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A NEW DIAGONAL SEPARATION AND ITS RELATIONS WITH THE HAUSDORFF PROPERTY

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ABSTRACT: Let $\mathcal{P}$ be a property of subobjects relevant in a category $\mathcal{C}$. An object $X \in \mathcal{C}$ is $\mathcal{P}$-separated if the diagonal in $X \times X$ has $\mathcal{P}$; thus e.g. closedness in the category of topological spaces (resp. locales) induces the Hausdorff (resp. strong Hausdorff) axiom. In this paper we study the locales (frames) in which the diagonal is fitted (i.e., an intersection of open sublocales – we speak about $\mathcal{F}$-separated locales). Recall that a locale is *fit* if each of its sublocales is fitted. Since this property is inherited by products and sublocales, fitness implies ($\mathcal{F}$sep) which is shown to be strictly weaker (one of the results of this paper). We show that ($\mathcal{F}$sep) is in a parallel with the strong Hausdorff axiom (sH):

(1) it is characterized by a Dowker-Strauss type property of the combinatorial structure of the systems of frame homomorphisms $L \to M$ (and therefore, in particular, it implies (T_U) for analogous reasons like (sH) does), and

(2) in a certain duality with (sH) it is characterized in L by all *almost homomorphisms* (frame homomorphisms with slightly relaxed join-requirement) $L \to M$ being frame homomorphisms (while one has such a characteristic of (sH) with *weak homomorphisms*, where meet-requirement is relaxed).

KEYWORDS: Frame, locale, sublocale, preframe, preframe homomorphism, weak homomorphism, binary coproduct of frames, diagonal map, strongly Hausdorff frame, fit frame, T_U-frame, simple extension.

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Introduction

Recall that the Hausdorff property of a topological space X is characterized by closedness of the diagonal in $X \times X$. This is a general phenomenon, the so called $\mathcal{P}$-separation (see [14, 3, 5], and also [18]): Given a property relevant in the category in question (typically of a topological nature), an object X is $\mathcal{P}$-separated if the diagonal in $X \times X$ has the property $\mathcal{P}$. Besides the Hausdorff property in classical spaces, relevant examples for $\mathcal{P}$ are e.g. the strong Hausdorff axiom in the category of locales (introduced by Isbell [9]), or the Boolean property in the same category (see [13, 18]). For a general

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treatment of separation on categories with closure operator see [4] and the references there.

In the context of locales (frames) there is an important property of fittedness. A sublocale (generalized subspace) of a locale is *fitted* if it is an intersection of open ones. Since the intersection $S^\circ = \bigcap\{T \mid S \subseteq T,\ T \text{ open}\}$ is an operation of closure type (see [6]), it is natural to ask about fitted diagonals; we will speak of the property ($\mathcal{F}$sep) or of the $\mathcal{F}$-separated locales (frames). The study of this property is the main topic of this paper.

A well-known (and relatively well-understood) property of a locale (frame) is the fitness ([9]). A locale is *fit* if each of its sublocales is fitted. Taking into account the fact that fitness is preserved under products and sublocales, we have an immediate observation that (fit) implies ($\mathcal{F}$sep). Hence the first question one may ask is whether this implication can be reversed (it cannot; ($\mathcal{F}$sep) is strictly weaker than fitness, which is one of the results we present).

There are two pleasant parallels of $\mathcal{F}$ ("diagonal is fitted") and the strong Hausdorff property (sH) ("diagonal is closed"). Dowker and Strauss proved in [7] that (sH) of a frame L can be characterized as the fact that a certain equivalence $\sim_0$ of frame homomorphisms $h, k : L \to M$ is trivial for any M (and, interestingly, $\sim_0$ has nothing to do with products). We prove that, similarly, there is an equivalence $h \sim_{\mathrm{q}} k$ of homomorphisms (again, unrelated to products) such that L is $\mathcal{F}$-separated iff for all frame homomorphisms $h, k : L \to M$, $h \sim_{\mathrm{q}} k$ only if $h = k$. Since both $\sim_0$ and $\sim_{\mathrm{q}}$ are (trivially) implied by $h \leq k$ we obtain as a consequence that both (sH) and ($\mathcal{F}$sep) imply the standard axiom T_U (a fact well-known for (sH), of course).

Another parallel concerns relaxed forms of frame homomorphisms. A *weak homomorphism* preserves all joins and zero meets ($x \wedge y = 0$) and an *almost homomorphism* preserves finite meets, directed joins and covers ($\bigvee x_i = 1$). It is known that a frame L is (sH) iff it is T_U and each weak homomorphism $L \to M$ is a frame homomorphism ([2, 17]). Here we prove that a frame L is ($\mathcal{F}$sep) iff it is T_U and each almost homomorphism $L \to M$ is a frame homomorphism.

The paper is organized as follows. After Preliminaries (divided into two sections, the first containing general notions and terminology, the second discussing some more specific facts) we prove, in Section 3, a technical characterization of ($\mathcal{F}$sep). In the next section we prove the Dowker-Strauss type characterization, and Section 5 is devoted to the characterization by almost homomorphisms. In the last section we prove that ($\mathcal{F}$sep) is strictly weaker than fitness, and that it does not coincide with any of the three standard axioms weaker than fitness: subfitness, weak subfitness and prefitness.

1. Preliminaries I: General

1.1. We will use the standard notation for posets; in particular we will write for subsets $A \subseteq (X, \leq)$

$$\downarrow A = \{x \mid \exists a \in A, x \leq a\},\ \downarrow a = \downarrow\{a\},$$

$$\uparrow A = \{x \mid \exists a \in A, x \geq a\},\ \uparrow a = \uparrow\{a\},$$

and speak of the A with $\downarrow A = A$ resp. $\uparrow A = A$ as of *down-sets* resp. *up-sets*. Our posets will be typically complete lattices; the suprema (joins) of subsets will be denoted by $\bigvee A$, $\bigvee_{i \in J} a_i$, $a \vee b$ etc., and infima (meets) by $\bigwedge A$, $\bigwedge_{i \in J} a_i$, $a \wedge b$ etc.

1.2. Adjoint maps. Recall that monotone maps $f\colon X \to Y$, $g\colon Y \to X$ are (Galois) *adjoint*, f to the left and g to the right, if $f(x) \leq y$ iff $x \leq g(y)$. We write $f \dashv g$, $g = f_*$, $f = g^*$.

