Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 20–31

EULERIAN IDEALS

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ABSTRACT: Let G be a simple graph and $I(X_G) = \varphi^{-1}(x_i^2 - x_j^2 : i, j \in V_G)$, where $\varphi\colon K[E_G] \to K[V_G]$ is the homomorphism that sends an edge to the product of its vertices. The ideal $I(X_G)$ is Cohen–Macaulay, one-dimensional and binomial. If G is bipartite, it is known that the Castelnuovo–Mumford regularity of $I(X_G)$ is equal to the maximum cardinality of a set of edges having no more than half of the edges of any Eulerian subgraph of G. Here, with respect to the grevlex order associated to an ordering of the edge set of G, we describe a Gröbner basis for $I(X_G)$, and we characterize the standard monomials of the ideal $(I(X_G), t_e)$ in terms of even sets of vertices marked with a parity. Using these results, we classify the case of $I(X_G)$ Gorenstein; we give a combinatorial interpretation of the degree of $I(X_G)$, via the set of even sets of vertices of G; and we show that the Castelnuovo–Mumford regularity of $I(X_G)$, for any graph, is the maximum cardinality of a set of edges having no more than half of the edges of any *even* Eulerian subgraph of G or, equivalently, the maximum cardinality of a minimum fixed parity T-join.

KEYWORDS: Binomial ideal, Castelnuovo–Mumford regularity, T-joins.
MATH. SUBJECT CLASSIFICATION (2010): 13A02, 13H10, 13P25, 05E40; 05C70; 05C45.

1. Introduction

The Eulerian ideal of a graph was introduced by the author, Vaz Pinto and Villarreal in [13]. The term Eulerian ideal, which we are introducing here, is owed to the relation between a generating set and the set of Eulerian subgraphs with an even cardinality edge set (even Eulerian subgraphs).

Let G be a simple graph without isolated vertices. Denote the set of vertices by V_G and the set of edges by E_G. Throughout, we will assume that E_G is a non-empty subset of the set of subsets of V_G of cardinality two. Let K be any field and let

$$K[V_G] = K[x_i : i \in V_G], \quad K[E_G] = K[t_h : h \in E_G]$$

be the rings of polynomials with coefficients in K whose variables are indexed by the vertices and edges of G, respectively. Let $\varphi\colon K[E_G] \to K[V_G]$ be given

Received November 9, 2020.
This work was partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

by

$$\varphi(t_h) = x_i x_j, \ \ \forall\, h = \{i,j\} \in E_G. \tag{1}$$

Definition 1.1. The *Eulerian ideal of* G is $I(X_G) = \varphi^{-1}(x_i^2 - x_j^2 : i,j \in V_G)$.

The motivation for this definition comes from the notion of vanishing ideal over a graph, for a *finite* field, introduced by Rentería, Simis and Villarreal in [14]. Note that, in the present case, no assumption is made for the field.

Let us briefly describe the main features of the Eulerian ideal. It is clear from the definition that $I(X_G)$ is a homogeneous ideal. Also, $t_h^2 - t_\ell^2 \in I(X_G)$, for every $h,\ell \in E_G$ and, moreover, one can show that any monomial is regular on $K[E_G]/I(X_G)$. From this we deduce that $I(X_G)$ has height $|E_G| - 1$, and that the quotient is a one-dimensional Cohen–Macaulay graded ring. Additionally, we know that the ideal is generated by binomials and these may be associated to even Eulerian subgraphs of the graph (see Definition 2.3 and Proposition 2.1, below).

These properties were studied in [13] including, also, the Castelnuovo–Mumford regularity of $I(X_G)$ in the *bipartite* case. It was shown that this invariant is equal to the maximum cardinality of a join, i.e., to the maximum cardinality of a set of edges that has no more than half of the edges of any Eulerian subgraph of the graph. This number was termed the *maximum vertex join number* by Solé and Zaslavsky and by Frank (*cf.* [17] and [5]). The starting point of this work was the extension of this result to the non-bipartite case. To achieve this, we define the notion of *parity join*; a set of edges that has no more than half of the edges of any *even* cardinality Eulerian subgraph of G. In Theorem 4.13 we show that the Castelnuovo–Mumford regularity of $I(X_G)$ is the maximum cardinality of a parity join.

To achieve this result, we start by showing that a set of homogeneous binomials obtained from the even Eulerian subgraphs, together with the set of binomials of the form $t_h^2 - t_\ell^2$, for every $h,\ell \in E_G$, form a Gröbner basis for $I(X_G)$ with respect to the graded reverse lexicographic order induced by a total order of the edges (*cf.* Theorem 3.3). The characterization of the outcome of the division of a monomial by this Gröbner basis has lead us to the notion of fixed parity T-joins (*cf.* Definition 4.4). More precisely, by associating a T-join to any monomial (*cf.* Definitions 2.2 and 4.2), we show that the remainder in a standard expression of the monomial, with respect to the aforementioned Gröbner basis, yields a T-join which has minimum cardinality among all T-joins of same parity cardinality (*cf.* Theorem 4.3).

We then describe a bijection between the set of standard monomials of the ideal $(I(X_G), t_e)$, with respect to a monomial order as above, and a set of T-sets marked with an element of $\mathbb{Z}/2$ (*cf.* Theorem 4.9). As the notion of minimum fixed parity T-joins and the notion of parity joins are just different ways of describing the same set of edges of a graph (cf. Lemma 4.12), the proof of Theorem 4.13, *i.e.*, the computation of the Castelnuovo–Mumford regularity of $I(X_G)$, is then carried out using fixed parity T-joins and the characterization of standard monomials of $(I(X_G), t_e)$ obtained.

Two additional independent results have arisen from this study. On the one hand, the explicit Gröbner basis has enabled a complete classification of Gorenstein Eulerian ideals; namely $K[E_G]/I(X_G)$ is a Gorenstein graded ring if and only if G does not contain any even Eulerian subgraphs (*cf.* Theorem 3.5). On the other hand, the characterization of standard monomials of $(I(X_G), t_e)$ has given us a reinterpretation of the degree of $I(X_G)$, computed in [13, Proposition 2.11], in terms even subsets of vertices of the graph (*cf.* Proposition 4.10).

The ideal $I(X_G)$ contains the toric ideal of the graph, $P(G)$, which is defined as the kernel of the map given by (1). These ideals have a longer history and a more intricate nature. Their systematic study started with the work of Simis, Vasconcelos and Villarreal [16]. We know that $P(G)$ is also a binomial ideal and that it is generated by the binomials associated to the even closed walks on the graph (*cf.* [18, Proposition 3.1]). In our case, a set of generators of $I(X_G)$ includes not only these binomials but also any binomial obtained from any even Eulerian subgraph of G, not necessarily connected, and a partition of its edge set into two equal cardinality parts (*cf.* Definitions 2.3 and 3.1). A contrasting feature to the Eulerian ideal is that, while the former always is, $P(G)$ may rarely be Cohen-Macaulay. By way of example, in the recent article [4], the authors show that for every pair of integers d and r satisfying $d \geq r \geq 4$, there exists a graph yielding a quotient $K[E_G]/P(G)$ with Castelnuovo–Mumford regularity r and h-polynomial of degree d. In recent years, several authors have studied the Castelnuovo–Mumford regularity of the quotient $K[E_G]/P(G)$. We know that, under mild assumptions, the matching number gives an upper bound for this invariant (*cf.* [11]) and that lower bounds can be produced from distinguished families of induced subgraphs of the graph (*cf.* [1, 9]).

The paper is structured as follows. In the next section we recall the main properties of $I(X_G)$. In Section 3 we describe a Gröbner basis for the Eulerian ideal and we give a characterization of the Gorenstein Eulerian ideals. In Section 4 we study the combinatorial properties of the division of monomials by a Gröbner basis and, as a result, we give a bijection between the set of standard monomials of the ideal $(I(X_G), t_e)$, with respect to the graded reverse lexicographic order, in terms of even sets of vertices marked with a parity. Finally, we apply these results to the computation of the degree and the Castelnuovo–Mumford regularity.

