Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 20–24

FULLY NONLINEAR DEAD-CORE SYSTEMS

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ABSTRACT: We study fully nonlinear dead-core systems coupled with strong absorption terms. We discover a chain reaction, exploiting properties of an equation along the system. The lack of both the classical Perron's method and comparison principle for the systems requires new tools for tackling the problem. By means of a fixed point argument, we prove existence of solutions, and obtain higher sharp regularity across the free boundary. Additionally, we prove a variant of a weak comparison principle and derive several geometric measure estimates for the free boundary, as well as a Liouville type theorems for entire solutions. These results are new even for linear dead-core systems.

KEYWORDS: Dead-core systems, elliptic systems, regularity, comparison principle, non-degeneracy, Liouville theorem for systems.
AMS SUBJECT CLASSIFICATION (2000): 35J57, 35J47, 35J67, 35B53, 35J60, 35B65.

1. Introduction

Due to applications in population dynamics, combustion processes, catalysis processes and in industry, in particular, in bio-technologies and chemical engineering, reaction-diffusion equations with strong absorption terms have been studied intensively in recent years. In literature, the regions where the density of a certain substance (liquid, gas) vanishes, are referred to as dead-cores. When the density of one substance is influenced of that of another one, we then deal with dead-core systems. The system modeling these densities of the reactants u, $v : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ with smooth boundary, is given by

$$\begin{cases} F(D^2u, x) &= f(u, v, x), \quad \text{in} \quad x \in \Omega, \\ G(D^2v, x) &= g(u, v, x), \quad \text{in} \quad x \in \Omega. \end{cases} \tag{1.1}$$

Received August 18, 2020.

DJA is supported by CNPq and grant 2019/0014 Paraiba State Research Foundation (FAPESQ). He thanks Centre for Mathematics of the University of Coimbra (CMUC) and the Abdus Salam International Centre for Theoretical Physics (ICTP) for great hospitality during his research visits.

RT is partially supported by FCT - Fundação para a Ciência e a Tecnologia, I.P., through projects PTDC/MAT-PUR/28686/2017 and UTAP-EXPL/MAT/0017/2017, as well as by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES. RT thanks Department of Mathematics of the Universidade Federal da Paraíba (UFPB) for hospitality and great working conditions during his research visit.

The operators F, $G : \mathbb{S}^n \times \Omega \to \mathbb{R}$, where $\mathbb{S}^n$ is the set of symmetric matrices of order n, model the diffusion processes, while the functions f and g are the convection terms of the model. Additionally, systems like (1.1) are related to Lane-Emden systems (see [14]) and also model games that combine the tug-of-war with random walks in two different boards (see [15]).

The goal of this paper is to study regularity of solutions for fully non-linear dead-core systems

$$\begin{cases} F(D^2u, x) = v_+^p, & \text{in } \ \Omega, \\ G(D^2v, x) = u_+^q & \text{in } \ \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$, p and q are non-negative constants such that $pq < 1$, and F and G are fully nonlinear uniformly elliptic operators with uniformly continuous coefficients, i.e., satisfy

$$\lambda\|N\| \leq \mathcal{F}(M+N, x) - \mathcal{F}(M, x) \leq \Lambda\|N\|, \ \ \forall N \geq 0, \tag{1.3}$$

with $M \in \mathbb{S}^n$, $0 < \lambda \leq \Lambda$. Here N is a non-negative definite symmetric matrix, therefore the norm $\|N\|$ is equal to the maximum eigenvalue of N. Solutions of (1.2) are understood in the viscosity sense, according to the following definition.

Definition 1.1. *A couple of functions u, $v \in C(\Omega)$ is called a viscosity sub-solution of (1.2), if whenever $\varphi, \psi \in C^2(\Omega)$ and $u - \varphi$, $v - \psi$ attain local maximum at $x_0 \in \Omega$, then*

$$\begin{cases} F\left(D^2\varphi(x_0), x_0\right) \geq v_+^p(x_0), & \text{in } \ \Omega, \\ G\left(D^2\psi(x_0), x_0\right) \geq u_+^q(x_0) & \text{in } \ \Omega. \end{cases}$$

Similarly, u, $v \in C(\Omega)$ is called a viscosity super-solution of (1.2), if whenever $\varphi, \psi \in C^2(\Omega)$ and $u - \varphi$, $v - \psi$ attain local minimum at $x_0 \in \Omega$, then

$$\begin{cases} F\left(D^2\varphi(x_0), x_0\right) \leq v_+^p(x_0), & \text{in } \ \Omega, \\ G\left(D^2\psi(x_0), x_0\right) \leq u_+^q(x_0) & \text{in } \ \Omega. \end{cases}$$

A couple of functions u, $v \in C(\Omega)$ is called a viscosity solution of (1.2), if its both a viscosity sub- and super-solution of (1.2).

In literature (see, for example, [6, 10, 11, 12]) viscosity solutions for systems are defined as locally bounded functions. Since our operators are uniformly elliptic (with continuous coefficients), using a fixed point argument, we are able to prove existence of continuous solutions. Based on this, we define viscosity solutions as continuous functions, which is in accordance with the classical definition for equations. Although, existence of solutions, Perron's method, as well as several geometric properties for systems are well studied in the literature, [5, 6, 10, 11, 12], however, they are valid only under certain superlinearity, monotonicity or growth conditions, which do not apply in our framework.

From Krylov-Safonov regularity theory (see [3]), one may imply that viscosity solutions of the system (1.2) are locally of the class $C^{0,\alpha}$, for $\alpha \in (0,1)$. The ideas introduced by Teixeira in [17], where dead-core equations were studied, inspire a hope for more regularity near the free boundary. They were implemented to derive higher regularity for the dead-core problems ruled by the infinity Laplacian, [1] (see also [7]). However, the tools used to obtain these regularity and geometric measure estimates for equations, do not apply to systems. Unlike dead-core problems for equations, when dealing with systems, one has two main challenges: the lack of a comparison principle and the lack of the classical Perron's method in the framework. The latter for systems is valid only under certain structural assumptions on the operators (see [5, 6, 10, 11, 12]). Without these tools, even proving existence of solutions is a challenging task, since for systems, in general, the comparison of a sub- and a super-solution does not hold, as shows the example below:

$$\begin{cases} -\Delta u + 2u + v = 0, & \text{in } \Omega, \\ -\Delta v + u - 1 = 0 & \text{in } \Omega. \end{cases}$$

The Dirichlet problem (with zero boundary data) for this system has unique solution (u, v), [11, Remark 3.4]. The pairs $(0,0)$ and (u,v) are a sub- and a super-solution respectively, but $(0,0) \leq (u,v)$ is not true, [11, Remark 4.2].