If $f \dashv g$ then f (resp. g) preserves all the existing suprema (resp. infima) and on the other hand,

1.2.1. *if X, Y are complete lattices then an $f\colon X \to Y$ preserving all suprema (a $g\colon Y \to X$ preserving all infima) has a right (left) adjoint.*

1.3. Heyting algebras. A bounded lattice L is called a *Heyting algebra* if there is a binary operation $x \to y$ (the *Heyting operation*) such that for all a, b, c in L,

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c. \qquad \text{(Hey)}$$

Thus

(H1) *for every $b \in L$ the mapping $b \to (-)\colon L \to L$ is a right adjoint to $(-) \wedge b\colon L \to L$*

and hence $\to$, if it exists, is uniquely determined. From 1.2 it immediately follows that

(H2) *in a complete Heyting algebra one has $(\bigvee A) \wedge b = \bigvee_{a \in A}(a \wedge b)$ for any $A \subseteq L$, $b \to (\bigwedge A) = \bigwedge_{a \in A}(b \to a)$, and $(\bigvee A) \to b = \bigwedge_{a \in A}(a \to b)$.*

1.3.1. From (Hey) we immediately obtain
(1) $a \le b \to a$, (2) $1 \to a = a$, (3) $a \to b = 1$ iff $a \le b$,
(4) $a \wedge (a \to b) \le b$ and consequently (using (1)) $a \wedge (a \to b) = a \wedge b$,
(5) $a \le b \to c$ iff $b \le a \to c$.
And also the three further useful *rules* are very simple.
(6) $a \to (b \to c) = (a \wedge b) \to c = b \to (a \to c)$
(we have $x \le a \to (b \to c)$ iff $x \wedge a \le b \to c$ iff $x \wedge a \wedge b \to c$ iff $x \le (a \wedge b) \to c$).
(7) $a \to b = a \to c$ iff $a \wedge b = a \wedge c$
($\Rightarrow$ by (4), $\Leftarrow$: By (3) and (H2), $a \to b = (a \to a) \wedge (a \to b) = a \to (a \wedge b) = a \to (a \wedge c) = a \to c$).
(8) $x = (x \vee a) \wedge (a \to x)$
(by (H1), (4) and (1), $(x \vee a) \wedge (a \to x) = (a \wedge (a \to x)) \vee (x \wedge (a \to x)) \le x$; by (1), $x \le (x \vee a) \wedge (a \to x)$).

1.4. Frames and preframes. A *frame* is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\} \qquad \text{(frm)}$$

for all $A \subseteq L$ and $b \in L$ (hence a complete Heyting algebra); a *frame homomorphism* preserves all joins and all finite meets. In a *preframe* only *directed joins* are required, and the distributivity (frm) is assumed on those; *preframe homomorphisms* preserve all directed joins and all finite meets.

The lattice $\Omega(X)$ of all open subsets of a topological space X is an example of a frame, and if $f\colon X \to Y$ is continuous we obtain a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ by setting $\Omega(f)(U) = f^{-1}[U]$. Thus we have a contravariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}$$

from the category of topological spaces into that of frames.

Since frames are complete, the equality (frm) makes by 1.2.1 every frame a Heyting algebra.

Note. In this paper we will work almost exclusively with frames, but preframe homomorphisms between frames will play a crucial role.

1.5. The concrete category of locales. The functor Ω from 1.4 is on a very substantial subcategory of **Top** (that of sober spaces*) a contravariant

*A space is *sober* if every completely prime filter $\mathcal{F}$ in $\Omega(X)$ (that is, an $\mathcal{F}$ such that $\bigcup_{i\in J} U_i \in \mathcal{F}$ only if $U_j \in \mathcal{F}$ for some $j \in J$) is $\{U \mid x \in U\}$ for some $x \in X$ – in other words, if every system of full embedding. This justifies to view the dual category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ as an extended category of spaces; one speaks of the *category of locales*.

It is of advantage to view it as a concrete category representing frame homomorphisms $h\colon M \to L$ by their right adjoints $f = h_*\colon L \to M$ (recall 1.2.1). In this context we often speak of frames as of *locales* and of the meet preserving maps $f\colon L \to M$ adjoint to frame homomorphisms as of the *localic maps*.

1.5.1. Here is a useful characterization (the so called *Frobenius equality*):

> *a meet preserving* $f\colon L \to M$ *is a localic map iff* $f(a) = 1$ *only for* $a = 1$, *and* $f(f^*(a) \to b) = a \to f(b)$.

1.5.2. Sublocales. The *sublocales*, subobjects of L in $\mathbf{Loc}$, are the $S \subseteq L$ such that

(S1) for every $M \subseteq S$, $\bigwedge M \in S$, and
(S2) for every $x \in L$ and every $s \in S$, $x \to s \in S$

(they are precisely the subsets constituting locales with the embedding maps extremal monomorphisms in $\mathbf{Loc}$). The *nucleus* associated with a sublocale S is the mapping $\nu_S\colon L \to L$ defined by $\nu_S(a) = \bigwedge\{s \in S \mid a \leq s\}$.

Any intersection of sublocales is a sublocale so that we have a complete lattice $\mathsf{S}(L)$ of sublocales of L with the join given by the formula $\bigvee_{i\in J} S_i = \{\bigwedge M \,|\, M \subseteq \bigcup_{i\in J} S_i\}$. One has that

> $\mathsf{S}(L)$ *is a coframe*,

that is, a complete lattice with the distributivity dual to (frm) above.

Each element $a \in L$ is associated with a *closed* sublocale $\mathfrak{c}(a)$ and an *open* sublocale $\mathfrak{o}(a)$,

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{and} \quad \mathfrak{o}(a) = \{x \in L \,|\, a \to x = x\} = \{a \to x \,|\, x \in a\}$$

(the equivalence of the two expressions for $\mathfrak{o}(a)$ immediately follows from 1.3.1(6)). These special sublocales extend the concepts of open and closed subspaces, and behave as they should: in $\Omega(X)$ they precisely correspond to the homonymous subspaces, they are complements of each other, all joins and finite meets of open sublocales are open, and similarly with finite joins and general meets of closed sublocales.

open sets that looks like a neighborhood system is really a neighborhood system of a point. For instance every Hausdorff space is sober.

1.5.3. Other special sublocales we will be concerned with are

the *fitted* ones, the intersections of open ones.

Recall that a frame L is said to be *fit* if each sublocale of L is fitted ([9]).

1.6. Images and preimages. If $f\colon L \to M$ is a localic map and if $S \subseteq L$ is a sublocale then the standard set-theoretical image $f[S]$ is also a sublocale. The set preimage $f^{-1}[T]$ of a sublocale $T \subseteq M$ is generally not one, but it is a subset closed under meets and hence (recall the formula for $\bigvee_i S_i$ in $\mathsf{S}(L)$ above) we have the sublocale

$$f_{-1}[T] = \bigvee\{S \mid S \in \mathsf{S}(L),\ S \subseteq f^{-1}[T]\},$$

the *localic preimage* of T under f. One has the adjunction

$$f[S] \subseteq T \quad \text{iff} \quad S \subseteq f_{-1}[T],$$

and $f_{-1}[-]\colon \mathsf{S}(M) \to \mathsf{S}(L)$ is a coframe homomorphism.

