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PROJECTION OF A POINT ONTO THE INTERSECTION OF SPHERES IN LINEAR VARIETIES

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ABSTRACT: We present a formula for the distance between a point and a sphere that is the intersection of two spheres in linear varieties in the Euclidean space. We also obtain formulae for the projection of a point onto the intersection of two spheres in linear varieties as well as for the centre and radius of the intersection.

KEYWORDS: Intersection of spheres, Projection onto intersections, Distance to intersections, Radical hyperplane, Reduction principle.
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1. Introduction

We are concerned with projections and distances. Projections of a point onto a subset, onto a subspace and onto a linear variety. Distances between a point and a subset, a subspace and a linear variety. Projections and distances are thoroughly studied in [4, 10].

The distance between a point and a sphere that is the intersection of two spheres being contained in two intersecting linear varieties is studied. We establish a formula that generalises one giving the distance between a point and the unit sphere in a linear subspace [2]. The distance between a point and the unit sphere has applications in perturbation theory through the concept of the gap (or aperture) between subspaces [[7], pp. 224-227], which is used in the study of convergence of linear operators [[8], pp. 197-200]. In [[2], pp. 1428-1429] a list of references is given in order to motivate consideration of the spherical gap between subspaces.

The space $\mathbb{R}^n$ is equipped with the usual inner product $\bullet$ and the associated Euclidean norm

$$\|a\| = \sqrt{a \bullet a}.$$

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Definition 1.1. *Let S be a nonempty subset of $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$. An element $\mathbb{P}_S(p) \in S$ in called a projection of p onto S if $\mathbb{P}_S(p)$ is a best approximation to p from S. $\mathbb{P}_S(p)$ is called a best approximation to p from S if*

$$\|p - \mathbb{P}_S(p)\| = d\left(p, S\right),$$

where

$$d\left(p, S\right) = \inf \|p - x\|\,, \quad x \in S,$$

is the distance between p and S.

We need some notations for linear varieties and for spheres.

Definition 1.2.

(1) *For linear varieties.*

$$\mathscr{L} = l_0 + \mathscr{L}_0,$$

where l_0 is a position vector and $\mathscr{L}_0$ is the director space of $\mathscr{L}$.

(2) *For spheres.*

(a) *Sphere in $\mathbb{R}^n$ with centre $c \in \mathbb{R}^n$ and radius r, $r > 0$,*

$$S\left(c, r\right) = \left\{x \in \mathbb{R}^n : \|x - c\| = r\right\}.$$

(b) *Sphere in $\mathbb{R}^n$ with centre c in the linear variety $\mathscr{L}$ and radius r, $r > 0$,*

$$S_{c\mathscr{L}}(r) = \left\{x \in \mathbb{R}^n \,:\, \|x - c\| = r\right\}, \quad c \in \mathscr{L}.$$

(c) *Sphere in the linear variety $\mathscr{L}$, with centre $c \in \mathscr{L}$ and radius r, $r > 0$,*

$$S_{\mathscr{L}}(c, r) = \left\{x \in \mathscr{L} \,:\, \|x - c\| = r, \quad c \in \mathscr{L}\right\}.$$

The projection onto the intersection of linear varieties is useful when studying the projection onto the intersection of spheres by using a generalisation of the Reduction Principle. So, useful properties of the projection of a point onto a linear variety are recalled.

Proposition 1.3.