As for the comparison principle, it is not valid even for linear systems. For example, when $F = G = \Delta$ in (1.1), i.e., considering the system for the Laplace operator, the solution of which can be interpreted as the Euler-Lagrange equation of the energy functional

$$I(u,v) := \int_\Omega \nabla u \cdot \nabla v + \mathcal{F}(u,v)\, dx \longrightarrow \min,$$

where $f(u,v,x) = \partial_u \mathcal{F}(u,v,x)$ and $g(u,v,x) = \partial_v \mathcal{F}(u,v)$, the comparison of solutions on the boundary does not imply comparison inside the domain.

The lack of a comparison principle does not allow to construct suitable barriers as to study geometric properties of viscosity solutions. Nevertheless, by means of a careful analysis of solutions of an equation in the system, we discover a chain reaction within the system, exploiting properties of an equation along the system. In particular, we show that if a pair (u, v) of non-negative functions is a viscosity solution of the system (1.2), and $x_0 \in \partial\{u + v > 0\}$, then

$$c\, r^{\frac{2}{1-pq}} \leq \sup_{B_r(x_0)} \left(u^{\frac{1}{1+p}} + v^{\frac{1}{1+q}} \right) \leq C\, r^{\frac{2}{1-pq}}$$

where $r > 0$ is small enough and c, $C > 0$ are constants depending on the dimension. This provides a regularity effect among equations. Observe that if $F = G$, $p = q$ and $u = v$ in (1.2), we recover the regularity result obtained in [17]. It is noteworthy that the regularity we derive for systems is higher than that of Hamiltonian elliptic systems studied in [16]. As shown in [16], the $C^{2,\alpha}$ regularity remains valid for systems with vanishing Naumann boundary condition, when $F = G = -\Delta$ and p and q are chosen in a certain way. Note also that one has higher regularity for solutions along the set $\partial\{u + v > 0\}$ compared to that coming from the Krylov-Safonov and Schauder regularity theory. Indeed, for example, for the Laplace operator, as $\Delta u \in C^{0,p}_{\text{loc}}$ and $\Delta v \in C^{0,q}_{\text{loc}}$, one has $u \in C^{2,p}_{\text{loc}}$ and $v \in C^{2,q}_{\text{loc}}$. On the other hand,

$$\frac{2(1+p)}{1-pq} > 2 + p \quad \text{and} \quad \frac{2(1+q)}{1-pq} > 2 + q,$$

for $p, q \in (0, 1)$. An application of this result yields that if a non-negative entire solution of the system (1.2) vanishes at a point and has a certain growth at infinity, then it must be identically zero. Additionally, we prove a variant of a weak comparison principle, showing that if one can compare viscosity sub- and super-solutions on the boundary, then at least one of the functions that make the pair of these sub- and super-solutions, still can be compared in the domain. The latter leads to several geometric measure estimates. Our results can be applied to systems with more than two equations.

The paper is organized as follows: in Section 2, using Schaefer's fixed point argument, we prove existence of solutions for the system (1.2) (Proposition 2.1). In Section 3, we prove regularity of solutions of the system (1.2) (see Theorem 3.1) across the free boundary. In Section 4, non-degeneracy of viscosity solutions (Theorem 4.1) is obtained. It is also in this section that we prove a weak variant of a comparison principle (Lemma 4.1). We close the paper with several applications, in Section 5 deriving geometric measure estimates for the free boundary (Lemma 5.1) and proving Liouville type results for entire solutions of (1.2) (Theorem 5.1 and Theorem 5.2).

2. Existence of solutions

In this section, assuming that the boundary of the domain is of class C^2, we prove existence of viscosity solutions of the system

$$\begin{cases} F(D^2u, x) = v_+^p, & \text{in} \quad \Omega, \\ G(D^2v, x) = u_+^q & \text{in} \quad \Omega, \\ \quad u = \varphi,\ v = \psi & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where φ, $\psi \in C^{0,1}(\partial\Omega)$. Throughout the paper we assume that

$$p \geq 0, \;\; q \geq 0, \;\; pq < 1,$$

and F and G satisfy (1.3). As commented above, neither the classical Perron's method nor the standard comparison principle apply in this framework. Thus, we need new tools to tackle the issue. For that purpose, we suitably construct a continuous and compact map, to use the following Schaefer's theorem (see, for example, [19]), extending the idea from [4].

Theorem 2.1. *If $T : X \to X$, where X is a Banach space, is continuous and compact, and the set*

$$\mathcal{E} = \{z \in X;\;\; \exists\, \theta \in [0, 1] \text{ such that } z = \theta T(z)\}$$

is bounded, then T has a fixed point.

Proposition 2.1. *There exists a viscosity solution (u, v) of system (2.1). Moreover, u and v are globally Lipschitz functions.*

Proof: Let $f \in C^{0,1}(\overline{\Omega})$ and let $u \in C^{0,1}(\overline{\Omega})$ be the unique viscosity solution of the following problem:

$$\begin{cases} F(D^2u, x) = f_+^p(x), & \text{in} \quad \Omega, \\ \qquad\quad u = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $\varphi \in C^{0,1}(\partial\Omega)$. Existence and uniqueness of such u is obtained by classical Perron's method (see [6, 9]). Similarly, for a function $g \in C^{0,1}(\overline{\Omega})$, there is a unique function $v \in C^{0,1}(\overline{\Omega})$ that solves the problem

$$\begin{cases} G(D^2v, x) = g_+^q(x), & \text{in} \quad \Omega, \\ \qquad\quad v = \psi & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

where $\psi \in C^{0,1}(\partial\Omega)$. For a given pair of functions $(f,g) \in C^{0,1}(\Omega) \times C^{0,1}(\Omega)$, we then define

$$T : C^{0,1}(\Omega) \times C^{0,1}(\Omega) \to C^{0,1}(\Omega) \times C^{0,1}(\Omega)$$

by $T(f,g) := (v,u)$, where v and u are the solutions of (2.3) and (2.2) respectively. Note that T is well defined, as (2.2) and (2.3) have unique, globally Lipschitz solutions. The aim is to use the Schaefer's theorem, so we need to check that T is continuous and compact.

