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THE COMMUTATION GRAPH FOR THE LONGEST SIGNED PERMUTATION

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ABSTRACT: Using the standard Coxeter presentation for the signed symmetric group $\mathfrak{S}_{n+1}^B$ on $n+1$ letters, two reduced expressions for a given signed permutation are in the same commutation class if one expression can be obtained from the other one by applying a finite sequence of commutations. The commutative classes of a given signed permutation can be seen as the vertices of a graph, called the commutation graph, where two classes are connected by an edge if there are elements in those classes that differ by a long braid relation. We define a ranking function for the commutation graph for the longest signed permutation, and use this function to compute the diameter and the radius of the graph. We also prove that the commutation graph for the longest signed permutation is not planar for $n > 2$, and identify the classes with a single element.

KEYWORDS: Signed permutations, reduced words, commutation graph, diameter, radius.

1. Introduction

A *Coxeter group* is a group W with a presentation

$$\left\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \right\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying $m(s,s) = 1$ for each $s \in S$, $m(s,t) = m(t,s)$ for each $s,t \in S$, and $m(s,t) \geq 2$ if $s \neq t$. Since the elements of S, called *simple reflections*, are involutions, one can write the relation $(st)^{m(s,t)} = 1$ as

$$\underbrace{stst\cdots}_{m(s,t)} = \underbrace{tsts\cdots}_{m(s,t)} \tag{1}$$

for $2 \leq m(s,t) < \infty$. When $m(s,t) = 2$, the relation $st = ts$ is called a *commutation* or a *short braid relation*, and if $m(s,t) \geq 3$, relation (1) is called a *long braid relation*.

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Since the set S generates W, every element $\mathrm{w} \in W$ can be written as a finite product $\mathrm{w} = s_{i_1}s_{i_2}\cdots s_{i_l}$, with $s_{i_j} \in S$. When l is minimal, we say that $l(\mathrm{w}) := l$ is the *length* of w and $s_{i_1}s_{i_2}\cdots s_{i_l}$ a *reduced expression* for w. The string of subscripts $i_1 i_2 \cdots i_l$ is called a *reduced word* for w and the collection of all reduced words for w is denoted by $R(\mathrm{w})$. Commutations and long braid relations have obvious analogues in reduced words, and we refer to these phenomena in reduced words as commutations and long braid relations, respectively, as well. Tits [12] showed that any reduced word can be transformed into any other by a sequence of commutations and long braid relations.

We define a relation $\sim$ on the set $R(\mathrm{w})$ by setting $a \sim b$ if and only if a and b differ by a sequence of commutations. This is an equivalence relation and the classes it defines are the *commutation classes* of w. We write $[a]$ to denote the commutation class of $a \in R(\mathrm{w})$, and let $C(\mathrm{w})$ be the set of all commutation classes of w.

The graph $G(\mathrm{w})$, with vertex set $C(\mathrm{w})$, and one edge between two commutation classes $[a]$ and $[b]$ whenever there are reduced words $a' \in [a]$ and $b' \in [b]$ such that a' and b' differ by a single long braid relation, is called the *commutation graph* of w. The *distance* $d([a],[b])$ between commutation classes $[a]$ and $[b]$ in $G(\mathrm{w})$ is the length of a shortest path joining $[a]$ and $[b]$. The *eccentricity* of $[a]$ is the distance to a farthest vertex of $[a]$, and the *radius* and *diameter* of $G(\mathrm{w})$ are, respectively, the minimum and maximum eccentricities.

The graph $G(\mathrm{w})$ has a rich combinatorial structure, and has been considered especially when W is the symmetric group $\mathfrak{S}_{n+1}$ of order $n+1$ [1, 7, 11, 10]. Elnitsky [8] showed that $G(\mathrm{w})$ is bipartite by establishing a bijection between $C(\mathrm{w})$ and rhombic tilings of a certain polygon that depends on w. A formula for the diameter of $G(\mathrm{w})$ was computed in [6] for any permutation $\mathrm{w} \in \mathfrak{S}_{n+1}$, and recursive formulas for the number of reduced words in each commutation class was obtained by Bédard [3]. Stembridge [9] investigated and enumerated permutations having a single commutation class. Connection of $G(\mathrm{w})$ to geometric representation theory where considered in [13]. The graph $G(\mathrm{w}_0)$ for the longest permutation w_0 of the symmetric group $\mathfrak{S}_{n+1}$, has been particularly well studied. Detailed properties of the commutation classes of w_0 where obtained in [3] and [5]. The radius of $G(\mathrm{w}_0)$ was computed and it was showed that this graph is not planar for $n > 4$.

In contrast to the symmetric group case, little is known about the structure of the graph $G(\mathrm{w})$ when w is an element of the Coxeter group $\mathfrak{S}_{n+1}^B$ of type B on $\{\pm 1, \ldots, \pm(n+1)\}$, also known as the hyperoctahedral group. In this paper, we focus on the commutation graph for the longest signed permutation w_0^B of $\mathfrak{S}_{n+1}^B$. We construct a ranking function on the commutation classes of $G(\mathrm{w}_0^B)$ which is invariant within each class, and differ by one unit between classes that share an edge. This ranking function is then used to compute the radius and the diameter of $G(\mathrm{w}_0^B)$. Wagners's Theorem [4] is used to prove that $G(\mathrm{w}_0^B)$ is not planar for $n > 2$. Finally, it is shown the existence of exactly two commutation classes in this graph having a single element, for all $n \geq 2$.

2. Definitions and Background

Given a positive integer $n \geq 2$, let $\mathfrak{S}_{n+1}$ denote the *symmetric group* of order $n+1$, formed by all permutations of the set $[n+1] := \{1, 2 \ldots, n+1\}$, with composition (read from the right) as group operation. A permutation $\mathrm{w} \in \mathfrak{S}_{n+1}$ will be written in one-line notation as $\mathrm{w} = (\mathrm{w}(1), \mathrm{w}(2), \ldots, \mathrm{w}(n+1))$. The symmetric group $\mathfrak{S}_{n+1}$ is generated by the simple reflections

$$\{s_1^A, s_2^A, \ldots, s_n^A\},$$

where s_i^A interchanges i and $i+1$. These reflections satisfy the following relations:

$$s_i^A s_j^A = s_j^A s_i^A, \; |i-j| > 1 \tag{2}$$

$$s_i^A s_{i+1}^A s_i^A = s_{i+1}^A s_i^A s_{i+1}^A, \; i \in [n-1]. \tag{3}$$

The symmetric group $\mathfrak{S}_{n+1}$ acts on itself by right multiplication, so that the product $\mathrm{w}s_i$ changes the values in positions i and $i+1$ in w. The permutation $\mathrm{w}_0^{A_n} := (n+1, n, \ldots, 1)$ is the longest permutations of the symmetric group $\mathfrak{S}_{n+1}$, with length $l(\mathrm{w}_0^{A_n}) = \binom{n+1}{2}$.

Let $\mathfrak{S}_{n+1}^B$ denote the Coxeter group of type B on $[\pm(n+1)] := \{\pm 1, \pm 2, \ldots, \pm(n+1)\}$. This group is formed by all signed permutations on $[\pm(n+1)]$ such that $\mathrm{w}(-k) = -\mathrm{w}(k)$ for all $k \in [n+1]$, with composition (read from the right) as group operation. $\mathfrak{S}_{n+1}^B$ can be seen as sub-group of $\mathfrak{S}_{2n+2}$, and we can use one-line notation $\mathrm{w} := (\overline{\mathrm{w}(n+1)}, \ldots, \overline{\mathrm{w}(1)}, \mathrm{w}(1), \ldots, \mathrm{w}(n+1))$ to represent a signed permutation w, where for ease of notation we write $\overline{\mathrm{w}(k)}$ to represent $-\mathrm{w}(k)$. Since w is completely determined by its values on the set $[n+1]$, we can also use the window notation to represent w as

$\mathrm{w} = [\mathrm{w}(1), \mathrm{w}(2), \ldots, \mathrm{w}(n+1)]$. The image $\mathrm{w}(i)$ will be referred to as the i-th letter of w in the the window notation.

The group $\mathfrak{S}^B_{n+1}$ is generated by the simple reflections $\{s_0, s_1, \ldots, s_n\}$, where $s_i := [1, \ldots, i+1, i, \ldots, n+1]$ for $1 \leq i \leq n$, and $s_0 := [\bar{1}, 2, \ldots, n+1]$. These reflections satisfy the following relations:

$$s_i s_j = s_j s_i,\ |i-j| > 1, \tag{4}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ i \in [n-1], \tag{5}$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0. \tag{6}$$

Relation (4) is a commutation, and long braid relations (5) and (6) are called *braid relation of type 1* and of *type 2*, respectively. As in the symmetric group case, $\mathfrak{S}^B_{n+1}$ acts on itself by right multiplication, meaning that for $i > 0$, $\mathrm{w}s_i$ changes the values in positions i and $i+1$ in the window notation of $\mathrm{w} \in \mathfrak{S}^B_{n+1}$. The reflection s_0 changes the sign of the first entry in the window notation of w, that is $\mathrm{w}s_0 = [\overline{\mathrm{w}(1)}, \mathrm{w}(2), \ldots, \mathrm{w}(n+1)]$. In this sense, if $\mathrm{w} = s_{i_1} \ldots s_{i_l}$ we say that s_{i_j} acts on the letters in positions i_j and $i_j + 1$ (resp. on the letter in the first position) in the window notation of $s_{i_1} s_{i_2} \ldots s_{i_{j-1}}$, when $i_j > 0$ (resp. $i_j = 0$). Abusing terminology slightly, we will say that the window notation of $\mathrm{w} = s_{i_1} \ldots s_{i_l}$ is the window notation of the corresponding word $i_1 \cdots i_l$, and identify the letter i_j with the generator s_{i_j}. To simplify notation, we denote by $k^{i_1 i_2 \cdots i_{j-1}}$ the k-th letter in the window notation of $i_1 i_2 \cdots i_{j-1}$, that is $k^{i_1 i_2 \cdots i_{j-1}} = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(k)$.