(1) *$\mathbb{P}_{\mathscr{L}}(p)$ is the orthogonal projection of p onto the linear variety $\mathscr{L} = l_0 + \mathscr{L}_0$ if and only if $p - \mathbb{P}_{\mathscr{L}}(p)$ is orthogonal to $\mathscr{L}_0$.*

(2) Let $\mathscr{M}_0$ be a vector subspace of $\mathbb{R}^n$ and let $\mathscr{M}_0^\perp$ be its orthogonal complement. Let B be a matrix whose columns are a basis for $\mathscr{M}_0$. Then the orthogonal projectors $\mathbb{P}_{\mathscr{M}_0}$ and $\mathbb{P}_{\mathscr{M}_0^\perp}$ are

$$\mathbb{P}_{\mathscr{M}_0} = B\left(B^T B\right)^{-1} B^T \quad \textit{and} \quad I - \mathbb{P}_{\mathscr{M}_0} = \mathbb{P}_{\mathscr{M}_0^\perp}.$$

(3) Let $\mathscr{L} = l_0 + \mathscr{L}_0$ be a linear variety in $\mathbb{R}^n$, where l_0 is a position vector and $\mathscr{L}_0$ is the director space, and let $p \in \mathbb{R}^n$. Then

$$\mathbb{P}_{\mathscr{L}}(p) = \mathbb{P}_{\mathscr{L}_0}(p - l_0) + l_0.$$

Proof: For: (1) see [[4], p. 215] and [[9], p. 45]; (2) see [[10], Chapter 5]; and (3) see [3]. ∎

This paper is organized as follows. 1. Introduction 2. Intersection of spheres in linear varieties 3. Projection onto the intersection of spheres 4. Distances between a point and the intersection of spheres 5. Conclusions.

2. Intersection of Spheres in Linear Varieties

In this section we deal with the centre and radius of the intersection of spheres. Intersecting linear varieties come into play. For details and conditions on the intersection of linear varieties, see [3] and the references therein.

We first consider a sphere in $\mathbb{R}^n$ intersected by two linear varieties.

Proposition 2.1. *Let $\mathscr{L}$ and $\mathscr{M}$ be two intersecting linear varieties in $\mathbb{R}^n$ given by*

$$\mathscr{L} = l_0 + \mathscr{L}_0$$

and

$$\mathscr{M} = m_0 + \mathscr{M}_0.$$

Suppose

$$S_{c\mathscr{L}}(r) = \{x \in \mathbb{R}^n : \|x - c\| = r\}, \quad c \in \mathscr{L},$$

is a sphere in $\mathbb{R}^n$ with centre c in $\mathscr{L}$ and such that $d(c, \mathscr{L} \cap \mathscr{M}) < r$ and

$$S_{\mathscr{L}}(c, r) = S_{c\mathscr{L}} \cap \mathscr{L}$$

is a sphere in $\mathscr{L}$. Then the intersection

$$\overline{S} = S_{\mathscr{L}}(c, r) \cap \mathscr{M} = S_{c\mathscr{L}}(r) \cap \mathscr{L} \cap \mathscr{M}$$

is a sphere in $\mathscr{L} \cap \mathscr{M}$,

$$\overline{S} = S_{\mathscr{L}\cap\mathscr{M}}(\bar{c}, \bar{r}),$$

with centre $\bar{c}$ given by

$$\bar{c} = \mathbb{P}_{\mathscr{L}\cap\mathscr{M}}(c) \tag{2.1}$$

and radius $\overline{r}$ satisfying

$$\overline{r}^2 = r^2 - d^2\left(c, \mathscr{L}\cap\mathscr{M}\right). \tag{2.2}$$

Proof: Let $x \in S_{c\mathscr{L}}(r) \cap \mathscr{L} \cap \mathscr{M}$. Since $\bar{c} = \mathbb{P}_{\mathscr{L}\cap\mathscr{M}}(c)$ the element $x - \bar{c}$ belongs to $\mathscr{L}_0 \cap \mathscr{M}_0$ and is orthogonal to $\bar{c} - c$, whence

$$\|x - \bar{c}\|^2 = \|x - c\|^2 - \|\bar{c} - c\|^2 = r^2 - d^2(c, \mathscr{L}\cap\mathscr{M}) = \overline{r}^2.$$

Conversely, if $x \in S_{\mathscr{L}\cap\mathscr{M}}(\bar{c}, \overline{r})$ then $x - \bar{c}$ is in $\mathscr{L}_0 \cap \mathscr{M}_0$ and is orthogonal to $\bar{c} - c$. Therefore we get

$$\|x - c\|^2 = \|x - \bar{c}\|^2 + \|\bar{c} - c\| = r^2,$$

and the result follows. ∎

The intersection of two spheres is a sphere. The radical hyperplane plays an important rôle when studying the intersections of spheres.