Let $(f_k, g_k) \to (f,g)$ in $C^{0,1}(\Omega) \times C^{0,1}(\Omega)$, and let

$$(v_k, u_k) := T(f_k, g_k).$$

We want to show that $T(f_k, g_k) \to T(f,g)$ in $C^{0,1}(\Omega) \times C^{0,1}(\Omega)$. By global Lipschitz regularity (see, for instance, [2]), for a universal constant $C > 0$ one has

$$\|u_k\|_{C^{0,1}(\overline{\Omega})} \leq C(\|(f_k)_+^p\|_\infty + \|\varphi\|_\infty),$$

and

$$\|v_k\|_{C^{0,1}(\overline{\Omega})} \leq C(\|(g_k)_+^q\|_\infty + \|\psi\|_\infty).$$

Since (f_k, g_k) converges, then it is limited, therefore, $\|f_k\|_\infty$ and $\|g_k\|_\infty$ are bounded uniformly. Thus, by Arzelá-Ascoli theorem, the sequence (v_k, u_k) is bounded in $C^{0,1}(\overline{\Omega}) \times C^{0,1}(\overline{\Omega})$ and hence, up to a sub-sequence, it converges to some (v,u). Using stability of viscosity solutions under uniform limits, [9, Proposition 2.1], and passing to the limit, we obtain

$$T(f_k, g_k) = (v_k, u_k) \to (v,u) = T(f,g).$$

The uniqueness of solutions of (2.2) and (2.3) guarantees that any other sub-sequence converges to (v,u). Thus, T is continuous.

To see that T is also compact, let (f_k, g_k) be a bounded sequence in $C^{0,1}(\Omega) \times C^{0,1}(\Omega)$. As above, $(v_k, u_k) = T(f_k, g_k) \in C^{0,1}(\Omega) \times C^{0,1}(\Omega)$ is bounded. Then there is a convergent sub-sequence.

To use Theorem 2.1, it remains to check that the set $\mathcal{E}$ of eigenvectors

$$\mathcal{E} := \{(f,g) \in C^{0,1}(\Omega) \times C^{0,1}(\Omega);\ \ \exists\, \theta \in [0,1] \text{ such that } (f,g) = \theta T(f,g)\}$$

is bounded. Observe that $(0,0) \in \mathcal{E}$ if and only if $\theta = 0$. Hence, we can assume $\theta \neq 0$. Let $(f,g) \in \mathcal{E}$. For any $0 < \theta \leq 1$, we have

$$\begin{cases} F_\theta\left(D^2 g, x\right) = \theta f_+^p(x), & \text{in} \quad \Omega, \\ \qquad g = \theta\varphi & \text{on} \quad \partial\Omega, \end{cases}$$

where $F_\theta(M,x) := \theta F(\theta^{-1}M,x)$ satisfies (1.3), and

$$\begin{cases} G_\theta\left(D^2 f, x\right) = \theta g_+^q(x), & \text{in} \quad \Omega, \\ f = \theta\psi & \text{on} \quad \partial\Omega, \end{cases}$$

where $G_\theta(M,x) := \theta G(\theta^{-1}M,x)$ satisfies (1.3). As above, from Krylov-Safonov regularity theory one has

$$\|g\|_{C^{0,1}(\overline{\Omega})} \leq C(\|f_+^p\|_\infty + \|\varphi\|_\infty),$$

and

$$\|f\|_{C^{0,1}(\overline{\Omega})} \leq C(\|g_+^q\|_\infty + \|\psi\|_\infty),$$

where $C > 0$ is a universal constant. However, since $F_\theta(D^2 g, x) \geq 0$, then g takes its maximum on the boundary, and hence, is bounded by $\|\varphi\|_\infty$.

$$\|g\|_\infty \leq \|\varphi\|_\infty.$$

Similarly,

$$\|f\|_\infty \leq \|\psi\|_\infty.$$

Thus,

$$\|g\|_{C^{0,1}(\overline{\Omega})} \leq C(\|\psi\|_\infty^p + \|\varphi\|_\infty)$$

and

$$\|f\|_{C^{0,1}(\overline{\Omega})} \leq C(\|\varphi\|_\infty^q + \|\psi\|_\infty),$$

where $C > 0$ is universal constant. This implies that $\mathcal{E}$ is bounded.

Finally, we can now apply Theorem 2.1 to conclude that T has a fixed point. This finishes the proof. ■

3. Regularity of solutions at the free boundary

In this section, we derive regularity for solutions of (1.2) across the free boundary, $\partial\{|(u,v)| > 0\}$, where

$$|(u,v)| := u^{\frac{1}{1+p}} + v^{\frac{1}{1+q}}. \tag{3.1}$$

As an auxiliary step, we show that bounded solutions of a system, that vanish at a point, can be flattened, meaning that they can be made smaller than a given positive constant, once the right hand side is suitably perturbed. As is seen below, there is a natural phenomenon, a *domino effect*, when a certain control over one of the diffusions dictates a wave across the system.

We say $(u,v) \geq 0$, if both u and v are non-negative. As our results are of local nature, we consider B_1 instead of Ω.