Acknowledgments. The author thanks Jens Vygen and András Sebő for a helpful discussion on the subject of T-joins. The relation between the notions of parity joins and of fixed parity T-joins in Lemma 4.12 was pointed out by András Sebő.

2. Preliminaries

Throughout, we will use the multi-index notation to denote monomials in $K[E_G]$. More precisely, for each $\alpha\colon E_G \to \mathbb{N}$, the monomial $\mathbf{t}^\alpha$ is the product of $t_h^{\alpha(h)}$ when h varies in E_G. We will employ interchangeably the terms *edge* and *variable*. In examples, V_G will be a subset of $\mathbb{N}$ and we will abbreviate $t_{\{i,j\}}$ to t_{ij}. For future reference, let us gather in the next proposition the main known properties of the Eulerian ideal of G.

Proposition 2.1. *Let $I(X_G)$ be as in Definition 1.1 and let $\mathbf{t}^\alpha$, $\mathbf{t}^\beta \in K[E_G]$ be relatively prime monomials of the same degree.*

(i) *$I(X_G)$ is generated by homogeneous binomials.*
(ii) *Any monomial is regular on $K[E_G]/I(X_G)$.*
(iii) *$K[E_G]/I(X_G)$ is a Cohen–Macaulay, one-dimensional graded ring.*
(iv) *$\mathbf{t}^\alpha - \mathbf{t}^\beta$ belongs to $I(X_G)$ if and only if the edge-induced subgraph given by the set of edges raised to odd powers in $\mathbf{t}^\alpha - \mathbf{t}^\beta$ has vertices of even degree.*

The proof of (i) uses a standard implicitization argument. One shows that $I(X_G)$ is the intersection with $K[E_G]$ of the ideal generated by

$$(t_h - x_i x_j z : h = \{i,j\} \in E_G) \cup (x_i^2 - x_j^2 : i,j \in V_G)$$

in the ring $K[E_G, V_G, z]$. The proof of (ii) and (iii) are straightforward if the characteristic of the field is not 2, in which case, $I(X_G)$ is the vanishing ideal of a set of points in projective space with nonzero coordinates (a projective toric subset). For the proof of (i) and (ii) in the general case, the proof of (iv) and details we refer the reader to [13, Propositions 2.1, 2.2, 2.5 and 2.8].

Definition 2.2. Given $\mathbf{t}^\alpha \in K[E_G]$, let $\mathcal{J}(\mathbf{t}^\alpha) = \{h : \alpha(h) \text{ is odd}\} \subset E_G$.

Using (ii) and (iv) of Proposition 2.1, we deduce that if $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is a homogeneous binomial then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G)$ if and only if the edge-induced subgraph of G given by symmetric difference $\mathcal{J}(\mathbf{t}^\alpha)\triangle\mathcal{J}(\mathbf{t}^\beta)$ has vertices of even degree, i.e., is an Eulerian subgraph. Since $\mathcal{J}(\mathbf{t}^\alpha)\triangle\mathcal{J}(\mathbf{t}^\beta) \equiv_2 \deg(\mathbf{t}^\alpha) + \deg(\mathbf{t}^\beta)$, the Eulerian subgraphs arising from homogeneous binomials of $I(X_G)$ have an edge set of even cardinality. Let us fix the terminology.

Definition 2.3. A subgraph of G is called *even Eulerian* if its vertices have even degrees and its edge set has even cardinality.

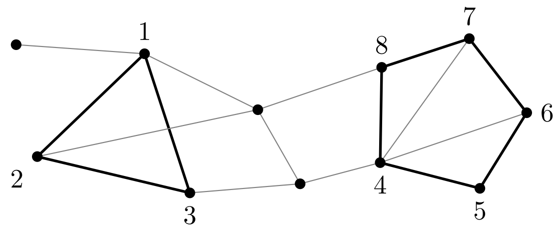


FIGURE 1. An even Eulerian subgraph of G

The subgraph in Figure 1 represented in bold is an even Eulerian subgraph. We emphasize that an even Eulerian subgraph is not assumed to be a connected graph, or a spanning subgraph.

Let M be a nonzero finitely generated graded module over a polynomial ring S, and let $F_i = \oplus_{j\in\mathbb{Z}} S(-j)^{\beta_{ij}}$ be the free graded modules in a minimal graded free resolution of M, $0 \to F_c \to \cdots \to F_1 \to F_0 \to M$. Then the Castelnuovo–Mumford regularity of M, referred to in the remainder of the text simply by *regularity*, is defined by $\operatorname{reg} M = \max_{i,j}\{j - i : \beta_{ij} \neq 0\}$. If $M = 0$, we adopt the convention $\operatorname{reg} M = 1$, so that, in particular, when $I(X_G) = 0$, $\operatorname{reg} I(X_G) = 1$; such is the case when G consists of a single edge. If M is Cohen–Macaulay and $\mathbf{f} \subset S$ is a regular sequence on M of maximum length, consisting of elements of degree one, then by [2, Proposition 4.14] the regularity of M is the largest degree of a nonzero homogeneous element of $M/(\mathbf{f})M$. In our case, given that $\operatorname{reg} I(X_G) = \operatorname{reg} K[E_G]/I(X_G) + 1$, we have the following result.

Proposition 2.4. *Let $e \in E_G$ and let $N = K[E_G]/(I(X_G), t_e)$. Then*

$$\operatorname{reg} I(X_G) = \max\{d : N_d \neq 0\} + 1.$$

In [13, Proposition 3.2], the regularity of $I(X_G)$ for some special families of graphs is given. We list them in Table 1.

G	$\operatorname{reg} I(X_G)$
Forest	$\lvert E_G\rvert$
Even cycle	$\lvert E_G\rvert/2$
Complete bipartite $\mathcal{K}_{a,b}$	$\max\{a,b\}$
Non-bipartite, uni-cyclic	$\lvert E_G\rvert$
Complete graph, $\mathcal{K}_n$, $n \geq 4$	$\lfloor \frac{n}{2} \rfloor + 1$

TABLE 1. $\operatorname{reg} I(X_G)$ for special families of graphs.

3. A Gröbner basis

Let G be as in Figure 1 and, associated to the even Eulerian subgraph in bold, consider the monomials $\mathbf{t}^\alpha = t_{12}t_{23}t_{13}t_{48}$ and $\mathbf{t}^\beta = t_{45}t_{56}t_{67}t_{78}$. Let φ be the map given by (1). Then, using Definition 1.1,

$$\varphi(\mathbf{t}^\alpha - \mathbf{t}^\beta) = (x_1^2x_2^2x_3^2 - x_5^2x_6^2x_7^2)x_4x_8 \implies \mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G).$$

By (iv) of Proposition 2.1, the conclusion is the same if we take any other even Eulerian subgraph of G and any other partition of its edge set in two parts of equal cardinality. This motivates the following definition.

Definition 3.1. A binomial $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is called *Eulerian* if $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are distinct, relatively prime, square-free monomials of same degree and the edge-induced subgraph given by $\mathcal{J}(\mathbf{t}^\alpha) \cup \mathcal{J}(\mathbf{t}^\beta)$ is an even Eulerian subgraph of G. Denoting the set of Eulerian binomials by $\mathscr{E}$ and $\mathscr{T} = \left\{t_h^2 - t_\ell^2 : h, \ell \in E_G\right\}$, define $\mathscr{G} = \mathscr{T} \cup \mathscr{E}$.

Of course, not all homogeneous binomials in $I(X_G)$ are Eulerian; it suffices that one of $\mathbf{t}^\alpha$ or $\mathbf{t}^\beta$ is not square-free. Clearly, $\mathscr{G}$ is a finite set. We will show that it is a Gröbner basis of $I(X_G)$ for any graded reverse lexicographic order on $K[E_G]$, in the following sense.

Definition 3.2. By a grevlex order on $K[E_G]$ we will mean the graded reverse lexicographic order on $K[E_G]$ induced by $t_{\epsilon(1)} \succ t_{\epsilon(2)} \succ \cdots \succ t_{\epsilon(s)}$, where ϵ is a bijection $\epsilon\colon \{1,\ldots,s\} \to E_G$. If $e = \epsilon_s$ is the last edge, then such a monomial order will be referred to as a grevlex order on $K[E_G]$ with t_e last.

