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PROPERTIES OF INCREASING ODDS RATE DISTRIBUTIONS WITH A STATISTICAL APPLICATION

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Abstract: We study the family of distributions characterised by an increasing odds rate (IOR), showing that this is a primary condition for being coherent with the notion of "adverse ageing". We prove some preservation properties of this class under several transformations that are often considered in reliability and life testing problems, including formation of order statistics. Moreover, the IOR assumption enables the derivation of survival bounds and tolerance limits, extending the scope of applicability of some known results, which are based, for instance, on the increasing hazard rate assumption. Finally, we propose a test for the IOR null hypothesis, establishing its approximate consistency and providing a table of simulated p-values.

Keywords: odds, hazard rate, failure rate, order statistics, bounds, nonparametric test.
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1. Introduction

Ageing properties of life distributions are typically characterised by the *hazard rate* (HR), which is also referred in the literature as the *failure rate*. Let X be an absolutely continuous random variable with cumulative distribution function (CDF) F, survival function $\overline{F} = 1 - F$, and probability density function (PDF) f. The HR of F, given by $h_F = \frac{f(x)}{\overline{F}(x)}$, is related to (and may be loosely interpreted as) the conditional probability of failure given that survival up to time x has occurred. The notion of a monotone HR plays a key role in reliability and survival analysis, as described, for example, in Barlow et al. (1963), Marshall and Olkin (2007) or Shaked and Shanthikumar (2007). In particular, F is said to have an *increasing hazard rate* (IHR) if h_F is increasing, an assumption widely-used in the literature to derive many

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useful characterisations and properties. However, different applications revealed that the IHR condition is too stringent in many practical situations, being in contrast with several important models, such as the bathtub and the heavy tailed distributions. Therefore, it seems natural to search for alternative models allowing for a broader applicability for describing the hazard of ageing, encompassing violations of the IHR condition.

We define *the odds of failure by time* x of the CDF F as

$$\Lambda_F(x) = \frac{F(x)}{\overline{F}(x)}, \tag{1}$$

which may be interpreted as the probability of failure over survival up to time x. In particular,

$$\Lambda_F(x) \approx \frac{P(X \in (x, x+\Delta] | X > x)}{P(X \in (x-\Delta, x] | X \leq x)}, \tag{2}$$

for sufficiently small $\Delta > 0$. This function has been considered by Marshall and Olkin (2007), Kirmani and Gupta (2001), Sankaran and Jayakumar (2008), Nair and Sankaran (2015), Kumar M. et al. (2019), Lando and Bertoli-Barsotti (2019), Lando et al. (2020). As the odds function is always increasing, its growth rate is of particular interest. For this purpose, we define the *odds rate* (OR) associated with F as

$$\lambda_F = \Lambda_F' = \frac{f}{\overline{F}^2}. \tag{3}$$

We are interested in studying the family of distributions that have an *increasing odds rate* (IOR), that is, the family

$$\mathcal{F}_{IOR} = \{F : \lambda_F \text{ is increasing}\}.$$

The IOR property is a basic condition that every distribution should satisfy in order to be coherent with the notion of "adverse ageing". Indeed, the increasingness of λ_F means that the ratio between probability of failure (or death) and probability of surviving by time x is accelerating with respect to time.

The paper is organized as follows. In Section 2, we present some preservation properties of the IOR family. In Section 3, we provide bounds for the survival function and tolerance limits for IOR distributions, and in Section 4, we propose a test of the IOR null hypothesis.

Note that the properties we have been defining only depend on the distribution and not on the random variable, so we refer to either one as more convenient.

2. Properties of the IOR family

The OR and the HR can be seen as special cases of the *generalized hazard rate* (Barlow et al., 1971) of an absolutely continuous CDF F with respect to another absolutely continuous CDF G, given by

$$h_F^G(x) = \frac{d}{dx} G^{-1} \circ F(x) = \frac{f(x)}{g \circ G^{-1} \circ F(x)}.$$

It is easily seen that we get $h_F^G = h_F$ for the choice $G(x) = 1 - e^{-x}$, namely, the CDF of a unit exponential, whereas we get $h_F^G = \Lambda_F$ when choosing $G(x) = \frac{x}{1+x}$, namely, the CDF of a log-logistic with both parameters equal to one (hereafter, denoted as log-logistic(1,1)). Correspondingly, $H_F^G = G^{-1} \circ F$ is referred to as the *generalized hazard function*. For $G(x) = 1 - e^{-x}$, $G^{-1}(p) = -\ln(1-p)$ and we obtain $H_F^G = -\ln \overline{F}$ which is simply referred to as the hazard function (Shaked and Shanthikumar, 2007), whereas, for $G(x) = \frac{x}{1+x}$, we obtain $G^{-1}(p) = \frac{p}{1-p}$ and $H_F^G = \Lambda_F$.

2.1. Relations with the IHR family. Since the OR can equivalently be expressed as $\lambda_F = \frac{h_F}{\overline{F}}$, taking into account that $\overline{F}$ is decreasing, the following result is straightforward.

Proposition 1. *If F is* IHR*, then F is* IOR*.*

It can be easily seen that the converse is not true. Below we present a list of relevant IOR distributions that are not IHR. Consider a distribution whose HR starts decreasing, then remains constant and, finally, becomes increasing, which is known in the literature as bathtub distribution (such models are particularly popular in life testing and reliability, see Rajarshi and Rajarshi (1988) or Nadarajah (2009)). It is obvious that the IHR condition is inconsistent with the bathtub model, yet, it can be seen that some bathtub distributions, such as the Hjorth's (Hjorth, 1980), the J-shaped (Topp and Leone, 1955), the Schäbe's (Schäbe, 1994) and the Haupt and Schäbe's (Haupt and Schäbe, 1997) distributions are IOR, at least under some conditions on the parameters. Moreover, several heavy tailed distributions, such as the log-logistic, the Pareto, the Fréchet, the Student's t (with shape parameter larger than or equal to 1 in each case), the lognormal and the Cauchy distributions, are IOR.

The IHR and the IOR conditions can also be expressed in terms of the *convex transform order* (van Zwet, 1964). We say that G dominates F in the convex transform order and write $F \leq_c G$ if the generalized hazard function $G^{-1} \circ F$ is convex. Then, the IHR family is the set of distributions that are dominated by the unit exponential, whereas the IOR family is the class of those that are dominated by the log-logistic(1,1). It can be easily argued, from Proposition 1 and from the transitivity of the $\leq_c$ order relation, that the unit exponential distribution is dominated by the log-logistic (1,1).

The IOR condition may still be equivalently expressed in terms of the *total time on test* (TTT) transform, given by

$$T^{-1}(p) = \int_0^{F^{-1}(p)} \overline{F}(y)\, dy, \quad p \in [0,1],$$

where F has a non-negative support.