Example 2.1. If $\mathrm{v} = [2, \bar{4}, 3, 1] \in \mathfrak{S}^B_4$, then $\mathrm{v}s_2 = [2, 3, \bar{4}, 1]$ and $\mathrm{v}s_0 = [\bar{2}, \bar{4}, 3, 1]$. If a is a reduced word for v, then we also have $1^{a2} = 2$, $2^{a2} = 3$, $3^{a2} = \bar{4}$ and $4^{a2} = 1$.

The length of a signed permutation w can be given by the formula (see [2])

$$l(\mathrm{w}) = inv(\mathrm{w}) - \sum_{\{j \in [n] : \mathrm{w}(j) < 0\}} \mathrm{w}(j), \tag{7}$$

where $inv(\mathrm{w}) = \#\{(i,j) \in [n]^2 : i < j,\ \mathrm{w}(i) > \mathrm{w}(j)\}$. From (7), it is easy to conclude that

$$l(\mathrm{w}s_i) = \begin{cases} l(\mathrm{w}) + 1, & if\ \mathrm{w}(i) < \mathrm{w}(i+1) \\ l(\mathrm{w}) - 1, & if\ \mathrm{w}(i) > \mathrm{w}(i+1) \end{cases}, \tag{8}$$

with $i \in [n]$ and

$$l(\mathrm{w}s_0) = \begin{cases} l(\mathrm{w}) + 1, & if\ \mathrm{w}(1) > 0 \\ l(\mathrm{w}) - 1, & if\ \mathrm{w}(1) < 0 \end{cases}. \tag{9}$$

The following result follows easily from (8) and (9).

Proposition 2.2. *Let $a \in R(\mathrm{w})$ be a reduced word of $\mathrm{w} \in \mathfrak{S}^B_{n+1}$ and $i \in [n]$. Then,*

(1) *the word $a \cdot i$ is a reduced word of $\mathrm{w}s_i$ if and only if $i^a < (i+1)^a$;*
(2) *the word $a \cdot 0$ is a reduced word of $\mathrm{w}s_0$ if and only if $1^a > 0$.*

The permutation $\mathrm{w}_0^{B_n} = [\bar{1}, \bar{2}, \ldots, \overline{n+1}]$ is the longest permutation of $\mathfrak{S}^B_{n+1}$, with length $l(\mathrm{w}_0^{B_n}) = (n+1)^2$. It is easy to check that the permutation associated to the word

$$w_0 = 0 \cdot 10 \cdot 210 \cdot \ldots \cdot (n+1)n \cdots 210 \cdot (n+1)n \cdots 21 \cdot \ldots \cdot (n+1)n \cdot n$$

is $\mathrm{w}_0^{B_n}$. Since w_0 has $(n+1)^2$ letters, it is a reduced word for $\mathrm{w}_0^{B_n}$. For instance, when $n = 3$, we obtain the reduced word $w_0 = 0 \cdot 10 \cdot 210 \cdot 3210 \cdot 321 \cdot 32 \cdot 3$.

The word w_0 will play an important role in the following sections. To simplify the handling of these words, we introduce some notation. A *subword* of a word a is a word obtained from a by deleting some of its letters, consecutive or not, and a *factor* of a is a subword of a formed by consecutive letters. If k is a nonnegative integer and $1 \le r \le k+1$, we define the *factorial* $k!$ as the word $k! = k(k-1)\cdots 10$, with $0! = 0$, and the *truncated factorial* $(k)_r$ as the word $(k)_r = k(k-1)\cdots(k-r+1)$. Note that $(k)_{k+1} = k!$. A k-th power of a word p is the word composed by the concatenation of k words p, *i.e.* $p^k = \underbrace{p \cdots p}_{k \text{ times}}$, with p^0 the empty word. With this notation, we can write the reduced word w_0 as

$$w_0 = \prod_{i=0}^{n} i! \prod_{i=1}^{n} (n)_{n-i+1}.$$

Given a reduced word $a = i_1 i_2 \ldots i_l$ we define $a^R := i_l i_{l-1} \cdots i_1$ as the *reversed word* of a. In particular, if a is a reduced word of $\mathrm{w}_0^{B_n}$, then a^R is also a reduced word of $\mathrm{w}_0^{B_n}$ since $l(a) = l(a^R)$ and $\mathrm{w}_0^{B_n}$ is an involution.

A reduced word $a = i_1 i_2 \cdots i_l$ of $\mathrm{w}_0^{B_n}$, can be seen as a set of operations that transform the identity in $\mathrm{w}_0^{B_n}$, more exactly a set of transpositions of letters in adjacent positions in the window notation and change of signs. This can easily be seen using the line diagram of a. The *line diagram* of $a = i_1 i_2 \cdots i_l$ is the array $[\pm 2(n+1)] \times [l]$ in Cartesian coordinates, which describes the trajectories of the letters $\overline{n+1}$, $\bar{n}$, …, $\bar{1}$, 1, …, $n+1$ as they are arranged into the permutation $\mathrm{w}_0^{B_n}$ by the successive action of the simple reflections s_{i_j}. The line diagram of the word $101210102 \in R(\mathrm{w}_0^{B_2})$ is displayed in Figure 1.

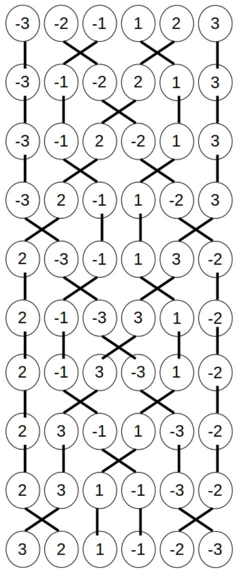

FIGURE 1. Line diagram of $101210102 \in R(\mathrm{w}_0^{B_2})$.

Let $a = i_1 i_2 \cdots i_l$ be a reduced word of $\mathrm{w}_0^{B_n}$, and consider a pair $(x, y) \in [\pm(n+1)]^2$, with $x < y$. Since in the one-line notation of $\mathrm{w}_0^{B_n}$ the letter y appears before letter x, there must be some reflection s_{i_j} such that x and y, or $\bar{x}$ and $\bar{y}$ are in positions i_j and $i_j + 1$ in the permutation associated with the word $i_1 i_2 \cdots i_{j-1}$. Then, the reflection s_{i_j} changes the values in these two pairs of positions, transposing the letters x and y, and $\bar{x}$ and $\bar{y}$. In this case, we say that i_j transposes the pairs (x, y) and $(\bar{y}, \bar{x})$. Moreover, since a is reduced, each pair $(x, y) \in [\pm(n+1)]^2$ with $x < y$ is transposed by exactly one reflection s_{i_j}. For instance, we can use the line diagram of the word $101210102 = i_1 i_2 \cdots i_9 \in R(\mathrm{w}_0^{B_2})$ in Figure 1 to check that the pair $(\bar{1}, 2)$ is transposed by the reflection $s_{i_3} = s_1$, while $(2, 3)$ is transposed by the reflection $s_{i_9} = s_2$.

Note also that s_0 is the only reflection that acts on a permutation transposing pairs $(\bar{x}, x)$, for $x \in [n+1]$. Since there are $n+1$ such pairs in $[\pm(n+1)]^2$, there are exactly $n+1$ reflections s_0 in any reduced decomposition for $\mathrm{w}_0^{B_n}$.

We define the lexicographic order on the set $R(\mathrm{w}_0^{B_n})$ of all reduced words of $\mathrm{w}_0^{B_n}$ as follows: given $a = a_1 \cdots a_l, b = b_1 \cdots b_l \in R(\mathrm{w}_0^{B_n})$ we say that a is *less than* b if exist an integer k such that $a_i = b_i$ for $1 \leq i \leq k-1$ and $a_k < b_k$. We can use commutation relations $i\ j \sim j\ i$, with $j < i$, to "move" each letter in a to their leftmost possible position, obtaining in this way the minimal word in the commutation class $[a]$ with respect to the lexicographic order.

Since the lexicographic order is a total order relation, we can check if two reduced words a, b are in the same commutation class by computing the minimal words, in the commutation class $[a]$ and $[b]$, with respect to the lexicographic order. That is, we have $[a] = [b]$ if and only if the minimal words for the lexicographic order of the two classes are the same.

3. Ranking function and the diameter of $G(\mathrm{w}_0^{B_n})$

In this section, we compute the diameter of the commutation graph $G(\mathrm{w}_0^{B_n})$. This is achieved through the construction of a ranking function on the commutation classes of $\mathrm{w}_0^{B_n}$. In the sequel, given a positive integer n, we let $[0, n] = [n] \cup \{0\}$.