The radical hyperplane is a generalisation of the concept of the radical plane in solid geometry as considered in [[5], pp. 461-462], [[11], pp. 103-104] and in [[1], p. 57]. The concepts of the power of a point with respect to a sphere and of the radical hyperplane of two spheres are related.

Definition 2.2. *For each sphere $S = S(c, r)$ and for each point $x \in \mathbb{R}^n$, the number*

$$\|x - c\|^2 - r^2$$

is called the power of x *with respect to $S(c, r)$. Let $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ be two spheres in $\mathbb{R}^n$. The* radical hyperplane *of S_1 and S_2 is the set of points with the same power with respect to the two spheres,*

$$\mathscr{H}\left(S_1, S_2\right) = \left\{x \in \mathbb{R}^n : \|x - c_1\|^2 - r_1^2 = \|x - c_2\|^2 - r_2^2\right\}.$$

Let the spheres $S_{\mathscr{L}}(c_1, r_1)$ and $S_{\mathscr{M}}(c_2, r_2)$ be subsets of the linear varieties $\mathscr{L}$ and $\mathscr{M}$ of $\mathbb{R}^n$. We will consider the radical hyperplane of these spheres as the radical hyperplane of the spheres $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ in $\mathbb{R}^n$,

$$\mathscr{H}\left(S_{\mathscr{L}}, S_{\mathscr{M}}\right) := \mathscr{H}\left(S_1, S_2\right).$$

The notation $\mathscr{H}$ stands either for $\mathscr{H}\left(S_1, S_2\right)$ or for $\mathscr{H}\left(S_{\mathscr{L}}, S_{\mathscr{M}}\right)$, depending on the context.

In the next proposition we recall some useful classical properties of the radical hyperplane of two spheres.

Proposition 2.3. *Let $S(c_1, r_1)$ and $S(c_2, r_2)$ be two spheres in $\mathbb{R}^n$ with distinct centres and $\mathscr{H}$ be their radical hyperplane. Then:*
(1) $\mathscr{H}$ is orthogonal to the line c_1c_2.
(2) If the spheres have non-empty intersection, then the radical hyperplane $\mathscr{H}$ contains the intersection of the two spheres.

In the present paper, we deal always with intersecting spheres.

Definition 2.4. *Two spheres $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ in $\mathbb{R}^n$ are said to have non-trivial intersection if*

$$|r_1 - r_2| < \|c_1 - c_2\| < r_1 + r_2.$$

Now, we are in a position to present formulae for the centre and the radius of the non-trivial intersection of two spheres.

Proposition 2.5. *Let $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ be spheres with non-trivial intersection in $\mathbb{R}^n$. Then the intersection $S^* = S_1 \cap S_2$ is a sphere contained in the radical hyperplane $\mathscr{H}$ and given by*

$$S^* = S^*_{\mathscr{H}}(c^*, r^*) = \{x \in \mathscr{H} : \|x - c^*\| = r^*\},$$

with centre

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2} c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2} c_2 \tag{2.3}$$

and radius

$$r^* = \frac{1}{2d}\sqrt{4d^2 r_1^2 - (d^2 + r_1^2 - r_2^2)^2}, \tag{2.4}$$

where $d = \|c_1 - c_2\|$.