Since T^{-1} is strictly increasing and continuous, its inverse function T is always defined. In particular, $T = F \circ I_F^{-1}$ is an absolutely continuous CDF with support $[0, \infty)$, where $I_F(x) = \int_0^x \overline{F}(y)\, dy$. It is well known that F is IHR if and only if T is convex (Barlow et al., 1971). With regard to the IOR condition, the following result holds.

Proposition 2. *Let F be a CDF defined on a non-negative support. F is* IOR *iff T is* IHR.

Proof: The HR of T is

$$h_T = \frac{T'}{1-T} = \frac{(I_F^{-1})'(f \circ I_F^{-1})}{\overline{F} \circ I_F^{-1}} = \frac{f \circ I_F^{-1}}{(\overline{F} \circ I_F^{-1})^2} = \lambda_F \circ I_F^{-1}. \tag{4}$$

Therefore, as I_F^{-1} is increasing, F is IOR if and only if T is IHR. ■

2.2. Closure under residual life distribution. Consider a CDF F such that $F(0) = 0$. The residual life distribution F_t of F at time t is defined by

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \tag{5}$$

for every $t \geq 0$ such that $\overline{F}(t) > 0$ (Marshall and Olkin, 2007). F_t is the conditional distribution of the remaining life given survival up to time t, which is of considerable practical interest in reliability and survival analysis. Since Λ_{F_t} is convex if and only if $\frac{1}{\overline{F}_t(x)} = \frac{\overline{F}(t)}{\overline{F}(t+x)}$ is convex, the following result is immediate.

Proposition 3. *Let F be a CDF. If F is* IOR *then F_t is also* IOR.

2.3. Preservation under weighting. Let F be a CDF and $w : \mathbb{R} \to \mathbb{R}^+$ a weight function for which $0 < E(w(X)) < \infty$. The weighted distribution associated with w and F is defined by

$$F_w(x) = \frac{\int_{-\infty}^{x} w(t)\, dF(t)}{\int_{\mathbb{R}} w(t)\, dF(t)} = \int_{-\infty}^{x} \tilde{w}(t)\, dF(t)$$

where $\tilde{w}$ is a normalized weight function. Distributions of this kind often occur in reliability and survival analysis. For this reason, the problem of characterizing ageing properties of the weighted distribution F_w in relation with the parent distribution F has been considered in Jain et al. (1989), Bartoszewicz and Skolimowska (2006), Misra et al. (2008) or Gupta and Arnold (2016), for example. The following result provides a condition for preservation of the IOR ageing property under weighting.

Theorem 4. *Let F be an* IOR *distribution, w be decreasing and $w\lambda_F$ increasing. Then F_w is* IOR.

Proof: A change of variable in the integral expression of F_w gives

$$F_w(x) = \int_0^{F(x)} \widetilde{w} \circ F^{-1}(p)\, dp = 1 - \int_0^{\overline{F}(x)} \widetilde{w} \circ \overline{F}^{-1}(p)\, dp = 1 - \Psi \circ \overline{F}(x)$$

where $\Psi(p) = \int_0^p \widetilde{w} \circ \overline{F}^{-1}(u)\, du$. Then, the OR of F_w is given by

$$\lambda_{F_w}(x) = \frac{\widetilde{w}(x) f(x)}{(\Psi \circ \overline{F}(x))^2} = \widetilde{w}(x)\lambda_F(x) \left(\frac{\overline{F}(x)}{\Psi \circ \overline{F}(x)} \right)^2.$$

Taking into account that w is decreasing, it follows that the function Ψ is convex and subsequently starshaped, that is, $\frac{x}{\Psi(x)}$ is decreasing. Therefore, $\frac{\overline{F}}{\Psi \circ \overline{F}}$ is increasing, so the conclusion follows. ■

It is known that the IHR property is preserved under weighting, if the weight function is increasing and concave (Jain et al., 1989). Example 5 below shows that the IOR property may not be preserved under this same condition.

Example 5. Take $F(x)=\frac{x}{x+1}$, the CDF of the log-logistic(1,1) and $w(x)=x^{\frac{1}{2}}$. Since $E(w(X))=\frac{\pi}{2}$, we obtain

$$F_w(x)=\frac{2}{\pi}\int_0^x \frac{t^{\frac{1}{2}}}{(t+1)^2}dt=\frac{2}{\pi}\left(\arctan\left(x^{\frac{1}{2}}\right)-\frac{x^{\frac{1}{2}}}{x+1}\right),$$

while λ_w is given by

$$\lambda_w(x)=\frac{2x^{\frac{1}{2}}}{(x+1)^2(\pi-2\left(\arctan\left(x^{\frac{1}{2}}\right)-\frac{x^{\frac{1}{2}}}{x+1}\right)}.$$

It is easy to check that $\lim_{x\to+\infty}\lambda_w(x)=0$ and $\lambda_w(0)=0$, meaning that $\lambda_w(x)$ is not monotone.

Theorem 4 can be used to obtain new distributions within the IOR family via the weighting method, or to establish the ageing behaviour of F_w when the conditions of the theorems of Jain et al. (1989), Bartoszewicz and Skolimowska (2006), Misra et al. (2008) or Gupta and Arnold (2016) are not verified, as shown in the following example.

Example 6. Take F to be the unit exponential distribution and $w(x)=\frac{1}{1+x}$, $x\geq 0$. Since $E(w(X))=-e\mathrm{Ei}(-1)$, where $\mathrm{Ei}(x)=\int_{-\infty}^x \frac{e^t}{t}dt$ is the exponential integral function, we obtain $F_w(x)=-\frac{\mathrm{Ei}(-x-1)}{\mathrm{Ei}(-1)}$. Suppose that we need information about the ageing pattern of F_w, yet, w does not satisfy the assumptions of the aforementioned theorems, so we cannot establish whether F_w is IHR. In fact, it can be seen that F_w is not IHR. However, w is decreasing and $w(x)\lambda_F(x)=\frac{e^x}{x+1}$ is increasing, so that Theorem 4 implies that F_w is IOR.

2.4. Closure under order statistics. Let $X_{(k)}$ be the k-th order statistic corresponding to an iid random sample of size n from X, a random variable with CDF F. We shall denote by $F_{(k)}$ the CDF of $X_{(k)}$ and by $\lambda_{(k)}$ its OR. It is well known that $F_{(k)}$ is given by

$$F_{(k)}(x)=F_\beta(F(x);k,n-k+1),$$

where $F_\beta(x;a,b)$, $x\in[0,1]$, $a,b>0$, is the CDF of a beta distribution with parameters a and b. In this subsection we prove that the order statistics in the iid case preserve the IOR property, that is, if F is IOR, then $F_{(k)}$ is IOR for every $1\leq k\leq n$. For technical reasons, the cases $k=1$ and $k=n$ need separated treatment.

