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ON EQUALIZERS IN THE CATEGORY OF LOCALES

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ABSTRACT: The fact that equalizers in the context of strongly Hausdorff locales (similarly like those in classical spaces) are closed is a special case of a standard categorical fact connecting diagonals with general equalizers. In this paper we analyze this and related phenomena in the category of locales. Here the mechanism of pullbacks connecting equalizers is based on natural preimages that preserve a number of properties (closedness, openness, fittedness, complementedness, etc.). Also, we have a new simple and transparent formula for equalizers in this category providing very easy proofs for some facts (including the general behavior of diagonals). In particular we discuss some aspects of the closed case (strong Hausdorff property), and the open and clopen one.

KEYWORDS: Frame, locale, sublocale, localic map, image and preimage, binary product of locales, strongly Hausdorff locale, diagonal map, equalizer, open diagonal.

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Introduction

The diagonal morphism is an equalizer of the projections, and pullbacks pull back equalizers, as in the following diagram

$$\begin{tikzcd}
P \arrow[r, "\mathrm{Equ}(f_1,f_2)"] \arrow[d, "g"'] & A \arrow[d, "\langle f_1,f_2\rangle"] \arrow[dr, "f_i"] & \\
B \arrow[r, "\delta_B"'] & B\times B \arrow[r, "p_i"'] & B
\end{tikzcd}$$

This simple categorical fact covers a number of facts connecting the behaviour of diagonals in categories with the general behaviour of equalizers (a

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very special example being the closed equalizers in Hausdorff spaces related to the fact that the Hausdorff property of X is characterized by the closed diagonal in $X \times X$, for more see [2]). In this paper we concentrate on some aspects of this phenomenon in the category of locales. In this category we can exploit some expedient concrete facts: subobjects, the so called sublocales, are very transparent entities, easy to work with, pulling back is being done by preimages, again transparent and closely reminiscent to the classical preimages both in the form and capability (preserving closedness and openness and other useful properties, some of them more relevant in the point-free context then classically), and a new very simple formula for equalizer.

In Preliminaries we recall the standard notation and facts, and add, for convenience of the reader, a brief description of binary coproducts of frames in the form it will be used. Then, we interpret the mentioned categorical fact and one of its straightforward extensions in the category of locales, and using the properties of preimages in this context list a number of properties of equalizers inherited from properties of the diagonal. In particular we turn to the closedness, concerning the famous Isbell's (strong) Hausdorff property, briefly mention the closed equalizer theorem, and discuss a certain criterion of this property. In the following section we then introduce a simple formula for equalizers in the category of locales and use it for (three-line) proofs of basic properties of the diagonal in general. Next, adding the closedness yields, again in a very simple way, basic characteristics of strongly Hausdorff frames, the T_U property and the Dowker-Strauss characteristic. In the last section we briefly discuss the cases of open and clopen diagonals.

1. Preliminaries

1.1. We will use the standard notation for posets; in particular we will write for subsets $A \subseteq (X, \leq)$

$$\downarrow A = \{x \mid \exists a \in A, x \leq a\},\ \downarrow a = \downarrow\{a\},$$

$$\uparrow A = \{x \mid \exists a \in A, x \geq a\},\ \uparrow a = \uparrow\{a\},$$

and speak of the A with $\downarrow A = A$ resp. $\uparrow A = A$ as of *down-sets* resp. *up-sets*. Our posets will be typically complete lattices; the suprema (joins) of subsets will be denoted by $\bigvee A$, $\bigvee_{i \in J} a_i$, $a \vee b$ etc., and infima (meets) by $\bigwedge A$, $\bigwedge_{i \in J} a_i$, $a \wedge b$ etc.

1.2. Adjoint maps. Monotone maps $f\colon X \to Y$, $g\colon Y \to X$ are said to be *adjoint, f to the left, g to the right* (we write $f \dashv g$, $g = f^*$, $f = g_*$), whenever

$$f(x) \le y \quad \text{iff} \quad x \le g(y).$$

This is characterized by $fg(y) \le y$ and $x \le gf(x)$, and if $f \dashv g$ then f (resp. g) preserves all the existing suprema (resp. infima). On the other hand,

1.2.1. *if X, Y are complete lattices then an $f\colon X \to Y$ preserving all suprema (a $g\colon Y \to X$ preserving all infima) has a right (left) adjoint.*

1.3. Heyting algebras. A bounded lattice (poset with finite suprema and infima) L is called a *Heyting algebra* if there is a binary operation $x \to y$ (the *Heyting operation*) such that for all a, b, c in L,

$$a \wedge b \le c \quad \text{iff} \quad a \le b \to c. \tag{Hey}$$

(Hey) says precisely that

(H1) *for every b the mapping $b \to (-)\colon L \to L$ is a right adjoint to $(-) \wedge b\colon L \to L$*

and hence the operation $\to$, if it exists, is uniquely determined. From 1.2 it immediately follows that

(H2) *in a Heyting algebra one has $(\bigvee A) \wedge b = \bigvee_{a\in A}(a \wedge b)$ for any $A \subseteq L$, $b \to (\bigwedge A) = \bigwedge_{a\in A}(b \to a)$, and $(\bigvee B) \to a = \bigwedge_{b\in B}(b \to a)$.*

1.3.1. A few Heyting rules. We will often use computation with the operation $\to$. Here are some immediate consequences of (Hey):

(1) $a \le b \to a$, (2) $1 \to a = a$, (3) $a \to b = 1$ iff $a \le b$,
(4) $a \wedge (a \to b) \le b$ and consequently (using (1)) $a \wedge (a \to b) = a \wedge b$, (5) $a \le b \to c$ iff $b \le a \to c$.
And also the three further useful *rules* are very simple.
(6) $a \to (b \to c) = (a \wedge b) \to c = b \to (a \to c)$
(we have $x \le a \to (b \to c)$ iff $x \wedge a \le b \to c$ iff $x \wedge a \wedge b \to c$ iff $x \le (a \wedge b) \to c$).
(7) $a \to b = a \to c$ iff $a \wedge b = a \wedge c$
($\Rightarrow$ by (4), $\Leftarrow$: By (3) and(H2), $a \to b = (a \to a) \wedge (a \to b) = a \to (a \wedge b) = a \to (a \wedge c) = a \to c$).
(8) $x = (x \vee a) \wedge (a \to x)$
(By (H1), (4) and (1), $(x \vee a) \wedge (a \to x) = (a \wedge (a \to x) \vee (x \wedge (a \to x) \le x$, by (1), $x \le (x \vee a) \wedge (a \to x)$).