Lemma 3.1. *For a given $\mu > 0$, there exists $\kappa = \kappa(\mu) > 0$ depending only on μ, λ, Λ, n, such that if u, $v \in [0,1]$ are such that $u(0) = v(0) = 0$ and*

$$\begin{cases} F(D^2u, x) &= \delta^2 v_+^p, \text{ in } B_1, \\ G(D^2v, x) &= \gamma u_+^q \text{ in } B_1. \end{cases}$$

in the viscosity sense for $\delta \in (0, \kappa)$ and $\gamma > 0$, then

$$\sup_{B_{1/2}} |(u, v)| \leq \mu.$$

Proof: We argue by contradiction and suppose the conclusion of the lemma is not true, i.e., for $\mu_0 > 0$ there exists sequence of functions u_i, v_i such that u_i, $v_i \in [0,1]$ and

$$\begin{cases} F_i(D^2u_i, x) &= i^{-2}\,(v_i)_+^p\,, \text{ in } B_1, \\ G_i(D^2v_i, x) &= \gamma\,(u_i)_+^q \text{ in } B_1, \end{cases}$$

where F_i, G_i are elliptic operators satisfying (1.3), but

$$\sup_{B_{1/2}} |(u_i, v_i)| > \mu_0. \tag{3.2}$$

By the Krylov-Safonov regularity theory (see, for example, [3]), up to subsequences, u_i and v_i converge locally uniformly in $B_{2/3}$ to respectively u_∞ and v_∞, as $i \to \infty$. Clearly, $u_\infty(0) = 0$, $u_\infty \in [0,1]$, and by stability of viscosity solutions, u_∞ in $B_{2/3}$ satisfies

$$F_\infty(D^2u_\infty, x) = 0,$$

where F_∞ satisfies (1.3). The maximum principle then implies that $u_\infty \equiv 0$. The latter provides that in addition to $v_\infty(0) = 0$ and $v_\infty \in [0,1]$, one also has

$$G_\infty(D^2v_\infty, x) = 0,$$

where G_∞ satisfies (1.3). This *chain reaction* allows one to apply once more the maximum principle and conclude that $v_\infty \equiv 0$, which contradicts (3.2). ∎

Next, we apply Lemma 3.1 to obtain regularity of solutions. Geometrically, it reveals that viscosity solutions of the system (1.2) touch the free boundary in a smooth fashion. In fact, it provides a quantitative information on the speed in which bounded viscosity solution detaches from the dead core. In the next section, we show that it is the exact speed in which viscosity solutions of (1.2) grow.

Theorem 3.1. *Let (u,v) be a non-negative viscosity solution of (1.2) in B_1. There exists a constant $C>0$, depending only on λ, Λ, p, q, $\|u\|_\infty, \|v\|_\infty$, such that for $x_0 \in \partial\{|(u,v)|>0\} \cap B_{1/2}$ there holds*

$$|(u(x),v(x))| \le C\,|x-x_0|^{\frac{2}{1-pq}}, \tag{3.3}$$

for any $x \in B_{1/4}$. In particular,

$$u(x) \le C\,|x-x_0|^{\frac{2(1+p)}{1-pq}} \quad \text{and} \quad v \le C\,|x-x_0|^{\frac{2(1+q)}{1-pq}}.$$

Proof: Without loss of generality, we assume that $x_0 = 0$.

Take now $\mu = 2^{-\frac{2}{1-pq}}$, and let $\kappa = \kappa(\mu) > 0$ be as in Lemma 3.1. We need to apply Lemma 3.1 to suitably chosen sequences of functions. We construct the first pair of functions in the following way. Set

$$u_1(x) := b \cdot u(ax) \quad \text{and} \quad v_1(x) := b \cdot v(ax),$$

where

$$a := \frac{\kappa}{2}\left(\|u\|_\infty + \|v\|_\infty\right)^{\frac{p-1}{4}} \quad \text{and} \quad b := \left(\|u\|_\infty + \|v\|_\infty\right)^{1/2}.$$

Note that $u_1,\ v_1 \in [0,1]$. As $0 \in \partial\{|(u,v)|>0\}$, then $u_1(0) = v_1(0) = 0$. Also $(u_1,v_1) \ge 0$ is a viscosity solution of the system

$$\left\{\begin{array}{rcl} \tilde{F}(D^2u_1,x) & = & \frac{\kappa^2}{4} v_1^p(x) \text{ in } B_1, \\ \tilde{G}(D^2v,x) & = & b^{-q}u_1^q(x) \text{ in } B_1, \end{array}\right. \tag{3.4}$$

where $\tilde{F}(M,x) := ba^2F(b^{-1}a^{-2}M, ax)$, $\tilde{G}(N,x) := G(b^{-1}a^{-2}N, ax)$ are (λ,Λ) - elliptic operators, i.e., satisfy (1.3). Applying Lemma 3.1, we deduce

$$\sup_{B_{1/2}} |(u_1,v_1)| \le 2^{-\frac{2}{1-pq}}. \tag{3.5}$$

Next, we define sequences of functions $u_2, v_2 : B_1 \to \mathbb{R}$ by

$$u_2(x) := 2^{\frac{2}{1-pq}(1+p)} u_1\left(\frac{x}{2}\right) \quad \text{and} \quad v_2(x) := 2^{\frac{2}{1-pq}(1+q)} v_1\left(\frac{x}{2}\right).$$

Using (3.4) we have that $u_2, v_2 \in [0,1]$. Also $u_2(0) = v_2(0) = 0$, and $(u_2,v_2) \ge 0$ is a viscosity solution of (3.4), for some uniformly elliptic operators $\tilde{F}$ and

$\tilde{G}$. Hence, Lemma 3.1 provides with

$$\sup_{B_{1/2}} |(u_2, v_2)| \leq 2^{-\frac{2}{1-pq}}.$$

Scaling back we have

$$\sup_{B_{1/4}} |(u_1, v_1)| \leq 2^{-2\cdot\frac{2}{1-pq}}.$$

Same way we define $u_i, v_i : B_1 \to \mathbb{R}$, $i = 3, 4, \ldots$, by

$$u_i(x) := 2^{\frac{2}{1-pq}(1+p)} u_{i-1}\left(\frac{x}{2}\right) \quad \text{and} \quad v_i(x) := 2^{\frac{2}{1-pq}(1+q)} v_{i-1}\left(\frac{x}{2}\right)$$

and deduce that

$$\sup_{B_{1/2}} |(u_i, v_i)| \leq 2^{-\frac{2}{1-pq}}.$$

Scaling back provides with

$$\sup_{B_{1/2^i}} |(u_1, v_1)| \leq 2^{-i\cdot\frac{2}{1-pq}}.$$

To finish the proof, for any given $r \in \left(0, \dfrac{A}{2}\right)$, we choose $i \in \mathbb{N}$ such that

$$2^{-(i+1)} < \frac{r}{A} \leq 2^{-i},$$

and estimate

$$\begin{aligned} \sup_{B_r} |(u, v)| &\leq \sup_{B_{r/A}} |(u_1, v_1)| \\ &\leq \sup_{B_{1/2^i}} |(u_1, v_1)| \\ &\leq 2^{-i\cdot\frac{2}{1-pq}} \\ &\leq \left(\tfrac{2}{A}\right)^{\frac{2}{1-pq}} r^{\frac{2}{1-pq}} \\ &= C r^{\frac{2}{1-pq}}. \end{aligned}$$