Let us denote by $S^-(u)$, or $S^-(u(x),\, x \in A)$, if one needs to restrict to some domain A, the number of sign changes of a function u (Shaked and Shanthikumar, 2007, p. 10). Moreover, for the sake of brevity, when needed, we denote in brackets the first sign in the sign pattern of u, for instance $S^-(u) \leq 1(+)$ means that the sign pattern of u is either +− or +. We shall need the following lemma.

Lemma 7. *Let a_i, $i = 0, \ldots, n$, be a decreasing sequence of real numbers such that $a_0 > 0$ and $a_n < 0$. Consider the polynomial function $P(x) = \sum_{i=0}^{n} a_i x^i$ defined on some interval $[0, b)$, where $b > 0$ or $b = \infty$. Then, $S^-(P^{(j)}(x),\, x \in [0, b)) \leq 1(+)$, for $0 \leq j \leq n$, where $P^{(j)}$ is the j-th derivative of P and $P^{(0)} = P$.*

Proof: As the sequence a_i, $i = 0, \ldots, n$, is decreasing and $a_0 a_n < 0$, there exists some k, such that $a_j > 0$, for every $j \leq k$ and $a_j \leq 0$, for every $j \geq k + 1$. Consequently, for every $j \geq k + 1$, and $x \geq 0$, we have that $P^{(j)}(x) \leq 0$, for $x > 0$. Taking into account that $P^{(k)}(0) = a_k > 0$, it follows that $S^-(P^{(k)}(x),\, x \in [0, b)) \leq 1(+)$. Moreover, when $j \leq k$ we have $P^{(j)}(0) = a_j > 0$, thus $S^-(P^{(j)}(x),\, x \in [0, b)) \leq 1(+)$ also in this case. ■

Theorem 8. *If F is* IOR, *then $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$ is* IOR.

Proof: Note that $F_{(n)} = F^n$. We need to prove that $\lambda_{(n)}$ is increasing. We may represent

$$\lambda_{(n)}(x) = \frac{nf(x)}{(1 - F(x))^2} \frac{F^{n-1}(x)}{(1 + F(x) + \cdots + F^{n-1}(x))^2}$$

The first ratio on the right-hand side of $\lambda_{(n)}$ is increasing as F is IOR. To prove that $\lambda_{(n)}$ is increasing, it is enough to prove that $\frac{F^{n-1}(x)}{(1+F(x)+\cdots+F^{n-1}(x))^2}$ is increasing, which is equivalent to establish that $L(x) = \frac{x^{n-1}}{(1+x+\cdots+x^{n-1})^2}$ is increasing in $[0, 1]$. Differentiating $L(x)$, we obtain

$$L'(x) = \frac{x^{n-2}}{(1 + x + \cdots + x^{n-1})^3} \sum_{j=0}^{n-1} (n - 1 - 2j) x^j.$$

It is obvious that $S^-(L'(x)) = S^-(P(x) = \sum_{j=0}^{n-1} (n - 1 - 2j) x^j)$ and the sign patterns coincide. Hence, it follows from Lemma 7 that $S^-(P'(x),\, x \in [0, 1]) \leq 1(+)$ as the sequence $a_j = n - 1 - 2j$ is decreasing and $a_0 = n - 1 > 0$, $a_{n-1} = -(n - 1) < 0$. Moreover, $P(1) = \sum_{j=0}^{n-1} (n - 1 - 2j) = 0$ and

$P(0) = a_0 > 0$, consequently, $P(x) \geq 0$, thus $L(x)$ is increasing in $[0,1]$, and the result follows. ■

Theorem 9. *If F is* IOR*, then $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$ is* IOR*.*

Proof: As $1 - F_{(1)} = \overline{F}^n$, it follows that $\lambda_{(1)}(x)$ is the product of two increasing functions, hence increasing itself:

$$\lambda_{(1)}(x) = \frac{nf(x)}{\overline{F}^2} \frac{1}{\overline{F}^{n-1}(x)}.$$

■

Theorem 10. *Let $2 \leq k \leq n-1$. If F is* IOR *then $X_{(k)}$ is* IOR*.*

Proof: We need to prove that $\lambda_{(k)}(x) = \frac{f_{(k)}(x)}{\overline{F}^2_{(k)}(x)}$ is increasing in x, where

$$\lambda_{(k)}(x) = n\binom{n-1}{k-1} \frac{f(x)F^{k-1}(x)\overline{F}^{n-k}(x)}{\left(\sum_{j=0}^{k-1}\binom{n}{j}F^j(x)\overline{F}^{n-j}(x)\right)^2}$$

$$= n\binom{n-1}{k-1} \frac{f(x)}{\overline{F}^2(x)} \frac{F^{k-1}(x)\overline{F}^{n-k+2}(x)}{\left(\sum_{j=0}^{k-1}\binom{n}{j}F^j(x)\overline{F}^{n-j}(x)\right)^2}.$$

Taking into account that $\frac{f}{\overline{F}^2}$ and F are increasing, it is enough to prove that $L(x) = \frac{x^{k-1}(1-x)^{n-k+2}}{\left(\sum_{j=0}^{k-1}\binom{n}{j}x^j(1-x)^{n-j}\right)^2}$ is increasing for $x \in [0,1]$. For simplicity, denote by $N(x)$ and $D(x)$ the numerator and the denominator of $L(x)$, respectively. Proving that L is increasing for $x \in [0,1]$ is equivalent to verifying that, for every $c > 0$, $S^-(L(x) - c,\, x \in [0,1]) \leq 1(\text{-})$. For the latter expression, note that the sign of $L(x) - c$ coincides with the sign of $M(x) = N(x) - cD(x)$, so we will describe the sign pattern of $M(x)$. Obviously, $M'(x) = N'(x) - cD'(x)$. Now, simple calculations give us

$$N'(x) = (-(n+1)x + (k-1))x^{k-2}(1-x)^{n-k+1}$$

and

$$D'(x) = -n\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k}D^{1/2}(x)$$

thus,

$$M'(x) = x^{k-1}(1-x)^{n-k}\left(2cn\binom{n-1}{k-1}D^{1/2}(x) - \frac{1-x}{x}((n+1)x-(k-1))\right). \tag{6}$$

Let us denote by $A(x)$ and $B(x)$ the first and second terms inside the large parenthesis in (6), and put $C(x) = A(x) - B(x)$, whose sign coincides with the sign of $M'(x)$. Simple differentiation shows that

$$A'(x) = -n\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k},$$
$$A''(x) = -n\binom{n-1}{k-1}(k-1-(n-1)x)x^{k-2}(1-x)^{n-k-1}.$$

