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ZERO-INFLATED BINOMIAL INTEGER-VALUED ARCH MODELS FOR TIME SERIES

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Abstract: An integer-valued ARCH process with a conditional zero-inflated binomial distribution is introduced. Stationarity, ergodicity and the autocovariance structure are studied as well as the estimation of parameters by conditional maximum likelihood. Numerical studies and an application to the poliomyelitis numbers recorded in USA illustrate the performance of this model when compared with others.

Keywords: Binomial ARCH model, time series, zero-inflation.
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1. Introduction

The integer-valued models present in literature have underlying distributions which allow the count time series in study to be zero, which means that zero is a possible value of the model.

It may happen, however, that the expected number of zeros according to the underlying distribution is not compatible with those actually occurring. We have in this case an inflation or deflation situation of the zero value and, in order to correct this situation, we have to provide for the possibility to mix the underlying distribution with a point probability. This is for example the case of integer-valued zero inflated models, studied in particular by Zhu ([9]), Lee, Lee and Chen ([4]) and Gonçalves, Mendes Lopes and Silva ([2]), involving Poisson, generalized Poisson, negative binomial and compound Poisson distributions, all of them with support $\mathbb{N}_0$.

Motivated by the wide presence of integer-valued time series with a finite range in diverse real-life applications, such as the monitoring of computer tools with n workstations (Weiß, [8]) or the daily number of hours in which the prices of electricity of Portugal and Spain are different ([3]), Ristic, Weiß and Janjić ([5]) introduce a binomial integer-valued ARCH model for time series with finite range $\{0, 1, ..., n\}$ where n denotes the (known) upper limit.

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So, in order to accommodate inflation at zero and finite support in the same work environment, we propose in this paper an integer-valued ARCH process with a conditional zero-inflated binomial distribution.

In Section 2 we recall the definition of the binomial integer-valued with autoregressive conditional heteroskedasticity process ([5]), denoted as BINARCH, and we introduce the zero-inflated binomial INARCH model definition, denoted as ZI_BINARCH. The probabilistic structure of this family of stochastic processes is discussed, including conditional and unconditional moments, strict and weak stationarity, ergodicity as well as the statement of Yule-Walker equations for the autocorrelation fonction of the ZI_BINARCH process. Section 3 includes the estimation of the parameters of the ZI_BINARCH model by conditional maximum likelihood and a simulation study that illustrates the properties of the estimation methodology in moderate and large samples. Section 4 concludes with a real-data application consisting of monthly counts of poliomyelitis cases recorded in the United States.

2. Zero-inflated Binomial INARCH processes

2.1. Binomial INARCH process. Let $X = (X_t,\, t \in \mathbb{Z})$ be a nonnegative integer-valued stochastic process and, for $t \in \mathbb{Z}$, let $\underline{X}_t$ denote the $\sigma-$ field generated by $(X_{t-j}, j \geq 0)$.

Definition 2.1. ([5]) *The process X follow a binomial integer-valued ARCH model if*

$$X_t|\underline{X}_{t-1} \sim Bin\,(n, \alpha_t)\,,\ t \in \mathbb{Z}, \tag{1}$$

with $n \in \mathbb{N}$ and

$$\alpha_t = a_0 + \frac{1}{n}\sum_{i=1}^{p} a_i X_{t-i},\ \ t \in \mathbb{Z}, \tag{2}$$

for constants $a_0 > 0$, $a_i \geq 0$, $i = 1, ..., p$, $p \in \{1, 2, ...\}$ such that

$$a_0 + \sum_{i=1}^{p} a_i < 1.$$

In a briefly way, the model is denoted $BINARCH(p)$.

So, from (1) we have

$$\begin{aligned} P\left(X_t = x \mid \underline{X}_{t-1}\right) &= \binom{n}{x} \alpha_t^x \left(1-\alpha_t\right)^{n-x} \ \mathbb{I}_{\{0,1,\ldots,n\}}\left(x\right) \\ E\left(X_t | \underline{X}_{t-1}\right) &= n\alpha_t \\ V\left(X_t | \underline{X}_{t-1}\right) &= n\alpha_t \left(1-\alpha_t\right). \end{aligned}$$

The $BINARCH$ model was studied by Ristic *et al* ([5]), stating in particular its ergodicity and strict and second order stationarity.

2.2. Zero inflated Binomial INARCH process. Now we introduce the definition of the zero inflated binomial integer-valued autoregressive conditional heteroscedastic model, briefly $ZI\_BINARCH(p)$.

Definition 2.2. *The stochastic process*

$$Z = \left(Z_t, t \in \mathbb{Z}\right)$$

follows a $ZI\_BINARCH(p)$ model if, for any $t \in \mathbb{Z}$,

$$P(Z_t = z \mid \underline{Z}_{t-1}) = \left[\beta + (1-\beta)\left(1-\beta_t\right)^n\right] \mathbb{I}_{\{0\}}\left(z\right) + (1-\beta) \binom{n}{z} \beta_t^z \ \left(1-\beta_t\right)^{n-z} \ \mathbb{I}_{\{1,\ldots,n\}}\left(z\right) \tag{3}$$

with $\underline{Z}_t$ the $\sigma-$ field generated by $\left(Z_{t-i},\ i \geq 0\right)$, n a positive integer, $\beta \in [0,1]$ and

$$\beta_t = b_0 + \frac{1}{n}\sum_{i=1}^{p} b_i Z_{t-i} \tag{4}$$

for constants $b_0 > 0$, $b_i \geq 0$, $i = 1, ..., p$, $p \in \{1, 2, ...\}$ such that

$$b_0 + \sum_{i=1}^{p} b_i < 1$$

so that $\beta_t \in \left]0,1\right[$ since the support of Z is $\{0, 1, ..., n\}$. If $\beta = 0$, the model reduces to the previous $BINARCH$ one.

In order to illustrate the probabilistic changes related with the zero inflation, we present in Figure 1 the histograms and basic descriptives of a series X following a $BINARCH(1)$ model with $\alpha_t = a_0 + \frac{1}{n}a_1 X_{t-1}$, and of a Z process following a $ZI\_BINARCH(1)$ model with $\beta_t = b_0 + \frac{1}{n}b_1 Z_{t-1}$, where $n = 5$, $a_0 = b_0 = 0.5$, $\beta = 0.5$ and $a_1 = b_1 = 0.4$. We note that the parameter β is quite large and the zero inflation is naturally notorious.

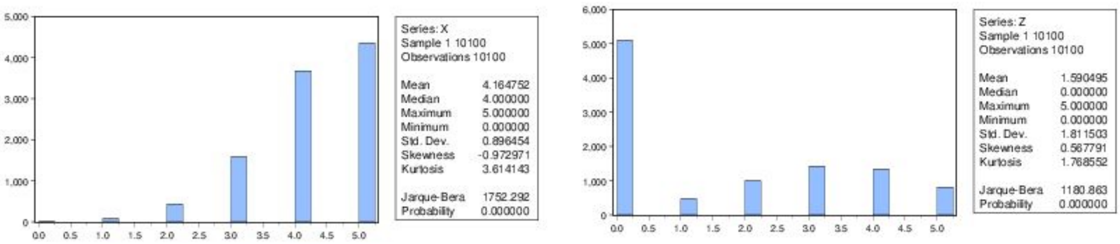


FIGURE 1. Histograms and descriptives of a $BINARCH(1)$ process X with $\alpha_t = a_0 + \frac{1}{n} a_1 X_{t-1}$, and of a $ZI\_BINARCH(1)$ process Z with $\beta_t = b_0 + \frac{1}{n} b_1 Z_{t-1}$ where $n = 5$, $a_0 = b_0 = 0.5$, $\beta = 0.5$ and $a_1 = b_1 = 0.4$.

2.3. Probabilistic structure of $ZI\_BINARCH(p)$ processes. The general probabilistic study presented in this Section includes the study of the strict and weak stationarity of $ZI\_BINARCH(p)$ processes as well as some properties of their temporal characteristics.