1.4. Frames and coframes. A *frame*, resp. *coframe*, is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}, \qquad \text{(frm)}$$

$$\text{resp.} \quad (\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}, \qquad \text{(cofrm)}$$

for all $A \subseteq L$ and $b \in L$; a *frame* (resp. *coframe*) *homomorphism* preserves all joins and all finite meets (resp. all meets and all finite joins). The lattice $\Omega(X)$ of all open subsets of a topological space X is an example of a frame, and if $f\colon X \to Y$ is continuous we obtain a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ by setting $\Omega(f)(U) = f^{-1}[U]$. Thus we have a contravariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}$$

from the category of topological spaces into that of frames.

Note that (frm) makes by 1.2.1 every frame to a Heyting algebra.

1.5. The concrete category of locales. The functor Ω is on a very substantial subcategory of **Top** (that of sober spaces[1]) a full embedding. Thus, but for the contravariance, we can view frames as a generalization of space, and the contravariance is mended by considering the *category of locales* $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$.

It is of advantage to represent it as a concrete category as follows. Since frame homomorphisms $h\colon M \to L$ preserve all joins they have uniquely defined right adjoints $f = h_*\colon L \to M$. We will represent **Loc** as the category with frames for objects (in this context we often speak of frames as of locales) and meet preserving maps $f\colon L \to M$ such that f^* are frame homomorphisms (the *localic maps*) for morphisms.

1.5.1. Here is a useful characterization (the so called *Frobenius equality*):

> *a meet preserving* $f\colon L \to M$ *is a localic map iff* $f(a) = 1$ *only for* $a = 1$ *and* $f(f^*(a) \to b) = a \to f(b)$.

1.6. Binary coproduct in Frm. In the sequel we will use an explicit description of the binary coproduct. For this we will need the following facts.

[1] A space is *sober* if every completely prime filter $\mathcal{F}$ in $\Omega(X)$ (that is, an $\mathcal{F}$ such that $\bigcup_{i\in J} U_i \in \mathcal{F}$ only if $U_j \in \mathcal{F}$ for some $j \in J$) is $\{U \mid x \in U\}$ for some $x \in X$ – in other words, if every system of open sets that looks like a neighborhood system is really a neighborhood system of a point. For instance every Hausdorff space is sober.

- In the category of bounded semilattices the cartesian product with the injections and projections as in
$$L_i \xrightarrow{\iota'_i} L_1 \times L_2 \xrightarrow{((a_1,a_2)\mapsto a_j)} L_j \ ,$$
with $\iota'_1 = (a \mapsto (a,1))$ and $\iota'_2 = (a \mapsto (1,a))$, constitutes a biproduct (easy to check).
- A quotient of a frame L by a (congruence induced by a) relation R can be obtained as $L/R = \{s \in L \mid s$ is R-saturated$\}$ where s is R-saturated if for all a, b, c, $aRb \Rightarrow \ (a \wedge c \leq s$ iff $b \wedge c \leq s)$, and that a homomorphism $h\colon L \to M$ such that $aRb \Rightarrow h(a) = h(b)$ factorizes to $\overline{h}\colon L/R \to M$ by taking the restriction (see e.g. [12, III.11]).

1.6.1. The down-set frame. For a bounded semilattice L (poset with finite meets and minimal element) consider
$$\mathfrak{D}(L) = (\{U \mid \emptyset \neq U = {\downarrow}U,\ U \subseteq L\}, \subseteq)$$
and define $\lambda = \lambda_L\colon L \to \mathfrak{D}(L)$ by setting $\lambda(a) = {\downarrow}a$. Obviously,

$\mathfrak{D}(L)$ is a frame, and λ is a semilattice homomorphism.

The pair $(\mathfrak{D}, \lambda)$ establishes a reflection of the category of bounded semilattices into **Frm**.

Proposition. *Let M be a frame and let $h\colon L \to M$ be a semilattice homomorphism. Then there is precisely one frame homomorphism $\widetilde{h}\colon \mathfrak{D}(L) \to M$ such that $\widetilde{h} \cdot \lambda = h$. It is given by the formula $\widetilde{h}(U) = \bigvee\{h(a) \mid a \in U\}$.*

1.6.2. Now a coproduct $L_1 \oplus L_2$ of frames L_i can be obtained as $\mathfrak{D}(L_1 \times L_2)/R$ with injections
$$\iota_i = \ L_i \xrightarrow{\iota'_i} L_1 \times L_2 \xrightarrow{\lambda} \mathfrak{D}(L_1 \times L_2) \longrightarrow \mathfrak{D}(L_1 \times L_2)/R = L_1 \oplus L_2$$
where the relation R is
$$R = \Big\{(\bigcup_{i\in J} {\downarrow}(a_i, b), {\downarrow}(\bigvee_{i\in J} a_i, b)) \mid a_i \in L_1, b \in L_2\Big\} \cup$$
$$\cup \ \Big\{(\bigcup_{i\in J} {\downarrow}(a, b_i), {\downarrow}(a, \bigvee_{i\in J} b_i)) \mid a \in L_1, b_i \in L_2\Big\}$$
and (hence) the R-saturated $U \in \mathfrak{D}(L_1 \times L_2)$ are precisely the down-sets such that for any (a_i, b), $i \in J$, in U we have also $(\bigvee_i a_i, b) \in U$, and similarly

for any (a, b_i), $i \in J$, in U also $(a, \bigvee_i b_i) \in U$. Such U will be referred to as *cp-ideals.*

In particular there are the cp-ideals

$$a \oplus b = {\downarrow}(a, b) \cup \{(x, y) \mid x = 0 \text{ or } y = 0\}.$$

In this notation obviously $\iota_1(a) = a \oplus 1$, $\iota_2(b) = 1 \oplus b$, $U = \bigvee\{a \oplus b \mid a \oplus b \subseteq U\}$ for all $U \in L_1 \oplus L_2$, and if $a, b \neq 0$ and $a \oplus b \subseteq a' \oplus b'$ then $a \leq a'$ and $b \leq b'$.