Hence, $A(x)$ is decreasing on $[0,1]$, concave on $[0,\frac{k-1}{n-1}]$ and convex on $[\frac{k-1}{n-1},1]$. On what concerns $B(x)$, we have that $B'(x) = \frac{k-1-(n+1)x^2}{x^2}$ and $B''(x) = -\frac{2(k-1)}{x^3}$. Hence, $B(x)$ is concave on $[0,1]$, increasing on $[0,(\frac{k-1}{n+1})^{1/2}]$ and decreasing on $[(\frac{k-1}{n+1})^{1/2},1]$. It is easy to verify that $(\frac{k-1}{n+1})^{1/2} > \frac{k-1}{n-1}$ if $k \leq n-2$ and $(\frac{k-1}{n+1})^{1/2} < \frac{k-1}{n-1}$ if $k = n-1$. To determine the sign pattern of $C(x)$, we distinguish two different cases:

Case $k \leq n-2$: Then $C(x)$ is decreasing in $[0,(\frac{k-1}{n+1})^{1/2}]$ and convex in $[(\frac{k-1}{n+1})^{1/2},1]$. Therefore, taking now into account that $\lim_{x\to 0} C(x) = +\infty$ and $\lim_{x\to 1} C(x) = 0$, it follows that $S^-(C(x),\, x \in [0,1]) \leq 1(+)$, meaning that the sign pattern of $M'(x)$, for $x \in [0,1]$, can be either +- or +. Consequently, remembering that $M(0) = -c$, it follows that $S(M(x),\, x \in [0,1]) \leq 1(-)$, hence $L(x)$ is increasing for $x \in [0,1]$ or, equivalently, $\lambda_{(k)}(x)$ is increasing.

Case $k = n-1$: In this case the function $C(x)$ may have many roots in $[0,1]$. Consequently, the approach used in the previous case does not allow for a conclusion. However, for this case we may handle directly the function $L(x)$ to prove that it is increasing in $[0,1]$. Remark that we may represent $L(x) = \frac{x^{n-2}(1-x)^3}{(1-(nx^{n-1}+(1-n)x^n))^2}$. Moreover, some simple algebraic calculations give

$$1-(nx^{n-1}+(1-n)x^n) = (1-x)^2\sum_{j=0}^{n-2}(j+1)x^j.$$

Inserting this into the expression of L, we find

$$L(x) = \frac{x^{n-2}}{(1-x)\left(\sum_{j=0}^{n-2}(j+1)x^j\right)^2},$$

that we use for differentiating. After collecting appropriately terms, we obtain

$$L'(x) = \frac{(n-1)^2x^{n-1} + \sum_{j=0}^{n-2}(n-2-3j)x^j}{\left(\sum_{j=0}^{n-2}(j+1)x^j\right)^3}. \tag{7}$$

As we are only interested in $x \in [0,1]$, the sign of $L'(x)$ is the same as its numerator, which we denote by $P_n(x)$. We aim to prove that $P_n(x) \geq 0$, for every $n \geq 3$. First, we prove that $P_n(x)$ is increasing with respect to n, that is, for every $n \geq 3$, and every $x \in [0,1]$, we have that $P_{n+1}(x) \geq P_n(x)$. For this purpose, write

$$\begin{aligned}
&P_{n+1}(x) - P_n(x) \\
&\quad = n^2x^n + \sum_{j=0}^{n-1}(n-2-3j)x^j - (n-1)^2x^{n-1} - \sum_{j=0}^{n-2}(n-2-3j)x^j \\
&\quad = n^2x^n - (n^2-1)x^{n-1} + \sum_{j=0}^{n-2} x^j \\
&\quad = n^2x^n - (n^2-1)x^{n-1} + \frac{x^{n-1}-1}{x-1} \\
&\quad = \frac{-n^2x^{n+1} + (2n^2-1)x^n - n^2x^{n-1} + 1}{1-x}
\end{aligned}$$

As before, denote by $N(x)$ the numerator in the last expression above. It is now immediate that the sign pattern of $P_{n+1}(x) - P_n(x)$ is the same as of $N(x)$. We aim to prove that $N(x)$ is positive for $x \in [0,1]$. The derivative of N may be represented as $N'(x) = nx^{n-2}D_n(x)$, where $D_n(x) = -n(n+1)x^2 + (2n^2-1)x - n(n-1)$. The polynomial D_n is easily seen to have two roots, both in $[0,1]$, $x_0 = \frac{n-1}{n}$ and $x_1 = \frac{n}{n+1}$. Therefore, the sign pattern of $D_n(x)$ when $x \in [0,1]$ is -+-. Taking into account that $N(0) > 0$, $N(1) = 0$ and $x_0 < x_1$, it follows that $N(x) \geq 0$, for every $x \in [0,1]$ if and only if $N(x_0) \geq 0$. We verify that this latter inequality holds. Indeed, after simplifications we obtain

$$N(x_0) = -\left(\frac{n-1}{n}\right)^{n-1}\frac{2n-1}{n} + 1 \geq -2\left(\frac{n-1}{n}\right)^{n-1} + 1.$$

It is easy to prove that $\left(\frac{n-1}{n}\right)^{n-1}$ is decreasing with respect to n, moreover $\lim_{n\to+\infty}\left(\frac{n-1}{n}\right)^{n-1} = e^{-1}$, therefore $N(x_0) \geq -2e^{-1} + 1 > 0$. Consequently, for $n \geq 3$ and every $x \in [0,1]$ we have proved that $P_{n+1}(x) \geq P_n(x)$. Finally, for $n = 3$, we have $P_3(x) = 1 - 2x + 4x^2 > 0$ for every $x \in [0,1]$. Putting everything together, we have proved that $P_n(x) > 0$, for every $n \geq 3$ and $x \in [0,1]$, meaning that $L(x)$ is indeed increasing, so the proof of this case is also concluded.

■

Note that, in the non iid case, the IOR property is not preserved under the formation of order statistics, as shown by the following counter-example.

Example 11. Let X and Y be independent log-logistic random variables with different shape parameters a_X and a_Y and same scale parameter. X and Y are IOR if $a_X, a_Y \geq 1$, but $\max(X, Y)$ is not IOR for appropriately chosen shape parameters (choose, for instance, $a_X = 3$ and $a_Y = 10$).

2.5. Non-closure under convolution. Differently from the IHR family (Marshall and Olkin, 2007, Chapter 4.B), the following counterexample establishes that the IOR family is not closed under convolution.