2.3.1 Conditional Moments

We have

$$E(Z_t|\underline{Z}_{t-1}) = (1-\beta)\, n\beta_t$$

and, analogously,

$$E(Z_t^2|\underline{Z}_{t-1}) = (1-\beta) \sum_{z=0}^{n} z^2 P(Z_t = z|\underline{Z}_{t-1}) = (1-\beta) \left[n\beta_t (1-\beta_t) + n^2\beta_t^2 \right],$$

deducing that

$$V(Z_t|\underline{Z}_{t-1}) = (1-\beta)\, n\beta_t \left[1 - \beta_t (1 - n\beta)\right].$$

The analysis of the conditional dispersion coefficient

$$\frac{E(Z_t|\underline{Z}_{t-1})}{V(Z_t|\underline{Z}_{t-1})} = 1 + \frac{\beta_t (1-n\beta)}{1 - \beta_t (1-n\beta)}$$

shows that the model is conditionally equal dispersed if and only if $(1 - n\beta) = 0$ (as, for example, if $n = 2$ and $\beta = 0.5$), it is overdispersed (resp. underdispersed) if and only if $(1 - n\beta) > 0$ (resp., $(1 - n\beta) < 0$).

2.3.2 Strict and weak stationarity and ergodicity of Z

Property 2.1. *The $ZI\_BINARCH(p)$ process Z given by (3) and (4) is ergodic, strictly stationary and also second order stationary.*

***Proof*.** The proof follows the lines of Ristic *et al* ([5]). We use the pth-order Markov process $Z=(Z_t, t\in\mathbb{Z})$ to build the vector-valued process

$$\mathbf{Z}_t=(Z_t,...,Z_{t-p+1}),t\in\mathbb{Z}$$

which is a finite first order Markov process, that is, a finite Markov chain. Its *one*-step-ahead transition probabilities are, with $\mathbf{k}=(k_1,...,k_p)$, $\mathbf{l}=(l_1,...,l_p)$ and $\delta_{ij}=\begin{cases}1, & i=j\\ 0 & i\neq j\end{cases}$,

$$\begin{aligned}&p_{\mathbf{k}|\mathbf{l}}=P(\mathbf{Z}_t=\mathbf{k}|\mathbf{Z}_{t-1}=\mathbf{l})=\delta_{k_2l_1}...\delta_{k_pl_{p-1}}P(Z_t=k_1|\mathbf{Z}_{t-1}=\mathbf{l})\\ &=\delta_{k_2l_1}...\delta_{k_pl_{p-1}}\left\{\left[\beta+(1-\beta)\left(1-\widetilde{\beta}_0\right)^n\right]\mathbb{I}_{\{0\}}(k_1)+(1-\beta)\binom{n}{k_1}\widetilde{\beta}_0^{k_1}\left(1-\widetilde{\beta}_0\right)^{n-k_1}\mathbb{I}_{\{1,...,n\}}(k_1)\right\}\end{aligned}$$

with $\widetilde{\beta}_0=b_0+\frac{1}{n}\sum_{i=1}^{p}b_il_i$.

Let $\mathbf{k}_i=(k_i,...,k_{i+p-1})$. The k-step-ahead transition probabilities are

$$\begin{aligned}&P(\mathbf{Z}_t=\mathbf{k}_0|\mathbf{Z}_{t-p}=\mathbf{k}_p)\\ =&\prod_{j=0}^{p-1}P(Z_{t-j}=k_j|Z_{t-j-1}=k_{j+1},...,Z_{t-j-p}=k_{j+p})\\ =&\prod_{j=0}^{p-1}\left\{\left[\beta+(1-\beta)\left(1-\widetilde{\beta}_j\right)^n\right]\mathbb{I}_{\{0\}}(k_j)+(1-\beta)\binom{n}{k_j}\widetilde{\beta}_j^{k_j}\left(1-\widetilde{\beta}_j\right)^{n-k_j}\mathbb{I}_{\{1,...,n\}}(k_j)\right\}\end{aligned}$$

where $\widetilde{\beta}_j=b_0+\frac{1}{n}\sum_{i=1}^{p}b_ik_{j+i}$, and they are larger than zero, since $0<\beta<1$ and $b_0>0$. The finite Markov chain $\mathbf{Z}_t$ is primitive, so irreductible and aperiodic and therefore ergodic with a unique stationary distribution ([7]). Since the range of Z_t is finite, any moments exist and so the strict stationarity of Z implies its second-order stationarity. ■

We note that any moment of β_t also exist.

2.3.3 Moments of $ZI\_BINARCH(p)$ processes

We have

$$E(Z_t)=E\left[E(Z_t|\underline{Z}_{t-1})\right]=E\left[(1-\beta)n\beta_t\right]=(1-\beta)nE(\beta_t).$$

Since

$$\begin{aligned}
& E\left(\beta_t\right)=E\left[b_0+\tfrac{1}{n}\sum_{i=1}^{p}b_iZ_{t-i}\right]=b_0+\tfrac{1}{n}\sum_{i=1}^{p}b_i\left(1-\beta\right)nE\left(\beta_t\right) \\
\Leftrightarrow\ & E\left(\beta_t\right)\left[1-\left(1-\beta\right)\sum_{i=1}^{p}b_i\right]=b_0 \\
\Leftrightarrow\ & E\left(\beta_t\right)=\frac{b_0}{1-(1-\beta)\sum_{i=1}^{p}b_i},
\end{aligned}$$

we also have

$$\begin{aligned}
V\left(Z_t\right) &= E\left[V(Z_t|\underline{Z}_{t-1})\right]+V\left[E(Z_t|\underline{Z}_{t-1})\right] \\
&= \left(1-\beta\right)nE\left[\beta_t\left[1-\beta_t\left(1-n\beta\right)\right]\right]+V\left[\left(1-\beta\right)n\beta_t\right]
\end{aligned}$$

that is,

$$V\left(Z_t\right)=\left(1-\beta\right)n\left[E\left(\beta_t\right)-\left(1-n\beta\right)E\left(\beta_t^2\right)\right]+\left(1-\beta\right)^2n^2V\left(\beta_t\right) \tag{5}$$

Let us evaluate now the dispersion coefficient $\frac{E(Z_t)}{V(Z_t)}$ in the case $p=1$, where $\beta_t=b_0+\frac{1}{n}b_1Z_{t-1}$.

In this case $E\left(Z_t\right)=\left(1-\beta\right)n\frac{b_0}{1-(1-\beta)b_1}$. Moreover,

$$E\left(\beta_t\right)=\frac{b_0}{1-\left(1-\beta\right)b_1}$$

and $V\left(\beta_t\right)=\left(\frac{b_1}{n}\right)^2V\left(Z_{t-1}\right)$. We deduce that

$$E\left(\beta_t^2\right)=\left(\tfrac{b_1}{n}\right)^2V\left(Z_{t-1}\right)+\left[\tfrac{b_0}{1-(1-\beta)b_1}\right]^2$$

and, replacing in (5) we obtain $V\left(Z_{t-1}\right)=\gamma_Z\left(0\right)$ given by

$$\begin{aligned}
\gamma_Z\left(0\right)=\left(1-\beta\right)n&\left\{\frac{b_0}{1-\left(1-\beta\right)b_1}-\left(1-n\beta\right)\left(\frac{b_1}{n}\right)^2\gamma_Z\left(0\right)-\left(1-n\beta\right)\left[\frac{b_0}{1-\left(1-\beta\right)b_1}\right]^2\right\} \\
&+\left(1-\beta\right)^2n^2\left(\frac{b_1}{n}\right)^2\gamma_Z\left(0\right) \\
\Leftrightarrow \gamma_Z\left(0\right)&\left\{1+\left(1-\beta\right)\left(1-n\beta\right)\frac{b_1^2}{n}-\left(1-\beta\right)^2b_1^2\right\} \\
&=\left(1-\beta\right)n\frac{b_0}{1-\left(1-\beta\right)b_1}\left[1-\left(1-n\beta\right)\frac{b_0}{1-\left(1-\beta\right)b_1}\right],
\end{aligned}$$

that is,

$$\begin{aligned}\gamma_Z(0) &= \frac{(1-\beta)\,n^2\frac{b_0}{[1-(1-\beta)b_1]^2}\left[1-(1-\beta)\,b_1-(1-n\beta)\,b_0\right]}{n+(1-\beta)(1-n\beta)\,b_1^2-n\,(1-\beta)^2\,b_1^2} \\ &= \frac{(1-\beta)\,n^2\frac{b_0}{[1-(1-\beta)b_1]^2}\left[1-(1-\beta)\,b_1-(1-n\beta)\,b_0\right]}{n+(1-\beta)\,b_1^2\,(1-n)}.\end{aligned}$$