1.6.1. (Localic) preimages of open resp. closed sublocales are open resp. closed and one has

$$f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a)) \quad \text{and} \quad f_{-1}[\mathfrak{c}(a)] = f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a)).$$

1.7. Binary coproduct in Frm. We will need the following facts.

- In the category of bounded semilattices the cartesian product with the injections and projections as in

$$L_i \xrightarrow{\ \iota'_i\ } L_1 \times L_2 \xrightarrow{((a_1,a_2)\mapsto a_j)} L_j\ ,$$

with $\iota'_1 = (a \mapsto (a,1))$ and $\iota'_2 = (a \mapsto (1,a))$, constitutes a biproduct (easy to check).
- A quotient of a frame L by a (congruence induced by a) relation R can be obtained as $L/R = \{s \in L \mid s \text{ is } R\text{-saturated}\}$ where s is R-saturated if for all a,b,c, $aRb \ \Rightarrow\ (a \wedge c \leq s \text{ iff } b \wedge c \leq s)$, and that a homomorphism $h\colon L \to M$ such that $aRb \Rightarrow h(a) = h(b)$ factorizes to $\overline{h}\colon L/R \to M$ by taking the restriction (see e.g. [15, III.11]).

1.7.1. The down-set frame. For a bounded meet semilattice L consider

$$\mathfrak{D}(L) = (\{U \mid \emptyset \neq U = {\downarrow}U,\ U \subseteq L\}, \subseteq)$$

with $\lambda_L = (a \mapsto {\downarrow}a)\colon L \to \mathfrak{D}(L)$. Obviously,

$\mathfrak{D}(L)$ is a frame, and λ is a semilattice homomorphism.

One has

Proposition. *For each semilattice homomorphism* $h\colon L\to M$ *into a frame* M *there is precisely one frame homomorphism* $\widetilde{h}\colon \mathfrak{D}(L)\to M$ *such that* $\widetilde{h}\cdot\lambda = h$. *It is given by the formula* $\widetilde{h}(U)=\bigvee\{h(a)\,|\,a\in U\}$.

1.7.2. The coproduct $L_1\oplus L_2$ of frames L_1, L_2 can now be obtained as $\mathfrak{D}(L_1\times L_2)/R$ with injections

$$\iota_i = \; L_i \xrightarrow{\iota_i'} L_1\times L_2 \xrightarrow{\lambda} \mathfrak{D}(L_1\times L_2) \longrightarrow \mathfrak{D}(L_1\times L_2)/R = L_1\oplus L_2$$

where the relation R is

$$R=\{(\bigcup_{i\in J}{\downarrow}(a_i,b), {\downarrow}(\bigvee_{i\in J}a_i,b))\,|\,a_i\in L_1, b\in L_2\}\cup$$
$$\cup\{(\bigcup_{i\in J}{\downarrow}(a,b_i), {\downarrow}(a,\bigvee_{i\in J}b_i))\,|\,a\in L_1, b_i\in L_2\}$$

so that R-saturated $U\in\mathfrak{D}(L_1\times L_2)$ are precisely the down-sets such that for any (a_i,b), $i\in J$, in U we have $(\bigvee_i a_i, b)\in U$, and for any (a,b_i), $i\in J$, in U also $(a,\bigvee_i b_i)\in U$.

In particular there are the R-saturated

$$a\oplus b = {\downarrow}(a,b)\cup\{(x,y)\,|\,x=0 \text{ or } y=0\}.$$

In this notation obviously $\iota_1(a)=a\oplus 1$, $\iota_2(b)=1\oplus b$, $U=\bigvee\{a\oplus b\,|\,a\oplus b\subseteq U\}$ for all $U\in L_1\oplus L_2$, and if $a,b\neq 0$ and $a\oplus b\subseteq a'\oplus b'$ then $a\le a'$ and $b\le b'$.

1.7.3. Thus, the codiagonal frame homomorphism $\delta_L\colon L\oplus L\to L$ satisfies $\delta(a\oplus b)=\delta(a\oplus 1)\wedge\delta(1\oplus b)=a\wedge b$ and hence

$$\delta(U)=\bigvee\{a\wedge b\,|\,a\oplus b\subseteq U\}=\bigvee\{a\wedge b\,|\,(a,b)\in U\}.$$

Consequently, the adjoint localic diagonal map $(\delta_L)_*\colon L\to L\oplus L$ and the diagonal sublocale are given by

$$\delta_*(a)=\{(u,v)\,|\,u\wedge v\le a\},\quad D_L=\delta_*[L].$$

2. Preliminaries II: Some more special facts

2.1. The coproduct as tensor product. 1. In the category $\bigvee$**Lat** of complete lattices with join-preserving mappings one has a tensor product $L\otimes M$ coinciding for L,M frames with the coproduct $L\oplus M$ ([13], for a pedestrian description see the appendix [17]). For our purposes we will just need the fact that if L_i, M_i, $i = 1, 2$, are frames and $f_i\colon L_i \to M_i$ are join preserving maps then there is precisely one join-preserving

$$f_1 \oplus f_2\colon L_1 \oplus L_2 \to M_1 \oplus M_2$$

determined by

$$(f_1 \oplus f_2)(a_1 \oplus a_2) = f_1(a_1) \oplus f_2(a_2).$$

2. Similarly in the category of preframes and preframe homomorphisms ([12, 19]) there is a tensor product $L \parr M$ generated by $a \parr b$ that can be represented for L, M frames as the coproduct $L \oplus M$ setting

$$a \parr b = (a \oplus 1) \vee (1 \oplus b) \tag{2.1.1}$$

which can be easily seen to be reversed as

$$a \oplus b = (a \parr 0) \wedge (0 \parr b) \quad [12,\ 19]. \tag{2.1.2}$$

For our purposes we will just need the fact that if L_i, M_i, $i = 1, 2$, are frames and $f_i\colon L_i \to M_i$ are preframe homomorphisms then there is precisely one preframe homomorphism

$$f_1 \parr f_2\colon L_1 \parr L_2 \to M_1 \parr M_2$$

determined by

$$(f_1 \parr f_2)(a_1 \parr a_2) = f_1(a_1) \parr f_2(a_2).$$

2.2. $\mathcal{P}$-separation. Let $\mathcal{P}$ be a property of monomorphisms in a category $\mathcal{C}$ with pullbacks and binary products. Recall that we speak of $\mathcal{P}$ being *pullback stable* if in every pullback

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "m'"'] & C \arrow[d, "m"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

with m a $\mathcal{P}$-monomorphism, m' is a $\mathcal{P}$-monomorphism.

If the diagonal $\Delta_B\colon B \to B \times B$ has the property $\mathcal{P}$ we say that B is $\mathcal{P}$*-separated* (according to [5]).

2.2.1. In the category **Loc** the pullback along the diagonal appears in this context as

$$\begin{tikzcd} f_{-1}[D_M] \arrow[r, "j=\subseteq"] \arrow[d, "g"'] & L \arrow[d, "f"] \\ D_M \arrow[r, "\subseteq"] & M \oplus M \end{tikzcd}$$

and one checks easily that $f_{-1}[D_M] = \mathrm{equ}(f_1, f_2)$ where $p_i f = f_i$ in $M \oplus M$ viewed as product (see [18]).