Recall that if $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are two monomials of the same degree and h is the last edge in the support of $\beta - \alpha$, then $\mathbf{t}^\alpha \succ \mathbf{t}^\beta$ in the grevlex order if and only if $\beta(h) > \alpha(h)$. In particular, if $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are also square-free, then $\mathrm{in}_\prec(\mathbf{t}^\alpha - \mathbf{t}^\beta) = \mathbf{t}^\alpha$ if and only if $\mathcal{J}(\mathbf{t}^\beta)$ contains the last edge of $\mathcal{J}(\mathbf{t}^\alpha) \triangle \mathcal{J}(\mathbf{t}^\beta)$.

Theorem 3.3. *Assume $|E_G| > 1$ and fix $\prec$ a grevlex order on $K[E_G]$. Then $\mathscr{G}$ is a Gröbner basis of $I(X_G)$ with respect to $\prec$.*

Proof: Let us begin by showing that $I(X_G) = (\mathscr{G})$. In view of Proposition 2.1, the inclusion $(\mathscr{G}) \subset I(X_G)$ is clear. Since $I(X_G)$ is generated by binomials, to prove the opposite inclusion it suffices to show that any homogeneous binomial $\mathbf{t}^\alpha - \mathbf{t}^\beta$ in $I(X_G)$ belongs to $(\mathscr{G})$. By (ii) of Proposition 2.1 we may assume $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. We will use induction on the degree of the binomial. By (iv) of Proposition 2.1, there are no homogeneous binomials in $I(X_G)$ of degree one, so the base case when degree is equal to two. In this case, either both $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are squares of variables, or neither of them is, in which case $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is an Eulerian binomial. Either way, we get $\mathbf{t}^\alpha - \mathbf{t}^\beta \in (\mathscr{G})$. Suppose now that $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(X_G)$ has degree larger than or equal to three. If $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are also square-free then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{E}$. Suppose that this is not the case, suppose that, say, $\mathbf{t}^\alpha$ is divisible by t_h^2, for some $h \in E_G$. Then, choose $\ell \in E_G$ such that t_ℓ divides $\mathbf{t}^\beta$ and write $\mathbf{t}^\alpha = t_h^2 \mathbf{t}^\gamma$ and $\mathbf{t}^\beta = t_\ell \mathbf{t}^\mu$. From

$$\mathbf{t}^\alpha - \mathbf{t}^\beta = t_\ell(t_\ell \mathbf{t}^\gamma - \mathbf{t}^\mu) + (t_h^2 - t_\ell^2)\mathbf{t}^\gamma,$$

and the fact that $(t_h^2 - t_\ell^2)\mathbf{t}^\gamma \in (\mathscr{G}) \subset I(X_G)$, we deduce that

$$t_\ell(t_\ell \mathbf{t}^\gamma - \mathbf{t}^\mu) \in I(X_G) \implies t_\ell \mathbf{t}^\gamma - \mathbf{t}^\mu \in I(X_G).$$

By induction, we get $t_\ell \mathbf{t}^\gamma - \mathbf{t}^\mu \in (\mathscr{G})$ and thus $\mathbf{t}^\alpha - \mathbf{t}^\beta \in (\mathscr{G})$. This concludes the proof that $I(X_G) = J = (\mathscr{G})$.

To prove that $\mathscr{G}$ is a Gröbner basis of $I(X_G)$ with respect to $\prec$, let us show that $S(f,g)$ reduces to zero with respect to $\mathscr{G}$, for every $f, g \in \mathscr{G}$. We only need to consider f and g for which $\gcd(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g)) \neq 1$. There are three cases.

If $f, g \in \mathscr{T}$ and $\gcd(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g)) \neq 1$, then, without loss of generality, we may assume f and g are of the form $f = t_h^2 - t_\ell^2$ and $g = t_h^2 - t_e^2$ with $\mathrm{in}_\prec(f) = \mathrm{in}_\prec(g) = t_h^2$. Then $S(f,g) = f - g = t_e^2 - t_\ell^2 \in \mathscr{T}$.

If $f \in \mathscr{T}$, $g \in \mathscr{E}$ and $\gcd(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g)) \neq 1$, then, without loss of generality, we may assume that $f = t_h^2 - t_e^2$ and $g = t_h \mathbf{t}^\gamma - t_\ell \mathbf{t}^\mu$ with $\mathrm{in}_\prec(f) = t_h^2$, $\mathrm{in}_\prec(g) = t_h \mathbf{t}^\gamma$ and ℓ the last variable of $\mathcal{J}(t_h\mathbf{t}^\gamma) \cup \mathcal{J}(t_\ell \mathbf{t}^\mu)$. Then, since $t_h\mathbf{t}^\mu - t_\ell \mathbf{t}^\gamma \in \mathscr{E}$, and $\mathrm{in}_\prec(t_h\mathbf{t}^\mu - t_\ell\mathbf{t}^\gamma) = t_h\mathbf{t}^\mu$,

$$S(f,g) = t_\ell t_h \mathbf{t}^\mu - \mathbf{t}_e^2 \mathbf{t}^\gamma \xrightarrow{\mathscr{E}} (t_\ell^2 - t_e^2)\mathbf{t}^\gamma \xrightarrow{\mathscr{T}} 0.$$

Finally, suppose that $f, g \in \mathscr{E}$ and $\gcd(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g)) \neq 1$. Let $f = \mathbf{t}^{\delta+\gamma} - \mathbf{t}^\epsilon$ and $g = \mathbf{t}^{\delta+\mu} - \mathbf{t}^\nu$, with $\mathrm{in}_\prec(f) = \mathbf{t}^{\delta+\gamma}$, $\mathrm{in}_\prec(g) = \mathbf{t}^{\delta+\mu}$ and $\gcd(\mathbf{t}^\gamma, \mathbf{t}^\mu) = 1$. Then

$$S(f,g) = \mathbf{t}^{\gamma+\nu} - \mathbf{t}^{\mu+\epsilon}. \tag{2}$$

Since $f, g \in \mathscr{E}$, each of the two sets $\mathcal{J}(\mathbf{t}^\delta\mathbf{t}^\gamma) \cup \mathcal{J}(\mathbf{t}^\epsilon)$ and $\mathcal{J}(\mathbf{t}^\delta\mathbf{t}^\mu) \cup \mathcal{J}(\mathbf{t}^\nu)$ defines an even Eulerian subgraph of G. It follows that the symmetric difference of the two sets also defines an even Eulerian subgraph of G. As

$$(\mathcal{J}(\mathbf{t}^\delta\mathbf{t}^\gamma) \cup \mathcal{J}(\mathbf{t}^\epsilon)) \triangle (\mathcal{J}(\mathbf{t}^\delta\mathbf{t}^\mu) \cup \mathcal{J}(\mathbf{t}^\nu)) = \mathcal{J}(\mathbf{t}^{\gamma+\nu}) \triangle \mathcal{J}(\mathbf{t}^{\mu+\epsilon}),$$

this means that $\mathcal{J}(\mathbf{t}^{\gamma+\nu}) \triangle \mathcal{J}(\mathbf{t}^{\mu+\epsilon})$ defines an even Eulerian subgraph of G. If $\mathbf{t}^{\gamma+\nu}$ and $\mathbf{t}^{\mu+\epsilon}$ are relatively prime and square-free we get $S(f,g) \in \mathscr{E}$ which, trivially, reduces to zero with respect to $\mathscr{G}$. Suppose this is not the case. Consider the monomials $\mathbf{t}^\zeta = \gcd(\mathbf{t}^{\gamma+\nu}, \mathbf{t}^{\epsilon+\mu})$, $\mathbf{t}^\phi = \gcd(\mathbf{t}^\gamma, \mathbf{t}^\nu)$ and $\mathbf{t}^\psi = \gcd(\mathbf{t}^\mu, \mathbf{t}^\epsilon)$. As, by assumption, $\gcd(\mathbf{t}^\gamma, \mathbf{t}^\epsilon) = \gcd(\mathbf{t}^\mu, \mathbf{t}^\nu) = 1$, we deduce that $\gcd(\mathbf{t}^\zeta, \mathbf{t}^\phi) = \gcd(\mathbf{t}^\zeta, \mathbf{t}^\psi) = 1$. Hence, there exist $\mathbf{t}^\alpha, \mathbf{t}^\beta \in K[E_G]$, relatively prime, square-free monomials, such that

$$\mathbf{t}^{\gamma+\nu-\zeta} = \mathbf{t}^\alpha (\mathbf{t}^\phi)^2 \quad \text{and} \quad \mathbf{t}^{\epsilon+\mu-\zeta} = \mathbf{t}^\beta (\mathbf{t}^\psi)^2. \tag{3}$$