The inverse of the dispersion coefficient is then equal to

$$\frac{V(Z_t)}{E(Z_t)}=\frac{\frac{n}{1-(1-\beta)b_1}\left[1-(1-n\beta)\,b_0-(1-\beta)\,b_1\right]}{(1-\beta)(1-n\beta)\,b_1^2+n\left[1-(1-\beta)^2\,b_1^2\right]}.$$

So, the model $ZI\_BINARCH(1)$ is equidispersed if and only if

$$\begin{aligned}&\tfrac{n}{1-(1-\beta)b_1}\left[1-(1-n\beta)\,b_0-(1-\beta)\,b_1\right]=(1-\beta)(1-n\beta)\,b_1^2+n\left[1-(1-\beta)^2\,b_1^2\right]\\ \Leftrightarrow\;& 1-(1-\beta)\,b_1-(1-n\beta)\,b_0=\tfrac{1-(1-\beta)b_1}{n}\left\{(1-\beta)(1-n\beta)\,b_1^2+n\left[1-(1-\beta)^2\,b_1^2\right]\right\}\\ \Leftrightarrow\;& -(1-n\beta)\,b_0=-1+(1-\beta)\,b_1+\tfrac{1-(1-\beta)b_1}{n}\left\{(1-\beta)(1-n\beta)\,b_1^2+n\left[1-(1-\beta)^2\,b_1^2\right]\right\}\\ \Leftrightarrow\;& b_0=\tfrac{1}{(1-n\beta)}-\tfrac{(1-\beta)b_1}{(1-n\beta)}-\tfrac{1-(1-\beta)b_1}{n(1-n\beta)}\left\{(1-\beta)(1-n\beta)\,b_1^2+n\left[1-(1-\beta)^2\,b_1^2\right]\right\}\end{aligned}$$

and, depending on the parameter values, it may also accomodate under or over dispersion.

Theorem 2.1. *The autocovariance function,* $\gamma_Z(k)=Cov(Z_t,Z_{t-k})$, $k\geq 0$, *of the* $ZI\_BINARCH(p)$ *process* Z *given by (3) and (4) satisfies the equations*

$$\gamma_Z(0)=\mu_Z-\tfrac{(1-n\beta)}{(1-\beta)n}\mu_Z^2+\left(1-\tfrac{1}{n}\right)\sum_{i=1}^{p}b_i\gamma_Z(i)$$
$$\gamma_Z(k)=(1-\beta)\sum_{i=1}^{p}b_i\gamma_Z(|k-i|)\,,\ k\geq 1,$$

where $\mu_Z=E(Z_t)$.

Proof. For $k\geq 0$,

$$\begin{aligned}\gamma_Z(k) &= Cov(Z_t,Z_{t-k})=E\left[(Z_t-\mu_Z)(Z_{t-k}-\mu_Z)\right]\\ &= E\left[(Z_{t-k}-\mu_Z)\,E(Z_t-\mu_Z|\underline{Z}_{t-1})\right]\\ &= E\left[(Z_{t-k}-\mu_Z)\left((1-\beta)\,n\beta_t-\mu_Z|\underline{Z}_{t-1}\right)\right]\end{aligned}$$

since $E(Z_t|\underline{Z}_{t-1})=(1-\beta)\,n\beta_t$. So,

$$
\begin{aligned}
\gamma_Z(k) &= Cov\left((1-\beta)\, n\beta_t, Z_{t-k}\right)\\
&= (1-\beta)\, nCov\left(\beta_t, Z_{t-k}\right)\\
&= (1-\beta)\, nCov\left(b_0+\frac{1}{n}\sum_{i=1}^{p} b_i Z_{t-i}, Z_{t-k}\right)\\
&= (1-\beta)\sum_{i=1}^{p} b_i\gamma_Z\left(|k-i|\right).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
V(\beta_t) &= V\left(b_0+\frac{1}{n}\sum_{i=1}^{p} b_i Z_{t-i}\right) = \frac{1}{n^2}\sum_{i=1}^{p} b_i\sum_{j=1}^{p} b_j\gamma_Z\left(|j-i|\right)\\
&= \frac{1}{(1-\beta)\, n^2}\sum_{i=1}^{p} b_i\gamma_Z(i)
\end{aligned}
$$

and so, from equality (5),

$$
\begin{aligned}
V(Z_t) &= (1-\beta)\, n\left[E(\beta_t)-(1-n\beta)\, E\left(\beta_t^2\right)\right]+(1-\beta)^2 n^2 V(\beta_t)\\
&= E(Z_t)-(1-\beta)\, n(1-n\beta)\left[V(\beta_t)+\left(E(\beta_t)\right)^2\right]+(1-\beta)^2 n^2 V(\beta_t)\\
&= E(Z_t)-(1-\beta)\, n(1-n\beta)\frac{\left(E(Z_t)\right)^2}{(1-\beta)^2 n^2}+\left[(1-\beta)^2 n^2-(1-\beta)\, n(1-n\beta)\right]V(\beta_t)\\
&= E(Z_t)-(1-n\beta)\frac{\left(E(Z_t)\right)^2}{(1-\beta)\, n}+(1-\beta)\, n\left[(1-\beta)\, n-1+\beta n\right]\frac{1}{(1-\beta)\, n^2}\sum_{i=1}^{p} b_i\gamma_Z(i)\\
&= \mu_Z-\frac{(1-n\beta)}{(1-\beta)\, n}\mu_Z^2+\left(1-\frac{1}{n}\right)\sum_{i=1}^{p} b_i\gamma_Z(i).
\end{aligned}
$$

$\blacksquare$

3. Parameter estimation of $ZI\_BINARCH(p)$ processes

3.1. Conditional maximum likelihood. Using the conditional maximum likelihood (CML) methodology we estimate in this Section the parameter vector $\Theta = (b_0, b_1, ..., b_p, \beta)^T = (\theta_0, \theta_1, ..., \theta_{p+1})^T$ of a stochastic process Z following a $ZI\_BINARCH(p)$ model, based on an N-sample $(Z_1, ..., Z_N)$. So, in the following we consider n known.

The probability function of the conditioned law is, from equations (3) and (4),

$$P\left(Z_t = z \mid \underline{Z}_{t-1}\right) = \left[\beta + (1-\beta)\left(1-\beta_t\right)^n\right]\ \mathbb{I}_{\{0\}}\left(z\right) + (1-\beta)\ \binom{n}{z}\ \beta_t^z\ \left(1-\beta_t\right)^{n-z}\ \mathbb{I}_{\{1,\dots,n\}}\left(z\right)$$

where $\beta_t = b_0 + \frac{1}{n}\sum\limits_{i=1}^{p} b_i Z_{t-i}$.

The conditional likelihood function associated to the N observations $Z_1, \dots, Z_N$ conditionally to the initial values is

$$L\left(\Theta\right) = \prod_{t=p+1}^{N} \left[\beta + (1-\beta)\left(1-\beta_t\right)^n\right]^{\ \mathbb{I}_{\{0\}}(Z_t)} \left[(1-\beta)\ \binom{n}{Z_t}\ \beta_t^{Z_t}\ \left(1-\beta_t\right)^{n-Z_t}\right]^{\ \mathbb{I}_{\{1,\dots,n\}}(Z_t)}.$$

The log-likelihood function is then given by

$$\begin{aligned}\mathcal{L}\left(\Theta\right) &= \log L\left(\Theta\right)\\ &= \sum_{t=p+1}^{N} \log\left\{\left[\beta + (1-\beta)\left(1-\beta_t\right)^n\right]^{\ \mathbb{I}_{\{0\}}(Z_t)} \left[(1-\beta)\ \binom{n}{Z_t}\ \beta_t^{Z_t}\ \left(1-\beta_t\right)^{n-Z_t}\right]^{\ \mathbb{I}_{\{1,\dots,n\}}(Z_t)}\right\}\\ &= \sum_{t=p+1}^{N} l_t\left(\Theta\right)\end{aligned}$$

with

$$\begin{aligned}l_t\left(\Theta\right) = &\ \mathbb{I}_{\{0\}}\left(Z_t\right)\log\left[\beta + (1-\beta)\left(1-\beta_t\right)^n\right] + \\ &+\ \mathbb{I}_{\{1,\dots,n\}}\left(Z_t\right)\left[\log\left(1-\beta\right) + \log\binom{n}{Z_t} + Z_t\log\beta_t + \left(n-Z_t\right)\log\left(1-\beta_t\right)\right]\end{aligned}$$