Example 12. If X is a log-logistic(1,1) and Y is a unit exponential, the CDF of the convolution $X + Y$ is

$$F^*(x) = \int_0^x \frac{e^{-t}}{1 + (x-t)^{-1}}\, dt = 1 - e^{-x-1}(\mathrm{Ei}(x+1) - \mathrm{Ei}(1) + e),$$

where Ei is the exponential integral (see Example 6). The corresponding OR,

$$\lambda_{F^*}(x) = \frac{e^{x+1}\left(\mathrm{Ei}(x+1) - \mathrm{Ei}(1) - \frac{e^{x+1}}{x+1} + e\right)}{(\mathrm{Ei}(x+1) - \mathrm{Ei}(1) + e)^2}$$

is not monotone.

3. Bounds and tolerance limits for IOR distributions

In this section, we use the IOR assumption to derive bounds for the survival function and lower tolerance limits, obtaining results that extend similar bounds, available in the literature, to this class of distributions.

3.1. Survival bounds. In reliability and life testing problems, it is often important to determine convenient bounds for the survival function, or other related notions, in some interval of interest. The monotonicity properties of the HR have been used extensively to establish lower bounds for the survival function (see Barlow and Marshall (1964a), Barlow and Marshall (1964b), Marshall and Olkin (2007)), while Zimmer et al. (1998) obtained similar results based on the monotonicity of the log-odds rate w.r.t log-time (see also Wang et al. (2003, 2005)). In the IOR case, the following theorem provides comparable bounds, given the knowledge of the survival function at two distinct points.

Theorem 13. *If F is* IOR, *then for every $x_1 < x_2$, there exist $a > 0$ and $b \in \mathbb{R}$ such that*

$$\overline{F}(x) \begin{cases} > \frac{1}{ax+b}, & x_1 < x < x_2, \\ = \frac{1}{ax+b}, & x = x_1 \text{ or } x = x_2, \\ < \frac{1}{ax+b}, & x < x_1 \text{ or } x > x_2. \end{cases} \tag{8}$$

Proof: It follows from Theorem 20 in Arab and Oliveira (2019, 2018) and the convexity of $\frac{1}{\overline{F}(x)}$ that for every line $\ell(x)$ with positive slope, the sign pattern of $\frac{1}{\overline{F}(x)} - \ell(x)$ is at most +−+. Now, given any $x_1 < x_2$, choose the line $\ell(x) = ax + b$ such that $\ell(x_1) = \frac{1}{\overline{F}(x_1)}$ and $\ell(x_2) = \frac{1}{\overline{F}(x_2)}$. For this particular choice, the sign pattern of $\frac{1}{\overline{F}(x)} - \ell(x)$ is exactly +−+, so (8) follows. ■

Remark 14. If F is IOR and $F(x_1)$ and $F(x_2)$ are known, then a and b are given explicitly as follows

$$a = \frac{1}{x_1 - x_2}\left(\frac{1}{\overline{F}(x_1)} - \frac{1}{\overline{F}(x_1)}\right), \quad b = \frac{1}{\overline{F}(x_1)} - ax_1.$$

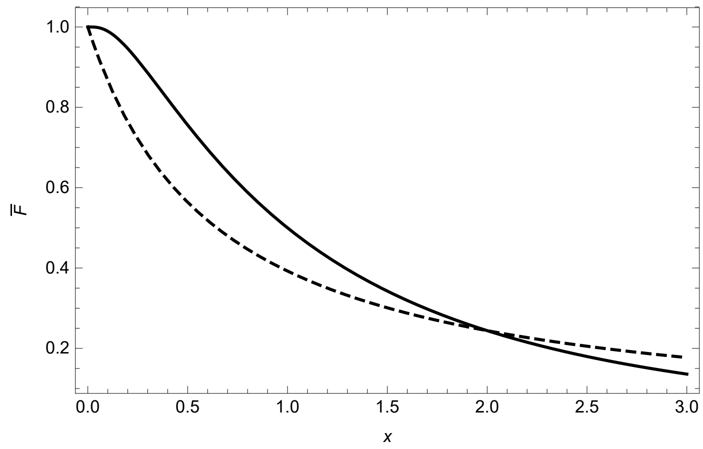

FIGURE 1. $\overline{F}(x) = 1 - \Phi(\ln(x))$, where Φ is the standard normal CDF, is a lognormal survival function (solid curve). The graph shows the corresponding lower survival bound in the interval $[0, 2]$ (dashed curve).

Corollary 15. *Let F be* IOR *and supported on $[0, +\infty)$, such that $F(0) = 0$, and let $x_p = F^{-1}(p)$, for some $p \in (0, 1)$. Then*

$$\overline{F}(x) \begin{cases} > \frac{1}{ax+b}, & 0 \leq x < x_p, \\ = \frac{1}{ax_p+b}, & x = x_p, \\ < \frac{1}{ax+b}, & x > x_p, \end{cases} \tag{9}$$

where $b \in [1, \frac{1}{1-p}]$ and $a = \frac{1}{x_p}\left(\frac{1}{1-p} - b\right)$.

Proof: If $b \in [1, \frac{1}{1-p}]$, then the sign pattern of $\frac{1}{\overline{F}(x)} - (ax + b)$ is either $-+$ or $-$. Since $\ell(x_p) = ax_p + b = \frac{1}{1-p} = \frac{1}{\overline{F}(x_p)}$, we have that, indeed, the sign pattern of $\frac{1}{\overline{F}(x)} - (ax + b)$ is equal to $-+$. ∎

Remark 16. On the interval $[0, x_p]$, we have a lower bound for $\overline{F}(x)$, that is $\overline{F}(x) \geq L_b(x) = \frac{1}{\frac{1}{x_p}(\frac{1}{1-p}-b)x+b}$, for $b \in [1, \frac{1}{1-p}]$. As $L_b(x)$is decreasing in b, then $\overline{F}(x) \geq \frac{1}{\frac{1}{x_p}(\frac{1}{1-p}-1)x+1}$. A graphical comparison of the survival function and this bound for the case of the lognormal distribution is shown in Figure 1.

The results of Theorem 13 and Corollary 15 may be used to determine lower bounds for the survival function in a given interval, if we can assume that we know the values of the distribution in at least two points x_1 and x_2 (it is often possible to set $x_1 = 0$ and $F(0) = 0$). It follows, as a consequence of (9), that if we design a company such that at time x_p the failure probability is equal to p, we immediately have a lower bound for the survival probability throughout the living time span of the company.

As mentioned above, Marshall and Olkin (2007) (see Chapter 6.B) provide similar bounds for the IHR family of distributions. In this regard, we note that the IOR bounds, given in Theorem 13 and Corollary 15, have broader applicability than the IHR ones, because, according to Proposition 1, the IOR condition is weaker and can be used when only little information on F is available. As the IOR condition does not require the existence of a finite mean (differently from the IHR condition, which requires the existence of all moments (Marshall and Olkin, 2007, p.109)), the IOR bounds can also be seen as an alternative to the Markov inequality when $E(X)$ is unknown or does not exist. It should be mentioned that the bounds proved in Marshall and Olkin (2007) may be tighter than ours, which is somewhat expectable, as we rely on less stringent conditions on F.