$\blacksquare$

4. Comparison and non-degeneracy estimates

In this section we prove a variant of a weak comparison principle for the system (1.2). With a careful analysis of radial super-solutions, we also derive non-degeneracy estimate for viscosity solutions. As was established by Theorem 3.1, the quantity $|(u,v)|$, defined by (3.1), grows not faster than $r^{\frac{2}{1-pq}}$. Here we prove that it grows exactly with that rate (see Figure 2).

4.1. A weak comparison principle. As commented in the introduction, the absence of the classical Perron's method in our framework forces us to find an alternative way to derive an information on relations between viscosity solutions and super-solutions of the system. For that purpose, we recall that uniform ellipticity in terms of extremal Pucci operators can be rewritten as follows:

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N) \le \mathcal{F}(M) - \mathcal{F}(N) \le \mathcal{M}^+_{\lambda,\Lambda}(M-N), \;\; \forall M, N \in \mathbb{S}^n,$$

with $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ being the classical extremal Pucci operators, i.e.,

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M, $0<\lambda\le\Lambda$. Since $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are uniformly elliptic operators (see [3, Lemma 2.10]), as defined by (1.3), with ellipticity constants λ and $n\Lambda$, from the comparison principle for $\mathcal{M}^-_{\lambda,\Lambda}$, we are able to conclude a weak form of comparison principle for the system (1.2), showing that at least one of the functions in viscosity super-solutions stays above a function in a pair that makes viscosity sub-solution of the system, as states the following lemma.

Lemma 4.1. *If (u^*,v^*) is a viscosity super-solution and (u_*,v_*) is a sub-viscosity solution of (1.2) such that $u^*\ge u_*$ and $v^*\ge v_*$ on $\partial\Omega$, then*

$$\max\{u^*-u_*, v^*-v_*\} \ge 0 \quad in \quad \Omega.$$

Proof: Suppose the conclusion of the lemma is not true, i.e., there exists $x_0\in\Omega$ such that

$$u_*(x_0) > u^*(x_0) \quad \text{and} \quad v_*(x_0) > v^*(x_0).$$

We then define the open set

$$\mathcal{D} := \{x \in \Omega;\ u_*(x) > u^*(x) \quad \text{and} \quad v_*(x) > v^*(x)\}.$$

Note that $\mathcal{D} \neq \emptyset$, since $x_0 \in \mathcal{D}$. On $\partial\mathcal{D}$ one has

$$\omega(x) := \max\{u^* - u_*, v^* - v_*\} = 0.$$

Also, since (u^*, v^*) is a viscosity super-solution and (u_*, v_*) is a viscosity sub-solution of the system (1.2), then

$$F(D^2u^*, x) \leq (v^*)_+^p(x) \leq (v_*)_+^p(x) \leq F(D^2u_*, x), \quad x \in \mathcal{D}.$$

As F is uniformly elliptic, the above inequality implies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(u^* - u_*)) \leq 0 \quad \text{in} \quad \mathcal{D}.$$

Similarly, we conclude that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(v^* - v_*)) \leq 0 \quad \text{in} \quad \mathcal{D}.$$

Hence,

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\omega) \leq 0 \quad \text{in} \quad \mathcal{D}.$$

On the other hand, $\omega = 0$ on $\partial\mathcal{D}$, therefore $\omega \geq 0$ in $\mathcal{D}$, which is a contradiction, since $\omega(x_0) < 0$. ■

4.2. Radial analysis. Here we construct a radial viscosity super-solution for the system (1.2). Understanding these radial solutions of the system plays a crucial role in proving non-degeneracy of solutions, as well as certain measure estimates. As was shown in Section 2, the system (1.2) has a solution. In certain cases solutions of the system (1.2) can be constructed explicitly, as shows the following example. One can easily see that the pair of functions

$$u(x) := A\left(|x| - \rho\right)_+^{\frac{2(1+p)}{1-pq}} \quad \text{and} \quad v(x) := B\left(|x| - \rho\right)_+^{\frac{2(1+q)}{1-pq}},$$

where A, B are universal constants depending only on p, q and n and $\rho \geq 0$, is a viscosity solution for the system (see Figure 1)

$$\begin{cases} \Delta u = v_+^p, & \text{in} \ \ \mathbb{R}^n, \\ \Delta v = u_+^q & \text{in} \ \ \mathbb{R}^n. \end{cases}$$

In general, as the operators F and G are uniformly elliptic, one can show that the system (1.2) has a radial viscosity sub- and super-solutions. Below we construct such functions. For that purpose, let us define

$$\tilde{u}(x) := A|x|^\alpha \quad \text{and} \quad \tilde{v}(x) := B|x|^\beta,$$

