_H + ((\nabla.(D_c(p(t), T(t))\nabla c(t))_H, w_H)_H.$$

Following the steps of Proposition 4, it is easy to prove that there exists a positive constant C, H and t independent, such that

$$|\tau_1(w_H)| \leq C\Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \big(L^2_{D_c}\big(\|T(t)\|^2_{H^2(\Delta)} + \|p(t)\|^2_{H^2(\Delta)}\big) \|c(t)\|^2_{C^1(\Delta)}\Big)^{1/2} \|\nabla_H w_H\|_H. \tag{63}$$

Moreover, considering Lemma 5.1 of [17], we easily get that

$$|\tau_2(w_H)| \leq C\Big(\sum_{\Delta\in\mathcal{T}_H} (diam\Delta)^4 \big(\|D_c(p(t), T(t))\|^2_{\infty, L^\infty(\Delta)} \|c(t)\|^2_{H^3(\Delta)} + \|D_c(p(t), T(t))\nabla c(t)\|^2_{[H^2(\Delta)]^2}\big)\Big)^{1/2} \|\nabla_H w_H\|_H, \tag{64}$$

where C is a positive constant, H and t independent, and we finish the proof using (63) and (64) in (62).

■

An estimate for $\tau_v(w_H)$ follows directly from Proposition 1 of [15] taking into account that

$$\begin{aligned}
\tau_v(w_H) = &- ((M_H((v^*_H(t) - \tilde{v}_H(t))R_H c(t)), \nabla_H w_H))_H \\
&- ((M_H(\tilde{v}_H(t)R_H c(t)) - \hat{v}_H(t)\hat{c}_H(t), \nabla_H w_H))_H \\
&- ((\hat{v}_H(t)\hat{c}_H(t), \nabla_H w_H))_H - ((\nabla \cdot (v(p(t), \nabla p(t))c(t)))_H, w_H)_H,
\end{aligned}$$

where $\tilde{v}_H(t) = (v_1(p(t), \frac{\partial p}{\partial x}(t)), v_2(p(t), \frac{\partial p}{\partial y}(t))$ and the components of $\hat{v}_H(t)$ at (x_i, y_j) are given by

$$(v_1(p(x_{i-1/2}, y_j, t), \frac{\partial p}{\partial x}(x_{i-1/2}, y_j, t)) \text{ and } v_2(p(x_i, y_{j-1/2}, t), \frac{\partial p}{\partial y}((x_i, y_{j-1/2}, t)),$$

respectively. $\hat{c}_H(t)$ is defined analogously.

Proposition 6 (Proposition 1 of [15])**.** *Let us suppose that v satisfies the assumption A_3 and $p(t) \in H^3(\Omega) \cap H_0^1(\Omega) \cap C^2(\overline{\Omega})$, $c(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v(t)c(t) \in [H^2(\Omega)]^2$. Then, for the functional $\tau_v : W_{H,0} \to \mathbb{R}$, defined by (56), there exists a positive constant C, H, t, p and c independent, such that*

$$\begin{aligned} |\tau_v(w_H)| \;&\leq C\Big(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^4 \Big(\|v(t)c(t)\|^2_{[H^2(\Delta)]^2} \\ &\quad + L_v^2 \|c(t)\|^2_{C(\Delta)} \|p(t)\|^2_{H^3(\Delta)} \Big)\Big)^{1/2} \|\nabla_H w_H\|_H, \end{aligned} \tag{65}$$

for $w_H \in W_{H,0}$, $H \in \Lambda$.

The following proposition gives an estimate for $\tau_c(w_H)$.

Proposition 7. *If $c(t) \in H^2(\Omega)$, for the functional $\tau_c : W_{H,0} \to \mathbb{R}$ defined by (57) we have*

$$|\tau_c(w_H)| \leq C \left(\sum_{\Delta \in \mathcal{T}_H} (diam\Delta)^4 \|c'(t)\|^2_{H^2(\Delta)} \right)^{1/2} \|\nabla_H w_H\|_H, \tag{66}$$

for $w_H \in W_{H,0}$, $H \in \Lambda$.