The first derivatives of l_t in order to θ_i, $i = 0, \dots, p$ are

$$\frac{\partial l_t}{\partial \theta_i} = -\ \mathbb{I}_{\{0\}}\left(Z_t\right)\frac{(1-\beta)\,n\left(1-\beta_t\right)^{n-1}}{\beta + (1-\beta)\left(1-\beta_t\right)^n}\frac{\partial \beta_t}{\partial \theta_i} + \mathbb{I}_{\{1,\dots,n\}}\left(Z_t\right)\left(\frac{Z_t}{\beta_t} - \frac{n-Z_t}{1-\beta_t}\right)\frac{\partial \beta_t}{\partial \theta_i} \tag{6}$$

and

$$\frac{\partial l_t}{\partial \theta_{p+1}} = \ \mathbb{I}_{\{0\}}\left(Z_t\right)\frac{1-\left(1-\beta_t\right)^n}{\beta + (1-\beta)\left(1-\beta_t\right)^n} -\ \mathbb{I}_{\{1,\dots,n\}}\left(Z_t\right)\left(\frac{1}{1-\beta}\right), \tag{7}$$

with

$$\frac{\partial\beta_t}{\partial\theta_0}=1, \qquad \frac{\partial\beta_t}{\partial\theta_i}=Z_{t-i},\ i=1,...,p, \qquad \frac{\partial\beta_t}{\partial\theta_{p+1}}=0.$$

As the derivative in order to θ_j of $\dfrac{(1-\beta_t)^{n-1}}{\beta+(1-\beta)(1-\beta_t)^n}$ is equal to

$$\frac{-(n-1)(1-\beta_t)^{n-2}\frac{\partial\beta_t}{\partial\theta_j}\left[\beta+(1-\beta)(1-\beta_t)^n\right]+(1-\beta)\,n\,(1-\beta_t)^{n-1}\frac{\partial\beta_t}{\partial\theta_j}(1-\beta_t)^{n-1}}{\left[\beta+(1-\beta)(1-\beta_t)^n\right]^2}$$

$$=\ (1-\beta_t)^{n-2}\frac{-n\beta+\beta+(1-\beta)(1-\beta_t)^n}{\left[\beta+(1-\beta)(1-\beta_t)^n\right]^2}\frac{\partial\beta_t}{\partial\theta_j},$$

the second derivatives are, for $0\le i,j\le p$,

$$\begin{aligned}\frac{\partial^2 l_t}{\partial\theta_j\partial\theta_i} &= -\,\mathbb{I}_{\{0\}}(Z_t)(1-\beta)\,n\left\{(1-\beta_t)^{n-2}\frac{-n\beta+\beta+(1-\beta)(1-\beta_t)^n}{\left[\beta+(1-\beta)(1-\beta_t)^n\right]^2}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}\right\}+\\ &\quad+\ \mathbb{I}_{\{1,\ldots,n\}}(Z_t)\left\{\frac{\partial\beta_t}{\partial\theta_i}\left[-\frac{Z_t}{\beta_t^2}\frac{\partial\beta_t}{\partial\theta_j}-\frac{n-Z_t}{(1-\beta_t)^2}\frac{\partial\beta_t}{\partial\theta_j}\right]\right\}\end{aligned}$$

that is,

$$\begin{aligned}\frac{\partial^2 l_t}{\partial\theta_j\partial\theta_i} &= -\,\mathbb{I}_{\{0\}}(Z_t)(1-\beta)\,n\,(1-\beta_t)^{n-2}\frac{-n\beta+\beta+(1-\beta)(1-\beta_t)^n}{\left[\beta+(1-\beta)(1-\beta_t)^n\right]^2}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}\\ &\quad-\,\mathbb{I}_{\{1,\ldots,n\}}(Z_t)\left[\frac{Z_t}{\beta_t^2}+\frac{n-Z_t}{(1-\beta_t)^2}\right]\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}\end{aligned}\tag{8}$$

Moreover, for $i=1,...,p$,

$$\frac{\partial^2 l_t}{\partial\theta_{p+1}\partial\theta_i}=\ \mathbb{I}_{\{0\}}(Z_t)\,n\,(1-\beta_t)^{n-1}\frac{\beta+(1-\beta)(1-\beta_t)^n+\left[1-(1-\beta_t)^n\right](1-\beta)}{\left[\beta+(1-\beta)(1-\beta_t)^n\right]^2}\frac{\partial\beta_t}{\partial\theta_i}\tag{9}$$

and

$$\frac{\partial^2 l_t}{\partial\theta_{p+1}^2}=-\,\mathbb{I}_{\{0\}}(Z_t)\frac{\left[1-(1-\beta_t)^n\right]^2}{\left[\beta+(1-\beta)(1-\beta_t)^n\right]^2}-\ \mathbb{I}_{\{1,\ldots,n\}}(Z_t)\frac{1}{(1-\beta)^2},\tag{10}$$

If N is large enough, the distribution of the conditional maximum likelihood estimator, $\widehat{\Theta}$, may be approached by the following distribution:

$$\widehat{\Theta}\ \dot{\sim}\ \mathcal{N}\left(\Theta_0,\left[N\mathcal{I}(\Theta_0)\right]^{-1}\right)$$

where $\mathcal{I}(\Theta_0)$ is the information matrix evaluated at Θ_0, the true value of Θ.

Let us present now consistent estimates for the information matrix. Taking expectations in both sides of the equation (8) we obtain

$$
\begin{aligned}
&E\Big(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}|\ \underline{Z}_{t-1}\Big) = \\
&\quad = E\left[-\mathbb{I}_{\{0\}}(Z_t)(1-\beta)n(1-\beta_t)^{n-2}\frac{-n\beta+\beta+(1-\beta)(1-\beta_t)^n}{[\beta+(1-\beta)(1-\beta_t)^n]^2}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}|\ \underline{Z}_{t-1}\right] \\
&\qquad - E\left[\mathbb{I}_{\{1,\ldots,n\}}(Z_t)\left(\frac{Z_t}{\beta_t^2}+\frac{n-Z_t}{(1-\beta_t)^2}\right)\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}|\ \underline{Z}_{t-1}\right] \\
&\quad = -(1-\beta)n(1-\beta_t)^{n-2}\frac{-n\beta+\beta+(1-\beta)(1-\beta_t)^n}{[\beta+(1-\beta)(1-\beta_t)^n]^2}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}P\left(Z_t=0\mid\underline{Z}_{t-1}\right) \\
&\qquad - E\left[\mathbb{I}_{\{1,\ldots,n\}}(Z_t)\left(\frac{Z_t}{\beta_t^2}+\frac{n-Z_t}{(1-\beta_t)^2}\right)\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}|\ \underline{Z}_{t-1}\right].
\end{aligned}
$$

So, as $P(Z_t=0|\ \underline{Z}_{t-1})=\beta+(1-\beta)(1-\beta_t)^n$,

$$
\begin{aligned}
E\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}|\ \underline{Z}_{t-1}\right) &= -(1-\beta)n(1-\beta_t)^{n-2}\frac{-n\beta+\beta+(1-\beta)(1-\beta_t)^n}{\beta+(1-\beta)(1-\beta_t)^n}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}+ \\
&\quad -E\left[\mathbb{I}_{\{1,\ldots,n\}}(Z_t)\left(\frac{Z_t}{\beta_t^2}+\frac{n-Z_t}{(1-\beta_t)^2}\right)\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}|\ \underline{Z}_{t-1}\right].
\end{aligned}
$$

But from $E(Z_t|\underline{Z}_{t-1})=(1-\beta)n\beta_t$ we deduce

$$
\begin{aligned}
E\left(\mathbb{I}_{\{1,\ldots,n\}}(Z_t)Z_t|\ \underline{Z}_{t-1}\right) &= (1-\beta)n\beta_t \\
E\left(\mathbb{I}_{\{1,\ldots,n\}}(Z_t)(n-Z_t)\mid\underline{Z}_{t-1}\right) &= nP(Z_t\neq 0|\ \underline{Z}_{t-1})-(1-\beta)n\beta_t \\
&= n\left[1-\beta-(1-\beta)(1-\beta_t)^n\right]-(1-\beta)n\beta_t \\
&= n(1-\beta)\left[1-(1-\beta_t)^n-\beta_t.\right]
\end{aligned}
$$