1.6.3. Diagonal. Thus, the codiagonal frame homomorphism $\delta^*\colon L \oplus L \to L$ (defined by $\delta^* \iota_i = \mathrm{id}$) is given by

$$\delta^*(a \oplus b) = \delta^*(a \oplus 1) \wedge \delta^*(1 \oplus b) = a \wedge b$$

resulting in

$$\delta^*(U) = \bigvee\{a \wedge b \mid a \oplus b \subseteq U\} = \bigvee\{a \wedge b \mid (a, b) \in U\}.$$

It is adjoint to the localic diagonal map $\delta\colon L \to L \oplus L$ given by

$$\delta(a) = \{(u, v) \mid u \wedge v \leq a\}.$$

1.6.4. Discrete coproducts. The functor Ω from 1.4 does not generally send products to coproducts. For locally compact spaces, however, it does, under the axiom of choice ([7], see also [10, 12]). We will need only the special case of the Boolean frames $\mathfrak{P}(X)$ (the "discrete case") and there it is constructive. We will present a proof; it is a short easy exercise of working with cp-ideals.

Denote by $p_i\colon X_1 \times X_2 \to X_i$ the projections, by $\iota_i\colon \mathfrak{P}(X_i) \to \mathfrak{P}(X_1) \oplus \mathfrak{P}(X_2)$ the coproduct injections, and by $\mu\colon \mathfrak{P}(X_1) \oplus \mathfrak{P}(X_2) \to \mathfrak{P}(X_1 \times X_2)$ the homomorphism defined by $\mu\, \iota_i = \Omega(p_i)$.

Proposition. *μ is an isomorphism.*

Proof: We easily see that

$$\mu(U) = \bigcup\{M \times N \mid M \oplus N \subseteq U\} = \bigcup\{M \times N \mid (M, N) \in U\}.$$

μ is obviously onto, and since all the frames in question are regular[2], to see that it is an isomorphism it suffices to prove that $\mu(U) = X_1 \times X_2$ only if $U = \mathfrak{P}(X_1) \times \mathfrak{P}(X_2)$. Thus, let $U = \{(M_i, N_i) \mid i \in J\}$ be a cp-ideal and let $\mu(U) = \bigcup_{i \in J} M_i \times N_i = X_1 \times X_2$. For a $y \in X_2$ set $J(y) = \{i \mid y \in N_i\}$. Then obviously: for every $i \in J(y)$, $(M_i, \{y\}) \in U$ (since U

[2] If L is regular then a homomorphism $h\colon M \to L$ is one-to-one whenever $h(a) = 1$ implies that $a = 1$ – see e.g. [12, V.5.6].

is a down-set), $(X_1, \{y\}) = (\bigcup_{i\in J(y)} M_i, \{y\}) \in U$ (since U is a cp-ideal), and $(X_1, X_2) = (X_1, \bigcup\{y \mid y \in X_2\}) \in U$ (since U is a cp-ideal). Thus, $U = {\downarrow}(X_1, X_2) = \mathfrak{P}(X_1) \times \mathfrak{P}(X_2)$. ■

1.7. Sublocales. The *sublocales*, subobjects of L in **Loc**, are, naturally, the subsets such that the embedding map is an extremal monomorphism. It turns out that they are precisely the $S \subseteq L$ such that

(S1) for every $M \subseteq S$, $\bigwedge M \in S$, and
(S2) for every $x \in L$ and every $s \in S$, $x \to s \in S$.

Any intersection of sublocales is a sublocale so that we have a complete lattice $\mathsf{S}(L)$ of sublocales of L with the join given by the formula $\bigvee_{i\in J} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i\in J} S_i\}$. One has that

$\mathsf{S}(L)$ is a coframe.

Each element $a \in L$ is associated with a *closed* sublocale $\mathfrak{c}(a)$ and an *open* sublocale $\mathfrak{o}(a)$,

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \in L \mid a \to x = x\}$$

(the equivalence of the two expressions for $\mathfrak{o}(a)$ immediately follows from 1.3.1(6)). These special sublocales extend the concepts of open and closed subspaces, and behave as they should: in $\Omega(X)$ they precisely correspond to the homonymous subspaces, they are complements of each other, all joins and finite meets of open sublocales are open, and similarly with finite joins and arbitrary meets of closed sublocales.

1.7.1. It is easy to check that the homomorphisms associated with the (localic) embeddings $\mathfrak{c}(a) \subseteq L$ resp. $\mathfrak{o}(a) \subseteq L$ are given by

$$x \mapsto a \vee x \quad \text{resp.} \quad x \mapsto a \to x.$$

1.8. Images and preimages. If $f\colon L \to M$ is a localic map and if $S \subseteq L$ is a sublocale then the standard set-theoretical image $f[S]$ is also a sublocale. The set preimage $f^{-1}[T]$ of a sublocale $T \subseteq M$ is generally not one, but it is a subset closed under meets and hence (recall the formula for $\bigvee_i S_i$ in $\mathsf{S}(L)$ above) we have the sublocale

$$f_{-1}[T] = \bigvee\{S \mid S \in \mathsf{S}(L),\ S \subseteq f^{-1}[T]\},$$

the *localic preimage* of T under f. One has the adjunction

$$f[S] \subseteq T \quad \text{iff} \quad S \subseteq f_{-1}[T],$$

and $f_{-1}[-]\colon \mathsf{S}(M) \to \mathsf{S}(L)$ is a coframe homomorphism.

(Localic) preimages of open resp. closed sublocales are open resp. closed and one has

$$f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a)) \quad \text{and} \quad f_{-1}[\mathfrak{c}(a)] = f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a)).$$

For more about frames and locales see, e.g., [10, 12].

2. Standard facts, and what they say in Loc

2.1. $\mathcal{P}$-separation. Let $\mathcal{P}$ be a property of monomorphisms in a category $\mathcal{C}$ with pullbacks and binary products. Recall that we speak of $\mathcal{P}$ being *pullback stable* if in every pullback

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "m'"'] & C \arrow[d, "m"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

with m a $\mathcal{P}$-monomorphism, m' is a $\mathcal{P}$-monomorphism.

If the diagonal $\Delta_B\colon B \to B \times B$ has the property $\mathcal{P}$ we say that B is *$\mathcal{P}$-separated* ([4]).