From the fact that $\mathcal{J}(\mathbf{t}^{\gamma+\nu}) \triangle \mathcal{J}(\mathbf{t}^{\mu+\epsilon})$ defines an even Eulerian subgraph we deduce that $\mathcal{J}(\mathbf{t}^\alpha) \triangle \mathcal{J}(\mathbf{t}^\beta)$ also defines an even Eulerian subgraph. Let $a = \deg(\mathbf{t}^\phi)$ and $b = \deg(\mathbf{t}^\psi)$. Then, letting e denote the last variable of E_G,

$$\begin{aligned} &\deg(\mathbf{t}^\alpha) + 2a = \deg S(f,g) - \deg(\mathbf{t}^\zeta) = \deg(\mathbf{t}^\beta) + 2b, \\ &S(f,g) = \mathbf{t}^\zeta(\mathbf{t}^\alpha(\mathbf{t}^\phi)^2 - \mathbf{t}^\beta(\mathbf{t}^\psi)^2) \xrightarrow{\mathscr{T}} \mathbf{t}^\zeta(\mathbf{t}^\alpha t_e^{2a} - \mathbf{t}^\beta t_e^{2b}). \end{aligned} \tag{4}$$

If $a = b$ then $\deg(\mathbf{t}^\alpha) = \deg(\mathbf{t}^\beta)$ and thus $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathscr{E}$. From (4) we deduce that $S(f,g)$ reduces to zero with respect to $\mathscr{E}$. Consider now the case $a \neq b$, and assume, without loss of generality, that $a < b$ and therefore

$\deg(\mathbf{t}^\alpha) > \deg(\mathbf{t}^\beta)$. By Lemma 3.4 (proved below) there exists $\mathbf{t}^\xi \in K[E_G]$ such that $\mathbf{t}^{\alpha-\xi} - \mathbf{t}^{\beta+\xi} \in \mathscr{E}$ and $\mathrm{in}_\prec(\mathbf{t}^{\alpha-\xi} - \mathbf{t}^{\beta+\xi}) = \mathbf{t}^{\alpha-\xi}$. It follows that

$$2\deg(\mathbf{t}^\xi) = \deg(\mathbf{t}^\alpha) - \deg(\mathbf{t}^\beta) = 2b - 2a.$$

Continuing from (4),

$$S(f,g) \xrightarrow{\mathscr{G}} \mathbf{t}^\zeta \mathbf{t}^\beta((\mathbf{t}^\xi)^2 t_e^{2a} - t_e^{2b}) \xrightarrow{\mathscr{T}} 0.$$

This finishes the proof of the theorem. ■

Lemma 3.4. *Let $\prec$ be a grevlex order on $K[E_G]$. Suppose that $\mathbf{t}^\alpha, \mathbf{t}^\beta \in K[E_G]$ are relatively prime, square-free monomials, with $\deg(\mathbf{t}^\alpha) > \deg(\mathbf{t}^\beta)$ and such that $\mathcal{J}(\mathbf{t}^\alpha) \cup \mathcal{J}(\mathbf{t}^\beta)$ defines an even Eulerian subgraph. Then there exists $\mathbf{t}^\xi \in K[E_G]$ dividing $\mathbf{t}^\alpha$ such that $\mathbf{t}^{\alpha-\xi} - \mathbf{t}^{\beta+\xi} \in \mathscr{E}$ and $\mathrm{in}_\prec(\mathbf{t}^{\alpha-\xi} - \mathbf{t}^{\beta+\xi}) = \mathbf{t}^{\alpha-\xi}$. In particular, $\mathbf{t}^\alpha$ is divisible by a leading term of an element of $\mathscr{E}$.*

Proof: Set $2d = \deg(\mathbf{t}^\alpha) - \deg(\mathbf{t}^\beta)$ and let $\mathbf{t}^\xi$ be the product of the d last variables in $\mathcal{J}(\mathbf{t}^\alpha)$. Then $\mathbf{t}^{\alpha-\xi}$ and $\mathbf{t}^{\beta+\xi}$ are relatively prime, square-free monomials of equal degree such that $\mathcal{J}(\mathbf{t}^{\alpha-\xi}) \cup \mathcal{J}(\mathbf{t}^{\beta+\xi}) = \mathcal{J}(\mathbf{t}^\alpha) \cup \mathcal{J}(\mathbf{t}^\beta)$ defines an even Eulerian subgraph. We deduce that $\mathbf{t}^{\alpha-\xi} - \mathbf{t}^{\beta+\xi} \in \mathscr{E}$. As the last edge in $\mathcal{J}(\mathbf{t}^{\alpha-\xi}) \cup \mathcal{J}(\mathbf{t}^{\beta+\xi})$ is in $\mathcal{J}(\mathbf{t}^{\beta+\xi})$ we get $\mathrm{in}_\prec(\mathbf{t}^{\alpha-\xi} - \mathbf{t}^{\beta+\xi}) = \mathbf{t}^{\alpha-\xi}$. ■

Gorenstein Eulerian ideals. Since $K[E_G]/I(X_G)$ is Cohen–Macaulay, for any graph, it is natural to inquire about the Gorenstein property of this graded ring. The Gröbner basis obtained in Theorem 3.3 enables a complete classification.

Theorem 3.5. *The following are equivalent:*

(i) *$K[E_G]/I(X_G)$ is Gorenstein;*
(ii) *G does not have any even Eulerian subgraphs;*
(iii) *$I(X_G)$ is a complete intersection.*

Proof: We may assume $|E_G| > 1$. Choose $e \in E_G$ and fix $\prec$ a grevlex order on $K[E_G]$ with t_e last. Let $\mathscr{G}$ be the Gröbner basis of $I(X_G)$ given in Definition 3.1. Let us prove that (i) implies (ii). Assume that $K[E_G]/I(X_G)$ is a Gorenstein graded ring. Then, as t_e is regular, the quotient

$$K[E_G]/(I(X_G), t_e)$$

is a Gorenstein graded ring of dimension zero. By [10, Corollary 3.3.5] the quotient $K[E_G]/\mathrm{in}_\prec(I(X_G), t_e)$ is also Gorenstein graded ring of dimension zero and hence, by [10, Proposition A.6.5], the ideal $\mathrm{in}_\prec(I(X_G), t_e)$ is generated by pure powers of variables. As no leading term of an element of $\mathscr{G}$ is divisible by the variable t_e, $\mathscr{G} \cup \{t_e\}$ is a Gröbner basis of $(I(X_G), t_e)$ and hence $\mathrm{in}_\prec(I(X_G), t_e)$ is generated by $\{t_e\} \cup \mathrm{in}_\prec \mathscr{G}$, where $\mathrm{in}_\prec \mathscr{G}$ denotes the set of leading terms of the elements of $\mathscr{G}$. Now, if there is an even Eulerian subgraph of G, then $\mathscr{E}$ is non-empty and therefore there exists a square-free monomial $\mathbf{t}^\alpha \in \mathrm{in}_\prec \mathscr{G}$, which, necessarily is not divisible by t_e. As no other variable, other than t_e, belongs to $\{t_e\} \cup \mathrm{in}_\prec \mathscr{G}$, this implies that $\mathrm{in}_\prec(I(X_G), t_e)$ is not generated by pure powers of variables. We conclude that G does not have any even Eulerian subgraphs.