2.2.2. ($\mathcal{P}$)-separatedness in **Loc** appeared as the (strong) Hausdorff property (sH) introduced by Isbell in [9], requiring closed diagonals; in [13] the case of Boolean ones was discussed (also see an analysis of such facts in [18]). In this paper we are concerned with the case of *fitted* diagonals; we will speak of

property $\mathcal{F}$, and $\mathcal{F}$*-separated frames* (*locales*).

As mentioned in [18, 2.4], it follows from the properties of preimage that fittedness is a pullback stable property. As a consequence, the full subcategory of $\mathcal{F}$-separated locales has good categorical behaviour. Indeed, from general categorical results for $\mathcal{P}$-separatedness ([5, Cor. 4.3]) (or alternatively from Theorem 4.5 below) it follows that property $\mathcal{F}$ is closed under all monomorphisms and limits.

2.3. Simple extensions. For a space Y and a subspace $X \subseteq Y$ one defines

$$E = E_{X,Y}$$

changing the topology of Y to $\{(U \cap X) \cup M \mid U \in \Omega(Y), M \subseteq Y \smallsetminus X, M \subseteq U\}$.

2.3.1. In [1] it was proved a. o. that

if Y is a regular space and both X and $Y \smallsetminus X$ are dense in Y then $E_{X,Y}$ is strongly Hausdorff but not fit.

(For (sH) and fitness recall 2.2.2 and 1.5.3; for more details see [17].)

3. A technical characterization of ($\mathcal{F}$sep)

3.1. Recall from 1.7.3 the diagonal localic map in **Loc** adjoint to the diagonal homomorphism

$$\delta_L = (U \mapsto \bigvee_{(a,b)\in U} a \wedge b)\colon L \oplus L \to L$$

in **Frm**, and the diagonal sublocale

$$D_L = (\delta_L)_*[L].$$

For $f_1, f_2\colon M \to L$ in **Loc** denote by $\langle f_1, f_2\rangle\colon M \to L \oplus L$ the localic map defined by $p_i\langle f_1, f_2\rangle = f_i$. Thus, in **Frm**,

$$\langle f_1, f_2\rangle^* \iota_i = f_i^* \quad \text{and hence} \quad \langle f_1, f_2\rangle^*(U) = \bigvee\{f_1^*(a) \wedge f_2^*(b) \mid (a,b) \in U\}.$$

We immediately obtain

3.2. Observations. 1. *If $f_i \le g_i$ then $\langle g_1, g_2\rangle^*(U) \le \langle f_1, f_2\rangle^*(U)$.*
2. $\langle p_1, p_2\rangle^*(U) = U$.
3. $\langle f_1 g, f_2 g\rangle^*(U) = g^*\langle f_1, f_2\rangle^*(U)$,
4. $\mathfrak{o}(\langle f, g\rangle^*(U)) = \langle f, g\rangle_{-1}[\mathfrak{o}(U)]$.
5. $\mathfrak{o}(\langle f, f\rangle^*(U)) = f^*(\delta_L(U))$.

3.3. Since one has (recall 2.2.1)

$$\mathrm{equ}(f,g) = \langle f, g\rangle_{-1}[D_L]$$

we have in an $\mathcal{F}$-separated L

$$\mathrm{equ}(f,g) = \bigcap\{\mathfrak{o}(\langle f, g\rangle^*(U)) \mid D_L \subseteq \mathfrak{o}(U)\}.$$

3.3.1. Observation. *For a localic $f\colon M \to L$ one has $f[M] \subseteq \mathfrak{o}(a)$ iff $f^*(a) = 1$.* (Indeed, $f[M] \subseteq \mathfrak{o}(a)$ iff $M \subseteq f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$ iff $f^*(a) = 1$.)

In particular, one has

$$D_L \subseteq \mathfrak{o}(U) \text{ iff } \delta_L(U) = 1. \tag{3.3.1}$$

3.4. As is well known (see e.g. [5, 4.3]), $\mathcal{P}$-separatedness is equivalent to the fact that every equalizer is closed (for the closure operator induced by property $\mathcal{P}$). By the formula in 3.3, the closure of the equalizer is the identity, hence

Theorem. *A frame L is $\mathcal{F}$-separated iff*

$$\langle f, g\rangle^*(U) = 1 \textit{ for all } U \textit{ with } D_L \subseteq \mathfrak{o}(U) \quad \Rightarrow \quad f = g. \quad \blacksquare$$

3.4.1. Note. This result also follows directly from 3.3 and Observation 3.3.1 above. Clearly, if L is $\mathcal{F}$-separated and $\langle f,g\rangle^*(U)=1$ for all U such that $D_L\subseteq \mathfrak{o}(U)$, then $\mathrm{equ}(f,g)=\bigcap\{\mathfrak{o}(\langle f,g\rangle^*(U))\,|\,D_L\subseteq\mathfrak{o}(U)\}=\mathfrak{o}(1)$. For the converse, it suffices to check that the condition makes

$$j\colon \bigcap\{\mathfrak{o}(U)\,|\,D_L\subseteq\mathfrak{o}(U)\}\subseteq L\oplus L$$

the equalizer of p_1,p_2. We have by 3.2, $\langle p_1j,p_2j\rangle^*(U)=j^*\langle p_1,p_2\rangle^*(U)=j^*(U)$ for all U. If $\mathfrak{o}(U)\supseteq D_L$ we have $j[\bigcap\{\mathfrak{o}(V)\,|\,D_L\subseteq\mathfrak{o}(V)\}]\subseteq\mathfrak{o}(U)$ and hence, by 3.3.1, $\langle p_1j,p_2j\rangle^*(U)=j^*(U)=1$, Thus, $p_1j=p_2j$. Now if $p_1f=p_2f$ and $D_L\subseteq\mathfrak{o}(U)$ we have by (2), (3) and (5) in 3.2 and (3.3.1), $f^*(U)=\langle p_1f,p_2f\rangle^*(U)=(p_1f)^*(\delta_L(U))=1$ and hence $f[M]\subseteq\mathfrak{o}(U)$ for any such U, and finally $f[M]\subseteq\bigcap\{\mathfrak{o}(V)\,|\,D_L\subseteq\mathfrak{o}(V)\}$.