Convention. In categories in which we have subobjects S of A represented by specific monomorphisms $m\colon S \to A$ we speak of S as having the property $\mathcal{P}$ if m has it. Indeed in the concrete categories we speak in this article, the property in question comes first as a property of concrete subobject (the diagonal subspace in $X \times X$ in **Top**, the diagonal sublocale in **Loc**) which is then viewed also as a property of the embedding morphism.

2.2. The following two facts from [4, Theorem and Corollary 4.3] (originally presented in [3, Propositions 10.1, 10.7] in terms of a closure operator $\mathcal{C}$ in the category) play a fundamental role.

2.2.1. Theorem. *Let $\mathcal{P}$ be pullback stable. Then B is $\mathcal{P}$-separated iff for any pair of morphisms $f_1, f_2\colon A \to B$, the equalizer* $\mathrm{Equ}(f_1, f_2) \subseteq A$ *has the property $\mathcal{P}$.*

2.2.2. Theorem. *Let $\mathcal{P}$ be pullback stable and let B be $\mathcal{P}$-separated. Then every A such that there is a monomorphism $m\colon A \to B$ is $\mathcal{P}$-separated.*

2.3. Recall 1.8. In any category with a factorization system $(\mathcal{E}, \mathcal{M})$ and pullbacks, the right adjoint to the direct image is given by pullback. Hence, we have

Proposition. *Let $f\colon L \to M$ be a localic map and let $S \subseteq M$ be a sublocale. Then*

$$\begin{tikzcd} f_{-1}[S] \arrow[r, "\subseteq"] \arrow[d, "g"'] & L \arrow[d, "f"] \\ S \arrow[r, "\subseteq"] & M \end{tikzcd}$$

is a pullback.

Note. It might be instructive to see this directly from the concrete form of the preimage in 1.8. If $f\alpha = j\beta$ in the diagram

$$\begin{tikzcd} X \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & f_{-1}[S] \arrow[r, "k=\subseteq"'] \arrow[d, "g"'] & L \arrow[d, "f"] \\ & S \arrow[r, "j=\subseteq"] & M \end{tikzcd}$$

then $\alpha(x) \in f^{-1}[S]$, but since $\alpha[X]$ is a sublocale, $\alpha(x) \in f_{-1}[S]$ and we can define $\gamma\colon X \to f_{-1}[S]$ by $\gamma(x) = \alpha(x)$. Then $k\gamma = \alpha$ and $jg\gamma = fk\gamma = f\alpha = j\beta$, and hence $g\gamma = \beta$. Since k is one-to-one, γ is uniquely defined.

2.4. Preimages in the category **Loc** are very close to set-theoretic preimages (recall 1.8), and behave very nicely with respect to the geometry of sublocales. Preimages of open resp. closed sublocales are open resp. closed (and joins and meets of such sublocales are preserved precisely as unions and intersections of such subsets in the classical topological spaces are). Furthermore (again similarly like the classical preimage), for every localic map f, $f_{-1}[-]\colon \mathsf{S}(M) \to \mathsf{S}(L)$ is a lattice (indeed coframe) homomorphism. Hence we have a number of pullback stable properties for which the Theorem 2.2.1 can be applied. This concerns, for instance,

(a) closedness,
(b) openness,
(c) clopenness,

(d) being a meet of open sublocales (fittedness [8, 5]), or a meet of at most κ many open sublocales,
(e) complementedness,
(f) being a finite combination of open and closed sublocales (a strong form of complementedness).

Just to give an example of a concrete fact: for the $\mathcal{P}$ from (d), if M is fit, then so is $M \oplus M$, and in particular the diagonal sublocale is fitted, that is, M is $\mathcal{P}$-separated. Hence, we obtain

Proposition. *Let M be fit and let L be arbitrary. Then for any pair of localic maps $f_1, f_2\colon L \to M$ the equalizer* $\mathrm{Equ}(f_1, f_2)$ *is fitted.*

2.5. Property (sH), strongly Hausdorff frames. The $\mathcal{P}$ from (a) above concerns the well known *strong Hausdorff property* suggested by Isbell in [8], abbreviated by (sH). Here we immediately obtain the fundamental property of this separation axiom

2.5.1. Corollary. *M is strongly Hausdorff iff for any pair of localic maps $f_1, f_2\colon L \to M$ the equalizer* $\mathrm{Equ}(f_1, f_2)$ *is closed.*

2.5.2. Note. The closedness of the equalizer $\mathrm{Equ}(f_1, f_2)$ was proved first by Banaschewski for regular M ([2]). The fact that it holds more generally for (sH) frames was later observed in Clementino's PhD Thesis ([3, Proposition 10.1, Example 10.6.4]), as a particular example of the general categorical stability fact (with the property $\mathcal{P}$ replaced by a closure operator $\mathcal{C}$). The novelty in the present paper is the technique of preimages taken as concrete sublocales.

In the following two sections we will analyze the closedness case in view of a new very transparent formula for the equalizer in **Loc**.

2.5.3. Openness. The case (b), openness has also already attracted attention ([9],[12]). In [9] it was shown that it characterizes discrete locales (atomic Boolean algebras). It will be analyzed (with a new proof of the characterization) in the last section.

2.6. Extending by monomorphisms. Recall 2.2.2. This may be not very conspicuous, but in a category with a non-trivial system of monomorphisms it is in fact a very strong criterion. In particular in the category of locales the structure of monomorphisms is fairly wild (the category **Loc** is not well-powered [10, II.2.10]): there are monomorphisms $m\colon A \to B$ such that the size of A arbitrarily exceeds the size of B (in the language of frames, epimorphisms $e\colon B \to A$ with huge A's).

Also, it presents us with a handy criterion of the strong Hausdorff property of spaces. A Hausdorff space is not necessarily strongly Hausdorff as a locale and it is not necessarily easy to distinguish the two properties without a laborious analysis of a localic square of the space. But a regular space is automatically strongly Hausdorff, and we can prove the strong Hausdorff property by showing that the topology in question is an epimorphic target in frames. See the following example (note that the one-to-one embedding onto the larger topology is an epimorphism in **Frm**).