The assertion that (ii) implies (iii) follows from the fact that if $\mathscr{E}$ is empty then $I(X_G)$ is generated by $\left\{t_h^2 - t_e^2 : h \in E_G \setminus \{e\}\right\}$, which implies that $I(X_G)$ is a complete intersection. By a well-known general result, (iii) implies (i). ■

4. Standard Monomials

Let $T \subset V_G$ be a (possibly empty) set of vertices. A T-join is a subset of edges $J \subset E_G$ such that T is precisely the set of odd degree vertices of the subgraph of G edge-induced by J. Since the number of odd degree vertices of a graph is even, the existence of a T-join implies that the intersection of T with the vertex set of every connected component of the graph has even cardinality.

Definition 4.1. A subset $T \subset V_G$ is called an *even subset of vertices* if $|T \cap V_H|$ is even, for every connected component $H \subset G$. We denote the set of all even subsets of vertices of a graph G by $\mathcal{E}(V_G)$.

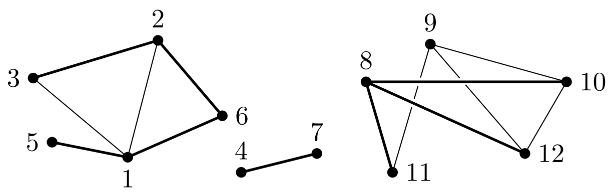


FIGURE 2. T-joins.

When T consists of two vertices in the same connected component, a T-join is an edge-disjoint union of a path between the two vertices together with cycles. A minimal cardinality T-join is then a shortest path between the two vertices. In the graph of Figure 2, the path in bold on the left is a $\{3, 5\}$-join, but not a minimal cardinality one. For a non-empty $T \in \mathcal{E}(V_G)$, a T-join may constructed by choosing paths between pairs of vertices of T in a same connected component and taking their symmetric difference. If $T = \emptyset$, then the empty set (of edges) is a T-join. Any non-empty T-join, in this case, is a subset of edges defining a subgraph of G with vertices of even degree or, according to Definition 2.3, an Eulerian subgraph of G. We see that a T-join always exists. Whether, for a given $T \in \mathcal{E}(V_G)$, a T-join with cardinality of a given parity exists is a different matter, which we will explore. We refer the reader to [12, Chapter 12] and [15, Chapter 29] for further properties of T-joins.

Definition 4.2. Given $\mathbf{t}^\alpha \in K[E_G]$, recalling from Definition 2.2, that $\mathcal{J}(\mathbf{t}^\alpha)$ is the subset of edges $\{h \in E_G : \alpha(h) \text{ is odd}\}$, define $\theta(\mathbf{t}^\alpha) \in \mathcal{E}(V_G)$ to be the set of odd degree vertices of the subgraph edge-induced by $\mathcal{J}(\mathbf{t}^\alpha)$.

Note that $\mathcal{J}(\mathbf{t}^\alpha)$ is a $\theta(\mathbf{t}^\alpha)$-join. Additionally, if $\mathbf{t}^\alpha, \mathbf{t}^\beta \in K[E_G]$, as

$$\mathcal{J}(\mathbf{t}^\alpha\mathbf{t}^\beta) = \mathcal{J}(\mathbf{t}^\alpha) \triangle \mathcal{J}(\mathbf{t}^\beta),$$

we deduce that $\mathcal{J}(\mathbf{t}^\alpha\mathbf{t}^\beta)$ is $(\theta(\mathbf{t}^\alpha)\triangle\theta(\mathbf{t}^\beta))$-join. This follows from an elementary property of T-joins (*cf.* [12, Proposition 12.6]). This implies that $\theta(\mathbf{t}^\alpha\mathbf{t}^\beta) = \theta(\mathbf{t}^\alpha)\triangle\theta(\mathbf{t}^\beta)$. In particular, $\mathcal{J}(\mathbf{t}^\alpha)\triangle\mathcal{J}(\mathbf{t}^\beta)$ is an Eulerian subgraph if and only if

$$\theta(\mathbf{t}^\alpha)\triangle\theta(\mathbf{t}^\beta) = \emptyset \iff \theta(\mathbf{t}^\alpha) = \theta(\mathbf{t}^\beta).$$

We will use this property in the sequel.

As we shall see, T-joins play an important role in the characterization of monomials. We start by showing that they can be used to characterize the process of division of monomials by the Gröbner basis that was introduced in Definition 3.1.

Theorem 4.3. *Assume $|E_G| > 1$, fix a grevlex order on $K[E_G]$ and let $\mathscr{G}$ be the associated Gröbner basis of $I(X_G)$. Let $\mathbf{t}^\delta, \mathbf{t}^\gamma \in K[E_G]$.*

(i) *If the monomial $\mathbf{t}^\delta$ is the remainder of the division of $\mathbf{t}^\gamma$ by an element of $\mathscr{G}$ then $\theta(\mathbf{t}^\delta) = \theta(\mathbf{t}^\gamma)$ and $|\mathcal{J}(\mathbf{t}^\delta)| \equiv_2 |\mathcal{J}(\mathbf{t}^\gamma)|$.*

(ii) *The remainder in a standard expression of $\mathbf{t}^\gamma$ with respect to $\mathscr{G}$ is a monomial, $\mathbf{t}^\delta$, with*

$$|\mathcal{J}(\mathbf{t}^\delta)| = \min\{|J| : J \textit{ is a } \theta(\mathbf{t}^\gamma)\textit{-join and } |J| \equiv_2 |\mathcal{J}(\mathbf{t}^\gamma)|\}.$$

Proof: Let $e \in E_G$ be the last edge. As $\mathscr{G} = \mathscr{T} \cup \mathscr{E}$, there are two cases in the proof of (i). If $\mathbf{t}^\gamma$ is divisible by t_h^2, for some $h \neq e$, the division of $\mathbf{t}^\gamma$ by the element $t_h^2 - t_\ell^2$ yields remainder $\mathbf{t}^\delta = \mathbf{t}^\gamma t_h^{-2} t_\ell^2$. In this case, $\mathcal{J}(\mathbf{t}^\delta) = \mathcal{J}(\mathbf{t}^\gamma)$ and (i) follows trivially. Suppose now that $\mathbf{t}^\gamma$ is divisible by $\mathbf{t}^\alpha$ where $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is an Eulerian binomial. Let $\mathbf{t}^\gamma = \mathbf{t}^\rho \mathbf{t}^\alpha$, for some $\mathbf{t}^\rho \in K[E_G]$. Then, division yields remainder $\mathbf{t}^\delta = \mathbf{t}^\rho \mathbf{t}^\beta$. Let $T_1 = \theta(\mathbf{t}^\alpha)$, $T_2 = \theta(\mathbf{t}^\beta)$ and $T_3 = \theta(\mathbf{t}^\rho)$. Since $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is Eulerian, $T_1 = T_2$. From

$$\begin{aligned}
\mathcal{J}(\mathbf{t}^\gamma) &= \mathcal{J}(\mathbf{t}^\rho) \triangle \mathcal{J}(\mathbf{t}^\alpha) \equiv_2 |\mathcal{J}(\mathbf{t}^\rho)| + |\mathcal{J}(\mathbf{t}^\alpha)| \\
\theta(\mathbf{t}^\gamma) &= \theta(\mathbf{t}^\rho) \triangle \theta(\mathbf{t}^\alpha) = T_3 \triangle T_1 = T_3 \triangle T_2,
\end{aligned}$$

we deduce that $\mathcal{J}(\mathbf{t}^\delta) = \mathcal{J}(\mathbf{t}^\rho) \triangle \mathcal{J}(\mathbf{t}^\beta)$ is a $\theta(\mathbf{t}^\gamma)$-join and, moreover,

$$\begin{aligned}
|\mathcal{J}(\mathbf{t}^\delta)| &= |\mathcal{J}(\mathbf{t}^\rho)| + |\mathcal{J}(\mathbf{t}^\beta)| - 2|\mathcal{J}(\mathbf{t}^\rho) \cap \mathcal{J}(\mathbf{t}^\beta)| \\
&\equiv_2 |\mathcal{J}(\mathbf{t}^\rho)| + |\mathcal{J}(\mathbf{t}^\alpha)| \equiv_2 |\mathcal{J}(\mathbf{t}^\gamma)|.
\end{aligned}$$