4. A Dowker-Strauss type characterization

4.1. A frame L satisfies T_U (T_U for *totally unordered* [11]; in [10] one speaks of *unordered locales*) if for any M and any frame homomorphisms $h,k:L\to M$,

$$h\le k\quad\Rightarrow\quad h=k. \tag{T_U}$$

4.1.1. Frame homomorphisms $h,k\colon L\to M$ are *connected* if there is a frame homomorphism h' such that

$$h\ge h'\le k.$$

We have a trivial

4.1.2. Observation. *A frame L satisfies (T_U) if and only if no distinct frame homomorphisms $h,k\colon L\to M$ are connected.*

4.2. Frame homomorphisms $h,k\colon L\to M$ are said to *respect covers* if for every cover C of L,

$$\bigvee_{c\in C}h(c)\wedge k(c)=1.$$

We have an obvious

4.2.1. Observation. *If frame homomorphisms are connected then they respect covers.*

4.3. Lemma. 1. *For a $U\in L\oplus L$ set $\widehat{U}=\{a\wedge b\,|\,(a,b)\in U\}$. Then*

$$\langle f,g\rangle^*(U)\ge\bigvee\{f^*(c)\wedge g^*(c)\,|\,c\in\widehat{U}\}.$$

2. *For a subset $C \subseteq L$ define $\widetilde{C} = \bigvee\{a \oplus a \mid a \in C\}$. Then*
$$\langle f, g\rangle^*(\widetilde{C}) = \bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\}.$$

Proof: 1 is trivial.
2: If $(x, y) \in \widetilde{C}$, that is, $x \oplus y \leq \widetilde{C}$ we have
$$f^*(x) \wedge g^*(y) = \delta_M((f^* \oplus g^*)(x \oplus y)) \leq \delta_M((f^* \oplus g^*)(\bigvee\{a \oplus a \mid a \in C\})) =$$
$$= \bigvee\{\delta_M((f^* \oplus g^*)(a \oplus a)) \mid a \in C\} = \bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\},$$
hence $\langle f, g\rangle^*(\widetilde{C}) \leq \bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\}$. The inequality $\geq$ is trivial. ■

4.4. Lemma. *$\langle f, g\rangle^*(U) = 1$ for all U with $D_L \subseteq \mathfrak{o}(U)$ iff f^* and g^* respect covers.*

Proof: $\Rightarrow$: If C is a cover then obviously $\delta_L(\widetilde{C}) = 1$, hence $\mathfrak{o}(\widetilde{C}) \supseteq D_L$ and $\bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\} = \langle f, g\rangle^*(\widetilde{C}) = 1$ by 4.3.2.
$\Leftarrow$: If $D_L \subseteq \mathfrak{o}(U)$ then $\delta_L(U) = 1$, that is, $\widehat{U}$ is a cover, hence $\bigvee\{f^*(c) \wedge g^*(c) \mid c \in \widehat{U}\} = 1$, and $\langle f, g\rangle^*(U) = 1$ by 4.3.1. ■

As an immediate consequence we obtain

4.5. Theorem. *A frame satisfies ($\mathcal{F}$sep) if and only if no distinct frame homomorphisms $h, k \colon L \to M$ respect covers.*

From 4.1.1 and 4.2.1 we now obtain an immediate

4.5.1. Corollary. *Every $\mathcal{F}$-separated frame satisfies (T_U).*

4.6. Notes. In [7], Dowker and Strauss proved the following characterization of the strong Hausdorff property.

4.6.1. Theorem. *A frame satisfies (sH) if and only if no distinct frame homomorphisms $h, k \colon L \to M$ satisfy the implication*
$$x \wedge y = 0 \ \Rightarrow h(x) \wedge k(y) = 0. \qquad \text{(DSeq)}$$

Note that

- (DSeq) immediately follows from $h \leq k$ and hence (sH) implies (T_U),
- and that, although the definitions of (sH) and ($\mathcal{F}$sep) are based on coproduct in **Frm**, the characterizations in 4.5 and 4.6.1 do not concern coproduct at all.

In the next section we will present another parallel characterizations of (sH) and ($\mathcal{F}$sep) concerning relaxed homomorphisms between frames combined with (T_U).

5. Another parallel with (sH): relaxed homomorphisms

5.1. Almost homomorphisms and the property (A). A mapping $h\colon L \to M$ between frames is an *almost homomorphism* if

(1) it is a preframe homomorphism,
(2) $h(0) = 0$, and
(3) if $\bigvee C = 1$ in L then $\bigvee h[C] = 1$.

We say that a frame L satisfies (A) if every almost homomorphism $h\colon L \to M$ is a frame homomorphism.

Note. There is a similar (already established [2]) property (W) from with which it will be in a certain parallel.

5.2. Proposition. *For frames satisfying (A), h, k are connected iff they respect covers.*

Proof: Let $h, k\colon L \to M$ respect covers. Consider the mapping

$$h \wedge k = (a \mapsto h(a) \wedge k(a))\colon L \to M.$$

It obviously preserves 0 and finite meets. Next, if $A \subseteq L$ is directed then

$$(h\wedge k)(\bigvee A) = \bigvee\{h(a)\wedge k(b) \mid a, b \in A\} = \bigvee\{(h\wedge k)(c) = h(c)\wedge k(c) \mid c \in A\}$$

because for any two $a, b \in A$ there is a $c \in A$ with $c \geq a, b$. Thus, $h \wedge k$ is a preframe homomorphism preserving 0. Since h, k respect covers it satisfies (3) in 5.1, hence it is an almost homomorphism, and by (A) a frame homomorphism. As $h \geq h \wedge k \leq k$, h and k are connected. ■

5.3. Recall from Section 2 the role of the frame coproduct as a tensor product in the category of preframes (as in [12, 19]). We will be using the

$$f \invamp f\colon L \invamp L \to M \invamp M$$

determined by a preframe homomorphism $f\colon L \to M$ between frames L, M.

5.4. Lemma. *Define* $\kappa_L\colon L\oplus L\to L\oplus L$ *by setting*

$$\kappa_L(U)=\bigvee\{V\to U\,|\,D_L\subseteq\mathfrak{o}(V)\}.$$

Then

1. κ_L *is a prenucleus with* $\kappa_L(U)=U$ *iff* $U\in\bigcap\{\mathfrak{o}(V)\,|\,D_L\subseteq\mathfrak{o}(V)\}$, *and*
2. *if a preframe homomorphism* $\phi\colon L\oplus L\to M\oplus M$ *satisfies the implication*

$$D_L\subseteq\mathfrak{o}(V)\;\Rightarrow D_M\subseteq\mathfrak{o}(\phi(V))$$

then

$$\phi\kappa_L\le\kappa_M\phi.$$

Proof: 1. Obviously κ_L is monotone and by the standard Heyting fact that $U\le V\to U$, $U\le\kappa_L(U)$, and $U=\kappa_L(U)$ iff for all V with $D_L\subseteq\mathfrak{o}(V)$ we have $U=V\to U$, that is, $U\in\mathfrak{o}(V)$.

2. The join $\bigvee\{V\to U\,|\,D_L\subseteq\mathfrak{o}(V)\}$ is obviously directed and hence

$$\phi\kappa_L(U)=\phi(\bigvee_{D_L\subseteq\mathfrak{o}(V)}(V\to U))=\bigvee_{D_L\subseteq\mathfrak{o}(V)}\phi(V\to U)\le$$
$$\overset{(*)}{\le}\bigvee_{D_L\subseteq\mathfrak{o}(V)}(\phi(V)\to\phi(U))\le\bigvee_{D_M\subseteq\mathfrak{o}(W)}(W\to\phi(U))=\kappa_M\phi(U)$$

where the inequality $(*)$ holds by (4) in 1.3.1 and because ϕ preserves finite meets. ■

5.5. Lemma. *Let* $f\colon L\to M$ *be an almost homomorphism. Then*

$$D_L\subseteq\mathfrak{o}(V)\;\Rightarrow\;D_M\subseteq\mathfrak{o}((f\invamp f)(V)).$$

Proof: By (3.3.1) we have to prove that if $\{a\wedge b\,|\,(a,b)\in V\}$ is a cover of L, then $\{u\wedge v\,|\,(u,v)\in(f\invamp f)(V)\}$ is a cover of M.