2.6.1. Example. In a topological space $(X, \mathcal{T})$ choose a closed set F and an open one, O, and define a new topology $\mathcal{T}_{OF}$ on X,

$$\mathcal{T}_{OF} = \{U \cup ((O \cup F) \cap V) \mid U, V \in \mathcal{T}\}.$$

The $\mathcal{T}$ is a subframe of $\mathcal{T}_{OF}$ and the embedding is an epimorphism in **Frm** by the following proposition (since $(X \smallsetminus F) \in \mathcal{T}$ satisfies $(O \cup F) \cup (X \smallsetminus F) = X \in \mathcal{T}$ and $(O \cup F) \cap (X \smallsetminus F) = O \cap (X \smallsetminus F) \in \mathcal{T}$).

Proposition. *Let L be a subframe of M and let M be generated by a subset S such that for each $s \in S$ there is a $t_s \in L$ such that both $s \vee t_s$ and $s \wedge t_s$ are in L. Then the embedding $j\colon L \subseteq M$ is an epimorphism.*

Proof: Let $f, g\colon M \to N$ be frame homomorphisms such that $fj = gj$. Consider an arbitrary $s \in S$ and the t_s from the assumption. Set $a = f(t_s) = g(f_s)$. Then $f(s) \vee a = g(s) \vee a$ and $f(s) \wedge a = g(s) \wedge a$, and since N is a distributive lattice we have $f(s) = g(s)$. Since S generates M, $f = g$. ■

Thus, if $\mathcal{T}$ is strongly Hausdorff (in particular, regular), then by 2.2.2 $\mathcal{T}_{OF}$ is strongly Hausdorff.

In particular consider the usual Euclidean topology $\mathcal{T}$ of the closed unit interval. Take

$$O = \bigcup_{n=1}^{\infty} \Big(\frac{1}{n+1}, \frac{1}{n}\Big) \quad \text{and} \quad F = \{0\}.$$

Then the unit interval equipped with $\mathcal{T}_{OF}$ is strongly Hausdorff. It is not regular, though: indeed, $G = \{\frac{1}{n} \mid n = 1, 2, \ldots\}$ is now closed but we easily see that it cannot be separated from 0.

2.6.2. Mono-regular frames. Let us call a locale M *mono-regular* if there is a regular L and a monomorphism $m\colon M \to L$. We have seen that a mono-regular locale is not necessarily regular, while by 2.2.2 it is always strongly Hausdorff. We have more.

Proposition. *Mono-regular locales constitute a reflective* (*in fact an epireflective*) *subcategory of* **Loc**.

Proof: Obviously, mono-regular locales are closed under limits (they are closed under products and equalizers). While **Loc** is not well-powered, it is co-well-powered, and we can use Theorem 16.8 from [1] to conclude the reflectivity. Indeed, let $\mathcal{E}$ be the class of epimorphisms and $\mathcal{M}$ the class of extremal monomorphisms in **Loc**. Then **Loc** is an $(\mathcal{E}, \mathcal{M})$-category that is $\mathcal{E}$-co-well-powered, and by the aforementioned theorem a (full, isomorphism-closed) subcategory of **Loc** is $\mathcal{E}$-reflective iff it is closed under products and $\mathcal{M}$-subobjects. ∎

2.7. Problems. (1) We have a reflective subcategory between that of regular locales and that of strongly Hausdorff ones. The question naturally arises:

Are there strongly Hausdorff locales that are not mono-regular?

(2) The fact that the category of regular locales is also reflective in **Loc** has a useful consequence: a locale L is mono-regular iff the regular reflection $\rho_L\colon L \to R(L)$ (see [12, p. 93] for an explicit description) is monic. The question above reduces then to:

Is there a strongly Hausdorff locale L with ρ_L not monic?

(3) For the property $\mathcal{P}$ in 2.4(d), we have seen that any fit locale is $\mathcal{P}$-separated. What about the converse? Is fitness equivalent to $\mathcal{P}$-separatedness? In case the answer is negative, a study of *mono-fit locales* (that is, locales M for which there is a fit L and a monomorphism $m\colon M \to L$) would be of interest.

3. Equalizers and diagonals in Loc: general formulas

3.1. Theorem. *Let $f_1, f_2\colon L \to M$ be localic maps. Then*

$$E = \mathrm{Equ}(f_1, f_2) = \{s \mid \forall x,\ f_1(x \to s) = f_2(x \to s)\}.$$

Proof: Using the standard Heyting formulas we immediately learn that E is a sublocale. Now let $g\colon K \to L$ be a localic map such that $f_1 g = f_2 g$. By 1.5.1 we obtain

$$f_1(x \to g(y)) = f_1(g(g^*(x) \to y)) = f_2(g(g^*(x) \to y)) = f_2(x \to g(y)). \quad \blacksquare$$

3.1.1. Corollary. *Let $f_1, f_2\colon L \to M$ be localic maps. Let B be a $\bigvee$-basis of L. Then*

$$E = \mathrm{Equ}(f_1, f_2) = \{s \mid \forall x \in B,\ f_1(x \to s) = f_2(x \to s)\}.$$

(Use (H2) in 1.3: $(\bigvee_i x_i) \to s = \bigwedge_i (x_i \to s)$.)

Technically it is often expedient to use the formula translated in the adjoint homomorphisms. We have

3.2. Corollary. *Let $f_1, f_2\colon L \to M$ be localic maps and let $h_1, h_2\colon M \to L$ be the adjoint frame homomorphisms. Let B be a $\bigvee$-basis of L. Then*

$$E = \{s \mid \forall x \in B\ \forall y \in M,\ h_1(y) \wedge x \leq s \textit{ iff } h_2(y) \wedge x \leq s\}.$$

(We have $f_1(x \to s) = f_2(x \to s)$ iff $\forall y,\ (y \leq f_1(x \to s)) \equiv (y \leq f_2(x \to s))$, and $y \leq f_i(x \to s)$ iff $h_i(y) \wedge x \leq s$.)