To prove (ii) we start by remarking that, since $\mathscr{G}$ consists of binomials, the remainder term in a standard expression of a monomial with respect to $\mathscr{G}$ is also a monomial. Let us denote

$$\Gamma = \{J : J \text{ is a } \theta(\mathbf{t}^\gamma)\text{-join and } |J| \equiv_2 |\mathcal{J}(\mathbf{t}^\gamma)|\}.$$

Since Γ is non-empty ($\mathcal{J}(\mathbf{t}^\gamma)$ belongs to it) we may consider $\mathbf{t}^\rho$, the product of the edges of a minimum cardinality element of Γ. Let $\mathbf{t}^\nu$ be the remainder in a standard expression of $\mathbf{t}^\rho$ with respect to $\mathscr{G}$. By (i), $J(\mathbf{t}^\nu) \in \Gamma$. Since

$$|\mathcal{J}(\mathbf{t}^\nu)| \leq \deg(\mathbf{t}^\nu) = \deg(\mathbf{t}^\rho),$$

by the minimality, we deduce that $\mathbf{t}^\nu$ is square-free. On the other hand, let $\mathbf{t}^\mu$ be a square-free monomial such that $\mathbf{t}^\delta = \mathbf{t}^\mu t_e^{2k}$, for some $k \geq 0$. Then $\mathcal{J}(\mathbf{t}^\delta) = \mathcal{J}(\mathbf{t}^\mu)$. Using (i),

$$\theta(\mathbf{t}^\gamma) = \theta(\mathbf{t}^\delta) = \theta(\mathbf{t}^\mu).$$

It suffices to prove that $\mathbf{t}^\mu = \mathbf{t}^\nu$. Arguing by contradiction, assume that $\mathbf{t}^\mu \neq \mathbf{t}^\nu$ and let $\mathbf{t}^\alpha = \mathbf{t}^\mu \gcd(\mathbf{t}^\mu, \mathbf{t}^\nu)^{-1}$ and $\mathbf{t}^\beta = \mathbf{t}^\nu \gcd(\mathbf{t}^\mu, \mathbf{t}^\nu)^{-1}$. Then, $\mathbf{t}^\alpha \neq \mathbf{t}^\beta$ are relatively prime, square-free monomials, satisfying $\theta(\mathbf{t}^\alpha) = \theta(\mathbf{t}^\beta)$. If they have equal degree then $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is Eulerian and hence one of $\mathbf{t}^\mu$ or $\mathbf{t}^\nu$ is divisible by a leading term of $\mathscr{G}$, which is absurd. If $\deg(\mathbf{t}^\alpha) \neq \deg(\mathbf{t}^\beta)$ then, Lemma 3.4 yields the same conclusion. ■

Minimum T-joins, i.e., T-joins with minimum cardinality are of special interest in questions of Combinatorial Optimization. Here, Theorem 4.3 is leading us to a refinement of this notion, namely, minimum cardinality

T-joins among T-joins of a fixed parity cardinality. As we show below, if G is bipartite the parity of the cardinality of T-joins (for a fixed T-join) does not change. However if G is non-bipartite, the sets of T-joins of even cardinality and and of odd cardinality are both non-empty.

Definition 4.4. Let $T \in \mathcal{E}(V_G)$, J a T-join and set $i = |J| + 2\mathbb{Z} \in \mathbb{Z}/2$. We will say that J is a T-join of parity i. Let us denote by $\mathcal{J}_i(G,T)$ the set of all T-joins of parity i and let $\tau_i(G,T)$ denote the minimum cardinality of an element of $\mathcal{J}_i(G,T)$.

To ease notation, we will denote the elements of $\mathbb{Z}/2$ by 0 and 1. Naturally, *parity zero* and *even* will be used as synonyms, and the same applies to *parity one* and *odd*. Note that $\tau_i(G,T)$ is defined only if $\mathcal{J}_i(G,T)$ is non-empty. We will refer to the minimum cardinality elements of $\mathcal{J}_i(G,T)$ as *minimum fixed parity* T-joins. Minimum fixed parity T-joins also appear in Combinatorial Optimization; they are solutions of the a so-called *Parity Join Problem* (*cf.* [6]).

Example 4.5. Let G be the graph in Figure 2. The path depicted in bold, on the left, is an even $\{3,5\}$-join. It is not a minimum even $\{3,5\}$-join as 3 and 5 are joined by a path of length two. We deduce that $\tau_0(G,\{3,5\}) = 2$. Other examples are: $\tau_1(G,\{3,5\}) = 3$, $\tau_0(G,\{4,7\}) = 4$ (take the edge between 4 and 7 and any triangle), $\tau_1(G,\{4,7\}) = 1$, $\tau_0(G,\{8,10,11,12\}) = 2$ and $\tau_1(G,\{8,10,11,12\}) = 3$.

For a fixed $T \in \mathcal{E}(V_G)$, the minimum cardinality of T-joins (without the parity constraint) is denoted in the literature by $\tau(G,T)$. If G admits T-joins of both parities then, of course, $\tau(G,T) = \min\{\tau_0(G,T), \tau_1(G,T)\}$.

Lemma 4.6. *Let $T \in \mathcal{E}(V_G)$. Then, G is non-bipartite if and only if $\mathcal{J}_0(G,T)$ and $\mathcal{J}_1(G,T)$ are both non-empty.*

Proof: If there exist T-joins, J_1 and J_2, such that $|J_1| \not\equiv_2 |J_2|$, then $J_1 \triangle J_2$ is non-empty and defines an Eulerian subgraph $C \subset G$ with

$$|E_C| = |J_1 \triangle J_2| \equiv_2 |J_1| + |J_2| \equiv_2 1.$$

As C decomposes into an edge-disjoint union of cycles, one of these cycles must be odd and thus G is non-bipartite. Conversely if G is non-bipartite and $C \subset G$ is an odd cycle then, for a T-join, $J \subset E_G$, the subset $J \triangle E_C$ is a T-join with $|J \triangle E_C| \not\equiv_2 |J|$. ■

Example 4.7. Let $G = \mathcal{K}_n$ be a complete graph on $n \geq 3$ vertices. Table 2 contains a list of minimum cardinality elements of $\mathcal{J}_0(G,T)$ and $\mathcal{J}_1(G,T)$, for T of varying cardinality. In this table the vertices in T are represented

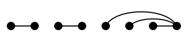
$\lvert T\rvert$	parity 0	parity 1
0	$\emptyset$	
2		
4		
6		
8		
$\vdots$	$\vdots$	$\vdots$

TABLE 2. Minimum even and odd T-joins in $\mathcal{K}_n$

in black, while for other vertices (in the cases of $|T| = 0$ and 1) we use gray. Since a T-join has at least $|T|/2$ edges it is easy to check that the T-joins listed in the two columns of Table 2 are minimum cardinality elements of $\mathcal{J}_0(G,T)$ and $\mathcal{J}_1(G,T)$, respectively.

Definition 4.8. Let I be an ideal in a polynomial ring and $\prec$ a monomial order. A monomial which does not belong to $\mathrm{in}_\prec(I)$ is called a standard monomial of I with respect to $\prec$. We denote the set of standard monomials by $\mathcal{B}_\prec(I)$.

The set of standard monomials of I with respect to a monomial order is a basis for the quotient of the polynomial ring by the ideal, as a vector space over the field (*cf.* Macaulay's Theorem, [3, §2.2.2]). Let $e \in E_G$ be a choice of an edge and $\prec$ a grevlex order with t_e last. In the final part of this work we will use the standard basis of the zero dimensional ideal $(I(X_G), t_e)$ to compute the degree and the regularity of $I(X_G)$. Because $(I(X_G), t_e)$ zero dimensional, $\mathcal{B}_\prec(I(X_G), t_e)$ is a finite set and, as we show next, can be described in a combinatorial way, using T-joins of fixed parity.