We have by (2.1.2)

$$(f\invamp f)(a\oplus b)=(f\invamp f)((a\invamp 0)\wedge(0\invamp b))=(f(a)\invamp 0)\wedge(0\invamp f(b))=f(a)\oplus f(b)$$

and hence for every $(a,b)\in V$ (that is, $a\oplus b\subseteq V$) we have $f(a)\oplus f(b)=(f\invamp f)(a\oplus b)\subseteq(f\invamp f)(V)$, that is, $(f(a),f(b))\in(f\invamp f)(V)$. Thus,

$$\bigvee\{u\wedge v\,|\,(u,v)\in(f\invamp f)(V)\}\ge\bigvee\{f(a)\wedge f(b)\,|\,(a,b)\in V\}=$$
$$=\bigvee\{f(a\wedge b)\,|\,(a,b)\in V\}=\bigvee f[\{a\wedge b\,|\,(a,b)\in V\}]=1$$

since f preserves covers. ■

5.6. Lemma. *Let ν resp. μ be the nuclei associated with $\bigcap_{D_L\subseteq\mathfrak{o}(V)}\mathfrak{o}(V)$ resp. $\bigcap_{D_M\subseteq\mathfrak{o}(V)}\mathfrak{o}(V)$. If $f\colon L\to M$ is an almost homomorphism then*

$$\mu(f⅋f)\nu=\mu(f⅋f).$$

Proof: By 5.4.2 we have $(f⅋f)\kappa_L\le\kappa_M(f⅋f)$. By 5.4.1, κ_L resp. κ_M are prenuclei generating ν resp. μ. Recall 5.4: by transfinite induction (using $(\kappa_K)_{\alpha+1}(U)=\kappa_K(\kappa_K)_\alpha(U)$ and $(\kappa_K)_\alpha(U)=\bigvee_{\beta<\alpha}(\kappa_K)_\beta(U)$ for limit α) we get $(f⅋f)\nu\le\mu(f⅋f)$. Finally

$$\mu(f⅋f)\nu\le\mu\mu(f⅋f)=\mu(f⅋f)\le\mu(f⅋f)\nu.\quad\blacksquare$$

5.7. Lemma. *If L is $\mathcal{F}$-separated then $\nu(a⅋b)=\nu((a\vee b)⅋0)$.*

Proof: If L is $\mathcal{F}$-separated then $\bigcap_{D_L\subseteq\mathfrak{o}(V)}\mathfrak{o}(V)=D_L$, hence $\nu=\alpha\delta_L$ for an isomorphism α, and hence it suffices to prove that $\delta(a⅋b)=\delta((a\vee b)⅋0)$. By (2.1.1) we have $\delta(a⅋b)=\delta((a\oplus1)\vee(1\oplus b))=a\vee b=\delta(((a\vee b)\oplus1)\vee(1\oplus0))=\delta((a\vee b)⅋0)$. $\blacksquare$

5.8. Proposition. *If L is $\mathcal{F}$-separated then each almost homomorphism $f\colon L\to M$ is a frame homomorphism.*

Proof: We will prove that f preserves finite joins.

D_M is a sublocale of $\bigcap_{D_M\subseteq\mathfrak{o}(V)}\mathfrak{o}(V)=D_L$ so that there is a frame homomorphism β such that $\delta_M=\beta\mu$. From 5.6 and 5.7 we now obtain

$$\begin{aligned}f(a)\vee f(b)&=\delta_M((f(a)\oplus1)\vee(1\oplus f(b))=\beta\mu(f(a)⅋f(b))=\\&=\beta\mu(f⅋f)(a⅋b)=\beta\mu(f⅋f)\nu(a⅋b)=\beta\mu(f⅋f)\nu((a\vee b)⅋0)=\\&=\beta\mu(f⅋f)((a\vee b)⅋0)=\delta_M(f(a\vee b)⅋f(0))=\\&=\delta_M((f(a\vee b)\oplus1)\vee(1\oplus0))=f(a\vee b).\quad\blacksquare\end{aligned}$$

5.9. Theorem. *L is $\mathcal{F}$-separated if and only if it is T_U and each almost homomorphism $f\colon L\to M$ is a frame homomorphism.*

Proof: $\Rightarrow$ is in 5.8 and 3.4.

$\Leftarrow$ follows from 5.2 and 3.4: if h,k respect covers then by 5.2 they are connected and by (T_U), $h=k$. $\blacksquare$

Note. It follows in particular from 5.9 that every fit frame has property (A). It might be worth showing that the proof of the corresponding fact for regular frames is much easier. Recall that a frame L is regular whenever $a=\bigvee\{b\in L\mid b\prec a\}$ for any $a\in L$ (where $b\prec a\equiv b^*\vee a=1$). Let

$h\colon L \to M$ be an almost homomorphism. Since any join can be computed as a directed join of finite joins and $h(0) = 0$, it suffices to check that h preserves binary joins. Let $a, b \in L$ and $x, y \in L$ such that $x \prec a$ and $y \prec b$. Then $h(x \vee y) \le h(a) \vee h(b)$. Indeed, for any $c, d \in L$ such that $c \wedge x = 0$, $c \vee a = 1$, $d \wedge y = 0$ and $d \vee y = 1$, we have $(c \wedge d) \vee (a \vee b) \ge (c \vee a) \wedge (d \vee b) = 1$ thus $h(c \wedge d) \vee h(a) \vee h(b) = 1$ (since h preserves that kind of joins). Hence

$$\begin{aligned} h(x \vee y) &= h(x \vee y) \wedge (h(c \wedge d) \vee h(a) \vee h(b)) \\ &= (h(x \vee y) \wedge h(c \wedge d)) \vee (h(x \vee y) \wedge (h(a) \vee h(b))) \\ &\le h(0) \vee h(a) \vee h(b) = h(a) \vee h(b). \end{aligned}$$

Finally, since $\{x \vee y \mid x \prec a, y \prec b\}$ is a directed set and $a \vee b = (\bigvee_{x \prec a} x) \vee (\bigvee_{y \prec b} b) = \bigvee\{x \vee y \mid x \prec a, y \prec b\}$ (by regularity), we get

$$h(a \vee b) = \bigvee\{h(x \vee y) \mid x \prec a, y \prec b\} \le h(a) \vee h(b).$$

5.10. Weak homomorphisms. Let L, M be frames. A *weak homomorphism* ([2]) $h\colon L \to M$ is a mapping such that

(1) $h(\bigvee A) = \bigvee h[A]$ for all $A \subseteq L$, and
(2) $x \wedge y = 0$ implies $h(x) \wedge h(y) = 0$, $h(1) = 1$.