Proof: The inequality follows immediately from Lemma 5.7 of [17]. ∎

We are now in position to estimate the error $e_{H,c}(t)$. From (54), assuming that $d_{c,i} \geq \beta_1$ in $\mathbb{R}^2$, $i = 1, 2$, and taking into account (58), (60), (61), (65), (66), we get

$$\begin{aligned} \frac{d}{dt}\|e_{H,c}(t)\|^2_H + 2(\beta_1 - 6\epsilon^2)\|\nabla_H e_{H,c}(t)\|^2_H &\leq \frac{4}{\epsilon^2} L^2_{D_c}\big(\|e_{H,p}(t)\|^2_H \\ &\quad + \|e_{H,T}(t)\|^2_H\big)\|\nabla_H R_H c(t)\|^2_\infty \\ &\quad + \frac{2}{\epsilon^2} C_v^2 \big(\|p_H(t)\|^2_\infty + \|\nabla_H p_H(t)\|^2_\infty\big)\|e_{H,c}(t)\|^2_H \\ &\quad + \frac{4}{\epsilon^2} L_v^2 \Big(\|e_{H,p}(t)\|^2_H + 2C_m \|\nabla_H e_{H,p}(t)\|^2_H\Big)\|R_H c(t)\|^2_\infty + \tau_c(t), \end{aligned} \tag{67}$$

where $\epsilon \neq 0$, and

$$\tau_c(t) = \frac{C}{2\epsilon^2}\sum_{\Delta\in\mathcal{T}_H}(diam\Delta)^4\Big(\|T(t)\|^2_{H^2(\Delta)}\|c(t)\|^2_{C^1(\Delta)} + \|D_c(t)\|^2_{\infty,L^\infty}\|c(t)\|^2_{H^3(\Delta)} + \|D_c(t)\nabla c(t)\|^2_{[H^2(\Delta)]^2} + \|v(t)c(t)\|^2_{[H^2(\Delta)]^2} + \|c(t)\|^2_{C^1(\Delta)}\|p(t)\|^2_{H^3(\Delta)} + \|c'(t)\|^2_{H^2(\Delta)}\Big). \quad (68)$$

Inequality (67) leads to the following result.

Theorem 3. *Let us assume that the sequence of grids $\overline{\Omega}_H, H \in \Lambda$, satisfies (44); $p(t) \in H^3(\Omega)\cap H^1_0(\Omega)$; $T(t) \in H^2(\Omega)\cap H^1_0(\Omega)$; the solution $c(t)$ of the IBVP (3), (6), (9) satisfies $R_Hc(t) \in C^1([0,T_f],W_{H,0})$ and $c(t) \in L^2(0,T_f,H^3(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T_f,H^2(\Omega))$; the solution $c_H(t)$ of the initial value problem (36), (37) belongs to $C^1([0,T_f],W_{H,0})$; v satisfies the assumption A_2 and D_c satisfies the assumption A_5 and the assumptions of Propositions 5 and 6 hold. Then, there exists a positive constant C, H and t independent, such that for the spatial error $e_{H,c}(t) = R_Hc(t) - c_H(t)$ holds the following*

$$\begin{aligned}
&\|e_{H,c}(t)\|^2_H + 2(\beta_1 - 6\epsilon^2)\int_0^t e^{\int_s^t g_H(p_H(\mu))d\mu}\|\nabla_H e_{H,c}(s)\|^2_H ds \\
&\quad\leq e^{\int_0^t g_H(p_H(\mu))d\mu}\|e_{H,c}(0)\|^2_H \\
&\quad+\frac{4}{\epsilon^2}L^2_{D_c}\int_0^t e^{\int_s^t g_H(p_H(\mu))d\mu}\big(\|e_{H,p}(s)\|^2_H + \|e_{H,T}(s)\|^2_H\big)\|\nabla_H R_Hc(s)\|^2_\infty ds \\
&\quad+\frac{4}{\epsilon^2}L^2_v\int_0^t e^{\int_s^t g_H(p_H(\mu))d\mu}\Big(\|e_{H,p}(s)\|^2_H + 2C_m\|\nabla_H e_{H,p}(s)\|^2_H\Big)\|R_Hc(s)\|^2_\infty ds \\
&\quad+\int_0^t e^{\int_s^t g_H(p_H(\mu))d\mu}\tau_c(s)ds,
\end{aligned} \quad (69)$$

for $t \in [0, T_f]$, $H \in \Lambda$ and H_{max} small enough. In (69), $\epsilon \neq 0$, $\tau_c(t)$ is defined by (68) and

$$g_H(p_H(t)) = \frac{2}{\epsilon^2} C_v^2 \big(\|p_H(t)\|_\infty^2 + \|\nabla_H p_H(t)\|_\infty^2\big).$$