Proof: Let c^* be the intersection of the radical hyperplane $\mathscr{H}$ with the line c_1c_2. Since $c^* \in c_1c_2$, we may write $c^* = c_1 + \lambda(c_2 - c_1) = c_2 + (\lambda - 1)(c_2 - c_1)$. On the other hand, $c^* \in \mathscr{H}$, meaning that $\|c^* - c_1\|^2 - r_1^2 = \|c^* - c_2\|^2 - r_2^2$. Hence we have

$$\lambda = \frac{r_1^2 - r_2^2 + d^2}{2d^2}$$

and

$$c^* = c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2}(c_2 - c_1) = \frac{r_2^2 - r_1^2 + d^2}{2d^2} c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2} c_2. \tag{2.5}$$

Now, if $x \in S_1 \cap S_2$ then $x \in \mathscr{H}$ and $x - c^*$ is perpendicular to c_1c_2. By Pythagoras' theorem,

$$\|x - c^*\|^2 = r_1^2 - \|c^* - c_1\|^2 .$$

By using relation (2.5), we obtain

$$\|x - c^*\| = \frac{1}{2d}\sqrt{4d^2r_1^2 - (d^2 + r_1^2 - r_2^2)^2}.$$

Conversely, let $x \in S^*(c^*, r^*) \cap \mathscr{H}$, with c^* and r^* given by (2.3) and (2.4), respectively. Since $x - c^*$ is orthogonal to c_1c^*, this leads to

$$\begin{aligned}
\|x - c_1\|^2 &= \|c^* - c_1\|^2 + \|x - c^*\|^2 \\
&= \left(\frac{r_1^2 - r_2^2 + d^2}{2d^2}\right)^2 \|c_2 - c_1\|^2 + r^{*2} \\
&= \frac{\left(r_1^2 - r_2^2 + d^2\right)^2}{4d^2} + \frac{1}{4d^2}\left(4d^2r_1^2 - \left(d^2 + r_1^2 - r_2^2\right)^2\right) = r_1^2.
\end{aligned}$$

Analogously we can prove that $\|x - c_2\| = r_2$ and we conclude that $x \in S_1 \cap S_2$. Therefore $S^*(c^*, r^*) \cap \mathscr{H} = S_1 \cap S_2$. ∎

Next we present formulae for the centre and for the radius of the intersection of spheres in linear varieties.

Proposition 2.6. *Consider two spheres with non-trivial intersection,*

$$S_{\mathscr{L}}(c_1, r_1) = \{x \in \mathscr{L} \ : \|x - c_1\| = r_1, \quad c_1 \in \mathscr{L}\}$$

and

$$S_{\mathscr{M}}(c_2, r_2) = \{x \in \mathscr{M} \ : \|x - c_2\| = r_2, \quad c_2 \in \mathscr{M}\},$$

where $\mathscr{L}$ and $\mathscr{M}$ are intersecting linear varieties.

Then the intersection $\widehat{S} = S_{\mathscr{L}} \cap S_{\mathscr{M}}$ of the spheres $S_{\mathscr{L}}$ and $S_{\mathscr{M}}$ is a sphere in $\mathscr{H} \cap \mathscr{L} \cap \mathscr{M}$,

$$\widehat{S} = \{x \in \mathscr{H} \cap \mathscr{L} \cap \mathscr{M} : \|x - \widehat{c}\| = \widehat{r}\},$$

with centre $\widehat{c} \in \mathscr{H} \cap \mathscr{L} \cap \mathscr{M}$ given by

$$\widehat{c} = \mathbb{P}_{\mathscr{H} \cap \mathscr{L} \cap \mathscr{M}}(c^*) \tag{2.6}$$

and radius $\widehat{r}$ satisfying

$$\widehat{r}^2 = \frac{1}{4d^2}\left(4d^2r_1^2 - \left(d^2 + r_1^2 - r_2^2\right)^2\right) - \left\|\widehat{c} - \frac{r_2^2 - r_1^2 + d^2}{2d^2}c_1 - \frac{r_1^2 - r_2^2 + d^2}{2d^2}c_2\right\|^2, \tag{2.7}$$

where $d = \|c_2 - c_1\|$ *and* c^* *is*

$$c^* = \frac{r_2^2 - r_1^2 + d^2}{2d^2} c_1 + \frac{r_1^2 - r_2^2 + d^2}{2d^2} c_2.$$