So, $E\left(\dfrac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}|\underline{Z}_{t-1}\right)=$

$$= -\left(1-\beta\right)n\left(1-\beta_t\right)^{n-2}\frac{-n\beta+\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}+$$
$$-\left[\frac{\left(1-\beta\right)n}{\beta_t}+\frac{n\left(1-\beta\right)\left[1-\left(1-\beta_t\right)^n-\beta_t\right]}{\left(1-\beta_t\right)^2}\right]\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}$$

$$= -n\left(1-\beta\right)\left\{\left(1-\beta_t\right)^{n-2}\left[\frac{-n\beta+\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\right]+\left[\frac{1}{\beta_t}+\frac{1-\left(1-\beta_t\right)^n-\beta_t}{\left(1-\beta_t\right)^2}\right]\right\}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}$$
$$= -n\left(1-\beta\right)\left\{\left(1-\beta_t\right)^{n-2}\left[\frac{-n\beta+\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\right]+\frac{1-\beta_t-\beta_t\left(1-\beta_t\right)^n}{\beta_t\left(1-\beta_t\right)^2}\right\}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}$$
$$= -n\left(1-\beta\right)\left\{\left(1-\beta_t\right)^{n-2}\left[\frac{-n\beta+\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\right]+\frac{1-\beta_t\left(1-\beta_t\right)^{n-1}}{\beta_t\left(1-\beta_t\right)}\right\}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i}.$$

Let us note that, with $A_t = \beta + \left(1-\beta\right)\left(1-\beta_t\right)^n$,

$$\begin{aligned}
& -\left(1-\beta_t\right)^{n-2}\left(\frac{-n\beta+A_t}{A_t}\right)+\frac{1-\beta_t\left(1-\beta_t\right)^{n-1}}{\beta_t\left(1-\beta_t\right)} \\
= \quad & -\frac{\beta_t\left(1-\beta_t\right)^{n-1}\left(-n\beta+A_t\right)+\left[1-\beta_t\left(1-\beta_t\right)^{n-1}\right]A_t}{A_t\beta_t\left(1-\beta_t\right)} \\
= \quad & \frac{-\beta_t\left(1-\beta_t\right)^{n-1}n\beta+A_t\beta_t\left(1-\beta_t\right)^{n-1}+A_t-\beta_t\left(1-\beta_t\right)^{n-1}A_t}{A_t\beta_t\left(1-\beta_t\right)} \\
= \quad & \frac{-\beta_t\left(1-\beta_t\right)^{n-1}n\beta+A_t}{A_t\beta_t\left(1-\beta_t\right).}
\end{aligned}$$

So,

$$E\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}|\underline{Z}_{t-1}\right) = -n\left(1-\beta\right)\frac{-\beta_t\left(1-\beta_t\right)^{n-1}n\beta+\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}{\left[\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n\right]\beta_t\left(1-\beta_t\right)}\frac{\partial\beta_t}{\partial\theta_j}\frac{\partial\beta_t}{\partial\theta_i} \qquad (11)$$

We also have from (9)

$$\begin{aligned}
& E\left(\frac{\partial^2 l_t}{\partial\theta_{p+1}\partial\theta_i}|\underline{Z}_{t-1}\right) \\
= \quad & n\left(1-\beta_t\right)^{n-1}\frac{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n+\left[1-\left(1-\beta_t\right)^n\right]\left(1-\beta\right)}{\left[\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n\right]^2}\frac{\partial\beta_t}{\partial\theta_i}E\left[\mathbb{I}_{\{0\}}\left(Z_t\right)\mid\underline{Z}_{t-1}\right] \\
= \quad & n\left(1-\beta_t\right)^{n-1}\frac{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n+\left[1-\left(1-\beta_t\right)^n\right]\left(1-\beta\right)}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\frac{\partial\beta_t}{\partial\theta_i}
\end{aligned}$$

that is,

$$E\left(\frac{\partial^2 l_t}{\partial\theta_{p+1}\partial\theta_i}|\underline{Z}_{t-1}\right)=n\left(1-\beta_t\right)^{n-1}\frac{1}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\frac{\partial\beta_t}{\partial\theta_i} \tag{12}$$

and from (10)

$$E\left(\frac{\partial^2 l_t}{\partial\theta_{p+1}^2}|\underline{Z}_{t-1}\right)=-\frac{\left[1-\left(1-\beta_t\right)^n\right]^2}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}-\frac{1}{1-\beta}\left[1-\left(1-\beta_t\right)^n\right] \tag{13}$$

Otherwise, from (6) we get

$$\begin{aligned}E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}|\underline{Z}_{t-1}\right)&=E\left\{\,\mathbb{I}_{\{0\}}\left(Z_t\right)\left[\frac{\left(1-\beta\right)n\left(1-\beta_t\right)^{n-1}}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\right]^2\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}+\right.\\&\qquad\left.+\,\mathbb{I}_{\{1,\ldots,n\}}\left(Z_t\right)\left(\frac{Z_t}{\beta_t}-\frac{n-Z_t}{1-\beta_t}\right)^2\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}|\,\underline{Z}_{t-1}\right\}\\&=\frac{\left[\left(1-\beta\right)n\left(1-\beta_t\right)^{n-1}\right]^2}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}+\\&\qquad+E\left(\mathbb{I}_{\{1,\ldots,n\}}\left(Z_t\right)\frac{\left(Z_t-n\beta_t\right)^2}{\beta_t^2\left(1-\beta_t\right)^2}|\,\underline{Z}_{t-1}\right)\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}.\end{aligned}$$

Taking into account that $E\left(Z_t^2|\underline{Z}_{t-1}\right)=\left(1-\beta\right)n\beta_t\left(1-\beta_t+n\beta_t\right)$ we deduce that

$$\begin{aligned}&E\left(\mathbb{I}_{\{1,\ldots,n\}}\left(Z_t\right)\left(Z_t-n\beta_t\right)^2|\underline{Z}_{t-1}\right)\\=\;&E\left[\mathbb{I}_{\{1,\ldots,n\}}\left(Z_t\right)\left(Z_t^2-2n\beta_tZ_t+n^2\beta_t^2|\underline{Z}_{t-1}\right)\right]\\=\;&\left(1-\beta\right)n\beta_t\left(1-\beta_t+n\beta_t\right)-2n^2\beta_t^2\left(1-\beta\right)+n^2\beta_t^2\left[1-\beta-\left(1-\beta\right)\left(1-\beta_t\right)^n\right]\\=\;&\left(1-\beta\right)n\beta_t\left\{1-\beta_t+n\beta_t-2n\beta_t+n\beta_t\left[1-\left(1-\beta_t\right)^n\right]\right\}\\=\;&\left(1-\beta\right)n\beta_t\left[1-\beta_t-n\beta_t\left(1-\beta_t\right)^n\right].\end{aligned}$$

Consequently

$$\begin{aligned}E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}|\underline{Z}_{t-1}\right)&=\left(\frac{\left[\left(1-\beta\right)n\left(1-\beta_t\right)^{n-1}\right]^2}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}+\frac{\left(1-\beta\right)n\left[1-\beta_t-n\beta_t\left(1-\beta_t\right)^n\right]}{\beta_t\left(1-\beta_t\right)^2}\right)\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}\\&=\left(1-\beta\right)n\left\{\frac{\left(1-\beta\right)n\left(1-\beta_t\right)^{2n-2}}{\beta+\left(1-\beta\right)\left(1-\beta_t\right)^n}+\frac{1-\beta_t-n\beta_t\left(1-\beta_t\right)^n}{\beta_t\left(1-\beta_t\right)^2}\right\}\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}.\end{aligned}$$