Similar bounds as the ones we discussed in this subsection have been proved, under a different set of assumptions, by Zimmer et al. (1998) or Wang et al. (2003, 2005). These authors assumed that the log-odds rate is increasing with respect to log-time. On one hand, this implies that only non-negative random variables may be considered. Moreover, even in such a case, this assumption excludes some important life distributions that are in the IOR family (like the lognormal or the Fréchet) or even in the IHR family (such as the gamma).

3.2. Tolerance limits. Many applications of interest in engineering (see NIST/SEMATECH (2020)) or in the medical context rely on the *tolerance limits*. In such a framework, one is interested in describing an interval that covers a fixed proportion of the possible values for the random variable. Recall a formal definition (see Barlow and Proschan (1966b)).

Definition 17. Let X have CDF F and $\mathbf{X} = (X_1, ..., X_n)$ be a random sample from X. Take $\alpha \in (0, 1)$, defining the *confidence coefficient*, $1 - q \in (0, 1)$, setting the *population coverage*. A tolerance limit is a function $L(\mathbf{X})$ such that

$$P(\overline{F}(L(\mathbf{X})) \geq 1 - q) \geq 1 - \alpha. \tag{10}$$

The IOR assumption provides a simple way to find tolerance limits. Indeed, let F be an IOR distribution such that $F(0) = 0$ and $Y_1, \ldots, Y_n$ a random sample form the log-logistic(1,1) with CDF $G(x) = \frac{x}{1+x}$. Because $G^{-1} \circ F$ is convex, Theorem 4.2 in Barlow and Proschan (1966a) yields

$$P\left(F\left(\sum_{i=1}^{n} a_i X_{(i)}\right) \leq x\right) \geq P\left(G\left(\sum_{i=1}^{n} a_i Y_{(i)}\right) \leq x\right), \tag{11}$$

where $\sum_{j=i}^{n} a_j \in [0,1]$ for $i = 1, \ldots, n$. Now, by setting $a_k = b \in [0,1]$ for some $1 \leq k \leq n$ and $a_i = 0$ for $i \neq k$, and noticing that $\overline{G}(u) \geq 1-q$ if and only if $u \leq G^{-1}(q)$, (11) implies that

$$P(\overline{F}(bX_{(k)}) \geq 1-q) \geq P(\overline{G}(bY_{(k)}) \geq 1-q) = P\left(bY_{(k)} \leq \frac{q}{1-q}\right) = 1-\alpha, \tag{12}$$

so that a tolerance limit is given by the choice $L(\mathbf{X}) = bX_{(k)}$. In particular, it is readily seen that

$$\begin{aligned} 1-\alpha = G_{(k)}\left(\frac{q}{b-bq}\right) = F_\beta\left(\frac{q}{b+q-bq}; k, n-k+1\right) = \\ = \left(\frac{b-bq}{q+b-bq}\right)^n \sum_{j=k}^{n} \binom{n}{j} \left(\frac{q}{b-bq}\right)^j, \end{aligned}$$

where, again, F_β is the beta CDF.

Zimmer et al. (1998) find the same tolerance limits under the assumption that F has an increasing log-odds rate with respect to log-time. Note that this assumption does not imply that F satisfies the IOR condition (as discussed above). Therefore, the discussion preceding Theorem 4 in Zimmer et al. (1998) and, subsequently, the theorem itself, are still subject to the verification of the convexity of $G^{-1} \circ F$, which does not follow automatically, as is the case of the IOR distributions. As a consequence of this observation, the IOR condition is the proper assumption to determine the tolerance limits of the form given by (12). Keeping this change of class in mind, more practical examples of how to use these tolerance limits are described in Zimmer et al. (1998). After specifying n, α and q, one could determine b for a given k, that is, at a given time. Conversely, it is also possible to determine, at a given confidence level, the time at which the survival function exceeds $1-q$.

4. Testing the IOR condition

In the literature, various methods have been proposed to test ageing properties of distributions. For example, Barlow and Proschan (1969), Bickel and Doksum (1969), Bickel (1969) and Proschan and Pyke (1967) test exponentiality against the IHR alternative, Tenga and Santner (1984) and Hall and Van Keilegom (2005) test the IHR null hypothesis against non-IHR alternatives, Sengupta and Paul (2005) test log-concavity against non-log-concave alternatives, Sahoo and Sengupta (2017) test the hypothesis of increasing ratio between hazard rates in two samples, Lando (2020) tests the null hypothesis of increasingness of the log-odds rate. In this section, we propose a test for the null hypothesis

$$\mathcal{H}_0 : F \text{ is IOR}, \tag{13}$$

against non-IOR alternatives, by checking the convexity of the odds function, Λ_F.

Denote by F_n the empirical CDF of a random sample $\mathbf{X} = (X_1, \dots, X_n)$ from F, that is, $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{X_i \le t}$, where $\mathbf{1}_A$ is the indicator of event A. The plug-in estimator of Λ_F is $\widehat{\Lambda} = \Lambda_{F_n}$, where, to simplify notations, we set

$$\widehat{\Lambda}_i = \widehat{\Lambda}(X_{i:n}) = \frac{i}{n-i}. \tag{14}$$

Drawing inspiration from Tenga and Santner (1984), we propose a test based on the distance between $\widehat{\Lambda}$ and an estimator of Λ that is, by construction, convex. Such an estimator is given by the greatest convex minorant (GCM) $\widehat{\Lambda}_c$, that is, the largest convex function that does not exceed $\widehat{\Lambda}$, defined by

$$\widehat{\Lambda}_c(x) = \sup\{u(x) : u \text{ is convex and } u(y) \le \widehat{\Lambda}(y), \forall y \in [X_{(1)}, X_{(n)}]\}.$$

Let us define $\widehat{\Lambda}_{c,i} = \widehat{\Lambda}_c(X_{i:n})$. Since, by construction, $\widehat{\Lambda}_{i-1} \ge \widehat{\Lambda}_{c,i}$, we consider the following weighted Kolmogorov-Smirnov test statistic

$$\mathrm{KS}_{\nu,n}(X_1, \dots, X_n) = \mathrm{KS}_{\nu,n}(\mathbf{X}) = \max_{S_{n,\nu}}\{w_i(\widehat{\Lambda}_{i-1} - \widehat{\Lambda}_{c,i})\}, \tag{15}$$

where the w_i's are positive weights, $\nu \ge 0$ is adequately small, and $S_{n,\nu} = \{i : x_i \le F_n^{-1}(1-\nu)\}$. The test statistic is evaluated over the restricted support $S_{n,\nu}$, avoiding the fact that the GCM $\widehat{\Lambda}_c$ is only defined until $X_{(n)}$, in order to establish large sample properties of the test from a theoretical point of view (see Theorem 18 below). Note also that $\mathrm{KS}_{\nu,n}$ is scale invariant.