Proof: Notice that

$$S_{\mathscr{L}} \cap S_{\mathscr{M}} = S(c_1, r_1) \cap S(c_2, r_2) \cap \mathscr{L} \cap \mathscr{M} = S(c^*, r^*) \cap \mathscr{H} \cap \mathscr{L} \cap \mathscr{M},$$

where the centre c^* and the radius r^* are given by Proposition 2.5. Then use Proposition 2.1. ■

3. Projection onto the Intersection of Spheres

In this section, we study the projection of a point $p \in \mathbb{R}^n$ onto a sphere which is either the intersection of two spheres in $\mathbb{R}^n$ or the intersection of two spheres in intersecting linear varieties.

We use the Reduction Principle as a fundamental tool. In [[4], pp. 80-81, p. 86; [9], p. 46], the following result has been stated in the context of convex sets. We here extend it to non-convex sets. One should notice, however, that for a non-convex set S, $\mathbb{P}_S$ is seen as a "set-valued" mapping since the projection may not be uniquely defined.

Proposition 3.1. *Let* S *be a (possibly non-convex) subset of* $\mathbb{R}^n$ *and let* $\mathscr{M}_0$ *be a subspace of* $\mathbb{R}^n$ *that contains* S.

Then, for every $p \in \mathbb{R}^n$*:*

(1) $\mathbb{P}_S\left(\mathbb{P}_{\mathscr{M}_0}(p)\right) = \mathbb{P}_S(p) = \mathbb{P}_{\mathscr{M}_0}\left(\mathbb{P}_S(p)\right)$.
(2) $d^2(p, S) = d^2(p, \mathscr{M}_0) + d^2\left(\mathbb{P}_{\mathscr{M}_0}(p), S\right)$.

Proof: (1) Since $S \subset \mathscr{M}_0$, it follows immediately that $\mathbb{P}_{\mathscr{M}_0}\mathbb{P}_S(p) = \mathbb{P}_S(p)$. Now let $s \in S$. Then for any $x \in \mathbb{P}_{\mathscr{M}_0}(p)$ we have $x - p \in \mathscr{M}_0^{\perp}$ and $x - s \in \mathscr{M}_0$. So

$$\|p - s\|^2 = \|p - x\|^2 + \|x - s\|^2.$$

Hence $s \in S$ minimizes $\|p - s\|$ if and only if it minimizes $\|x - s\|$. This means that $s \in \mathbb{P}_S(p)$ if and only if $s \in \mathbb{P}_S\left(\mathbb{P}_{\mathscr{M}_0}(p)\right)$, and therefore we obtain $\mathbb{P}_S = \mathbb{P}_S\left(\mathbb{P}_{\mathscr{M}_0}\right)$.

(2) This is an immediate consequence of (1).

■

Since translations are isometries, we have the following generalisation to linear varieties.

Corollary 3.2. *Let S be a (possibly non-convex) subset of $\mathbb{R}^n$ and let $\mathscr{M}$ be a linear variety of $\mathbb{R}^n$ that contains S. Then, for every $p \in \mathbb{R}^n$,*

$$\mathbb{P}_S\left(\mathbb{P}_{\mathscr{M}}(p)\right) = \mathbb{P}_S(p) = \mathbb{P}_{\mathscr{M}}\left(\mathbb{P}_S(p)\right).$$

Proposition 3.3. *Let $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ be two non-trivially intersecting spheres in $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. Then:*

(1) If $p \in c_1c_2$ then $\mathbb{P}_{S_1 \cap S_2}(p) = S_1 \cap S_2$;