A frame L satisfies (W) if

> *every weak homomorphism $h\colon L \to M$ is a frame homomorphism.*

In [2] there was proved (implicitly, see also [17]), this time using the fact that *$L \oplus M$ is a tensor product in the category of $\bigvee$-lattices* (recall [13], see 2.1 above) that

Theorem. *L is strongly Hausdorff if and only if it is T_U and each weak homomorphism $f\colon L \to M$ is a frame homomorphism.*

5.10.1. Note. The parallel of $((\mathcal{F}\mathrm{sep}) \equiv (\mathrm{A})\&(T_U))$ and $((\mathrm{sH}) \equiv (\mathrm{W})\&(T_U))$, and also a certain complementarity of (W) relaxing the meet part of frame homomorphism while (A) relaxing the join one is in fact deeper.

The crucial lemma for 5.9 was that

$$\nu_\circ(a \parr b) = \nu_\circ((a \vee b) \parr 0)$$

(the role of the subscript $\circ$ will be apparent shortly) and it was instrumental for proving that h preserved *binary joins*. In [2] the crucial lemma was that $(a \oplus b) \vee d_L = ((a \wedge b) \oplus 1) \vee d_L$ where $d_L = \bigwedge D_L$, instrumental for proving that h preserved *binary meets*. Now realize that $(-) \vee d_L$ is the nucleus ν_- associated with the closure $\overline{D_L}$ that is, with the intersection of all *closed* sublocales containing D_L, hence it amounted to

$$\nu_-(a \oplus b) = \nu_-((a \wedge b) \oplus 1).$$

In the equality above we have the nucleus associated with the "*other closure*" $(D_L)^\circ$ (see [6]), the intersection of all *open* sublocales containing D_L, another link between the two results.

There are, of course, differences necessarily preventing something like full duality between the facts: The whole situation is non-symmetric (biased in the direction of joins, the two closures are both intersections, albeit of dually connected entities, and the two tensor products are also not quite so closely linked ($\otimes$ follows the construction generating $\oplus$ everywhere, and $a \otimes b$ are quite like $a \oplus b$, $\parr$ coincides with $\oplus$ for frames, and has to be specially represented).

6. ($\mathcal{F}$sep) is strictly weaker than fitness

6.1. Recall that elements a, b of a distributive lattice L are *normally separated* ([2]) if

$$\exists u, v,\ \ u \wedge v = 0, a \leq u \vee b \ \text{ and } \ b \leq a \vee v.$$

Dually, we say that elements a, b are *extremally separated* if

$$\exists u, v,\ \ u \vee v = 1, a \wedge v \leq b \ \text{ and } \ u \wedge b \leq a.$$

Dualizing Proposition 3.2 from [2] we get

6.1.1. Lemma. *Let L, M be distributive lattices. Then an $h\colon L \to M$ preserving meets, finite covers and 0, preserves joins of extremally separated elements.*

Proof: Since $h(u) \vee h(v) = 1$, we have

$$h(a) \vee h(b) \geq h((a \vee b) \wedge u) \vee h((a \vee b) \wedge v) = h(a \vee b) \wedge (h(u) \vee h(v)) = h(a \vee b)$$

and the other inequality is trivial. ■

6.1.2. Notes. Elements a, b of a frame L are extremally separated if and only if

$$(a \to b) \vee (b \to a) = 1. \qquad (*)$$

It is straightforward to check that every pair in L is extremally separated if and only if L is hereditarily extremally disconnected ([8]), i.e. iff every closed sublocale of L is extremally disconnected. It then follows from 6.1.1 that every hereditarily extremally disconnected frame has property (A). Indeed, if h is an almost homomorphism, it will be a frame homomorphism iff it preserves binary joins (as arbitrary joins are directed joins of finite joins).

6.2. We will consider a simple extension $E = E_{X,Y}$ (recall 2.3) with X dense in Y.

6.2.1. Lemma. *Let $M, N \subseteq Y \smallsetminus X$ be disjoint. Then $X \cup M$ and $X \cup N$ are extremally separated in $\Omega(E)$.*

Proof: We check $(*)$ for $a = X \cup M$ and $b = X \cup N$. We have

$$\begin{aligned}(X \cup M) \to (X \cup N) &= int_E((X \cup N) \cup (X \cup M)^c)\\ &= int_E((X \cup N) \cup ((Y \smallsetminus X) \cap (Y \smallsetminus M)))\\ &= int_E(X \cup N \cup (Y \smallsetminus M)) = int_E(X \cup (Y \smallsetminus M))\end{aligned}$$

(by disjointness). Now $X \cup (Y \smallsetminus M)$ is open in E ($X\cup$ anything is open), so $(X \cup M) \to (X \cup N) = X \cup (Y \smallsetminus M)$. Similarly $(X \cup N) \to (X \cup M) = X \cup (Y \smallsetminus N)$ and hence

$$[(X\cup M) \to (X\cup N)]\cup[(X\cup N) \to (X\cup M)] = X\cup(Y\smallsetminus N)\cup(Y\smallsetminus M) = Y. \quad \blacksquare$$

6.2.2. Lemma. *If $h\colon \Omega(E) \to L$ is an almost homomorphism, then for any $M, N \subseteq Y \smallsetminus X$, $h(X \cup M) \vee h(X \cup N) = h(X \cup M \cup N)$.*

Proof: $h(X\cup M\cup N) = h(X\cup M\cup(N\smallsetminus M)) = h(X\cup M)\vee h(X\cup(N\smallsetminus M)) \leq h(X \cup M) \vee h(X \cup N)$ by 6.2.1 and 6.1.1. The other inequality is trivial. $\blacksquare$

6.2.3. Lemma. *Let $\Omega(Y)$ satisfy (A) and let $h\colon \Omega(E) \to M$ be an almost homomorphism. Let $U_1, U_2 \in \Omega(E)$ with $U, V \in \Omega(Y)$ and $M \subseteq U \cap (Y \smallsetminus X)$, $N \subseteq V \cap (Y \smallsetminus X)$ such that $U_1 = (U \cap X) \cup M$ and $U_2 = (V \cap X) \cup N$. If, moreover*

$$U \cap N \subseteq M \quad \textit{and} \quad V \cap M \subseteq N,$$

then $h(U_1 \cup U_2) = h(U_1) \vee h(U_2)$.