Given two words $a, b \in R(\mathrm{w}_0^{B_n})$ we write $a \overset{\mathrm{L}_1}{\approx} b$ (respectively, $a \overset{\mathrm{L}_2}{\approx} b$) if a and b differ from a long braid relation of type 1 (respectively, type 2). We also write $[a] \overset{\mathrm{L}_1}{\approx} [b]$ (respectively, $[a] \overset{\mathrm{L}_2}{\approx} [b]$) if there is $a' \in [a]$ and $b' \in [b]$ such that $a' \overset{\mathrm{L}_1}{\approx} b'$ (respectively, $a' \overset{\mathrm{L}_2}{\approx} b'$).

Figure 2 depicts the graph $G(\mathrm{w}_0^{B_2})$, where the solid edges represent long braid relations of type 1 and the dashed edges represent long braid relations of type 2. Let $a = 102101210$, $b = 121010210$ and $c = 210210210$. Since $a = 1\mathbf{02}101210 \sim 1\mathbf{20}101210$ and $12\mathbf{0101}210 \overset{\mathrm{L}_2}{\approx} 12\mathbf{1010}210 = b$, we have $[a] \overset{\mathrm{L}_2}{\approx} [b]$. We also have that $b = \mathbf{121}010210 \overset{\mathrm{L}_1}{\approx} \mathbf{212}010210 = c'$ and $c' = 21\mathbf{20}10210 \sim 21\mathbf{02}10210 = c$, so $[b] \overset{\mathrm{L}_1}{\approx} [c]$.

Definition 3.1. Let $a = i_1 i_2 \cdots i_l$ be a reduced word for $\mathrm{w}_0^{B_n}$. We define the word $\widetilde{a} := \widetilde{i_1}\widetilde{i_2} \cdots \widetilde{i_l}$, where $\widetilde{0} = 0$ and if $i_j > 0$ then

$$\widetilde{i_j} = \begin{cases} i_j, & \text{if } (i_j)^{i_1 i_2 \cdots i_{j-1}} > 0 \text{ and } (i_j+1)^{i_1 i_2 \cdots i_{j-1}} > 0 \\ \overline{i_j}, & \text{otherwise} \end{cases}$$

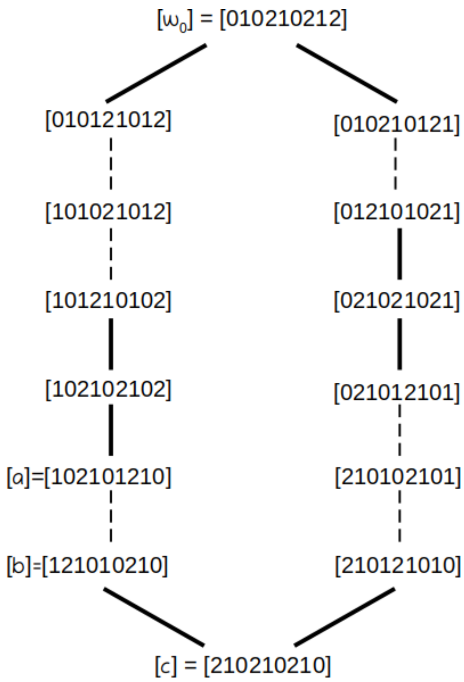

FIGURE 2. The graph $G(\mathrm{w}_0^{B_2})$.

We also define the *sum function*, S, and the *negative number function*, *neg*, on $R(\mathrm{w}_0^{B_n})$ setting $S(a) := \sum_{j=1}^{l} \widetilde{i_j}$, and $neg(a) := \#\left\{i_j | \widetilde{i_j} < 0\right\}$, for each $a = i_1 i_2 \cdots i_l \in R(\mathrm{w}_0^{B_n})$.

Thus, given a reduced word $a = i_1 i_2 \cdots i_l$ and $j \in [l]$ such that $i_j \neq 0$, we have $\widetilde{i_j} = i_j$ if and only if i_j transposes two positive letters in the window notation of $i_1 i_2 \cdots i_{j-1}$. In this case we say that the letter i_j (and the corresponding reflection s_{i_j}) is *positive*. Otherwise, if i_j transposes one or two negative letters in the window notation of $i_1 i_2 \cdots i_{j-1}$, we say that the letter i_j (and the corresponding reflection s_{i_j}) is *negative*. The *neg* function counts the negative generators of a, and the S function is the sum of the indices of the positive generators minus the sum of the indices of the negative generators.

Example 3.2. Let $a = 210102101$. The sequence of left factors r_i, with length $1 \leq i \leq 9$, of a, the window notation for the respective signed permutations, and their correspondent words $\widetilde{r_i}$ are presented bellow.

$$\begin{aligned}
r_1 &= 2,\ [1, \mathbf{3}, \mathbf{2}],\ \widetilde{r_1} = 2 \\
r_2 &= 21,\ [\mathbf{3}, \mathbf{1}, 2],\ \widetilde{r_2} = 21 \\
r_3 &= 210,\ [\bar{\mathbf{3}}, 1, 2],\ \widetilde{r_3} = 210 \\
r_4 &= 2101,\ [\mathbf{1}, \bar{\mathbf{3}}, 2],\ \widetilde{r_4} = 210\bar{1} \\
r_5 &= 21010,\ [\bar{\mathbf{1}}, \bar{3}, 2],\ \widetilde{r_5} = 210\bar{1}0 \\
r_6 &= 210102,\ [\bar{1}, \mathbf{2}, \bar{\mathbf{3}}],\ \widetilde{r_6} = 210\bar{1}0\bar{2} \\
r_7 &= 2101021,\ [\mathbf{2}, \bar{\mathbf{1}}, \bar{3}],\ \widetilde{r_7} = 210\bar{1}0\bar{2}\bar{1} \\
r_8 &= 21010210,\ [\bar{\mathbf{2}}, \bar{1}, \bar{3}],\ \widetilde{r_8} = 210\bar{1}0\bar{2}\bar{1}0 \\
a = r_9 &= 210102101,\ [\bar{\mathbf{1}}, \bar{\mathbf{2}}, \bar{3}],\ \widetilde{r_9} = 210\bar{1}0\bar{2}\bar{1}0\bar{1} = \widetilde{a}.
\end{aligned}$$

We have $S(a) = 2 + 1 - 1 - 2 - 1 - 1 = -2$ and $neg(a) = 4$.

Proposition 3.3. *Let $a, b \in R(\mathrm{w}_0^{B_n})$. If $a \sim b$, then $S(a) = S(b)$ and $neg(a) = neg(b)$.*

Proof: Suppose without loss of generality that a and b differ by one commutation. We can write a and b as

$$\begin{aligned}
a &= p_1 \cdot i\ j \cdot p_2 \\
b &= p_1 \cdot j\ i \cdot p_2,
\end{aligned}$$

for some $i, j \in [0, n]$ such that $|i - j| > 1$, with p_1 and p_2 words on the alphabet $[0, n]$. Obviously, the sign of the generators in p_1 are the same for both words a and b. Moreover, since i and j act on distinct letters in the window notation of p_1, the signs of i and j are the same in both a and b. Finally, since $p_1 \cdot i\ j$ and $p_1 \cdot j\ i$ represent the same permutation, the signs of the generators in p_2 are also the same in both words. It follows that $S(a) = S(b)$ and $neg(a) = neg(b)$. ■

In the next corollaries, which are consequence of Proposition 2.2, we analyze the signs of the letters of a reduced word for $\mathrm{w}_0^{B_n}$ before a given factor. This results will allows us to characterize the behavior of the functions S and neg on words that differ by a single long braid relation.

Corollary 3.4. *Let $i \in [n-1]$, and $a = p_1 \cdot i \cdot p_2$, with p_1, p_2 words on the alphabet $[0,n]$, be a reduced word for $\mathrm{w}_0^{B_n}$. If $i^{p_1} > 0$ then $(i+1)^{p_1} > 0$.*

Corollary 3.5. *Let $a = p_1 \cdot 1\ 0\ 1\ 0 \cdot p_2$, with p_1, p_2 words on the alphabet $[0,n]$, be a reduced word for $\mathrm{w}_0^{B_n}$. Then $1^{p_1} > 0$ and $2^{p_1} > 0$.*

Proof: Since a is reduced, we must have $1^{p_1} = 1^{p_1 \cdot 101} > 0$, and by Corollary 3.4, we get $2^{p_1} > 0$. ∎

Proposition 3.6. *Let $a, b \in R(\mathrm{w}_0^{B_n})$.*

(1) *If $a \overset{L_1}{\sim} b$ then $|S(a) - S(b)| = 1$ and $neg(a) = neg(b)$,*
(2) *If $a \overset{L_2}{\sim} b$ then $|S(a) - S(b)| = 2$, $|neg(a) - neg(b)| = 1$ and $|S(a) + neg(a) - (S(b) + neg(b))| = 1$.*

Proof: Supposing that $a \overset{\mathrm{L}_1}{\sim} b$ and we may write a and b as

$$a = p_1 \cdot i(i+1)i \cdot p_2$$
$$b = p_1 \cdot (i+1)i(i+1) \cdot p_2,$$

for some $i \in [n-1]$ and p_1, p_2 words on the alphabet $[0,n]$.

Since $p_1 \cdot i(i+1)i$ and $p_1 \cdot (i+1)i(i+1)$ represent the same permutation, the sign of the generators in p_2 are the same in both words. This means that the sum function and the negative number will only depend on the sign of the letters in the factor $i(i+1)i$ and $(i+1)i(i+1)$ in a and b, respectively.