(2) If $p \notin c_1c_2$ then

$$\mathbb{P}_{S_1\cap S_2}(p) = c^* + \frac{1}{2d}\sqrt{4d^2r_1^2 - (d^2 + r_1^2 - r_2^2)^2}\,\frac{\mathbb{P}_{\mathscr{H}_0}(p - c^*)}{\|\mathbb{P}_{\mathscr{H}_0}(p - c^*)\|},$$

where $\mathscr{H}_0$ is the director subspace of the radical hyperplane $\mathscr{H}$ and c^* is the centre of the sphere $S_1 \cap S_2$ in $\mathscr{H}_0$ defined by (2.3), with $d = \|c_2 - c_1\|$.

Proof: (1) First notice that $\mathbb{P}_{S_1\cap S_2}(c^*) = S_1 \cap S_2$ because c^* is the centre of $S_1 \cap S_2$. Also, if $p \in c_1c_2$ then $p - c^*$ is orthogonal to $\mathscr{H}$ and $\mathbb{P}_{\mathscr{H}}(p) = c^*$. Now, since $S_1 \cap S_2$ is contained in $\mathscr{H}$, Corollary 3.2 guarantees

$$\mathbb{P}_{S_1\cap S_2}(p) = \mathbb{P}_{S_1\cap S_2}\mathbb{P}_{\mathscr{H}}(p) = \mathbb{P}_{S_1\cap S_2}(c^*) = S_1 \cap S_2.$$

(2) Let $p \notin c_1c_2$. Proposition 2.5 gives $S_1 \cap S_2 = S(c^*, r^*)$, so $S_1 \cap S_2 - c^* = S(0, r^*) \subset \mathscr{H}_0$. Therefore

$$\begin{aligned}\mathbb{P}_{S_1\cap S_2}(p) &= \mathbb{P}_{S_1\cap S_2 - c*}(p - c^*) + c^* \\ &= c^* + \mathbb{P}_{S(0,r^*)}\mathbb{P}_{\mathscr{H}_0}(p - c^*) = c^* + r^*\frac{\mathbb{P}_{\mathscr{H}_0(p-c^*)}}{\left\|\mathbb{P}_{\mathscr{H}_0(p-c^*)}\right\|}\end{aligned}$$

and the result follows.

■

We finally consider the projection of a point onto the intersection of two spheres belonging to intersecting linear varieties. By using Proposition 1.3 and Proposition 3.1, we get the following.

Proposition 3.4. *Let $S_{\mathscr{L}} = S_{\mathscr{L}}(c_1, r_1)$ and $S_{\mathscr{M}} = S_{\mathscr{M}}(c_2, r_2)$ be two non-trivially intersecting spheres in the intersecting linear varieties $\mathscr{L}$ and $\mathscr{M}$. Let $p \in \mathbb{R}^n$ be such that $\mathbb{P}_{\mathscr{H}\cap\mathscr{L}\cap\mathscr{M}}(p) \neq \widehat{c}$. Then*

$$\mathbb{P}_{S_{\mathscr{L}}\cap S_{\mathscr{M}}}(p) = \widehat{c} + \frac{\widehat{r}}{\|\mathbb{P}_{\mathscr{H}_0\cap\mathscr{L}_0\cap\mathscr{M}_0}(p - \widehat{c})\|}\mathbb{P}_{\mathscr{H}_0\cap\mathscr{L}_0\cap\mathscr{M}_0}(p - \widehat{c}),$$

where $\mathscr{H}_0$, $\mathscr{L}_0$ and $\mathscr{M}_0$ are, respectively, the director subspaces of the linear varieties $\mathscr{H}$, $\mathscr{L}$ and $\mathscr{M}$ and with $\widehat{c}$ and $\widehat{r}$ given by (2.6) and (2.7).