3.3. The diagonal in Loc. It is the equalizer $D = \mathrm{Equ}(p_1, p_2)$ with p_i the right adjoints to the frame coproduct injections

$$\iota_1 = (y \mapsto y \oplus 1)\colon L \to L \oplus L, \quad \iota_2 = (y \mapsto 1 \oplus y)\colon L \to L \oplus L.$$

Thus we obtain from 3.2 for the cp-ideals U, elements of $L \oplus L$ as in 1.6.2, the following result:

3.3.1. Theorem. *Any of the following two formulas characterize the cp-ideals that are elements of the diagonal $D \subseteq L \oplus L$:*

$$\forall a, b, c, \qquad (a \wedge b) \oplus c \subseteq U \textit{ iff } a \oplus (b \wedge c) \subseteq U \qquad (*)$$

$$\textit{resp.}\ \forall a, b \qquad a \oplus b \subseteq U \textit{ iff } (a \wedge b) \oplus (a \wedge b) \subseteq U. \qquad (**)$$

Proof: Consider the join-basis $\{a \oplus c \mid a, c \in L\}$ of $L \oplus L$. We have $\iota_1(b) \cap (a \oplus c) = (b \oplus 1) \cap (a \oplus c) = (b \wedge a) \oplus c = (a \wedge b) \oplus c$ and similarly $\iota_2(b) \cap (a \oplus b) = a \oplus (b \wedge c)$, translating 3.3 into $(*)$.

Now if $(*)$ holds we have $(a \wedge b) \oplus (a \wedge b) \subseteq U$ iff $a \oplus (a \wedge b \wedge b) = a \oplus (b \wedge a) \subseteq U$ iff $(a \wedge a) \oplus b \subseteq U$, and if we have $(**)$ both $(a \wedge b) \oplus c \subseteq U$ and $a \oplus (b \wedge c) \subseteq U$ are equivalent with $(a \wedge b \wedge c) \oplus (a \wedge b \wedge c) \subseteq U$. $\blacksquare$

3.3.2. Notes. 1. The formulas $(*)$ resp. $(**)$ say, of course, the same as that $(a \wedge b, c) \in U$ iff $(a, b \wedge c) \in U$ resp. $(a, b) \in U$ iff $(a \wedge b, a \wedge b) \in U$.

2. The formulas appear in the literature usually for the closed diagonals in connection with the strong Hausdorff axiom (see 4.1 below). Note that they have nothing to do with any special property. Also note how immediate consequences of the extremely easy formula for the equalizer above they are.

3.4. Proposition. *The smallest cp-ideal in $D \subseteq L \oplus L$ is*

$$d_L = \{(a, b) \mid a \wedge b = 0\}.$$

Proof: If $(a \wedge y) \wedge b = 0$ then $a \wedge (b \wedge y) = 0$, hence $d_L \in D$. On the other hand, if $U \in D$ and $a \wedge b = 0$ then $U \ni (a, a \wedge b)$ and by rule $(*)$ we obtain $U \ni (a, b) = (a \wedge a, b)$; hence $d_L \subseteq U$. ■

3.5. The codiagonal map $M \oplus M \to M$ given by $\delta^*(U) = \bigvee\{u \wedge v \mid (u, v) \in U\}$ yields the adjoint diagonal map $\delta(a) = \{(u, v) \mid u \wedge v \leq a\}$ and the diagonal in the sublocale form as

$$D = D_M = \{\delta(a) \mid a \in M\}$$

with minimum $d = d_M = \delta(0) = \{(u, v) \mid u \wedge v = 0\}$. Thus, the closure $\overline{D}$ is equal to $\uparrow d_M$ and we have, in the notation above and with $c(h_1, h_2) = \bigvee\{h_1(x) \wedge h_2(y) \mid x \wedge y = 0\}$,

$$\mathrm{Equ}(f_1, f_2) = f_{-1}[D_M] \subseteq f_{-1}[\overline{D}_M] = f^{-1}[\uparrow d_M] = \uparrow h(d_M) = \uparrow c(h_1, h_2).$$

Hence for every $s \in \mathrm{Equ}(f_1, f_2)$ we have $s \geq c(h_1, h_2)$, more explicitly

$$\begin{aligned} &\forall x \in L, f_1(x \to s) = f_2(x \to s) \quad \Rightarrow \\ &\qquad \Rightarrow \quad \forall u, v \text{ such that } u \wedge v = 0,\ s \geq h_1(u) \wedge h_2(v). \end{aligned} \tag{$*$}$$

3.5.1. The implication can be written as

$$\min \mathrm{Equ}(f_1, f_2) \geq c(h_1, h_2). \tag{$**$}$$

If M is strongly Hausdorff we have the equality (but of course much more than that).

Here is a simple example showing that $(**)$ need not be an equality. Let L be the chain $\{0, 1, \ldots, n\}$ and define $f \colon L \to L$ by

$$f(x) = \begin{cases} x + 1 & \text{for } x \leq n - 2 \\ x & \text{otherwise} \end{cases}$$

(the adjoint frame homomorphism $h\colon L \to L$ sends 0 to 0, the x with $1 \le x \le n-1$ to $x-1$, and n to n). Then $\mathrm{Equ}(\mathrm{id}, f) = \{n-1, n\}$ while $c(\mathrm{id}, f) = 0$ because $x \wedge y = 0$ only if x or y is 0.

3.5.2. Question. Under what condition on M (or perhaps on M and L) one has the equality in $(**)$? Does it hold for a class of frames M larger than the strongly Hausdorff ones?

4. More concretely in the strongly Hausdorff locales

As an immediate corollary of 3.3.1 we obtain

4.1. Theorem. *The following statements are equivalent.*

(1) *D_L is closed, that is, L is strongly Hausdorff.*
(2) *For every cp-ideal U such that $U \supseteq d_L$,*

$$(a \wedge b) \oplus c \subseteq U \quad \textit{iff} \quad a \oplus (b \wedge c) \subseteq U.$$

(3) *For every cp-ideal U such that $U \supseteq d_L$,*

$$a \oplus b \subseteq U \quad \textit{iff} \quad (a \wedge b) \oplus (a \wedge b) \subseteq U.$$

4.1.1. Note. The characterizations of strongly Hausdorff L from (2) and (3) are usually replaced by the equivalent

(2') $((a \wedge b) \oplus c) \vee d_L = (a \oplus (b \wedge c)) \vee d_L$, *resp.*
(3') $(a \oplus b) \vee d_L = ((a \wedge b) \oplus (a \wedge b)) \vee d_L$.

4.2. The following is, in essence, the *Banaschewski coequalizer theorem* ([2, Lemma 1]) extended from regular frames to the strongly Hausdorff ones.