The generators corresponding to $i(i+1)i$ and $(i+1)i(i+1)$ will act on the letters i^{p_1}, $(i+1)^{p_1}$ and $(i+2)^{p_1}$, so we only have to check the possible signs of those letters. If $i^{p_1} > 0$ then Corollary 3.4 imply $(i+1)^{p_1} > 0$ and also $(i+2)^{p_1} > 0$, and all generators in each factor $i(i+1)i$ and $(i+1)i(i+1)$ are positive. In this case, we have

$$S(a) - S(b) = i + (i+1) + i - (i+1+i+i+1) = -1. \tag{10}$$

Suppose now that $i^{p_1} < 0$. If only this letter is negative, then exactly two generators in each factor $i(i+1)i$ and $(i+1)i(i+1)$ are negative. In this case, we have

$$S(a) - S(b) = -i - (i+1) + i - (i+1-i-(i+1)) = -1. \tag{11}$$

Finally, if two or more letters are negative, then the generators of the factors $i(i+1)i$ and $(i+1)i(i+1)$ of a and b, respectively, are all negative, and we have

$$S(a) - S(b) = -i - (i+1) - i - (-(i+1) - i - (i+1)) = 1. \tag{12}$$

It follows from equations (10), (11) e (12), that $|S(a) - S(b)| = 1$, and $neg(a) = neg(b)$.

Suppose now that $a \overset{\mathrm{L}_2}{\sim} b$. We may write a and b as

$$a = p_1 \cdot 1\ 0\ 1\ 0 \cdot p_2$$
$$b = p_1 \cdot 0\ 1\ 0\ 1 \cdot p_2,$$

with p_1, p_2 words on the alphabet $[0, n]$. As in the previous case, we only need to check the signs of the generators 1 0 1 0 and 0 1 0 1 of a and b respectively. From Corollary 3.5, only the leftmost 1 of the factor 1 0 1 0 of a is positive, and the two letters 1 of the factor 0 1 0 1 of b are negative. Thus we have

$$S(a) - S(b) = 1 - 1 - (-1) - (-1) = 2$$

and $neg(a) - neg(b) = -1$. It follows that $S(a) + neg(a) - (S(b) + neg(b)) = 1$. ■

Definition 3.7. The *rank* of a reduced word $a \in R(\mathrm{w}_0^{B_n})$ is defined as the integer $rank(a) := S(a) + neg(a)$.

Next proposition shows that the map $rank : R(\mathrm{w}_0^{B_n}) \to \mathbb{N}$ is invariant in each commutation class, and varies by one unit between classes that differ by a single long braid relation. It generalizes the function used in [5] to compute the radius and diameter of the commutation graph for the longest permutation of the symmetric group.

Proposition 3.8. *Let $a, b \in R(\mathrm{w}_0^{B_n})$.*

(1) *If $a \sim b$, then $rank(a) = rank(b)$.*
(2) *If $a \overset{L_1}{\sim} b$ or $a \overset{L_2}{\sim} b$, then $|rank(a) - rank(b)| = 1$.*

Proof: Condition (1) is a consequence of the invariance of the functions S and neg inside a commutation class, proved in Proposition 3.3, and (2) follows from Proposition 3.6. ■

Since the *rank* function is a class invariant, we define the rank of $[a]$ as $rank(a)$. We conclude that $G(\mathrm{w}_0^{B_n})$ is a layered graph, with each layer defined by the rank value, and two classes are connected by an edge only if their ranks differ by one unit.

The ranks of w_0 and w_0^R will have an important role in the computation of the diameter of $G(\mathrm{w}_0^{B_n})$. Since $w_0 = \prod_{i=0}^n i! \prod_{i=1}^n (n)_{n-i+1}$, every non-zero generator transposes a negative letter, and therefore every non-zero generator is negative. Moreover, since there are $n(n+1)$ non-zero generators in w_0, we have

$$\begin{aligned} rank(w_0) &= S(w_0) + neg(w_0) = \\ &= -\frac{n(n+1)(n+2)}{6} - \frac{n(n+1)(2n+1)}{6} + n(n+1) \\ &= \frac{n(n+1)(1-n)}{2}. \end{aligned}$$

In the case of w_0^R, we can write $w_0^R = \left(\prod_{i=1}^n (n)_{n-i+1}\right)^R \cdot \left(\prod_{i=0}^n i!\right)^R$. Since all letters 0 are in the factor $\left(\prod_{i=0}^n i!\right)^R$, it follows that every generator in the factor $\left(\prod_{i=1}^n (n)_{n-i+1}\right)^R$ is positive. Moreover, since $\left(\prod_{i=0}^n i!\right)^R = 012 \cdot \ldots \cdot n \cdot 012 \cdots (n-1) \cdot \ldots \cdot 012 \cdot 01 \cdot 0$, each one of the $\frac{(n+1)n}{2}$ non-zero generators in this factor is negative. Therefore,

$$\begin{aligned} rank(w_0^R) &= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)(n+2)}{6} + \frac{(n+1)n}{2} \\ &= \frac{n(n+1)(n+2)}{6}. \end{aligned}$$

Next, we will show that the rank of the commutative class of w_0 is the minimum for the function *rank*. First, however, we need some auxiliar results.

Definition 3.9. Consider a word $a = p_1 \cdot i! \cdot p_2 \in R(\mathrm{w}_0^{B_n})$, where $i \in [0,n]$, and p_1 and p_2 are words over the alphabet $[0,n]$. We say that the factor $i!$ of a is a *negative factorial* if all of its non-zero generators are negative. We consider $0!$ to be a negative fatorial.

Lemma 3.10. *Let $a = p_1 \cdot i! \cdot p_2 \in R(\mathrm{w}_0^{B_n})$, where $i \in [n]$, and p_1 and p_2 are words over the alphabet $[0,n]$. Then, the factor $i!$ is a negative factorial if and only if $k^{p_1} < 0$ for all $k \in [i]$.*

Proof: The factor $i!$ acts on p_1 by "moving" the letter $(i+1)^{p_1}$ to the first position, that is $(i+1)^{p_1} = 1^{p_1 \cdot i(i-1)\cdots 21}$, and then changing its sign. Thus, by Proposition 2.2 we have $(i+1)^{p_1} > 0$. Since the factorial $i!$ is negative, all its generators are negative and all transposes the letter $(i+1)^{p_1}$. Therefore, $k^{p_1} < 0$ for all $k \in [i]$.

Conversely, if $k^{p_1} < 0$ for all $k \in [i]$, all non-zero generators of $i!$ will act only on negative letters, and thus the factorial $i!$ is negative. ∎

Lemma 3.11. *Let $a = p_1 \cdot j \cdot i! \cdot p_2 \in R(\mathrm{w}_0^{B_n})$, with $i \in [n]$, $j \in [0, i-1]$, and p_1 and p_2 words over the alphabet $[0, n]$, such that a is the minimal word in $[a]$ for the lexicographic order, and $rank(a)$ is the minimum value for the rank function. If the factorial $i!$ is negative, then $j = 0$.*

Proof: For $i = 1$ the result is obvious since a is the minimal word in $[a]$ for the lexicographic order. If $i > 1$, we must have $0 \leq j < i$ and we can use commutation relations to "move" the rightmost letter j of the factor $p_1 \cdot j$ to the rightmost possible position inside the factorial $i!$. Thus,

$$a \sim a' = p_1 \cdot i(i-1) \cdots j(j+1)j \cdot (j-1)! \cdot p_2.$$

Notice that the factor $i!$ being negative forces the generator j to also be negative. Thus, if $j \neq 0$, we have $a' \overset{\mathrm{L}_1}{\sim} b = p_1 \cdot i(i-1) \cdots (j+1)j(j+1) \cdot (j-1)! \cdot p_2$, which by Proposition 3.6 satisfy $rank(b) < rank(a)$, contradicting the minimality of $rank(a)$. Therefore, we must have $j = 0$. ∎

Note that in the conditions of Lemma 3.11, we may write $a = p_1 \cdot j! \cdot i! \cdot p_2$ for some $j \in [n]$. Next lemma shows that if $i!$ is negative, then $j!$ is also negative and satisfy $j < i$.

Lemma 3.12. *Let $a = p_1 \cdot j! \cdot i! \cdot p_2$ with $i, j \in [0, n]$, and p_1 and p_2 words over the alphabet $[0, n]$, such that a is the minimal word in $[a]$ for the lexicographic order, and $rank(a)$ is the minimum value for the rank function. If the factorial $i!$ is negative, then $j!$ is also negative and $j < i$.*

Proof: If $i = 1$ and $j \geq 1$ then we have

$$a = p_1 \cdot j(j-1) \cdots 10 \cdot 10 \cdot p_2 \overset{\mathrm{L}_2}{\sim} p_1 \cdot j(j-1) \cdots 01 \cdot 01 \cdot p_2 = b.$$

But then, the proof of Proposition 3.6 implies that $rank(b) < rank(a)$, contradicting the minimality of $rank(a)$. Suppose now that $j \geq i > 1$. Then, we can write a as

$$a = p_1 \cdot j(j-1) \cdots (i+1)i!i! \cdot p_2.$$

The leftmost factor $i!$ of a is positive since the factor $i!i!$ acts on the letters $(i+1)^{p_1\cdot j(j-1)\cdots(i+1)}$ and $i^{p_1\cdot j(j-1)\cdots(i+1)}$ by changing its signs. We have

$$\begin{aligned} a &\sim p_1 \cdot j(j-1)\cdots(i+1)i(i-1)i\cdot(i-2)!(i-1)!\cdot p_2 = a' \\ &\overset{\mathrm{L}_1}{\sim} p_1 \cdot j(j-1)\cdots(i+1)(i-1)i(i-1)\cdot(i-2)!(i-1)!\cdot p_2 = b. \end{aligned}$$

We already know that the leftmost generator i of the factor $i(i-1)i$ of a' is positive, and the rightmost generator i of this factor is negative, since it belongs to a negative factorial. But then, Proposition 3.6 implies that $rank(b) < rank(a)$, contradicting the minimality of $rank(a)$. Thus, we must have $j < i$.