In Theorem 3, we fix $\epsilon \neq 0$ such that $\beta_1 - 6\epsilon^2 > 0$ and Corollary 1 guarantees the uniform boundness of $g_H(p_H(t))$, $H \in \Lambda$ and H_{max} small enough. The next corollary follows from Theorems 1 and 2.

Corollary 4. *Under the assumptions of Theorems 1, 2 and 3, there exists a positive constant C, H and t independent, such that*

$$\|e_{H,c}(t)\|_H^2 + \int_0^t \|\nabla_H e_{H,c}(s)\|_H^2 ds \leq C H_{max}^4, t \in [0, T_f],$$

for $H \in \Lambda$ and H_{max} small enough.

5. Numerical experiments

The aim of this section is to illustrate the theoretical convergence rates. For that, we make use of the FDM formulation. We start by defining the numerical strategy used for the time discretization of (1) $-$ (9). Let $[0, T_f]$ be the temporal domain and let us define the time mesh $t_m = m\Delta t$, for $m = 0, \dots, M_t$, with $t_{M_t} = T_f$ and Δt the uniform time step. By p_H^m, T_H^m, and c_H^m we denote the numerical approximations for $p_H(t_m)$, $T_H(t_m)$, and $c_H(t_m)$, respectively. The proposed numerical strategy is of iterative type and consists in solving equation (1), followed by equation (2), and equation (3). In particular, we look for p_H^m, T_H^m, and c_H^m, such that,

$$a_H \frac{p_H^{m+1} - 2p_H^m + p_H^{m-1}}{\Delta t^2} + b_H \frac{p_H^{m+1} - p_H^m}{\Delta t} = \nabla_H^* \cdot (E_H \nabla_H p_H^{m+1}) + f_{3,H}^{m+1} \text{ in } \Omega_H, \tag{70}$$

for $m = 1, \dots, M_t - 1$,

$$\frac{T_H^{m+1} - T_H^m}{\Delta t} = \nabla_H^* \cdot \big(D_{T_H}^m \nabla_H T_H^{m+1}\big) + k T_H^{m+1} + f_2(p_H^{m+1}) + g_{2,H}^{m+1} \text{ in } \Omega_H, \tag{71}$$

for $m = 0, \dots, M_t - 1$,

$$\frac{c_H^{m+1} - c_H^m}{\Delta t} + \nabla_{c,H} \cdot (c_H^{m+1} v_H^{m+1}) = \nabla_H^* \cdot (D_{c_H}^{m+1} \nabla_H c_H^{m+1}) + f_{1,H}^{m+1} \text{ in } \Omega_H, \tag{72}$$

for $m = 0, \ldots, M_t - 1$, complemented with the initial conditions

$$\frac{p_H^1 - p_H^0}{\Delta t} = R_H p_{v,0},\ p_H^0 = R_H p_0,\ T_H^0 = R_H T_0,\ \text{and}\ c_H^0 = R_H c_0,\ \text{in}\ \Omega_H, \tag{73}$$

and the boundary conditions

$$p_H^m = 0, \quad T_H^m = 0, \quad \text{and} \quad c_H^m = 0, \quad \text{on}\ \partial\Omega_H,\ m = 0, \ldots, M_t. \tag{74}$$

The numerical errors associated with this fully discrete approximation are defined by

$$e_{H,p}^m = R_H p(t_m) - p_H^m,\ e_{H,T}^m = R_H T(t_m) - T_H^m,\ \text{and}\ e_{H,c}^m = R_H c(t_m) - c_H^m.$$

Let us note that the theoretical analysis of the time strategy (70) – (74) can be obtained by extending the results provided in [19] and [21].

5.1. Numerical convergence test. In the following, we give two numerical examples that provide numerical validation of the theoretical convergence rates established in the previous section.