Proposition. *Let M be strongly Hausdorff. For frame homomorphisms $h_1, h_2\colon M \to L$ recall the*

$$c(h_1, h_2) = \bigvee\{h_1(x) \wedge h_2(y) \mid x \wedge y = 0\}.$$

from 3.5. Then the equalizer of $f_1, f_2\colon L \to M$ can be written as

$$\uparrow \bigvee\{h_1(x) \wedge h_2(y) \mid x \wedge y = 0\} \subseteq L$$

where h_i are the left adjoint homomorphisms to f_i.

Proof: The homomorphism $h\colon M \oplus M \to L$ defined by $h\iota_i = h_i$ is the right adjoint to the f defined by $p_i f = f_i$. It is standard (and an immediate consequence of the fact that $U = \bigvee\{x \oplus y \mid (x,y) \in U\}$) that we have generally $h(U) = \bigvee\{h_1(x) \wedge h_2(y) \mid (x,y) \in U\}$ and hence in particular $h(d_M) = \bigvee\{h_1(x) \wedge h_2(y) \mid x \wedge y = 0\}$. By 2.3 and 1.5 we obtain the equalizer as $f_{-1}[\mathfrak{c}(d_M)] = \mathfrak{c}(h(d_M))$. ∎

4.3. The axiom T_U. A frame L is said to be T_U[3] if for any two frame homomorphisms $h_1, h_2\colon L \to M$, $h_1 \leq h_2$ implies $h_1 = h_2$ (equivalently, if for any two localic maps $f_1, f_2\colon M \to L$, $f_1 \leq f_2$ implies $f_1 = f_2$). This is an interesting condition: the formula applied for spaces and continuous maps is just the very weak axiom of symmetry (that is, $x \in \overline{\{y\}}$ iff $x \in \overline{\{y\}}$), but already if we consider general M and frame homomorphisms $h_i\colon \Omega(X) \to M$ it is much stronger: for instance it is not implied by the (plain) Hausdorff axiom.

4.3.1. Proposition. (*sH*) *implies* T_U.

Proof: Let $f_1, f_2\colon L \to M$ be such that $f_1 \leq f_2$. Then $h_2 \leq h_1$ and hence

$$\begin{aligned}\bigvee\{h_1(x) \wedge h_2(y) \mid x \wedge y = 0\} &\leq \bigvee\{h_1(x) \wedge h_1(y) \mid x \wedge y = 0\} = \\ &= \bigvee\{h_1(x \wedge y) \mid x \wedge y = 0\} = 0\end{aligned}$$

and thus the equalizer of f_1, f_2 is the whole of L, and $f_1 = f_2$. ∎

4.4. Dowker-Strauss characterization. In [6], the following condition was shown to be equivalent with (sH) for L:

> *For frame homomorphisms $h_1, h_2\colon L \to M$ we have the implication $x \wedge y = 0 \;\Rightarrow\; h_1(x) \wedge h_2(y) = 0$ only if $h_1 = h_2$.*

Using the c from 4.2 we can reformulate this condition as the implication

$$c(h_1, h_2) = 0 \quad\Rightarrow\quad h_1 = h_2. \tag{DS}$$

Now we can prove the characterization very easily (new is the implication $\Rightarrow$, the $\Leftarrow$ is standard in the literature).

4.4.1. Theorem. *A frame M is strongly Hausdorff iff it satisfies* (DS) *for any $h_1, h_2\colon M \to L$.*

Proof: Let M be strongly Hausdorff. As in 4.2 we have the equalizer of f_1, f_2 in the form of $\uparrow c(h_1, h_2)$. If $c(h_1, h_2) = 0$, the equalizer is the whole of L.

[3] This axiom was introduced in [9]; there Isbell speaks of *unordered frames* (see also [10, III.1.5]).

The other implication is as in the literature. One proves that the adjoint to the equalizer embedding, the coequalizer of h_1, h_2 is

$$\gamma = (x \mapsto c \vee x) \colon M \to {\uparrow}x,$$

hence closed. If $x \wedge y = 0$ then

$$\gamma h_1(a) \wedge \gamma h_2(b) = (c \vee h_1(a)) \wedge (c \vee h_2(b)) = c \vee (h_1(a) \wedge h_2(b)) = c = 0_{\uparrow c}$$

and hence, by (DS), $\gamma h_1 = \gamma h_2$. If for a $g \colon M \to K$ we have $gh_1 = gh_2$ then

$$g(c) = \bigvee \{gh_1(x) \wedge gh_2(y) \mid x \wedge y = 0\} = 0$$

and hence we can define $g' \colon {\uparrow}c \to K$ to obtain $g'\gamma = g$. ∎

5. Open and clopen diagonals

In this section we will discuss the openness of the diagonal. This case (coinciding with the clopen one) has already attracted attention ([11, 14]); it was shown that it characterizes atomic Boolean algebras. We will present a simple proof of this fact.

5.1. Recall the formulas for the diagonal and codiagonal morphisms

$$\delta \colon L \to L \oplus L \quad \text{and} \quad \delta^* \colon L \oplus L \to L$$

from 1.6.3. The diagonal sublocale $D = \delta[L]$ will be now an open sublocale $\mathfrak{o}(W)$ for a fixed cp-ideal $W \in L \oplus L$, and for the inclusion map $j \colon D \subseteq L \oplus L$ we have the adjunction

$$j^*(U) \subseteq V \quad \text{iff} \quad U \subseteq j(V) = V$$

with $j^*(U) = W \to U$ (recall 1.7.1). We have an isomorphism

$$\alpha = (a \mapsto \delta(a)) \colon L \cong D$$

such that $\delta = j \cdot \alpha$ and $\delta^* = \alpha^{-1} \cdot j^*$.

5.1.1. Furthermore, if we set $\delta_!(a) = \alpha(a) \cap W$ we obtain

$$\delta_!(a) \subseteq U \quad \text{iff} \quad \alpha(a) \subseteq W \to U = j^*(U) \quad \text{iff} \quad a \leq \alpha^{-1} j^*(U) = \delta^*(U),$$

hence an adjunction $\delta_! \dashv \delta^*$. Note that $\delta_!(x) \subseteq x \oplus x$ (since $\delta^*(x \oplus x) = x$), hence

$$a \oplus a \subseteq \delta_!(x) \;\Rightarrow\; a \leq x$$

by 1.6.2.