4. Distances between a Point and the Intersection of Spheres

The distance between a point and the intersection of two spheres is considered. The following result generalises a formula, in [[2], p. 1428], giving the distance between a point and the unit sphere in a subspace of $\mathbb{R}^n$.

Proposition 4.1. *Let $S_{\mathscr{L}} = S_{\mathscr{L}}(c_1, r_1)$ and $S_{\mathscr{M}} = S_{\mathscr{M}}(c_2, r_2)$ be non-trivially intersecting spheres in the intersecting linear varieties $\mathscr{L}$ and $\mathscr{M}$, respectively. Let $\widehat{S}\left(\widehat{c}, \widehat{r}\right) = \widehat{S}_{\mathscr{H}\cap\mathscr{L}\cap\mathscr{M}}(\widehat{c}, \widehat{r}) = S_{\mathscr{L}} \cap S_{\mathscr{M}}$ and let $\mathscr{H}_0 \cap \mathscr{L}_0 \cap \mathscr{M}_0$ be the director subspace of $\mathscr{H} \cap \mathscr{L} \cap \mathscr{M}$. Then:*

(1) If $\mathbb{P}_{\mathscr{H}\cap\mathscr{L}\cap\mathscr{M}}(p) = \widehat{c}$, then

$$d\left(p, \widehat{S}\left(\widehat{c}, \widehat{r}\right)\right) = \sqrt{\left\|p - \widehat{c}\right\|^2 + \widehat{r}^2}.$$

(2) If $p \in \mathbb{R}^n$ is such that $\mathbb{P}_{\mathscr{H}\cap\mathscr{L}\cap\mathscr{M}}(p) \neq \widehat{c}$, then the distance $d\left(p, S_{\mathscr{L}} \cap S_{\mathscr{M}}\right)$ between the point $p \in \mathbb{R}^n$ and the sphere $\widehat{S}\left(\widehat{c}, \widehat{r}\right)$ satisfies

$$d^2\left(p, S_{\mathscr{L}} \cap S_{\mathscr{M}}\right) = \left\|p - \widehat{c}\right\|^2 + \widehat{r}^2 - 2\widehat{r}\left\|\mathbb{P}_{\mathscr{H}_0\cap\mathscr{L}_0\cap\mathscr{M}_0}\left(p - \widehat{c}\right)\right\|,$$

where $\widehat{c}$ and $\widehat{r}$ are given in (2.6) and in (2.7).

Proof: (1) Notice that if $\mathbb{P}_{\mathscr{H}\cap\mathscr{L}\cap\mathscr{M}}(p) = \widehat{c}$, then $p - \widehat{c}$ is orthogonal to $\mathscr{F}_0 = \left(\mathscr{H} \cap \mathscr{L} \cap \mathscr{M}\right)_0 = \mathscr{H}_0 \cap \mathscr{L}_0 \cap \mathscr{M}_0$ and $\mathbb{P}_{\mathscr{F}_0}(p - \widehat{c}) = o$. By Pythagoras' theorem, for every $x \in \widehat{S}(\widehat{c}, \widehat{r})$,

$$\left\|p - x\right\|^2 = \left\|p - \widehat{c}\right\|^2 + \left\|\widehat{c} - x\right\|^2 = \left\|p - \widehat{c}\right\|^2 + \widehat{r}^2,$$

hence

$$d\left(p, \widehat{S}\left(\widehat{c}, \widehat{r}\right)\right) = \sqrt{\left\|p - \widehat{c}\right\|^2 + \widehat{r}^2}.$$