Finally, note that since $i!$ is negative, by Lemma 3.10 we have $k^{p_1\cdot j!} < 0$ for all $k \in [i]$. Thus, the condition $j < i$ implies that $k^{p_1\cdot j!} < 0$ for all $k \in [j+1]$, proving that $j!$ is negative. ■

Proposition 3.13. *Let $a \in R(\mathrm{w}_0^{B_n})$. If $[a] \neq [w_0]$, then there exists $b \in R(\mathrm{w}_0^{B_n})$ such that $[a] \overset{L_1}{\sim} [b]$ or $[a] \overset{L_2}{\sim} [b]$ and $rank(a) > rank(b)$.*

Proof: We prove the contrapositive assertion. Suppose that $[a]$ is such that for all commutation class $[b]$ with $[a] \overset{L_1}{\sim} [b]$ or $[a] \overset{L_2}{\sim} [b]$ we have $rank(b) > rank(a)$. Moreover, assume without loss of generality that a is the minimal word in the commutation class $[a]$, with respect to the lexicographic order. We will show that in this case, we have $[a] = [\mathrm{w}_0^{B_n}]$. To achieve this aim, we start by showing that $\prod_{i=0}^{n} i!$ is a left factor of a.

It is easy to see that every letter 0 in a belongs to a factorial factor, and so a has exactly $n+1$ factorials. Moreover, the rightmost factorial of a must be negative since it acts on a permutation with only one positive letter in their window notation. Therefore, the successive application of Lemma 3.12 shows that we can write $a = i_0!i_1!\cdots i_n!\cdot p$, where each $i_j!$ is a negative factorial with $i_j < i_{j+1}$, and p is a word over the alphabet $[n]$. It follows that

$$a = 0!1!\cdots n!\cdot p = \prod_{i=0}^{n} i!\cdot p.$$

It is easy to check that the permutation associated to the left factor $\prod_{i=0}^{n} i!$ of a is $[\overline{n+1}, \bar{n}, \ldots, \bar{1}]$. Therefore, the permutation associated with p must be $[n+1, n, \ldots, 1]$. That is, the restriction of p to the set $[n]$ corresponds to the longest permutation in the symmetric group $\mathfrak{S}_{n+1}$. Moreover, since p has $\frac{(n+1)n}{2}$ letters, it is a reduced word for $\mathrm{w}_0^{A_n}$.

All letters in p are negative, and thus by the proof of Proposition 3.6, the word p cannot contain a factor $i(i+1)i$, for $i \in [n-1]$. It was proven in [5] that amongst all reduced words for $\mathrm{w}_0^{A_n}$, the ones having the largest value for the sum function, when restricted to type A words, are precisely the elements of the commutation class of $\prod_{i=1}^{n}(n)_{n-i+1}$, which are reduced words for $\mathrm{w}_0^{A_n}$ without any factor $i(i+1)i$. It follows that $p = \prod_{i=1}^{n}(n)_{n-i+1}$, and therefore, $a = w_0$. ■

Proposition 3.13 shows that any commutation class distinct from $[w_0]$ is linked to a class with a smaller *rank*-value, which means that every commutation class is linked to $[w_0]$. This gives a new proof that $G(\mathrm{w}_0^{B_n})$ is a connected graph [12]. From Proposition 3.8 it follows that $G(\mathrm{w}_0^{B_n})$ is a bipartite graph [8], with the partition of the commutation classes given by the parity of the *rank*-values of its vertices.

Proposition 3.14. *Let $a, b \in R(\mathrm{w}_0^{B_n})$ such that $a \overset{L_1}{\sim} b$ (resp. $a \overset{L_2}{\sim} b$), and $rank(a) < rank(b)$. Then $a^R \overset{L_1}{\sim} b^R$ (resp. $a^R \overset{L_2}{\sim} b^R$), and $rank(a^R) > rank(b^R)$.*

Proof: It is clear that if $a \overset{\mathrm{L}_1}{\sim} b$ (respectively, $a \overset{\mathrm{L}_2}{\sim} b$), then $a^R \overset{\mathrm{L}_1}{\sim} b^R$ (respectively, $a^R \overset{\mathrm{L}_2}{\sim} b^R$). We prove the result for the case $a \overset{\mathrm{L}_1}{\sim} b$. The other case is similar. Supposing that $a \overset{\mathrm{L}_1}{\sim} b$ we can write

$$\begin{aligned} a &= p_1 \cdot i\ (i+1)\ i \cdot p_2 \\ b &= p_1 \cdot (i+1)\ i\ (i+1) \cdot p_2, \end{aligned}$$

for some $i \in [n-1]$ and p_1, p_2 words on the alphabet $[0, n]$. Since we are assuming that $rank(a) < rank(b)$, by the proof of Proposition 3.6 we must have either $(i+1)^{p_1} > 0$ and $(i+2)^{p_1} > 0$. This implies that the generators in p_2 change the signs of $(i+1)^{p_1}$ and $(i+2)^{p_1}$, which means that $(i+1)^{p_2^R} < 0$ and $(i+2)^{p_2^R} < 0$. Again by Proposition 3.6, it follows that $rank(a^R) > rank(b^R)$. ■

From Propositions 3.13 and 3.14, we can conclude that if $[a] \neq [w_0^R]$, then exist $b \in R(\mathrm{w}_0^{B_n})$ such that $[a] \overset{L_1}{\sim} [b]$ or $[a] \overset{L_2}{\sim} [b]$ and $rank(a) > rank(b)$, so $rank(w_0^R)$ is the maximum value for the map *rank*, and that $rank(a) = rank(w_0^R)$ if and only if $a \in [w_0^R]$. This means that the function $rank : C(\mathrm{w}_0^{B_n}) \to \mathbb{N}$ is a rank function for the graph $G(\mathrm{w}_0^{B_n})$, making it into a ranked partially ordered set with a unique minimal and a unique maximal element.

Proposition 3.15. *Let $a \in R(\mathrm{w}_0^{B_n})$. Then, $d([w_0],[a]) = rank(a) - rank(w_0)$ and $d([a],[w_0^R]) = rank(w_0^R) - rank(a)$. In particular,*

$$d([w_0],[w_0^R]) = \frac{n(n+1)(4n-1)}{6}.$$

Proof: Since $G(\mathrm{w}_0^{B_n})$ is a layered connected graph, with edges between classes in consecutive layers, having minimal class $[w_0]$ and maximal class $[w_0^R]$, it follows that $d([w_0],[a]) = rank(a) - rank(w_0)$ and $d([a],[w_0^R]) = rank(w_0^R) - rank(a)$ for any class $[a]$. In particular, we have

$$d([w_0],[w_0^R]) = rank(w_0^R) - rank(w_0) = \frac{n(n+1)(4n-1)}{6}. \qquad \blacksquare$$

Teorema 3.16. *The diameter of $G(\mathrm{w}_0^{B_n})$ is the number*

$$rank(w_0^R) - rank(w_0) = \frac{n(n+1)(4n-1)}{6}.$$

Proof: To prove that the diameter is $\dfrac{n(n+1)(4n-1)}{6}$, we need to show that this number is the maximum distance between any two commutation classes in the graph $G(\mathrm{w}_0^{B_n})$. Consider two classes $[a]$ and $[b]$. Since

$$d([w_0],[a]) + d([a],[w_0^R]) + d([w_0],[b]) + d([b],[w_0]) = 2(rank(w_0^R) - rank(w_0))$$

from Proposition 3.15, using the triangle inequality, we conclude that,

$$\begin{aligned} d([a],[b]) &\leq min\{d([w_0],[a]) + d([w_0],[b]), d([a],[w_0^R]) + d([b],[w_0^R])\} \\ &\leq rank(w_0^R) - rank(w_0), \end{aligned}$$

proving that the distance between any two commutation classes $[a]$ and $[b]$ is at most $rank(w_0^R) - rank(w_0)$. It follows that the maximum eccentricity of a commutation class in the graph $G(\mathrm{w}_0^{B_n})$ is $\dfrac{n(n+1)(4n-1)}{6}$. $\blacksquare$

4. The Radius of $G(\mathrm{w}_0^{B_n})$

Consider the symmetric group $\mathfrak{S}_{2(n+1)}$ on the alphabet $[\pm(n+1)]$ generated by the reflections $\{s_i^A : i \in [\pm n] \cup \{0\}\}$ where s_0^A interchanges $\bar{1}$ and 1, and s_i^A interchanges i and $i+1$, for $i > 0$, or $\bar{i}$ and $\overline{i+1}$, if $i < 0$. The group $\mathfrak{S}_{n+1}^B$ is a sub-group of $\mathfrak{S}_{2(n+1)}$ so we can consider an element $\mathrm{w} \in \mathfrak{S}_{n+1}^B$ as permutation in $\mathfrak{S}_{2(n+1)}$ through the injection

$$\begin{aligned} \phi\colon \mathfrak{S}_{n+1}^B &\to \mathfrak{S}_{2(n+1)} \\ \mathrm{w} &\mapsto \phi(\mathrm{w}) = \left(\overline{\mathrm{w}(n+1)}, \ldots, \overline{\mathrm{w}(1)}, \mathrm{w}(1), \ldots, \mathrm{w}(n+1)\right). \end{aligned}$$

Note that $\phi(\mathrm{w}_0^{B_n}) = \mathrm{w}_0^{A_{2n+1}}$, *i.e.* the longest signed permutation of $\mathfrak{S}_{n+1}^B$ corresponds to the longest permutation in $\mathfrak{S}_{2(n+1)}$. Also, we have $\phi(s_i) = s_i^A s_{\bar{i}}^A$, for $i \in [n]$, and $\phi(s_0) = s_0^A$. If $a = i_1 i_2 \cdots i_l$ is a reduced word in $\mathfrak{S}_{n+1}^B$, we write $\phi(a)$ to denote the corresponding word on $\mathfrak{S}_{2(n+1)}$, that is, $\phi(a) = \widetilde{i_1}\widetilde{i_2}\ldots\widetilde{i_l}$, with $\widetilde{i_j} = i_j\overline{i_j}$ if $i_j \neq 0$ and $\widetilde{i_j} = 0$ otherwise. It is easy to check that $[\phi(a^R)] = [\phi(a)^R]$.