5.2. Observations. 1. We have (by 1.3.1 (3) and the adjunction in 5.1.1)

$$W = \bigcap\{U \mid W \to U = 1\} = \bigcap\{U \mid j^*(U) = 1\} =$$
$$= \bigcap\{U \mid \delta^*(U) = 1\} = \delta_!(1).$$

2. The minimal $d = \bigwedge D$ from 3.5 is equal to the pseudocomplement W^* (since $\bigwedge \mathfrak{o}(W) = W \to 0$).

5.3. Atoms. The set of all atoms of a lattice L, that is, of the $0 \neq a \in L$ such that $0 \neq x \leq a$ only if $x = a$, will be denoted by $\mathsf{At}(L)$. We will use the standard fact that

for any $a \in \mathsf{At}(L)$, $a \leq \bigvee_{i \in J} x_i$ *only if* $a \leq x_i$ *for some* i.

Atoms in a locale L are characterized by the following property:

5.3.1. Proposition. *An element* $0 \neq a \in L$ *is an atom iff* $\mathfrak{o}(a \oplus a) \subseteq D_L$.

Proof: The condition $\mathfrak{o}(a \oplus a) \subseteq D_L$ is equivalent to $(a \oplus a) \to U \in D_L$ for every cp-ideal U. By 3.3.1, this is equivalent to $x \oplus y \subseteq (a \oplus a) \to U$ iff $(x \wedge y) \oplus (x \wedge y) \subseteq (a \oplus a) \to U$ for every $x, y \in L$, that is,

$$(a \wedge x) \oplus (a \wedge y) \subseteq U \quad \text{iff} \quad (a \wedge x \wedge y) \oplus (a \wedge x \wedge y) \subseteq U \qquad (*)$$

for every $x, y \in L$.

Let a be an atom. If $(a \wedge x \wedge y) \oplus (a \wedge x \wedge y) \subseteq U$, $a \wedge x \neq 0$ and $a \wedge y \neq 0$ then $a \leq x \wedge y$ and thus $a \oplus a \subseteq U$, that is, $(a \wedge x) \oplus (a \wedge y) \subseteq U$ and we have $(*)$.

Conversely, let $0 \neq x \leq a$ and let $(*)$ hold. Setting in particular $U = x \oplus x$ and $y = 1$ in $(*)$ we obtain $(a \wedge x) \oplus a \subseteq x \oplus x$ iff $(a \wedge x) \oplus (a \wedge x) \subseteq x \oplus x$; thus $x = a$. ■

Hence, in the particular case that the diagonal is an open sublocale $\mathfrak{o}(W)$, we have that

5.3.2. Corollary. *An element* $0 \neq a \in L$ *is an atom iff* $a \oplus a \subseteq W$.

5.4. $\mathfrak{P}(X)$ is the Boolean algebra of all subsets of X, and

$$\phi\colon L \to \mathfrak{P}(\mathsf{At}(L)$$

is the monotone mapping defined by $\phi(x) = \{a \in \mathsf{At}(L) \mid a \leq x\}$. For $M \subseteq \mathsf{At}(L)$ obviously $\bigvee M \leq x$ iff $M \subseteq \phi(x)$ and hence ϕ is a right adjoint to

$$\upsilon\colon \mathfrak{P}(\mathsf{At}(L)) \to L$$

defined by $\upsilon(M) = \bigvee M$.

5.4.1. Theorem. *Let the diagonal $D \subseteq L \oplus L$ be open. Then L is an atomic Boolean algebra.*

Proof: We will prove that the maps ϕ and υ are mutually inverse and hence provide an isomorphism between L and $\mathfrak{P}(\mathsf{At}(L))$.

By the adjunction above, $\phi\upsilon \geq \mathrm{id}$ and $\upsilon\phi \leq \mathrm{id}$. By the atomicity, for every $a \in M \in \mathfrak{P}(\mathsf{At}(L))$, $a \leq \bigvee M$ yields $a \in M$ and hence $\phi\upsilon = \mathrm{id}$. Thus, it suffices to prove that for all $x \in L$, $x \leq \bigvee \phi(x)$. By the adjunction $\delta_! \dashv \delta^*$ we have

$$x \leq \delta^*\delta_!(x) = \delta^*(\bigvee\{a \oplus b \mid a \oplus b \subseteq \delta_!(x)\}) = \bigvee\{a \wedge b \mid a \oplus b \subseteq \delta_!(x)\},$$

and since $a \wedge b$ can be obtained also from $(a \wedge b) \oplus (a \wedge b)$, and $(a \wedge b) \oplus (a \wedge b) \subseteq a \oplus b$ we can proceed, using 5.1.1 and 5.3.2,

$$\cdots = \bigvee\{a \mid a \oplus a \subseteq \delta_!(x)\} \leq \bigvee\{a \mid a \oplus a \subseteq \delta_!(1) = W, a \leq x\} = \bigvee \phi(x). \quad \blacksquare$$

5.4.2. Corollary. *The following statements about a frame L are equivalent.*

(1) *The diagonal $\delta \colon L \to L \oplus L$ is open.*
(2) *The diagonal $\delta \colon L \to L \oplus L$ is clopen.*
(3) *L is an atomic Boolean algebra.*

(If L is Boolean then it is regular, and the diagonal is (also) closed. (3) implies the other two by 1.6.4.)

5.4.3. Notes. 1. Thus, if D is open (if the frame is atomic Boolean), we have by 5.2 that W and D are complements of each other, that is, that

$$\{(u, v) \mid u \wedge v = 0\} \text{ is the complement of}$$
$$\bigcap\{U \mid \bigvee\{u \wedge v \mid (u, v) \in U\} = 1\}.$$

2. Just an observation to the property "being complemented": A very simple example shows that an L with a complemented $D \subseteq L \oplus L$ does not have to be Boolean. Consider $L = \Omega(X)$ with X consisting of a convergent sequence and its limit, say

$$\Big\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0\Big\}.$$

Then $X \times X$ is scattered (and hence $S(\Omega(X \times X))$ is Boolean – see e.g. [13]) and locally compact (and hence $L \oplus L \cong \Omega(X \times X)$ – see e.g. [7, 10] or [12]). Thus, all the sublocales of $L \oplus L$ are complemented and L is, of course, not Boolean.

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