We reject the null hypothesis for "large" values of $\mathrm{KS}_{\nu,n}$. Using the same approach as Theorem 3.1 in Tenga and Santner (1984), it can be seen that the least favorable distribution of $\mathrm{KS}_{\nu,n}$ is obtained by taking $\mathrm{KS}_{\nu,n}(\mathbf{Y})$, where $\mathbf{Y}$ is a random sample from the log-logistic(1,1). Such a distribution enables the determination of the critical values or the p-values of the test, which can be computed through simulation. Let $\mathbf{x}$ be a realisation of $\mathbf{X}$. The test proposed rejects $\mathcal{H}_0$ when $\mathrm{KS}_{\nu,n}(\mathbf{x}) \geq c_{\alpha,\nu,n}$, where $c_{\alpha,\nu,n}$ is the solution of $P(\mathrm{KS}_{\nu,n}(\mathbf{Y}) \geq c_{\alpha,\nu,n}) \geq \alpha$ and α is the size of the test. Alternatively, we can compute the p-value of the test, that is, $p = P(\mathrm{KS}_{\nu,n}(\mathbf{Y}) \geq \mathrm{KS}_{\nu,n}(\mathbf{x}))$.

Theorem 18. *Let* $\mathrm{KS}_{\nu,n}$ *be the test statistic (15),* $\nu > 0$ *and the weights* w_i*,* $i = 1, \dots, n$*, satisfy* $0 < \underline{w} < w_i \leq \overline{w} < \infty$ *(for every sample size* n*). If* $\mathcal{H}_0$ *is false, then*

$$\lim_{\nu\to 0^+} \lim_{n\to\infty} P(\mathrm{KS}_{\nu,n}(\mathbf{X}) > c_{\alpha,\nu,n}) = 1. \tag{16}$$

Proof: First, we need to prove that, under $\mathcal{H}_0$,

$$\lim_{\nu\to 0^+} \lim_{n\to\infty} \sup_{S_{n,\nu}} |\widehat{\Lambda}_c - \Lambda_F| = 0. \tag{17}$$

The empirical quantile $F_n^{-1}(1-\nu)$ is a consistent estimator of $\xi_\nu = F^{-1}(1-\nu)$. Moreover, the Glivenko-Cantelli Theorem and the uniform continuity of the transformation $\frac{p}{1-p}$ in intervals bounded away from 1, implies that, for each fixed $\nu, \eta > 0$, $\sup_{x\leq\xi_\nu+\eta} |\widehat{\Lambda}(x) - \Lambda_F(x)| \to 0$ with probability 1. Then, for any small $\nu, \eta, \epsilon > 0$,

$$\lim_{m\to\infty} P(\sup_{x\leq\xi_\nu+\eta} |\widehat{\Lambda} - \Lambda_F| \leq \epsilon, \forall n > m) = 1.$$

If $\mathcal{H}_0$ is true, by definition of the GCM and by convexity of Λ_F and $\Lambda_F + \epsilon$ we obtain, as $m \to \infty$,

$$\Lambda_F(t) - \epsilon \leq \widehat{\Lambda}_c(t) \leq \widehat{\Lambda}(t) \leq \Lambda_F(t) + \epsilon, \tag{18}$$

uniformly, for every $t \leq \xi_\nu + \eta$ and $n > m$. Then, for each $\nu > 0$,

$$\lim_{m\to\infty} P(\sup_{x\leq\xi_\nu+\eta} |\Lambda_F - \widehat{\Lambda}_c| \leq \epsilon, \forall n > m) \geq \lim_{m\to\infty} P(\sup_{x\leq\xi_\nu+\eta} |\widehat{\Lambda} - \Lambda_F| \leq \epsilon, \forall n > m) = 1,$$

which implies that the iterated limit in (17) is 0.

We now prove that, under $\mathcal{H}_0$, the critical value of the test, $c_{\alpha,\nu,n}$, tends to 0 for every fixed $\nu > 0$. Choose m large enough such that

$$P(\sup_{x \leq \xi_\nu + \eta} |\widehat{\Lambda} - \Lambda_F| \leq \epsilon, \forall n > m) =$$
$$= P(\Lambda_F(t) - \epsilon \leq \widehat{\Lambda}(t) \leq \Lambda_F(t) - \epsilon, \forall t \in S_{\nu,n}, \forall n > m) \geq 1 - \alpha$$

Since, when $\mathcal{H}_0$ is true, (18) holds and $\Lambda_F(t) + \epsilon - (\Lambda_F(t) - \epsilon) = 2\epsilon$, we obtain

$$P(\sup_{x \leq \xi_\nu + \eta} |\widehat{\Lambda} - \widehat{\Lambda}_c| \leq 2\epsilon, \forall n > m) \geq 1 - \alpha \quad \Rightarrow$$
$$P(\sup_{S_{n,\nu}} w_i |\widehat{\Lambda}_{i-1} - \widehat{\Lambda}_{c,i}| \leq 2\epsilon \overline{w}, \forall n \geq m) \geq 1 - \alpha$$

Since ϵ is arbitrarily small we get $c_{\alpha,\nu,n} \to 0$.

If Λ_F is not convex, we know that there exist at least 3 values $\{a, b, c\}$ in the support of F that violate the convexity condition. However,

$$\lim_{\nu \to 0} \lim_{n \to \infty} P(\{a, b, c\} \in S_{n,\nu}) = 1,$$

therefore we assume that it exists some m and some $\nu_0 > 0$ such that, for $n > m$ and for $\nu < \nu_0$, $\{a, b, c\} \in S_{\nu,n}$. Let $n > m$ and $\nu < \nu_0$: using the same argument of Theorem 4.3 in Tenga and Santner (1984), one can show that in this case $c_{\alpha,\nu,n}$ does not tend to 0, which is a necessary condition for the acceptance of $\mathcal{H}_0$ when $n \to \infty$. Then, $P(\sup_{S_{n,\nu}} w_i |\widehat{\Lambda}_{i-1} - \widehat{\Lambda}_{c,i}| > c_{\alpha,\nu,n}) \to 1$, for every fixed $\nu < \nu_0$, and the thesis follows. ∎