Example 1. Regarding the coefficient functions of system (1) – (9) we set

$$a(x,y) = 1 + x,\ b(x,y) = 2xy, ,\ e_1(x,y) = x + y,\ \text{and}\ e_2(x,y) = y$$

in the acoustic pressure equation (1),

$$d_{T,1}(T) = 1 + 2T,\ d_{T,2}(t)(T) = 1 + T, \quad k = 1, ,\ \text{and}\ f_2(p) = p$$

in the temperature equation (2), and

$$v(p, \nabla p) = (p + \frac{\partial p}{\partial x}, p + \frac{\partial p}{\partial y}),\ d_{c,1}(p,T) = 1 + p + T,\ \text{and}\ d_{c,2}(p,T) = 2 + p^2 + T^2$$

in the concentration equation (3). In order to obtain a problem with known analytic solution, the initial conditions (4) – (6) and the functions f_1, g_2, and f_3 are defined such that the exact solution of the coupled system (1) – (9) is given by

$$p(x,y,t) = e^t xy(1-x)(1-\cos(2\pi y)), \quad T(x,y,t) = e^t x \sin(2\pi y)(x-1)(y-1)$$
$$\text{and} \quad c(x,y,t) = e^t xy \sin(2\pi x - \pi)(1-y).$$

■

To estimate the rate of convergence we use the quantities

$$\begin{aligned} E_p &= \max_{m=1,\ldots,M_t} \|D_{-t}e^m_{H,p}\|_H + \|\nabla_H e^m_{H,p}\|_H, \\ E_T &= \max_{m=1,\ldots,M_t} \|e^m_{H,T}\|_H + \|\nabla_H e^m_{H,T}\|_H, \\ E_c &= \max_{m=1,\ldots,M_t} \|e^m_{H,c}\|_H + \|\nabla_H e^m_{H,c}\|_H, \end{aligned}$$

for acoustic pressure, temperature and concentration variables, respectively. Here, by D_{-t}, we denote the standard first-order backward operator in time.

For the numerical calculations we consider an initial random mesh H_1 of size $N \times M$. The size of this mesh is successively increased (by two in each direction) by adding to the new mesh the midpoints of the current mesh. On each mesh H_j, $j \in \mathbb{N}$, we measure the errors $E_{p,j}$, $E_{T,j}$, and $E_{c,j}$, and the convergence rates are estimated from the expression

$$\text{rate}_\lambda = \frac{\ln\left(\dfrac{E_{\lambda,j+1}}{E_{\lambda,j}}\right)}{\ln\left(\dfrac{H_{\max,j+1}}{H_{\max,j}}\right)}, \quad \text{with } \lambda = p,\ T,\ c.$$

We also set $\Omega = [0,1]^2$ and $T_f = 0.1$. The time step is given by $\Delta t = H^2_{\min,j}$, which is small enough to ensure that the error of the time discretization is negligible. The results are given in Table 1, and they illustrate the theoretical second-order convergence rate for the three variables: acoustic pressure p, temperature T and concentration c. Plots of the numerical solutions are shown in Figure 1.

N	M	H_{max}	E_p	rate$_p$	E_T	rate$_T$	E_c	rate$_c$
6	7	2.0563e-01	4.0833e-02	-	5.7739e-02	-	6.2992e-02	
12	14	1.0281e-01	1.3835e-02	1.5614	1.4315e-02	2.0120	1.6351e-02	1.9458
24	28	5.1407e-02	3.6414e-03	1.9258	3.5720e-03	2.0027	4.1212e-03	1.9882
48	56	2.5703e-02	8.9986e-04	2.0167	8.9256e-04	2.0007	1.0323e-03	1.9972
96	112	1.2852e-02	2.2531e-04	1.9978	2.2311e-04	2.0002	2.5820e-04	1.9993
192	224	6.4258e-03	5.6635e-05	1.9922	5.5777e-05	2.0000	6.4553e-05	1.9999

TABLE 1. Numerical convergence rates for Example 1.