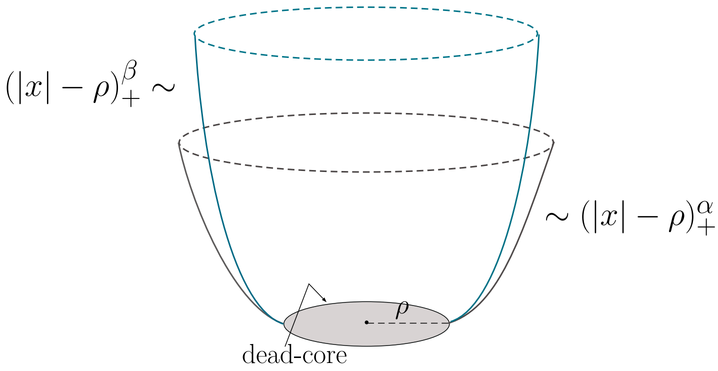

FIGURE 1. Radial solution

where

$$\alpha := \frac{2(1+p)}{1-pq}, \quad \beta := \frac{2(1+q)}{1-pq}$$

and A, B are positive constants to be chosen a posteriori. Direct computation gives

$$\tilde{u}_{ij}(x) = A\alpha\left[(\alpha-2)|x|^{\alpha-4}x_i x_j + \delta_{ij}|x|^{\alpha-2}\right],$$

where $\delta_{ij} = 1$, if $i = j$ and is zero otherwise. Thus, using the ellipticity of F, we estimate

$$F(D^2\tilde{u}, x) \leq \Lambda\left[A\alpha(\alpha-1) + (n-1)A\alpha\right]|x|^{\alpha-2}. \tag{4.1}$$

The aim is to choose the constants A and B such that $(\tilde{u}, \tilde{v})$ is a viscosity super-solution of the system (1.2). If B is such that

$$\Lambda\left[A\alpha(\alpha-1) + (n-1)A\alpha\right] = B^p, \tag{4.2}$$

then (4.1) yields

$$F(D^2\tilde{u}, x) \leq \left[B|x|^{\frac{2(1+q)}{1-pq}}\right]^p.$$

Similarly,

$$G(D^2\tilde{v}, x) \leq \Lambda\left[B\beta(\beta-1) + (n-1)B\beta\right]|x|^{\beta-2}, \tag{4.3}$$

so if A is such that

$$\Lambda\left[B\beta(\beta-1) + (n-1)B\beta\right] = A^q, \tag{4.4}$$

then (4.3) gives

$$G(D^2\tilde{v}, x) \leq \left[A|x|^{\frac{2(1+p)}{1-pq}}\right]^q.$$

Thus, the constants A and B are chosen to satisfy (4.2) and (4.4), which provides

$$A = \Lambda^{\frac{p+1}{pq-1}} \left[\alpha(\alpha-1)+\alpha(n-1)\right]^{\frac{1}{pq-1}} \left[\beta(\beta-1)+\beta(n-1)\right]^{\frac{p}{pq-1}} \tag{4.5}$$

and

$$B = \Lambda^{\frac{q+1}{pq-1}} \left[\beta(\beta-1)+\beta(n-1)\right]^{\frac{1}{pq-1}} \left[\alpha(\alpha-1)+\alpha(n-1)\right]^{\frac{q}{pq-1}}. \tag{4.6}$$

In conclusion, $(\tilde{u}, \tilde{v})$ is a radial viscosity super-solution of the system (1.2). We state it below for a future reference.

Proposition 4.1. *The pair of functions*

$$\tilde{u}(x) := A|x|^{\frac{2(1+p)}{1-pq}} \quad \text{and} \quad \tilde{v}(x) := B|x|^{\frac{2(1+q)}{1-pq}},$$

where A and B are defined by (4.5) and (4.6) respectively, is a viscosity super-solution of the system (1.2). Similarly, $(\tilde{u}, \tilde{v})$ is a radial viscosity sub-solution of (1.2), if in the definition of A, B the constant Λ is substituted by λ.

4.3. Non-degeneracy of solutions. Using Lemma 4.1 and Proposition 4.1, we obtain non-degeneracy of solutions.

Theorem 4.1. *If (u, v) is a non-negative bounded viscosity solution of (1.2) in B_1, and $y \in \overline{\{|(u,v)| > 0\}} \cap B_{1/2}$, then for a constant $c > 0$, depending on the dimension, one has*

$$\sup_{\overline{B}_r(y)} |(u,v)| \geq c r^{\frac{2}{1-pq}},$$

for any $r \in \left(0, \frac{1}{2}\right)$.

Proof: Since u and v are continuous, it is enough to prove the theorem for the points $y \in \{|(u,v)| > 0\} \cap B_{1/2}$, i.e. $u(y) > 0$ and $v(y) > 0$. By translation, we may assume, without loss of generality, that $y = 0$. Let now $\tilde{u}$ and $\tilde{v}$ be as in Proposition 4.1. Note that if for a $\xi \in \partial B_r(y)$ one has

$$(u(\xi), v(\xi)) \geq (\tilde{u}(\xi), \tilde{v}(\xi)), \tag{4.7}$$

then

$$\sup_{\overline{B}_r} |(u,v)| \geq |(u(\xi), v(\xi))| \geq c r^{\frac{2}{1-pq}}.$$

Hence, it is enough to check (4.7). Suppose it is not true, i.e.,

$$(u(\xi), v(\xi)) < (\tilde{u}(\xi), \tilde{v}(\xi)), \quad \forall \xi \in \partial B_r.$$

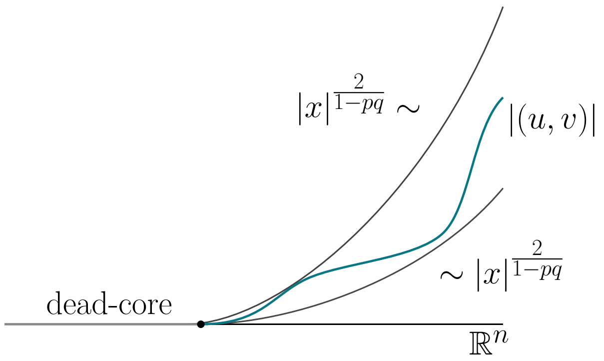

FIGURE 2. The growth of $|(u,v)|$ near the free boundary

Define

$$\hat{u} = \begin{cases} \min\{u, \tilde{u}\} & \text{in} \quad B_r \\ u & \text{in} \quad B_r^c \end{cases}$$

and

$$\hat{v} = \begin{cases} \min\{v, \tilde{v}\} & \text{in} \quad B_r \\ v & \text{in} \quad B_r^c. \end{cases}$$

Since $(\tilde{u}, \tilde{v})$ is a super-solution and (u, v) is a solution of (1.2), then $(\hat{u}, \hat{v})$ also is a super-solution (see [12, Lemma 3.1]). On the other hand, Lemma 4.1 implies

$$0 < u(0) \leq \hat{u}(0) = 0 \quad \text{or} \quad 0 < v(0) \leq \hat{v}(0) = 0,$$

which is a contradiction. ■

Corollary 4.1. *If (u, v) is a non-negative viscosity solution of (1.2), and $x_0 \in \partial\{|(u,v)| > 0\}$, then*

$$c\, r^{\frac{2}{1-pq}} \leq \sup_{B_r(x_0)} |(u,v)| \leq C\, r^{\frac{2}{1-pq}}$$

where $r > 0$ is small enough and c, $C > 0$ are constants depending on the dimension.