Proof: We can write $U_1 = U \cap (X \cup M)$ and $U_2 = V \cap (X \cup N)$ with the advantage that $U, V, X \cup M, X \cup N$ are all open in $\Omega(E)$. Now

$$\begin{aligned}h(U_1 \cup U_2) &= h((U \cup V) \cap (X \cup (M \cup N)))\\ &= h(U \cup V) \wedge h(X \cup (M \cup N)).\end{aligned} \qquad (*)$$

($(U \cup V)$ and $(X \cup (M \cup N))$ are in $\Omega(E)$ and h preserves $\wedge$). Now, consider the subframe embedding $\iota\colon \Omega(Y) \subseteq \Omega(E)$. The composite $h \circ \iota\colon \Omega(Y) \to M$ is obviously an almost homomorphism and Y satisfies (A), hence it is a frame homomorphism. Thus $h(U \cup V) = h(U) \vee h(V)$. Moreover, by 6.2.2 we also have $h(X \cup (M \cup N)) = h(X \cup M) \vee h(X \cup N)$. Following $(*)$ we get

$$\begin{aligned} h(U_1 \cup U_2) &= (h(U) \vee h(V)) \wedge (h(X \cup M) \vee h(X \cup N)) \\ &= \big(h(U) \wedge h(X \cup M)\big) \vee \big(h(U) \wedge h(X \cup N)\big) \\ &\qquad \vee \big(h(V) \wedge h(X \cup M)\big) \vee \big(h(V) \wedge h(X \cup N)\big) \\ &= h(U_1) \vee h(U \cap (X \cup N)) \vee h(V \cap (X \cup M)) \vee h(U_2). \end{aligned}$$

Now since $U \cap N \subseteq M$, we have $U \cap (X \cup N) = (U \cap X) \cup (U \cap N) \subseteq (U \cap X) \cup M = U_1$, and similarly, $V \cap (X \cup M) \subseteq U_2$, and the statement follows. ■

6.2.4. Lemma. *Let $\Omega(Y)$ satisfy (A) and let Y be T_1. Furthermore, let $h\colon \Omega(E) \to M$ be an almost homomorphism and let $U_1, U_2 \in \Omega(E)$ with $U, V \in \Omega(Y)$ and $M \subseteq U \cap (Y \smallsetminus X)$, $N \subseteq V \cap (Y \smallsetminus X)$ such that $U_1 = (U \cap X) \cup M$ and $U_2 = (V \cap X) \cup N$. Moreover, assume that M and N are finite.*

Then $h(U_1 \cup U_2) = h(U_1) \vee h(U_2)$.

Proof: Decompose $U_i = U_i' \cup U_i''$, $i = 1, 2$ with

$$U_1' = (U \cap X) \cup (M \cap N), \qquad U_2' = (V \cap X) \cup (M \cap N),$$
$$U_1'' = (U \cap X) \cup (M \smallsetminus N), \qquad U_2'' = (V \cap X) \cup (N \smallsetminus M).$$

Obviously $M \cap N \subseteq U \cap V$, $M \smallsetminus N \subseteq U$ and $N \smallsetminus M \subseteq V$ and hence "they are in their typical form" so $U_i', U_i'' \in \Omega(E)$. Now

$$\begin{aligned} h(U_1 \cup U_2) &= h\big((U \cup V) \cap \big(X \cup (M \cap N) \cup (M \smallsetminus N) \cup (N \smallsetminus M)\big)\big) \\ &= h(U \cup V) \wedge \big(h(X \cup (M \cap N)) \vee h(X \cup (M \smallsetminus N) \cup (N \smallsetminus M))\big) \\ &= h(U_1' \vee U_2') \vee h(U_1'' \vee U_2'') \end{aligned}$$

where we have used 6.2.2 and the fact that h preserves meets. Now since $U_1' \le U_1$, $U_2' \le U_2$, $U_1'' \le U_1$ and $U_2'' \le U_2$, it suffices to show that

$$h(U_1' \vee U_2') = h(U_1') \vee h(U_2') \quad \text{and} \quad h(U_1'' \vee U_2'') = h(U_1'') \vee h(U_2'').$$

The first follows from 6.2.3 since U_1', U_2' clearly satisfy the additional condition in the statement.

For the latter note that we can write

$$U_1'' = ((U \smallsetminus N) \cap X) \cup (M \smallsetminus N) \quad \text{and} \quad U_2'' = ((V \smallsetminus M) \cap X) \cup (N \smallsetminus M)$$

(the equalities simply follow because $N \cap X = \emptyset = M \cap X$). Moreover, since N and M are finite and Y is T_1, then $U \smallsetminus N, V \smallsetminus M \in \Omega(Y)$. Now we see that U_1'', U_2'' also satisfy the additional condition in 6.2.3 and the result follows. ∎

6.3. Theorem. *Let Y be a T_1-space satisfying (A). Then $E = E_{X,Y}$ also satisfies (A).*

Proof: Let $h\colon \Omega(E) \to M$ be an almost homomorphism. We will show that it preserves binary joins. Let $U_1 = (U \cap X) \cup M$ and $U_2 = (V \cap X) \cup N$ with $U, V \in \Omega(Y)$, $M \subseteq U \cap (Y \smallsetminus X)$ and $N \subseteq V \cap (Y \smallsetminus X)$. Then

$$\begin{aligned} h(U_1 \cup V_1) &= h\Big(\bigcup_{F \in \mathcal{P}_f(M),\, G \in \mathcal{P}_f(N)} ((U \cap X) \cup F) \cup (V \cap X) \cup G\Big) \\ &= \bigvee_{F \in \mathcal{P}_f(M),\, G \in \mathcal{P}_f(N)} h\big(((U \cap X) \cup F) \cup ((V \cap X) \cup G)\big) \\ &= \bigvee_{F \in \mathcal{P}_f(M),\, G \in \mathcal{P}_f(N)} h((U \cap X) \cup F) \vee h((V \cap X) \cup G) \\ &\leq h(U_1) \vee h(U_2), \end{aligned}$$

by 6.2.4 and the fact that the union is directed. ∎

As a consequence we now obtain from the result of Banaschewski [1] presented in 2.3.1 that

6.4. Theorem. *There exists a strongly Hausdorff $\mathcal{F}$-separated spatial frame which is not fit.* ∎

6.5. ($\mathcal{F}$sep) and conditions akin to fitness. Since we know that, for trivial reasons fitness implies ($\mathcal{F}$sep) we have now learned that

fitness is strictly stronger than ($\mathcal{F}$sep).

This solves Problem 3 of [18, 2.7] in the negative.

The question naturally arises what is the relation of ($\mathcal{F}$sep) to other often used properties strictly weaker than fitness. In particular there is the *subfitness* (arguably even more important than fitness itself)

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \neq b \vee c \qquad \text{(sfit)}$$

or the weaker *weak subfitness*

$$a \neq 0 \quad \Rightarrow \quad \exists c \neq 1,\ a \vee c = 1 \tag{wsfit}$$

or, finally, the *prefitness*

$$a \neq 0 \quad \Rightarrow \quad \exists c = c^{**} \neq 1,\ a \vee c = 1. \tag{pfit}$$

6.5.1. Note. The formula for prefitness looks deceptively similar to that of weak prefitness. In fact it is quite a strong property (although strictly weaker than fitness), incomparable with subfitness ([16, 17]).

6.5.2. Proposition. *None of the properties* (*sfit*), (*wsfit*) *or* (*pfit*) *coincides with* ($\mathcal{F}$*sep*).

Proof: None of them is hereditary while each ($\mathcal{P}$)-separation is even closed under all monomorphisms ([5, 18]). ■

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