Example 2. In this second example we test the sharpness of the conditions of our convergence theorems. Namely, we set our problem $(1)-(9)$ such that the solution of the concentration equation (3) is given by

$$c(x,y,t) = 2e^t x^2 y(x-1)(y-1)|y-0.5|^{2.1}.$$

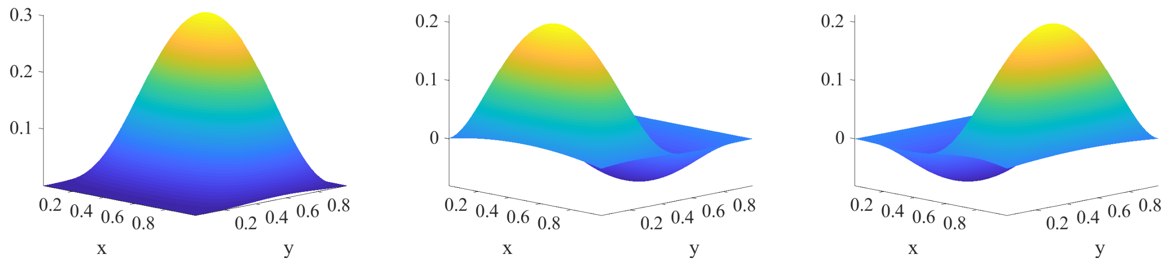

FIGURE 1. From left to right: numerical approximations p_H^m, T_H^m and c_H^m at $T = 0.1$ with $N = 192$ and $M = 224$.

This solution function belongs only to $H^2(\Omega)$ and does not satisfies the conditions of Theorem 3, which requires solutions with higher regularity, at least $H^3(\Omega)$. Thus, a decrease in the convergence rate for the variable c may occur. All the other parameters are the same as the ones used in Example 1.

■

The results for Example 2 are given in Table 2, and they shown that the convergence rate for the concentration variable c is only one. The regular enough variables (p and T) keep the second-order convergence rate. These results suggest that the regularity conditions imposed on the solutions of (1) – (9) are sharp, in the sense that if these restrictions do not hold, then a loss in convergence rate may appear. This behavior is in line with the findings of [17]. In that work the authors studied similar numerical schemes for elliptic equations, and they proved that the order of convergence is one if the solution of the problem belongs to $H^2(\Omega)$.

N	M	H_{max}	E_p	rate$_p$	E_T	rate$_T$	E_c	rate$_c$
6	8	2.0463e-01	4.1853e-02	-	5.8298e-02	-	2.9100e-03	-
12	16	1.0232e-01	1.4535e-02	1.5258	1.4342e-02	2.0231	1.5597e-03	0.8998
24	32	5.1158e-02	3.8642e-03	1.9113	3.5696e-03	2.0064	8.1279e-04	0.9403
48	64	2.5579e-02	9.4836e-04	2.0267	8.9163e-04	2.0013	3.8138e-04	1.0916
96	128	1.2789e-02	2.3696e-04	2.0008	2.2285e-04	2.0004	1.8334e-04	1.0567
192	256	6.3947e-03	5.9137e-05	2.0025	5.5709e-05	2.0001	8.6387e-05	1.0856

TABLE 2. Numerical convergence rates for Example 2.

6. Conclusion

The use of enhancers like ultrasound, light and electrical and magnetic fields, is becoming popular in drug delivery systems. The role of an enhancer can be three-fold, to break biological barriers (e.g., the stratum corneum in transdermal delivery), to increase drug transport (e.g., of large drug molecules), and to control drug release (e.g., using thermoresponsive drug nanocarriers). In the last case, the goal is to avoid drug side effects and maintain an optimal therapeutic level.

To develop efficient drug delivery systems several parameters need to be optimized and tuned. Mathematical models play an important role during this optimization process. Here, we are interested in ultrasound enhanced drug transport. We consider a mathematical model that takes into account all variables of interest: drug concentration, acoustic pressure and temperature. The model under analysis is defined by a coupled system of partial differential equations (IBVP (1)-(9)).

To obtain a reliable numerical simulation, it is crucial to design an accurate numerical method. In this paper, we propose and present a detailed convergence analysis of a piecewise linear FEM with quadrature. Such detailed analysis is absent in most simulation studies. One of the main numerical challenges is that the drug advective velocity depends on the gradient of the acoustic pressure and on temperature. Key highlights of this work are the second order approximation for concentration with respect to a L^2-discrete norm, the low regularity assumptions on the exact solution and no requirement of mesh uniformity.

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