Lemma 4.1. *If $a = i_1 i_2 \cdots i_l$ is a reduced word for $\mathrm{w}_0^{B_n}$, then $\phi(a)$ is a reduced word for $\mathrm{w}_0^{A_{2n+1}}$.*

Proof: Since every non-zero generator of $\mathfrak{S}_{n+1}^B$ corresponds to two generators in $\mathfrak{S}_{2(n+1)}$, and a has exactly $n+1$ generators 0, it follows that the length of $\phi(a)$ is $2(n+1)n + (n+1) = (n+1)(2n+1)$. We conclude that $\phi(a)$ is a reduced word for $\mathrm{w}_0^{A_{2n+1}}$. ■

Given two words $a, b \in R(\mathrm{w}_0^{A_{2n+1}})$ we write $a \overset{\mathrm{L}_A}{\sim} b$ if a and b differ by a long braid relation.

Lemma 4.2. *Let $a, b \in R(\mathrm{w}_0^{B_n})$.*

(1) *If $a \sim b$, then $\phi(a) \sim \phi(b)$.*
(2) *If $a \overset{L_1}{\sim} b$, then the word $\phi(a)$ differ from $\phi(b)$ by two braid relations.*
(3) *If $a \overset{L_2}{\sim} b$, then $\phi(a)$ differ from $\phi(b)$ by four braid relations.*

Proof: We only prove (1) and (3). The proof of (2) is analogous to the proof of (3). Suppose without loss of generality that a and b differ by one commutation. Then, we can write $a = p_1 \cdot i\ j \cdot p_2$ and $b = p_1 \cdot j\ i \cdot p_2$, for some $i, j \in [n]$ such that $|i - j| > 1$, with p_1 and p_2 words on the alphabet $[0, n]$. It follows that

$$\begin{aligned}\phi(a) &= \phi(p_1) \cdot i\ \bar{\imath}\ j\ \bar{\jmath} \cdot \phi(p_2)\\ &\sim \phi(p_1) \cdot j\ i\ \bar{\imath}\ \bar{\jmath} \cdot \phi(p_2)\\ &\sim \phi(p_1) \cdot j\ \bar{\jmath}\ i\ \bar{\imath} \cdot \phi(p_2) = \phi(b).\end{aligned}$$

The case $i = 0$ and $j > 1$ is analogous.

Suppose now that $a \overset{\mathrm{L}_2}{\sim} b$. Then we can write $a = p_1 \cdot 0\ 1\ 0\ 1 \cdot p_2$ and $b = p_1 \cdot 1\ 0\ 1\ 0 \cdot p_2$, with p_1, p_2 words on the alphabet $[0, n]$. It follows that

$$\begin{aligned}\phi(a) &= \phi(p_1) \cdot 0\ 1\ \bar{1}\ 0\ 1\ \bar{1} \cdot \phi(p_2)\\ &\sim \phi(p_1) \cdot 0\ 1\ \bar{1}\ 0\ \bar{1}\ 1 \cdot \phi(p_2)\\ &\overset{L_A}{\sim} \phi(p_1) \cdot 0\ 1\ 0\ \bar{1}\ 0\ 1 \cdot \phi(p_2)\\ &\overset{L_A}{\sim} \phi(p_1) \cdot 1\ 0\ 1\ \bar{1}\ 0\ 1 \cdot \phi(p_2)\\ &\sim \phi(p_1) \cdot 1\ 0\ \bar{1}\ 1\ 0\ 1 \cdot \phi(p_2)\\ &\overset{L_A}{\sim} \phi(p_1) \cdot 1\ 0\ \bar{1}\ 0\ 1\ 0 \cdot \phi(p_2)\\ &\overset{L_A}{\sim} \phi(p_1) \cdot 1\ \bar{1}\ 0\ \bar{1}\ 1\ 0 \cdot \phi(p_2) = \phi(b).\end{aligned}$$

■

Let T_n be the set of all triples $(x, y, z) \in [\pm(n+1)]^3$ such that $x < y < z$. Recalling that each pair of integers $x < y$ in $[\pm(n+1)]^2$ is transposed by the permutation $\mathrm{w}_0^{B_n}$, we can give the following definition (see also [5]):

Definition 4.3. Given $a \in R(\mathrm{w}_0^{B_n})$ and a triple $(x, y, z) \in T_n$ we define $T(a, xyz) = 1$ if, by the action of the generators of a on the identity permutation, in the process of transforming it into $\mathrm{w}_0^{B_n}$, the transposition of the pair (x, y) occurs before the transposition of the pair (y, z), and define $T(a, xyz) = -1$ otherwise.

The number $T(a, xyz)$ can be easily obtained by the line diagram of a. For instance, for the reduced word $a = 101210102$, whose line diagram is displayed in Figure 1, we have $T(a, \bar{2}\bar{1}1) = 1$ and $T(a, \bar{3}12) = -1$. It was proved in [5] that over the graph $G(\mathrm{w}_0^{A_{2n+1}})$, the map T is invariant for words in the same commutation class. Moreover, $[a] \overset{\mathrm{L}_A}{\sim} [b]$ if and only if $T(a, xyz) = T(b, xyz)$ for all triples $(x, y, z) \in T_n$ except for one (see also [6]). From Lemma 4.2 we obtain the following properties.

Lemma 4.4. *Let a, b be two reduced words of $\mathrm{w}_0^{B_n}$.*

- *$[a] = [b]$ if and only if $T(a, xyz) = T(b, xyz)$ for all triples $(x, y, z) \in T_n$.*
- *If $a \overset{L_1}{\sim} b$, then $T(a, xyz) = T(b, xyz)$ for all triples $(x, y, z) \in T_n$ except for two.*
- *If $a \overset{L_2}{\sim} b$, then $T(a, xyz) = T(b, xyz)$ for all triples $(x, y, z) \in T_n$ except for four.*

Lemma 4.5. *Let a be a reduced word. For all triples $(x, y, z) \in T_n$ we have $T(a, xyz) = -T(a, \bar{z}\bar{y}\bar{x})$.*

Proof: Let $(x, y, z) \in T_n$ and suppose without loss of generality that $T(a, xyz) = 1$. First note that $(\bar{z}, \bar{y}, \bar{x}) \in T_n$ because $x < y < z$. Since $T(a, xyz) = 1$, the pair (x, y) was transposed before the pair (y, z). The generators that transpose those pairs also transpose the pairs $(\bar{x}, \bar{y})$ and $(\bar{y}, \bar{z})$, respectively, so the transposition of $(\bar{x}, \bar{y})$ happened before the transposition of $(\bar{y}, \bar{z})$, which implies that $T(a, \bar{z}\bar{y}\bar{x}) = -1$. ■

Proposition 4.6. *Given a word $a \in R(\mathrm{w}_0^{B_n})$ and a triple $(x, y, z) \in T_n$ we have $T(a, xyz) = -T(a^R, xyz)$.*

Proof: The line diagram of a^R corresponds to a horizontal reflection of the symmetric of all components in the line diagram of a. Given a triple (x, y, z), if the pair (x, y) was transposed before the pair (y, z) by the generators of a, then the pair (y, z) was transposed before the pair (x, y) by the generators of a^R, which means that $T(a, xyz) = -T(a^R, xyz)$. The other case is analogous. ■

Given a reduced word $a \in R(\mathrm{w}_0^{B_n})$, it follows from Proposition 4.6 that the length of a path between $[\phi(a)]$ and $[\phi(a^R)]$ is at most equal to $|T_n| = \binom{2n+2}{3} = \frac{(2n+2)(2n+1)2n}{6}$. It was proved in [5] that this number is the distance between $[\phi(a)]$ and $[\phi(a^R)]$ in the graph $G(\mathrm{w}_0^{A_{2n+1}})$. We will use this fact to compute the radius of $G(\mathrm{w}_0^{B_n})$. Consider the set $T'_n = \{(x, y, z) \in T_n : \ x = \bar{y} \text{ or } x = \bar{z} \text{ or } y = \bar{z}\}$.