Let us note that, again with $A_t = \beta + (1-\beta)(1-\beta_t)^n$,

$$\begin{aligned}
& \frac{(1-\beta)\,n\,(1-\beta_t)^{2n-2}}{A_t} + \frac{1-\beta_t - n\beta_t(1-\beta_t)^n}{\beta_t(1-\beta_t)^2} \\
= \; & \frac{(1-\beta)\,n\beta_t(1-\beta_t)^{2n}}{A_t\beta_t(1-\beta_t)^2} + \\
& + \frac{\beta - \beta\beta_t - n\beta\beta_t(1-\beta_t)^n + (1-\beta)(1-\beta_t)^n - (1-\beta)\beta_t(1-\beta_t)^n - (1-\beta)\,n\beta_t(1-\beta_t)^{2n}}{A_t\beta_t(1-\beta_t)^2} \\
= \; & \frac{\beta - \beta\beta_t - n\beta\beta_t(1-\beta_t)^n + (1-\beta)(1-\beta_t)^n - (1-\beta)\beta_t(1-\beta_t)^n}{A_t\beta_t(1-\beta_t)^2} \\
= \; & \frac{\beta - \beta\beta_t - n\beta\beta_t(1-\beta_t)^n + (1-\beta)(1-\beta_t)^n - (1-\beta)\beta_t(1-\beta_t)^n}{A_t\beta_t(1-\beta_t)^2} \\
= \; & \frac{\beta + (1-\beta)(1-\beta_t)^n - \beta_t\left[\beta + (1-\beta)(1-\beta_t)^n\right] - n\beta\beta_t(1-\beta_t)^n}{A_t\beta_t(1-\beta_t)^2} \\
= \; & \frac{A_t(1-\beta_t) - n\beta\beta_t(1-\beta_t)^n}{A_t\beta_t(1-\beta_t)^2} = \frac{A_t - n\beta\beta_t(1-\beta_t)^{n-1}}{A_t\beta_t(1-\beta_t)\,.}
\end{aligned}$$

So

$$E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}|\underline{Z}_{t-1}\right) = n(1-\beta)\frac{\beta + (1-\beta)(1-\beta_t)^n - n\beta\beta_t(1-\beta_t)^{n-1}}{\left[\beta + (1-\beta)(1-\beta_t)^n\right]\beta_t(1-\beta_t)}\frac{\partial\beta_t}{\partial\theta_i}\frac{\partial\beta_t}{\partial\theta_j}. \tag{14}$$

Also for $i = 1, ..., p$, from (6) and (7)

$$\begin{aligned}
E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_{p+1}}|\underline{Z}_{t-1}\right) &= -\frac{(1-\beta)\,n\,(1-\beta_t)^{n-1}}{\beta + (1-\beta)(1-\beta_t)^n}\left[1 - (1-\beta_t)^n\right]\frac{\partial\beta_t}{\partial\theta_i} \\
&\quad - E\left[\mathbb{I}_{\{1,\dots,n\}}(Z_t)\left(\frac{Z_t}{\beta_t} - \frac{n-Z_t}{1-\beta_t}|\underline{Z}_{t-1}\right)\right]\left(\frac{1}{1-\beta}\right)\frac{\partial\beta_t}{\partial\theta_i} \\
&= -\frac{(1-\beta)\,n\,(1-\beta_t)^{n-1}}{\beta + (1-\beta)(1-\beta_t)^n}\left[1 - (1-\beta_t)^n\right]\frac{\partial\beta_t}{\partial\theta_i} \\
&\quad - \frac{1}{\beta_t(1-\beta_t)(1-\beta)}E\left[\mathbb{I}_{\{1,\dots,n\}}(Z_t)\left(Z_t - n\beta_t|\underline{Z}_{t-1}\right)\right]\frac{\partial\beta_t}{\partial\theta_i}.
\end{aligned}$$

So

$$E\left(\frac{\partial l_t}{\partial \theta_i}\frac{\partial l_t}{\partial \theta_{p+1}}|\underline{Z}_{t-1}\right) = -\frac{(1-\beta)\,n\,(1-\beta_t)^{n-1}}{\beta+(1-\beta)\,(1-\beta_t)^n}\left[1-(1-\beta_t)^n\right]\frac{\partial \beta_t}{\partial \theta_i}$$
$$-\frac{1}{\beta_t\,(1-\beta_t)\,(1-\beta)}\left[(1-\beta)\,n\beta_t - n\beta_t\left[1-\beta-(1-\beta)\,(1-\beta_t)^n\right]\right]\frac{\partial \beta_t}{\partial \theta_i}$$
$$= -\frac{(1-\beta)\,n\,(1-\beta_t)^{n-1}}{\beta+(1-\beta)\,(1-\beta_t)^n}\left[1-(1-\beta_t)^n\right]\frac{\partial \beta_t}{\partial \theta_i}$$
$$-\frac{(1-\beta)\,n}{\beta_t\,(1-\beta_t)\,(1-\beta)}\left[\beta_t-\beta_t\left[1-(1-\beta_t)^n\right]\right]\frac{\partial \beta_t}{\partial \theta_i}$$

that is,

$$E\Big(\frac{\partial l_t}{\partial \theta_i}\frac{\partial l_t}{\partial \theta_{p+1}}|\underline{Z}_{t-1}\Big) =$$
$$= -(1-\beta)\,n\left\{\frac{(1-\beta_t)^{n-1}}{\beta+(1-\beta)\,(1-\beta_t)^n}\left[1-(1-\beta_t)^n\right]+\frac{(1-\beta_t)^n}{(1-\beta_t)\,(1-\beta)}\right\}\frac{\partial \beta_t}{\partial \theta_i}$$
$$= -(1-\beta)\,n\,(1-\beta_t)^{n-1}\left\{\frac{1-(1-\beta_t)^n}{\beta+(1-\beta)\,(1-\beta_t)^n}+\frac{1}{1-\beta}\right\}\frac{\partial \beta_t}{\partial \theta_i}$$
$$= -n\,(1-\beta_t)^{n-1}\left\{\frac{1-\beta-(1-\beta)\,(1-\beta_t)^n+\beta+(1-\beta)\,(1-\beta_t)^n}{\left[\beta+(1-\beta)\,(1-\beta_t)^n\right]}\right\}\frac{\partial \beta_t}{\partial \theta_i},$$

simplifying to

$$E\left(\frac{\partial l_t}{\partial \theta_i}\frac{\partial l_t}{\partial \theta_{p+1}}|\underline{Z}_{t-1}\right) = -\frac{n\,(1-\beta_t)^{n-1}}{\beta+(1-\beta)\,(1-\beta_t)^n}\frac{\partial \beta_t}{\partial \theta_i}, \tag{15}$$

and

$$E\Big(\frac{\partial l_t}{\partial \theta_{p+1}}\frac{\partial l_t}{\partial \theta_{p+1}}|\underline{Z}_{t-1}\Big) =$$
$$= E\left(\mathbb{I}_{\{0\}}\left(Z_t\right)\left[\frac{1-(1-\beta_t)^n}{\beta+(1-\beta)\,(1-\beta_t)^n}\right]^2+\mathbb{I}_{\{1,\ldots,n\}}\left(Z_t\right)\left(\frac{1}{1-\beta}\right)^2 \mid \underline{Z}_{t-1}\right)$$
$$= \left[\frac{1-(1-\beta_t)^n}{\beta+(1-\beta)\,(1-\beta_t)^n}\right]^2 P\left(Z_t=0|\underline{Z}_{t-1}\right)+\left(\frac{1}{1-\beta}\right)^2 P\left(Z_t\neq 0|\ \underline{Z}_{t-1}\right)$$
$$= \frac{\left[1-(1-\beta_t)^n\right]^2}{\beta+(1-\beta)\,(1-\beta_t)^n}+\left(\frac{1}{1-\beta}\right)^2\left[1-\beta-(1-\beta)\,(1-\beta_t)^n\right]$$

which gives

$$E\left(\frac{\partial l_t}{\partial \theta_{p+1}}\frac{\partial l_t}{\partial \theta_{p+1}}|\underline{Z}_{t-1}\right) = \frac{\left[1-(1-\beta_t)^n\right]^2}{\beta+(1-\beta)(1-\beta_t)^n}+\frac{1}{1-\beta}\left[1-(1-\beta_t)^n\right]. \quad (16)$$

Comparing now the expectation of (14), (15) and (16) with that of (11), (12) and (13) respectively, we conclude that this model satisfy the information matrix equality:

$$-E\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\right) = E\left(\frac{\partial l_t}{\partial\theta_i}\frac{\partial l_t}{\partial\theta_j}\right),\ 0\le i,j\le p+1.$$

So, the matrices

$$\widehat{S}_n = \frac{1}{N}\sum_{t=p+1}^{N}\nabla l_t \nabla^T l_t \quad \text{and} \quad \widehat{D}_n = -\frac{1}{N}\sum_{t=p+1}^{N}\nabla\left[\nabla^T l_t\right]$$

are consistent estimates for the information matrix, where ∇l_t is the gradient of l_t , and both can be used to estimate the asymptotic covariance matrix of the conditional maximum likelihood estimator.

3.2. Simulation study. In this Section we illustrate by several forms the finite sample performance of the CML estimation methodology previously presented considering a stochastic process Z following a $ZI\_BINARCH(1)$ model. We assume that n is known and consider $n=5$.