For the sake of clarity of notation, recall that E_G is regarded as set of (unordered) pairs of vertices and therefore if $e = \{a, b\}$ is an edge and $T \in \mathcal{E}(V_G)$ is an even subset of vertices, we may refer to $T \triangle e = T \triangle \{a, b\} \in \mathcal{E}(V_G)$.

Theorem 4.9. *Let $e \in E_G$ and $\prec$ a grevlex order on $K[E_G]$ with t_e last. Let $\vartheta\colon \mathcal{B}_\prec(I(X_G), t_e) \to \mathcal{E}(V_G) \times (\mathbb{Z}/2)$ be given by $\mathbf{t}^\alpha \mapsto (\theta(\mathbf{t}^\alpha), \deg(\mathbf{t}^\alpha) + 2\mathbb{Z})$. Then ϑ is injective and*

$$\operatorname{Im}\vartheta = \{(T,i) : \mathcal{J}_i(G,T) \neq \emptyset \text{ and } \tau_{i+1}(G, T\triangle e) = \tau_i(G,T) + 1\}\,. \tag{5}$$

Proof: Consider the Gröbner basis of $(I(X_G), t_e)$ given by $\mathscr{G} \cup \{t_e\}$, where $\mathscr{G}$ is the Gröbner basis of $I(X_G)$ given in Definition 3.1. Let $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$, belonging to $\mathcal{B}_\prec(I(X_G), t_e)$, be such that $\vartheta(\mathbf{t}^\alpha) = \vartheta(\mathbf{t}^\beta)$. Denote

$$\begin{aligned}
&T = \theta(\mathbf{t}^\alpha) = \theta(\mathbf{t}^\beta),\\
&i = \deg(\mathbf{t}^\alpha) + 2\mathbb{Z} = \deg(\mathbf{t}^\beta) + 2\mathbb{Z}.
\end{aligned}$$

Since $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are standard elements of $I(X_G)$ with respect to $\prec$ that, additionally, are not divisible by t_e, both $\mathbf{t}^\alpha$ and $\mathbf{t}^\beta$ are square-free. Moreover by (ii) of Theorem 4.3, $\mathcal{J}(\mathbf{t}^\alpha)$ and $\mathcal{J}(\mathbf{t}^\beta)$ are minimum cardinality elements of $\mathcal{J}_i(G,T)$. We conclude that

$$\deg(\mathbf{t}^\alpha) = |\mathcal{J}(\mathbf{t}^\alpha)| = |\mathcal{J}(\mathbf{t}^\beta)| = \deg(\mathbf{t}^\beta)$$

Assume, arguing by contradiction, that $\mathbf{t}^\alpha \neq \mathbf{t}^\beta$. Then, as $\theta(\mathbf{t}^\alpha) = \theta(\mathbf{t}^\beta)$ and $\deg(\mathbf{t}^\alpha) = \deg(\mathbf{t}^\beta)$, the binomial $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is Eulerian and thus one of $\mathbf{t}^\alpha$ or $\mathbf{t}^\beta$ is a leading term of $\mathscr{G} \cup \{t_e\}$, which is a contradiction.

Let us now prove (5). Suppose $(T,i) = \vartheta(\mathbf{t}^\alpha)$, for some $\mathbf{t}^\alpha \in \mathcal{B}_\prec(I(X_G), t_e)$. As $\mathbf{t}^\alpha$ and $\mathbf{t}^\alpha t_e$ are square-free standard monomials of $I(X_G)$ and

$$\theta(\mathbf{t}^\alpha t_e) = T\triangle e,$$

by (ii) of Theorem 4.3,

$$\tau_i(G,T) = \deg(\mathbf{t}^\alpha) = \deg(\mathbf{t}^\alpha t_e) - 1 = \tau_{i+1}(G, T\triangle e) - 1.$$

Conversely, suppose $\mathcal{J}_i(G,T)$ is non-empty and

$$\tau_{i+1}(G, T\triangle e) = \tau_i(G,T) + 1.$$

Let J be a minimum cardinality element of $\mathcal{J}_i(G,T)$ and let $\mathbf{t}^\alpha$ be the remainder in a standard expression with respect to $\mathscr{G}$ of the monomial given by the product of all edges in J. By (i) of Theorem 4.3, $\mathcal{J}(\mathbf{t}^\alpha) \in \mathcal{J}_i(G,T)$ and by (ii) of the same proposition, $|\mathcal{J}(\mathbf{t}^\alpha)| = \tau_i(G,T) = |J|$. This proves that $\mathbf{t}^\alpha$ is square-free, so that if t_e divides $\mathbf{t}^\alpha$, $|\mathcal{J}(\mathbf{t}^\alpha t_e^{-1})| = |\mathcal{J}(\mathbf{t}^\alpha)| - 1$. However, as $\mathcal{J}(\mathbf{t}^\alpha t_e^{-1})$ is a $(T\triangle e)$-join, this would imply that

$$\tau_{i+1}(G, T\triangle e) \leq \tau_i(G,T) - 1$$

which is not true. We deduce that $\mathbf{t}^\alpha$ is not divisible by t_e and therefore belongs to $\mathcal{B}_\prec(I(X_G), t_e)$. By what was said, it is clear that $\vartheta(\mathbf{t}^\alpha) = (T, i)$. ■

Degree. The next result was first proved in [13, Proposition 2.11] by reducing to $K = \mathbb{Z}/3$ and to the vanishing ideal of the projective set parameterized by G. We can now give an alternative proof, drawing closely on the properties of the graph. Below, $b_0(G)$ denotes the number of connected components of G.

Proposition 4.10. *The degree of* $K[E_G]/I(X_G)$ *is*

$$\begin{cases} 2^{|V_G|-b_0(G)}, & \textit{if } G \textit{ is non-bipartite,} \\ 2^{|V_G|-b_0(G)-1}, & \textit{if } G \textit{ is bipartite.} \end{cases}$$

Proof: Fix $e \in E_G$ and $\prec$ a grevlex order on $K[E_G]$ with t_e last. As t_e is regular on $K[E_G]/I(X_G)$ and its degree is one, the degree of $K[E_G]/I(X_G)$ is equal to the dimension of $K[E_G]/(I(X_G), t_e)$ as a vector space over K, which is then given by the number of elements of $\mathcal{B}_\prec(I(X_G), t_e)$. Let ϑ be the map of Theorem 4.9. Given $T \in \mathcal{E}(V_G)$, consider the subset of $\mathcal{E}(V_G) \times (\mathbb{Z}/2)$ given by

$$\mathcal{F}_T = \{(T, 0), (T, 1), (T \triangle e, 0), (T \triangle e, 1)\}.$$

We claim that $|\operatorname{Im}\vartheta \cap \mathcal{F}_T| = 1$, if G is bipartite, and $|\operatorname{Im}\vartheta \cap \mathcal{F}_T| = 2$, if G is non-bipartite. From this claim we deduce that the degree of

$$K[E_G]/(I(X_G), t_e)$$

is equal to $\frac{|\mathcal{E}(V_G)|}{2}$ if G is bipartite or $|\mathcal{E}(V_G)|$, otherwise. To ease notation, let $r = b_0(G)$ and denote by $n_1, \ldots, n_r$ the cardinalities of the sets of vertices of the connected components. Then $|\mathcal{E}(V_G)| = 2^{n_1-1} \cdots 2^{n_r-1} = 2^{|V_G|-r}$ and thus the result follows.