Of course, the value of ν should be taken as small as possible (e.g., $\nu \leq 0.01$ if $n \geq 100$). Yet, smaller ν's correspond to slower convergence. In fact, as shown in the proof, we will need more observations in order to get $c_{\alpha,\nu,n}$ close enough to 0. For instance, for $n = 10, \ldots, 100$ we would need to take $\nu = 0.1$ (too large) in order to show that the sequence $c_{0.1,n,\alpha}$ decreases in n. In the limit case $\nu = 0$, the critical values do not go to zero and we cannot establish consistency of the test. The choice of the weights has also an impact on the performance of the test. If Λ_F is convex, the increments $\widehat{\Lambda}_i - \widehat{\Lambda}_{i-1}$ are increasing, for $2 \leq i \leq n$. Therefore, the weights may be tailored to downsize the effect of larger differences due to larger i's. Three choices of weights were considered: (1) $w_i = 1$, (2) $w_i = \delta + k(1 - \frac{i}{n})^k$ ($k \geq 1$, for convenience we set $\delta = 10^{-3}$, but any $\delta > 0$ ensures the existence of $\underline{w}$) and (3) $w_i = \frac{1}{\widehat{\Lambda}_{i-1}} = \frac{n}{i-1} - 1$; we denote the test statistic as $\mathrm{KS}_{\nu,n}^{(s)}$, where the superscript

$s = 1, 2, 3$, denotes the weights discussed above. $\mathrm{KS}^{(1)}_{\nu,n}$ performs well against alternatives with decreasing OR, but may provide poor performance against alternatives with non-monotone OR. Overall, $\mathrm{KS}^{(2)}_{\nu,n}$ performs well against alternatives with both decreasing or non-monotone OR, especially for $k > 2$. $\mathrm{KS}^{(3)}_{0,n}$ was studied by Lando et al. (2020), although it does not satisfy the large sample property of Theorem 18 (since the weights are unbounded and $\nu = 0$), $\mathrm{KS}^{(3)}_{0,n}$ has a good performance, as shown by the simulation study of Lando et al. (2020). Based on our simulation results and on the consistency property of Theorem 18, we recommend using the test $\mathrm{KS}^{(2)}_{0.01,n}$, where $k = 3$ (clearly, the parameter $\nu = 0.01$ has an effect just for $n \geq 100$). A table of simulated p-values for this test is provided.

n \ p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$n = 10$	0.095	0.126	0.146	0.164	0.184	0.203	0.223	0.241	0.256	0.269
$n = 20$	0.131	0.16	0.185	0.211	0.23	0.25	0.271	0.296	0.321	0.335
$n = 30$	0.144	0.17	0.194	0.217	0.245	0.268	0.292	0.316	0.344	0.362
$n = 40$	0.135	0.164	0.192	0.214	0.241	0.27	0.298	0.33	0.362	0.381
$n = 50$	0.137	0.164	0.191	0.218	0.248	0.273	0.301	0.329	0.367	0.385
$n = 75$	0.125	0.152	0.179	0.211	0.239	0.267	0.298	0.34	0.38	0.401
$n = 100$	0.11	0.137	0.157	0.177	0.203	0.226	0.25	0.287	0.337	0.372
$n = 150$	0.105	0.123	0.146	0.173	0.2	0.236	0.27	0.308	0.345	0.365
$n = 200$	0.093	0.115	0.132	0.149	0.168	0.196	0.22	0.26	0.301	0.326

TABLE 1. Simulated quantiles of $\mathrm{KS}^{(2)}_{0.01,n}$ ($k = 3$). The number of simulation runs is 3000, for $n \leq 40$, and 1000, for $n \geq 50$.

4.1. Simulation. We analysed the performance of $\mathrm{KS}^{(2)}_{0.01,n}$, $k = 3$. Table 1 reports the average p-values (over 500 simulation runs) and the rejection rates ($\alpha = 0.1$) for sampe sizes $n = 10, 30, 50, 100$. If the data are drawn from an IOR distribution (such as the lognormal), the test mostly leads to acceptance of $\mathcal{H}_0$, thus, the case when $\mathcal{H}_0$ is false is particularly interesting. We considered the following alternatives to the IOR model: 1) log-logistic distribution (shape parameter less than 1) which has a decreasing OR, 2) Weibull distribution (shape parameter less than 1) which has a U-shaped OR, 3) Birnbaum-Saunders distribution Birnbaum and Saunders (1969) (shape parameter ≥ 2), which has an increasing-decreasing-increasing OR. Obviously, the non-convexity of the odds function is much easier to detect in case 1) than in cases 2) and, especially, in case 3), thus, the overall performance of

Distribution	**CDF**	θ **range**	$n = 10$	$n = 30$	$n = 50$	$n = 100$
lognormal	$\Phi(\ln(x) - \theta)$	[-2,2]	0.77(0.5%)	0.9(0%)	0.93(0%)	0.96(0%)
log-logistic	$\frac{1}{1+x^{-\theta}}$	$[0.01, 0.7]$	0.07(85%)	0.02(94%)	0.01(99%)	0.008(100%)
Weibull	$1 - \exp\left(-x^{\theta}\right)$	$[0.01, 0.7]$	0.11(69%)	0.07(83%)	0.06(85%)	0.03(91%)
Birnbaum-Saunders	$\Phi\left(\frac{1}{\theta}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right)$	$[2, 4]$	0.33(20%)	0.26(19%)	0.25(21%)	0.16(40%)

TABLE 2. Simulation. Average p-values (over 500 runs) and rejection rates ($\alpha = 0.1$). Φ denotes the CDF of a standard normal.

the test is good, as it can be seen by the average p-values, rejection rates and corresponding empirical power. As a rule of thumb, p-values below 0.3 may provide evidence against the null hypothesis.

4.2. Application. Wang et al. (2005) (p. 13) provide a table containing 30 failure times, based on the pull strength of wires from a substrate. By means of statistical testing, it can be seen that the underlying distribution is not IHR. However, by applying $\mathrm{KS}^{(2)}_{0.01,n}$, we find $\mathrm{KS}^{(2)}_{0.01,30} = 0.21$ and an approximate p-value within 0.6 and 0.7 (see Table 2), which leads to acceptance of $\mathcal{H}_0$. In order to establish a lower bound on the reliability at a pull strength of 223 grams, Wang et al. (2005) set $\overline{F}(208.8) = 0.7$ and $\overline{F}(387.3) = 0.7$ and obtain $\overline{F}(223) > 0.663$. Such a result is obtained using the bounds given by Zimmer et al. (1998) and assuming that the distribution has an increasing log-odds rate. Using the same crossing points and assuming that the distribution is IOR (in this case, the assumption is supported by statistical testing), we apply the bounds of Theorem 13 and we obtain $\overline{F}(223) > 0.639$ ($a = 0.0095038$, $b = -0.555822$).

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