A numerical study was carried out in order to evaluate the finite sample performance of the CML estimators of b_0, b_1 and β, generating a sample of size N of the $ZI\_BINARCH(1)$ model with parameters $b_0=0.5$, $b_1=0.4$ and $\beta=0.5$. For this sample, we obtained the CML estimates following the previous theoretical approach. We repeated this procedure 100 times and the mean values of the estimates, along with the standard deviations in parenthesis, are presented in Table 1. The used sample sizes are $N=1000$ and 10000.

Table 1. CML estimates for the $ZI\_BINARCH(1)$ model with $b_0=0.5, b_1=0.4, \beta=0.5$

	$E_{est}\left(\widehat{b_0}\right)$	$E_{est}\left(\widehat{b_1}\right)$	$E_{est}\left(\widehat{\beta}\right)$
$N=1000$	0.496262 (0.015351)	0.406347 (0.021799)	0.499498 (0.017069)
$N=10000$	0.500122 (0.004532)	0.400024 (0.006691)	0.499416 (0.004974)

These simulations show that, as expected, the estimates of the parameters seem to converge to the corresponding true parameter values as the sample size increases. Further, the standard deviations of the estimates decrease when the sample size increases.

Figure 2 displays the Box-plots of the CML estimates for the three parameters for $N = 1000$ and 10000. We observe, in all cases, a strong concentration near the true value of the parameter and we notice that this concentration increases with N.

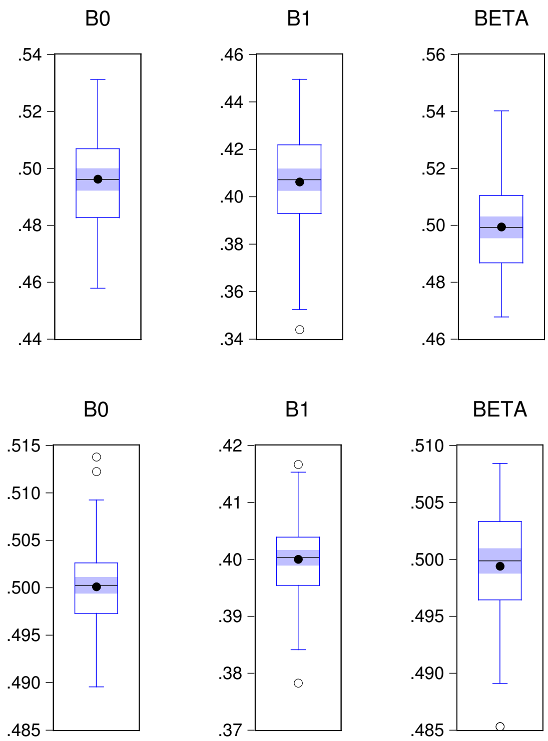


FIGURE 2. Box-plots of the CML estimates of b_0,b_1 and β, with $N = 1000$ (up) and $N = 10000$ (down).

The Q-Q plots of the CML estimates corresponding to $N = 1000$ and 10000 are presented in Figure 3. We note that the empirical quantiles approach the Gaussian distribution ones when N increases, for all the parameters in study. Moreover, the Jarque-Bera, Cramer-von Mises, Watson and Anderson-Darling statistics and p-values presented in Table 2 for $N = 100$ show the clear compatibility of the empirical distributions of CML estimates of all parameters with the Gaussian distribution.

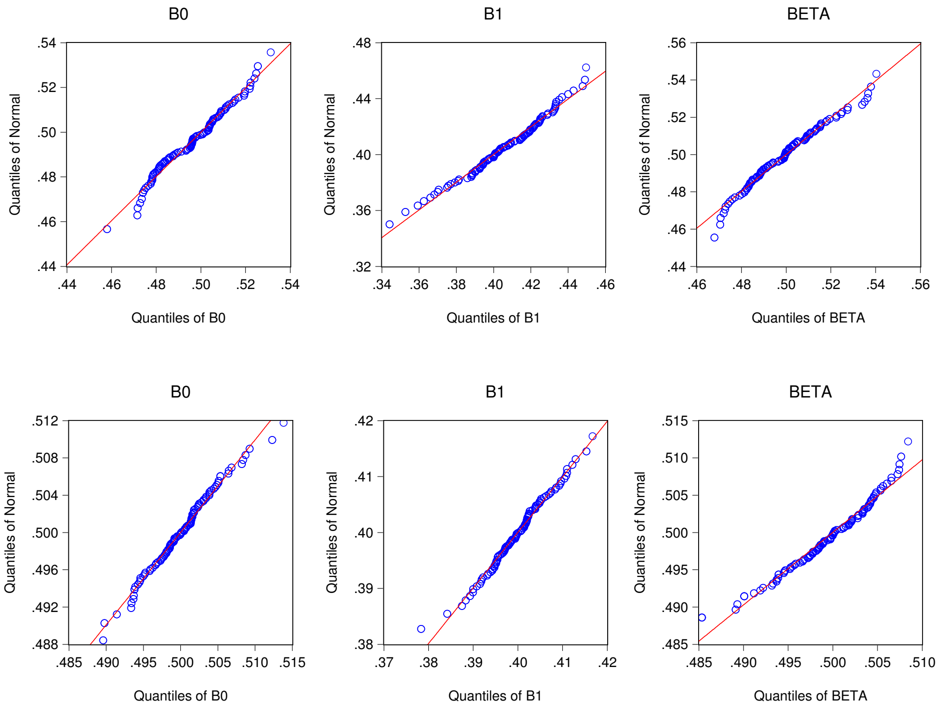


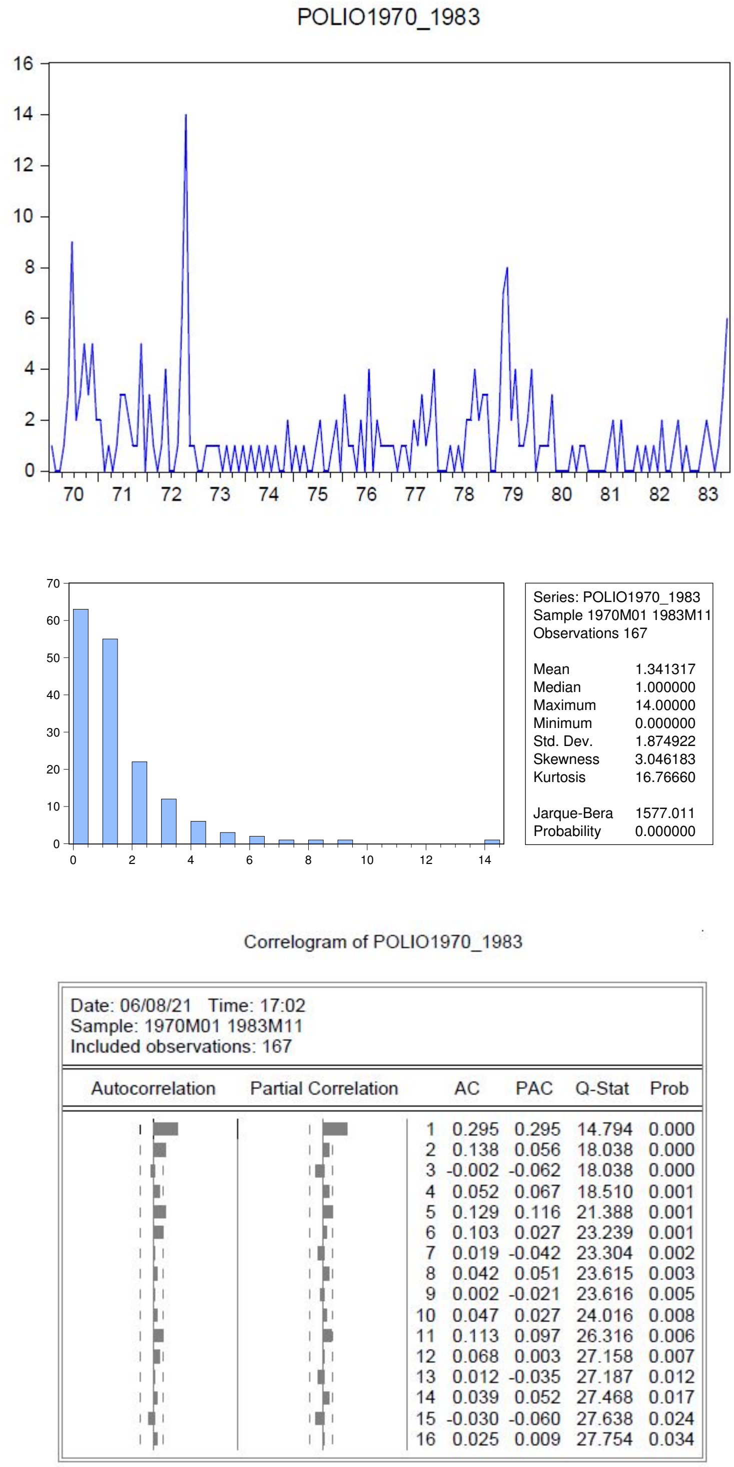
FIGURE 3. Q-Q plots of the CML estimates of b_0, b_1 and β, for $N = 1000$ (up) and 10000 (down).