Lemma 4.7. *Let $a, b \in R(\mathrm{w}_0^{B_n})$.*

(1) *If $a \overset{L_1}{\sim} b$ then $T(a, xyz) = T(b, xyz)$ for all triples $(x, y, z) \in T'_n$.*
(2) *If $a \overset{L_2}{\sim} b$, then $T(a, xyz) = T(b, xyz)$ for all triple $(x, y, z) \in T_n \setminus T'_n$*

Proof: Supposing $a \overset{L_1}{\sim} b$, we may write

$$\begin{aligned} a &= p_1 \cdot i\ (i+1)\ i \cdot p_2 \\ b &= p_1 \cdot (i+1)\ i\ (i+1) \cdot p_2 \end{aligned}$$

for some $i \in [n-1]$ and p_1, p_2 words on the alphabet $[0, n]$. From Lemma 4.4, $T(a, xyz) = T(b, xyz)$ for all triples $(x, y, z) \in T_n$, except for two. In fact, those triples are (x', y', z') and $(\bar{z}', \bar{y}', \bar{x}')$, with $x' = i^{p_1}$, $y' = (i+1)^{p_1}$ $z' = (i+2)^{p_1}$. Since $i \neq 0$, the generators of the factors $i\ (i+1)\ i$ and $(i+1)\ i\ (i+1)$ of a and b, respectively, do not transpose pairs of the form $(\bar{k}, k)$, and therefore the triples (x', y', z') and $(\bar{z}', \bar{y}', \bar{x}')$ are not in T_n'.

Analogous arguments shows that if $a \overset{L_2}{\sim} b$, then $T(a, xyz) = T(b, xyz)$ for all triple $(x, y, z) \in T_n \setminus T_n'$ ■

Lemma 4.8. *If $a \in R(\mathrm{w}_0^{B_n})$, then a and a^R differ by at least $\frac{n(n+1)}{2}$ braid relations of type 2, and at least $\frac{n(n+1)(4n-4)}{6}$ braid relation of type 1.*

Proof: From Proposition 4.6, $T(a, xyz) = -T(a^R, xyz)$ for all $(x, y, z) \in T_n$, and thus by Lemmas 4.4 and 4.7, there are necessarily at least $\frac{1}{4}|T_n'| + \frac{1}{2}|T_n \setminus T_n'|$ long relations in a path between a and a^R. It is easy to check that $|T_n'| = \frac{4n(n+1)}{2}$, so that a path between a and a^R has at least $\frac{n(n+1)}{2}$ long braid relations of type 2. Since

$$|T_n \setminus T_n'| = \frac{(2n+2)(2n+1)2n}{6} - \frac{4n(n+1)}{2} = \frac{2n(n+1)(4n-4)}{6},$$

there are at least $\frac{n(n+1)(4n-4)}{6}$ long braid relations of type 1 in a path between a and a^R. ■

The next result is a consequence of the previous lemma and Theorem 3.16.

Teorema 4.9. *The radius of the graph $G(\mathrm{w}_0^{B_n})$ is $\dfrac{n(n+1)(4n-1)}{6}$.*

Proof: From the previous lemma, for any $a \in R(\mathrm{w}_0^{B_n})$ we have

$$d([a], [a^R]) \geq \frac{n(n+1)}{2} + \frac{n(n+1)(4n-4)}{6} = \frac{n(n+1)(4n-1)}{6}. \qquad (13)$$

Since the right hand side of equation (13) was proved in Theorem 3.16 to be the diameter of $G(\mathrm{w}_0^{B_n})$, it follows that $d([a], [a^R]) = \dfrac{n(n+1)(4n-1)}{6}$. That is, the eccentricity of any commutation class is equal to the diameter, which proves that this number is also the radius of the graph. ■

5. Planarity and Atoms in $G(\mathrm{w}_0^{B_n})$

Figure 2 shows that $G(\mathrm{w}_0^{B_2})$ is a planar graph. We will show that for $n > 2$, $G(\mathrm{w}_0^{B_n})$ is not planar, using Wagners's Theorem [4]. An *edge contraction* of an edge $e = \{u, v\}$ in a graph is the graph that we obtain by combining the vertices a and b into a single vertex, which is adjacent to every vertex that was adjacent to u and v in the original graph. A *graph minor* of a graph is a new graph obtained by deleting vertices, deleting edges, and/or contracting edges of the original graph. Wagner's theorem says that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as minor.

Lemma 5.1. *The graph $G(\mathrm{w}_0^{B_3})$ is not planar.*

Proof: The minor of $G(\mathrm{w}_0^{B_3})$ having as vertices the sets:
$A = \{[1012101032101323]\}$,
$B = \{[1012101232101023]\}$,
$C = \{[1021021023210123]\}$,
$D = \{[1012101023210123]\}$,
$E = \{[1021021032101323], [1021021321010323], [1012101321010323]\}$,
$F = \{[1021021232101023], [1021012132101023], [1210102132101023],$
$[2102102132101023], [2101210132101023], [2101210103210123],$
$[2101021013210123], [0210121013210123], [0210210213210123],$
$[0121010213210123], [0102101213210123], [0102102123210123],$
$[0101210123210123], [0101210132101323], [1010210132101323]\}$,
where E and F are the edge contractions of their vertices, is isomorphic to $K_{3,3}$ since each of the vertices A, B, C is connected to all the vertices D, E, F. By Wagner's theorem, we conclude that $G(\mathrm{w}_0^{B_3})$ is not planar. ■

Figure 3 displays the minor of $G(\mathrm{w}_0^{B_3})$ isomorphic to $K_{3,3}$, where solid edges represent long braid relations of type 1, dashed edges represent long braid relations of type 2, and the vertices are given in previous lemma with E_i and F_j the i-th and j-th elements of the sets E and F, respectively, with $E_0 = E$ and $F_0 = F$.

Lemma 5.2. *If $n > 1$, then the graph $G(\mathrm{w}_0^{B_{n-1}})$ is a sub-graph of $G(\mathrm{w}_0^{B_n})$.*

Proof: Note that given a word $a \in R(\mathrm{w}_0^{B_{n-1}})$, the word $a \cdot n! \cdot 123 \cdots n$ is a reduced word of $\mathrm{w}_0^{B_n}$ since it has length $n^2 + 2n + 1 = (n + 1)^2$, and its

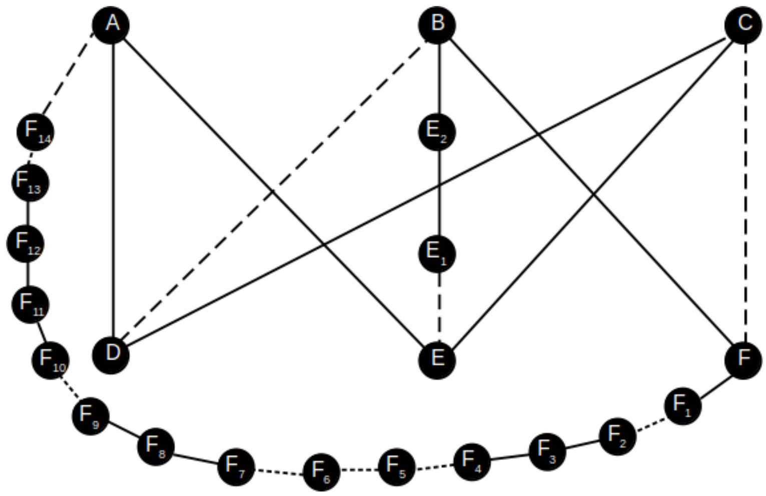

FIGURE 3. Minor of $G(\mathrm{w}_0^{B_3})$ isomorphic to $K_{3,3}$.

associated permutation is $\mathrm{w}_0^{B_n}$. Thus, the sub-graph of $G(\mathrm{w}_0^{B_n})$ formed by the commutation classes of the words $a \cdot n! \cdot 123 \cdots n$ with $a \in R(\mathrm{w}_0^{B_{n-1}})$ is isomorph to $G(\mathrm{w}_0^{B_{n-1}})$. ∎

From Lemmas 5.1 and 5.2 we have the following result.

Teorema 5.3. *For $n > 2$ the graph $G(\mathrm{w}_0^{B_n})$ is not planar.*

The reduced word $010121012 \in R(\mathrm{w}_0^{B_2})$ has the property that $[010121012] = \{010121012\}$, *i.e.* its commutation class has only one element. A commutation class formed by exactly one word is called an *atom*. It is easy to see that a reduced word is an atom if and only if each factor of length two is formed by generators with consecutive indices. In [5] it was proved that $G(\mathrm{w}_0^{A_n})$ has exactly 4 atoms for any $n > 2$. Next, we compute the number of atoms in $G(\mathrm{w}_0^{B_n})$.

Lemma 5.4. *The word $n!(12...n)$ is the only reduced word amongst the set of all words of length $\geq 2n+1$ on the alphabet $[0, n]$ having rightmost and leftmost generator equal to n, at least one generator 0, and where each factor of length 2 is formed by generators with consecutive indices.*

Proof: We start by considering the set of words with only one letter 0 in the conditions of the lemma. If a is such word, then we can write $a = p_1\ 0\ p_2$, with p_1 and p_2 words in the alphabet $[n]$, where the leftmost generator of p_1 is n and the rightmost generator of p_1 is 1, and the leftmost generator of

p_2 is 1 and the rightmost generator of p_2 is n. It was proved in [5] that the increasing (resp. decreasing) word $12\cdots n$ (resp. $n(n-1)\cdots 1$) is the only reduced word amongst the set of all words of length $\geq n$ over the alphabet $[n]$, having leftmost generator 1 (resp. n) and rightmost generator n (resp. 1), and where each factor of length 2 is formed by generators with consecutive indices. Since the lengtht of p_1 and p_2 is $\geq n$ and they are words in the alphabet $[n]$, it follows that $p_1 = n(n-1)\cdots 1$ and $p_2 = 12\cdots n$, and thus $a = n!(12...n)$.