5. Applications

As a consequence of the regularity and non-degeneracy results, we obtain geometric measure estimates and derive a Liouville type theorem for entire solutions of the system.

5.1. Geometric measure estimates. We establish positive density and porosity results for the free boundary. Moreover, we show that its $(n-\varepsilon)$-dimensional Hausdorff measure is finite, where $\varepsilon > 0$. We use $|E|$ for the n-dimensional Lebesgue measure of the set E. We also recall the definition of a porous set.

Definition 5.1. *The set $E \subset \mathbb{R}^n$ is called porous with porosity σ, if there is $R > 0$ such that $\forall x \in E$ and $\forall r \in (0, R)$ there exists $y \in \mathbb{R}^n$ such that*

$$B_{\sigma r}(y) \subset B_r(x) \setminus E.$$

A porous set of porosity σ has Hausdorff dimension not exceeding $n - c\sigma^n$, where $c > 0$ is a constant depending only on dimension. In particular, a porous set has Lebesgue measure zero (see [13, Theorem 2.1], for instance).

Lemma 5.1. *If (u, v) is a non-negative bounded viscosity solution of (1.2) in B_1, and $y \in \partial\{|(u, v)| > 0\} \cap B_{1/2}$, then*

$$|B_\rho(y) \cap \{|(u, v)| > 0\}| \geq c\rho^n, \quad \forall \rho \in \left(0, \frac{1}{2}\right),$$

where $c > 0$ is a constant depending only on λ, Λ, $\|u\|_\infty$, $\|v\|_\infty$, p, q and n. Moreover, the free boundary is a porous set, and as a consequence

$$\mathcal{H}^{n-\varepsilon}\left(\partial\{|(u, v)| > 0\} \cap B_{1/2}\right) < \infty,$$

where $\varepsilon > 0$ is a constant depending only on λ, Λ, p, q and n.

Proof: For a $\rho \in (0, 1/2)$, Theorem 4.1 guarantees the existence of a point ξ_ρ such that

$$|\,(u(\xi_\rho), v(\xi_\rho))\,| \geq c\rho^{\frac{2}{1-pq}}. \tag{5.1}$$

On the other hand, for a $\tau > 0$, take

$$y_0 \in B_{\tau\rho}(\xi_\rho) \cap \partial\{|(u, v)| > 0\} \neq \emptyset.$$

Recalling Theorem 3.1 and using (5.1), we have

$$c\rho^{\frac{2}{1-pq}} \leq |(u(\xi_\rho), v(\xi_\rho))| \leq \sup_{B_{\rho\tau}(y_0)} \leq C\,(\rho\tau)^{\frac{2}{1-pq}},$$

which is a contradiction once

$$\tau < \left(\frac{c}{C}\right)^{\frac{1-pq}{2}}.$$

Therefore, for $\tau > 0$ small enough

$$B_{\tau\rho}(\xi_\rho) \subset \{|(u,v)| > 0\},$$

and hence

$$|B_\rho(y) \cap \{|(u,v)| > 0\}| \geq |B_\rho(y) \cap B_{\tau\rho}(\xi_\rho)| \geq c\rho^n.$$

It remains to check the finiteness of the $(n-\varepsilon)$-dimensional Hausdorff measure of the free boundary, which, as observed above, is a consequence of its porosity. To show the latter, it is enough to take

$$y^* := t\xi_\rho + (1-t)y,$$

where t is close enough to 1 to guarantee

$$B_{\frac{\tau}{2}\rho}(y^*) \subset B_\tau(\xi_\rho) \cap B_\rho(y) \subset B_\rho(y) \setminus \partial\{|(u,v)| > 0\},$$

i.e., the set $\partial\{|(u,v)| > 0\} \cap B_{1/2}$ is a $\frac{\tau}{2}$-porous set, and the result follows. ■

5.2. Liouville type results for systems. As another application of the regularity result above, exploiting the ideas from [18], we obtain a Liouville type theorem for solutions of (1.2) in $\Omega = \mathbb{R}^n$. We refer to them as entire solutions of the system. Although Theorem 3.1 provides with regularity information only across the free boundary, it is enough to show that the only entire solution, which vanishes at a point and has a growth suitably controlled at infinity, is the trivial one.

Theorem 5.1. *Let (u,v) is a non-negative viscosity solution of*

$$\begin{cases} F(D^2u, x) = v_+^p & \text{in} \quad \mathbb{R}^n \\ G(D^2v, x) = u_+^q & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{5.2}$$

and $u(x_0) = v(x_0) = 0$. If

$$|(u(x), v(x))| = o\left(|x|^{\frac{2}{1-pq}}\right), \quad \text{as} \quad |x| \to \infty, \tag{5.3}$$

then $u \equiv v \equiv 0$.