Let us now prove the claim. For a fixed T, at least one of the sets $\mathcal{J}_0(G, T)$, $\mathcal{J}_1(G, T)$ is non-empty. Fix $i \in \mathbb{Z}/2$ such that $\mathcal{J}_i(G, T) \neq \emptyset$ and $J \in \mathcal{J}_i(G, T)$ of minimum cardinality. Then, as $J \triangle \{e\} \in \mathcal{J}_{i+1}(G, T \triangle e)$, the set $\mathcal{J}_{i+1}(G, T \triangle e)$ is also non-empty. Moreover, since $|J \triangle \{e\}| \leq |J| + 1$,

$$\tau_{i+1}(G, T \triangle e) \leq \tau_i(G, T) + 1.$$

Repeating this argument with $T \triangle e$ and $i + 1 \in \mathbb{Z}/2$ and combining the results,

$$\tau_{i+1}(G, T \triangle e) - 1 \leq \tau_i(G, T) \leq \tau_{i+1}(G, T \triangle e) + 1.$$

As $\tau_i(G,T)$ and $\tau_{i+1}(G,T\triangle e)$ have opposite parities, either

$$\tau_i(G,T) = \tau_{i+1}(G,T\triangle e) - 1 \text{ or}$$
$$\tau_i(G,T) = \tau_{i+1}(G,T\triangle e) + 1.$$

In either case, using Theorem 4.9, we deduce that

$$|\,\{(T,i),(T\triangle e,i+1)\}\cap \operatorname{Im}\vartheta| = 1.$$

If G is bipartite then, by Lemma 4.6, only one of $\mathcal{J}_0(G,T)$ or $\mathcal{J}_1(G,T)$ is non-empty and thus $|\operatorname{Im}\vartheta\cap\mathcal{F}_T| = 1$. If G is non-bipartite then both of these sets are non-empty and hence $|\operatorname{Im}\vartheta\cap\mathcal{F}_T| = 2$. The claim is proved. ■

Regularity. Theorem 4.9 will be used to express $\operatorname{reg} I(X_G)$ in a combinatorial way. Before we do this, and to explain the connection with the initial results in this direction contained in [13], we need the following notion.

Definition 4.11. $J\subset E_G$ is called a *parity join* if and only if $|J\cap E_C|\leq \frac{|E_C|}{2}$, for every *even* Eulerian subgraph of G.

This definition is related to the notion of *join* (*cf.* [5]); a subset $J\subset E_G$ is called a join if and only if $|J\cap E_C|\leq\frac{|E_C|}{2}$, for every Eulerian subgraph $C\subset G$. A join is always a parity join but not the way around. The relation between joins and T-joins is established in Guan's Theorem (*cf.* [8]); if J is a minimum cardinality T-join then J is a join and, vice-versa, if J is a join then it is a minimum cardinality T-join, for T equal to the set of odd degree vertices of the induced subgraph. A similar result holds for parity joins.

Lemma 4.12. *If T is an even subset of vertices and $i\in\mathbb{Z}/2$, then any element of $\mathcal{J}_i(G,T)$ of minimum cardinality is a parity join. Conversely, any parity join, J, is a minimum cardinality element of $\mathcal{J}_i(G,T)$, where T is the set of odd degree vertices of the subgraph induced by J and $i=|J|+2\mathbb{Z}$.*

Proof: Let J be a minimum cardinality element of $\mathcal{J}_i(G,T)$ and let C be an even Eulerian subgraph of G. Then $J\triangle E_C$ is a T-join with $|J\triangle E_C|\equiv_2|J|$ hence

$$|J|\leq|J\triangle E_C| \iff |J\cap E_C|\leq\tfrac{|E_C|}{2},$$

i.e., J is a parity join. Conversely, let $J\subset E_G$ be a parity join and $T\subset V_G$ be the set of odd degree vertices of the subgraph of G induced by J. Then

J is a T-join. Set $i = |J| + 2\mathbb{Z}$ and let $J' \in \mathcal{J}_i(G,T)$. Then $J \triangle J'$ defines an even Eulerian subgraph and therefore,

$$\begin{aligned} &|J \cap (J \triangle J')| \leq \tfrac{|J \triangle J'|}{2} \iff \\ &|J| - |J \cap J'| \leq \tfrac{|J|+|J'|}{2} - |J \cap J'| \iff |J| \leq |J'|. \end{aligned}$$

We deduce that J is a minimum cardinality element of $\mathcal{J}_i(G,T)$. ■

Theorem 4.13. *The regularity of $I(X_G)$ is the maximum cardinality of minimum fixed parity T-joins or, equivalently, the maximum cardinality of parity joins.*

Proof: Let $e \in E_G$ and fix $\prec$ a grevlex order on $K[E_G]$ with t_e last. By Proposition 2.4, we get $\operatorname{reg} I(X_G) = \max\{d : N_d \neq 0\} + 1$, where

$$N = K[E_G]/(I(X_G), t_e).$$

Another way of expressing this, using $\mathcal{B}_\prec(I(X_G), t_e)$, is

$$\operatorname{reg} I(X_G) = \max\{\deg(\mathbf{t}^\alpha) : \mathbf{t}^\alpha \in \mathcal{B}_\prec(I(X_G), t_e)\} + 1.$$

Let $\mathbf{t}^\alpha \in \mathcal{B}_\prec(I(X_G), t_e)$ be of maximum degree and let ϑ be as in Theorem 4.9. Denote $(T,i) = \vartheta(\mathbf{t}^\alpha)$. Then, by (ii) of Theorem 4.3 and Theorem 4.9,

$$\deg(\mathbf{t}^\alpha) + 1 = \tau_i(G,T) + 1 = \tau_{i+1}(G, T \triangle e)$$

and so $\operatorname{reg} I(X_G) = \tau_{i+1}(G, T \triangle e)$. We conclude that

$$\operatorname{reg} I(X_G) \leq \max\{\tau_i(G,T) : T \in \mathscr{E}(V_G),\ i \in \mathbb{Z}/2,\ \mathcal{J}_i(G,T) \neq \emptyset\}.$$

To prove the opposite inequality, let $T_0 \in \mathcal{E}(V_G)$, $k \in \mathbb{Z}/2$ be such that $\tau_k(G,T_0)$ is the maximum of the set above. Fix $J \in \mathcal{J}_k(G,T_0)$ with

$$|J| = \tau_k(G,T_0)$$

and let $\mathbf{t}^\alpha$ be the remainder in a standard expression with respect to $\mathscr{G}$ of the monomial given by the product of all edges in J. Then, arguing as in the proof of Theorem 4.9, we deduce that $\mathbf{t}^\alpha$ is square-free, $\mathcal{J}(\mathbf{t}^\alpha) \in \mathcal{J}_k(G,T_0)$ and $\deg(\mathbf{t}^\alpha) = \tau_k(G,T_0)$. By the maximality of $\tau_k(G,T_0)$ we get

$$\tau_{k+1}(G, T_0 \triangle e) \leq \tau_k(G,T_0)$$

which, by Theorem 4.9, means that $(T_0,k) \notin \operatorname{Im}\vartheta$. This implies that $\mathbf{t}^\alpha$ is not a standard element of $(I(X_G), t_e)$; hence t_e divides $\mathbf{t}^\alpha$ and

$$\mathbf{t}^\alpha t_e^{-1} \in \mathcal{B}_\prec(I(X_G), t_e).$$

Accordingly, $\operatorname{reg} I(X_G) \geq \deg(\mathbf{t}^\alpha t_e^{-1}) + 1 = \deg(\mathbf{t}^\alpha) = \tau_k(G,T_0)$. ■

In [13, Theorem 4.5] it was shown that for a bipartite graph the regularity of $I(X_G)$ is equal to the maximum cardinality of a join. Theorem 4.13 is therefore a generalization of this result.

Example 4.14. By Theorem 4.13, the value of $\operatorname{reg} I(X_G)$, for $G = \mathcal{K}_n$, can now be obtained by an analysis of the minimum fixed parity T-joins. (This was done in [13] by reducing to $K = \mathbb{Z}/3$ and using the results of [7].) From Table 2, if $n = 3$, $\operatorname{reg} I(X_G) = 3$; obtained by taking a minimum cardinality element of $\mathcal{J}_1(G, \emptyset)$. (Note that $G = \mathcal{K}_3$ is listed in Table 1 in the family of non-bipartite uni-cyclic graphs.) If $n \geq 4$, the maximum cardinality of a fixed parity T-join is $r = \lfloor n/2 \rfloor + 1$. Denoting $i = r + 2\mathbb{Z}$, we see that $r = \tau_i(G, V_G)$, if n is even, and, if n is odd, $r = \tau_i(G, T)$ with $T = V_G \setminus \{v\}$, for any choice of $v \in V_G$.

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