Suppose now that a is a word in the conditions of the lemma with exactly two generators 0. We will must show that in this case a is not reduced. We can write a as

$$a = p_1 \cdot 0 \cdot p_2 \cdot 0 \cdot p_3,$$

with p_1, p_2 and p_3 words on the alphabet $[n]$. Using the previous arguments we have that $p_1 = n(n-1)\cdots 1$ and $p_3 = 12\cdots n$, and so

$$a = n! \cdot p_2 \cdot 0 \cdot (12\cdots n).$$

Since $1^{n!} = \overline{n+1}$, this letter can not be in the first position of the window notation of $n! \cdot p_2$. Thus there exists $k \in [n+1] \setminus \{1\}$ such that $k^{n! \cdot p_2} = \overline{n+1}$, and therefore $k^{n! \cdot p_2 \cdot 01\cdots(k-1)} = \overline{n+1} < (k-1)^{n! \cdot p_2 \cdot 01\cdots(k-1)}$. By Proposition 2.2, the word $n! \cdot p_2 \cdot 01\cdots(k-1)$ is not reduced, and so a is not reduced. If a has more than two generators 0, an inductive argument shows that a is not reduced. ∎

Lemma 5.5. *Let $a \in R(\mathrm{w}_0^{B_n})$. If $a = p_1 \cdot n!(12...n) \cdot p_2$, for some words p_1 and p_2 over the alphabet $[0, n]$, then, $p_1 \cdot p_2 \in R(\mathrm{w}_0^{B_{n-1}})$.*

Proof: We start by proving that $(n+1)^{p_1} = n+1$. If $(n+1)^{p_1} = k$ with $k < n+1$ we have two cases:

- $i^{p_1} = n+1$ for some $i \in [n]$.
- $i^{p_1} = \overline{n+1}$ for some $i \in [n]$.

In the first case we have $i^{p_1 \cdot n(n-1)\cdots(i+1)} = n+1 > (i+1)^{p_1 \cdot n(n-1)\cdots(i+1)} = k$, since the factor $n(n-1)\cdots(i+1)$ "moves" the letter $(n+1)^{p_1} = k$ into position $i+1$, and in the second case, we have $i^{p_1 \cdot n!1\cdots(i-1)} = \bar{k} > (i+1)^{p_1 \cdot n!1\cdots(i-1)} = \overline{n+1}$. Proposition 2.2 implies that in the first case the word $p_1 \cdot n(n-1)\cdots(i+1)i$ is not reduced, and in the second case the word $p_1 \cdot n!1\cdots(i-1)i$ is not reduced, contradicting our assumption. Thus, we must have $(n+1)^{p_1} = n+1$ and the permutation associated to $n!(12...n)$ acts on p_1 changing the sign of the letter $n+1$. This means that $p_1 \cdot p_2$ acts on the identity permutation changing the sign of the letters 1, 2, …, n. Since $p_1 \cdot p_2$ has $(n+1)^2 - (2n+1) = n^2$ letters, we conclude that it is a reduced word for $\mathrm{w}_0^{B_{n-1}}$. ■

Lemma 5.6. *If a is an atom of $G(\mathrm{w}_0^{B_n})$, then there are words p_1, p_2 over the alphabet $[0, n-1]$ such that $a = p_1 \cdot n!(12...n) \cdot p_2$ with $p_1 \cdot p_2 \in R(\mathrm{w}_0^{B_{n-1}})$.*

Proof: Consider the first and last occurrence of a generator n in a. Between those two generators there is a generator 0 that is responsible for the change of the sign of $n+1$. Thus, we can write a as $a = p_1 \cdot n \cdot p_2 \cdot 0 \cdot p_3 \cdot n \cdot p_4$ with p_1, p_4 words in the alphabet $[0, n-1]$ and p_2, p_3 words in the alphabet $[0, n]$. Since a is an atom, each factor of length 2 of a is formed by generators with consecutive indices, and thus, by Lemma 5.4, we have $n \cdot p_2 \cdot 0 \cdot p_3 \cdot n = n!(12 \cdots n)$. Finally, by the previous lemma, we find that $p_1 \cdot p_4$ is a reduced word for $\mathrm{w}_0^{B_{n-1}}$. ■

Definition 5.7. Given a permutation $\mathrm{v} \in \mathfrak{S}_{n+1}$ let $w_{\mathrm{v}} = \prod_{i=1}^{n+1} \mathfrak{f}_{\mathrm{v}(i)}$, where $\mathfrak{f}_{j+1} = j!(12 \cdots j)$ for $j \in [0, n]$. We say that w_{v} is a *ordered word*, and set $O(n) = \{w_{\mathrm{v}} : \ \mathrm{v} \in \mathfrak{S}_{n+1}\}$, the set of all ordered words.

For instance, if $\mathrm{v} = (2, 4, 1, 3) \in \mathfrak{S}_4$, then $w_{\mathrm{v}} = \mathfrak{f}_2\mathfrak{f}_4\mathfrak{f}_1\mathfrak{f}_3 = 101 \cdot 3210123 \cdot 0 \cdot 21012$.

Proposition 5.8. *Every ordered word $w \in O(n)$ is a reduced word of $\mathrm{w}_0^{B_n}$.*

Proof: We have that $\mathfrak{f}_{i+1}$ as lenght $2i + 1$, so every ordered word has length

$$\sum_{i=0}^{n} (2i+1) = 2\sum_{i=0}^{n} i + n + 1 = \frac{2n(n+1)}{2} + n + 1 = (n+1)^2.$$

The permutation associated to $\mathfrak{f}_{i+1}$ acts on a permutation by changing the sign of the $(i+1)$-th letter of its window notation. Since $\mathrm{v} \in \mathfrak{S}_{n+1}$, the factor $\mathfrak{f}_{i+1}$ appears exactly once in w_{v}, for all $i \in [0, n]$, and thus it follows that w_{v} is a reduced word of $\mathrm{w}_0^{B_n}$. ■

Definition 5.9. Given $a \in R(\mathrm{w}_0^{B_n})$ define $ord(a) := (k_1, k_2, \ldots, k_{n+1})$ where $k_i \in [n+1]$ is the i-th letter to change sign in the process of transforming the identity into $\mathrm{w}_0^{B_n}$ using a.

Lemma 5.10. *Let $a, b \in R(\mathrm{w}_0^{B_n})$ such that $a \sim b$. Then $ord(a) = ord(b)$.*

Proof: Suppose, without loss of generality, that $a = p_1 \cdot i\ j \cdot p_2$ and $b = p_1 \cdot j\ i \cdot p_2$, for some $i, j \in [0, n]$ such that $|i - j| > 1$, with p_1 and p_2. If $i \neq 0 \neq j$, then all $n + 1$ generators 0 are in the factors p_1 and p_2 in the alphabet $[0, n]$. In this case, we have $ord(a) = ord(b)$, since the permutation associated with $p_1 \cdot i\ j$ is the same as the one associated with $p_1 \cdot j\ i$. Otherwise, or i or j is equal to 0 and in that case $ord(a) = ord(b)$ since i and j acts on distinct letters in the window notation of p_1. ∎

Proposition 5.11. *The set $O(n)$ has $(n+1)!$ reduced words, each belonging to a different commutation class.*

Proof: It is easy to see that by construction, $|O(n)| = |\mathfrak{S}_{n+1}| = (n+1)!$. To show that each ordered word belong to a different commutation class just notice that the order in which the letters change sign is different for each ordered word, *i.e.* $ord(w_{\mathrm{u}}) = ord(w_{\mathrm{v}})$ if and only if $\mathrm{u} = \mathrm{v}$. The result now follows from Lemma 5.10. ∎

Teorema 5.12. *For $n \geq 1$ there are exactly two atoms in $G(\mathrm{w}_0^{B_n})$, namely w_{id} and $w_{\mathrm{w}_0^{A_n}}$.*

Proof: We proceed by induction. The graph $G(\mathrm{w}_0^{B_1})$ has only two commutation classes, [0101] and [1010], and both are atoms, with $0101 = w_{id}$ and $1010 = w_{\mathrm{w}_0^{A_1}}$. Assume the result for $n = k - 1$, and consider the graph $G(\mathrm{w}_0^{B_k})$. From Lemma 5.6, if a is an atom of $G(\mathrm{w}_0^{B_k})$, then $a = p_1 k!(12 \cdots k)p_2$ with $p_1 p_2$ a reduced word of $\mathrm{w}_0^{B_{k-1}}$. Since a is an atom, the rightmost generator of p_1 and the leftmost generator of p_2 must both be equal to $k-1$. But this implies that $p_1 p_2$ would not be reduced, so either p_1 or p_2 must be the empty word. If p_1 is the empty word, then $a = k!(12 \cdots k)p_2$ with p_2 a reduced word of $\mathrm{w}_0^{B_{k-1}}$. Each factor of length 2 of a is formed by generators with consecutive indices, which implies that p_2 is an atom of $G(\mathrm{w}_0^{B_{k-1}})$. By the induction hypothesis, p_2 is equal to $w_{\mathrm{w}_0^{A_{k-1}}}$, so that $a = k!(12 \cdots k)w_{\mathrm{w}_0^{A_{k-1}}} = w_{\mathrm{w}_0^{A_k}}$. If p_2 is the empty word, an analogous argument shows that $a = w_{id}$. ∎

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