Proof: Without loss of generality, we may assume that $x_0 = 0$. We then define

$$u_k(x) := k^{\frac{-2(1+p)}{1-pq}} u(kx) \quad \text{and} \quad v_k(x) := k^{\frac{-2(1+q)}{1-pq}} v(kx),$$

for $k \in \mathbb{N}$ and note that the pair (u_k, v_k) is a viscosity solution of the system (1.2) in B_1. Moreover, $u_k(0) = v_k(0) = 0$, since $0 \in \partial\{|(u,v)| > 0\}$. Therefore, one can apply Theorem 3.1 to estimate $|(u_k, v_k)|$. More precisely, let

$x_k \in \overline{B}_r$ be such that $|(u_k, v_k)|$ reaches its supremum at that point, for $r > 0$ small. Applying Theorem 3.1, we obtain

$$|(u_k(x_k), v_k(x_k))| \leq C_k |x_k|^{\frac{2}{1-pq}}, \tag{5.4}$$

where $C_k > 0$ goes to zero, as $k \to 0$. Thus, if $|kx_k|$ is bounded as $k \to \infty$, then $|(u_k(kx_k), v_k(kx_k))|$ remains bounded. The latter implies

$$|(u_k, v_k)| \to 0, \quad \text{as} \quad k \to \infty. \tag{5.5}$$

Note that due to (5.3), (5.5) remains true also in the case when $|kx_k| \to \infty$, as $k \to \infty$. In fact, from (5.3) one gets

$$|(u_k(x_k), v_k(x_k))| \leq |kx_k|^{-\frac{2}{1-pq}} k^{-\frac{2}{1-pq}} \to 0.$$

Our aim is to show that both u and v are identically zero. Let us assume this is not the case. If $y \in \mathbb{R}^n$ is such that $|(u(y), v(y))| > 0$, then by choosing $k \in \mathbb{N}$ large enough so $y \in B_{kr}$, and using (5.4), (5.5), we estimate

$$\begin{aligned} \frac{|(u(y), v(y))|}{|y|^{\frac{2}{1-pq}}} &\leq \sup_{B_{kr}} \frac{|(u(x), v(x))|}{|x|^{\frac{2}{1-pq}}} \\ &= \sup_{B_r} \frac{|(u_k(x), v_k(x))|}{|x|^{\frac{2}{1-pq}}} \\ &\leq \frac{|(u(y), v(y))|}{2|y|^{\frac{2}{1-pq}}}, \end{aligned}$$

which implies $|(u(y), v(y))| = 0$, a contradiction. ∎

Additionally, if

$$F(0, x) = G(0, x) = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{5.6}$$

then Theorem 5.1 can be improved by relaxing (5.3) and not requiring $\partial\{|(u, v) > 0|\} \neq \emptyset$.

Theorem 5.2. *Let (5.6) hold. If (u, v) is a non-negative viscosity sub-solution of (5.2) and*

$$\limsup_{|x| \to \infty} \frac{|(u(x), v(x))|}{|x|^{\frac{2}{1-pq}}} < \min\{A^{\frac{1}{1+p}}, B^{\frac{1}{1+q}}\}, \tag{5.7}$$

where A and B are the constants defined by (4.5) amd (4.6) respectively, then $u \equiv v \equiv 0$.

Proof: Set

$$\mathcal{S}_R := \sup_{\partial B_R} |(u,v)| = \sup_{\partial B_R} \left(u^{\frac{1}{1+p}} + v^{\frac{1}{1+q}} \right).$$

Using (5.7), we choose $\theta < 1$ and $R \gg 1$ such that

$$R^{-\frac{2}{1-pq}} \mathcal{S}_R \leq \theta m, \tag{5.8}$$

where $m := \min\{A^{\frac{1}{1+p}}, B^{\frac{1}{1+q}}\}$. Recall that the pair of functions

$$\tilde{u}(x) := A(|x|-r)_+^{\frac{2(1+p)}{1-pq}} \quad \text{and} \quad \tilde{v}(x) := B(|x|-r)_+^{\frac{2(1+q)}{1-pq}}, \;\; x \in \mathbb{R}^n,$$

where $r > 0$, is a viscosity super-solution of (5.2), Proposition 4.1, and the corresponding dead-core is the ball of radius r centered at 0, that is, $\{|(\tilde{u},\tilde{v})| = 0\} = B_r$. For $R \gg 1$, taking

$$r := R - \left[\frac{1}{m} \mathcal{S}_R \right]^{\frac{1-pq}{2}} \geq (1 - \theta^{\frac{1-pq}{2}}) R, \tag{5.9}$$

from (5.8) on ∂B_R we have

$$\tilde{u} = A(R-r)_+^{\frac{2(1+p)}{1-pq}} = A \left[\frac{1}{m} \mathcal{S}_R \right]^{1+p} \geq \sup_{\partial B_R} u$$

and

$$\tilde{v} = B(R-r)_+^{\frac{2(1+q)}{1-pq}} = B \left[\frac{1}{m} \mathcal{S}_R \right]^{1+q} \geq \sup_{\partial B_R} v.$$

Then, by Lemma 4.1, we deduce

$$\max\{\tilde{u} - u, \tilde{v} - v\} \geq 0 \quad \text{in } B_R.$$

Thus, for a fixed point $x \in \mathbb{R}^n$, choosing R large enough and using (5.9), we conclude that either $u(x) = 0$ or $v(x) = 0$, i.e.,

$$u(x)v(x) = 0, \;\; x \in \mathbb{R}^n.$$

The latter implies

$$\{u > 0\} \subset \{v = 0\} \quad \text{and} \quad \{v > 0\} \subset \{u = 0\}. \tag{5.10}$$

Suppose there exists $y \in \mathbb{R}^n$ such that $u(y) > 0$. Since u is continuous, then it remains positive in a neighborhood of y. From (5.10) we obtain $v = 0$ in that neighborhood. On the other hand, from (5.6) we have $0 = G(0,y) \geq u_+^q$, which implies that $u(y) = 0$, a contradiction. ∎

Remark 5.1. *Observe that the constant on the right hand side of* (5.7) *is sharp. Indeed, for the pair of functions* $(\tilde{u}, \tilde{v})$ *defined above, one has equality in* (5.7)*, but neither* $\tilde{u}$ *nor* $\tilde{v}$ *is not identically zero.*

Acknowledgments. DJA is partially supported by CNPq and grant 2019/0014 Paraiba State Research Foundation (FAPESQ). He thanks Centre for Mathematics of the University of Coimbra (CMUC) and the Abdus Salam International Centre for Theoretical Physics (ICTP) for great hospitality during his research visits. RT is partially supported by FCT - Fundação para a Ciência e a Tecnologia, I.P., through projects PTDC/MAT-PUR/28686/2017 and UTAP-EXPL/MAT/0017/2017, as well as by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES. RT thanks Department of Mathematics of the Universidade Federal da Paraíba (UFPB) for hospitality and great working conditions during his research visit.

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