Table 2. CML estimates of b_0, b_1 and β and Gaussian distribution.

$N = 100$	b_0		b_1		β	
Method	Value	Probab.	Value	Probab.	Value	Probab.
Jarque-Bera	2.062752	0.3565	2.624497	0.2692	2.988610	0.2267
Cramer-von Mises (W2)	0.039846	0.6811	0.045249	0.5818	0.076325	0.2287
Watson (U2)	0.037944	0.6642	0.045247	0.5311	0.057642	0.3625
Anderson-Darling (A2)	0.288669	0.6097	0.274759	0.6548	0.544131	0.1583

4. Real-data example: Poliomyelitis cases in USA

We apply the estimation methodology to the polio data presented in Zeger (1988). The data consists of 167 monthly counts of poliomyelitis cases recorded in the United States from january 1970 to november 1983 by the Centres for Desease Control. Figure 4 presents this series, its descriptive summaries and empirical autocorrelation and partial autocorrelation values. The empirical mean and standard deviation are 1.341317 and 1.874922 respectively. There is a large number of zero observations, the maximum observed is 14, the first autocorrelation is 0.295 and those of higher order are not significant.



	AC	PAC	Q-Stat	Prob
1	0.295	0.295	14.794	0.000
2	0.138	0.056	18.038	0.000
3	-0.002	-0.062	18.038	0.000
4	0.052	0.067	18.510	0.001
5	0.129	0.116	21.388	0.001
6	0.103	0.027	23.239	0.001
7	0.019	-0.042	23.304	0.002
8	0.042	0.051	23.615	0.003
9	0.002	-0.021	23.616	0.005
10	0.047	0.027	24.016	0.008
11	0.113	0.097	26.316	0.006
12	0.068	0.003	27.158	0.007
13	0.012	-0.035	27.187	0.012
14	0.039	0.052	27.468	0.017
15	-0.030	-0.060	27.638	0.024
16	0.025	0.009	27.754	0.034

FIGURE 4. Polio series: plot, descriptive summaries and autocorrelation and partial autocorrelation values.

The results related to the estimation of BINARCH and ZI BINARCH of order one with $n = 14$ are present in Table 3. Despite the non-significance of the autocorrelations of higher orders, we study also the effect of modeling the same observations by models BINARCH and ZI BINARCH with order two. For the comparison of the fitting quality we use, besides the log-likelihood function and Akaike ([1]) and Schwarz ([6]) criterion values, the mean square error given by

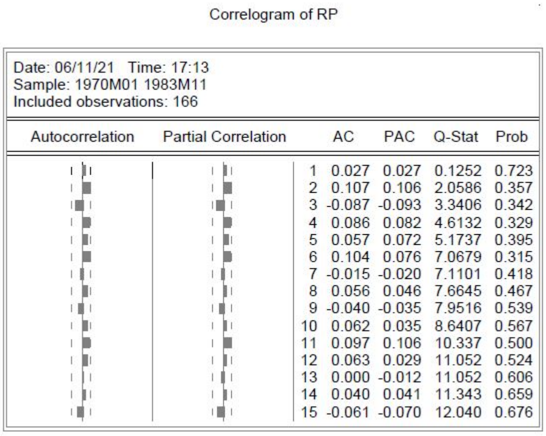
$$RMS^2 = \frac{1}{N}\sum_{t=1}^{N}\left(Z_t - E\left(Z_t|\underline{Z_{t-1}}\right)\right)^2.$$

Taking into account the sistematic smaller values of the criterions, we note that the ZI_BINARCH models perform better than the BINARCH ones of equal order. Comparing now the ZI_BINARCH(1) and ZI_BINARCH(2) models, we observe that the first one has better values for all criterions excepting the negative log-likelihood. We point out the similarity of the estimates in these two modellings and also the expected non-significance of the parameter b_2 (with p-value equal to 0.7661) in the ZI_BINARCH(2) fitting. So, we retain the ZI_BINARCH(1) model and then analyse the residuals series produced.

Table 3. Conditional maximum likelihood estimates of the parameters of the models, with the corresponding standard errors and probabilities, and the negative log-likelihood functions, Akaike and Schwarz criterions and root mean square errors

Model	Estimates				- log L	Akaike	Schwarz	RMS
BINARCH(1)	b_0 0.0628 (0.0059) 0.0000	b_1 0.3470 (0.0290) 0.0000			296.7372	3.5992	3.6367	1.7895
ZI_BINARCH(1)	b_0 0.0856 (0.0100) 0.0000	b_1 0.4262 (0.0342) 0.0000	β 0.2473 (0.0489) 0.0000		283.5904	3.4529	3.5091	1.7883
BINARCH(2)	b_0 0.0569 (0.0064) 0.0000	b_1 0.3273 (0.0295) 0.0000	b_2 0.0866 (0.0465) 0.0626		294.0457	3.6006	3.6570	1.7905
ZI_BINARCH(2)	b_0 0.0839 (0.0106) 0.0000	b_1 0.4214 (0.0363) 0.0000	b_2 0.0175 (0.0587) 0.7661	β 0.2407 (0.0492) 0.0000	282.5818	3.4737	3.5490	1.7899

Figure 5 shows the correlogram and partial correlogram of the Pearson residuals. The compatibility with an white noise is clear. Figure 6 presents the polio series and the series estimated by the ZI_BINARCH(1) model that was chosen in the previous discussion.



Correlogram of RP

Date: 06/11/21 Time: 17:13
Sample: 1970M01 1983M11
Included observations: 166

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.027	0.027	0.1252	0.723
		2	0.107	0.106	2.0586	0.357
		3	-0.087	-0.093	3.3406	0.342
		4	0.086	0.082	4.6132	0.329
		5	0.057	0.072	5.1737	0.395
		6	0.104	0.076	7.0679	0.315
		7	-0.015	-0.020	7.1101	0.418
		8	0.056	0.046	7.6645	0.467
		9	-0.040	-0.035	7.9516	0.539
		10	0.062	0.035	8.6407	0.567
		11	0.097	0.106	10.337	0.500
		12	0.063	0.029	11.052	0.524
		13	0.000	-0.012	11.052	0.606
		14	0.040	0.041	11.343	0.659
		15	-0.061	-0.070	12.040	0.676

FIGURE 5. Pearson residuals: autocorrelation and partial autocorrelation values.

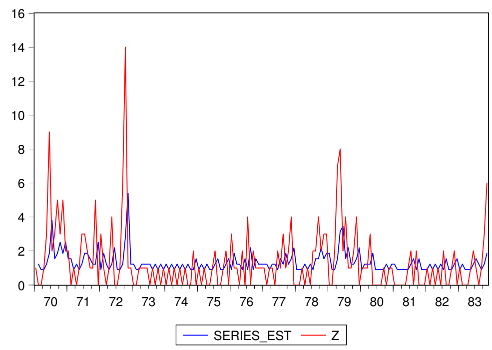


FIGURE 6. Polio series and fitted conditional mean from the estimated ZI_BINARCH(1) model.

5. Conclusion

In order to deal with zero inflation within the finite-range conditionally heteroscedastic count series, we define and study the zero inflated binomial INGARCH process. The probabilistic structure of this model is developed by establishing its stationarity and ergodicity properties, and obtaining closed-form expressions for its mean and autocovariance functions. Furthermore, a statistical analysis is performed by estimating the model parameters and establishing the corresponding asymptotic behaviour. The simulation study carried out shows the good performance of such estimators with finite samples. Additionally, we highlight the best fit of these models to time series with a large number of zeros, as illustrated by the real-data example considered. Future developments of this study may consider other discrete conditional distributions with finite support, such as general discrete truncated ones.

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