(2) If $\mathbb{P}_{\mathscr{H}\cap\mathscr{L}\cap\mathscr{M}}(p) := \mathbb{P}_{\mathscr{F}}(p) \neq \widehat{c}$, then

$$\begin{aligned} d^2\left(p, \widehat{S}\left(\widehat{c}, \widehat{r}\right)\right) &= \left\|p - \mathbb{P}_{\widehat{S}(\widehat{c},\widehat{r})}\left(p\right)\right\|^2 \\ &= \left\|p\right\|^2 + \left\|\mathbb{P}_{\widehat{S}(\widehat{c},\widehat{r})}\left(p\right)\right\|^2 - 2p \bullet \mathbb{P}_{\widehat{S}(\widehat{c},\widehat{r})}\left(p\right), \end{aligned}$$

with

$$\begin{aligned}\left\|\mathbb{P}_{\widehat{S}(\widehat{c},\widehat{r})}(p)\right\|^2 &= \left\|\widehat{c}+\widehat{r}\frac{\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})}{\|\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\|}\right\|^2\\ &= \|\widehat{c}\|^2+\widehat{r}^2+2\frac{\widehat{r}}{\|\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\|}\widehat{c}\bullet\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\end{aligned}$$

and

$$-2p\bullet\mathbb{P}_{\widehat{S}(\widehat{c},\widehat{r})}(p) = -2p\bullet\widehat{c}-\frac{2\widehat{r}}{\|\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\|}p\bullet\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c}).$$

Thus, by taking into account the idempotency and the self-adjointness of the projector [[4], p. 79], we obtain

$$\begin{aligned}d^2\left(p,\widehat{S}(\widehat{c},\widehat{r})\right) &= \|p\|^2+\|\widehat{c}\|^2+\widehat{r}^2-2p\bullet\widehat{c}-\frac{2\widehat{r}}{\|\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\|}(p-\widehat{c})\bullet\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\\ &= \|p\|^2+\|\widehat{c}\|^2+\widehat{r}^2-2p\bullet\widehat{c}-2\widehat{r}\,\|\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\|\\ &= \|p-\widehat{c}\|^2+\widehat{r}^2-2\widehat{r}\,\|\mathbb{P}_{\mathscr{F}_0}(p-\widehat{c})\|.\end{aligned}$$

■

For two spheres in $\mathbb{R}^n$, from Proposition 3.3, we have the following.

Proposition 4.2. *Let $S_1 = S(c_1, r_1)$ and $S_2 = S(c_2, r_2)$ be two spheres in $\mathbb{R}^n$ with non-trivial intersection and let $p \in \mathbb{R}^n$, $p \notin c_1c_2$. Then*

$$\begin{aligned}d^2(p, S_1\cap S_2) = {}& \|p\|^2+\frac{1}{4d^2}\left[4d^2r_1^2-\left(d^2+r_1^2-r_2^2\right)^2\right]\\ &-\frac{\frac{1}{d}\sqrt{4d^2r_1^2-\left(d^2+r_1^2-r_2^2\right)^2}}{\|\mathbb{P}_{\mathscr{H}_0}(p-c^*)\|}\left(p\bullet\mathbb{P}_{\mathscr{H}_0}(p-c^*)\right),\end{aligned}$$

where $\mathscr{H}_0$ is the director subspace of the radical hyperplane $\mathscr{H}$ and c^* is the centre of the sphere $S_1\cap S_2$ in $\mathscr{H}_0$ defined by

$$c^* = \frac{r_2^2-r_1^2+d^2}{2d^2}c_1+\frac{r_1^2-r_2^2+d^2}{2d^2}c_2.$$

5. Conclusions

The distance between a point and the unit sphere in a subspace is useful when dealing with the gap between two subspaces. See [2] and the references therein. We established a formula for the distance between a point and the intersection of two spheres in linear varieties [Proposition 4.1].

The projection of a point onto a sphere is essentially the projection of a point onto a ball. We derived a generalisation for non-convex sets [Proposition 3.1] of the Reduction Principle [[4], pp. 80-81, p. 86; [9], p. 46], usually stated for convex sets.

In general, finding the projection of a point onto a convex set might not be computationally easy [[6], p. 198]. We obtained an explicit formula for the projection of a point onto a sphere that is the intersection of two spheres in linear varieties [Propositions 3.3 and 3.4].

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