Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 21–26

FRAME PRESENTATIONS OF COMPACT HEDGEHOGS AND THEIR PROPERTIES

IGOR ARRIETA, JAVIER GUTIÉRREZ GARCÍA AND JORGE PICADO

Dedicated to Professor Bernhard Banaschewski on the occasion of his 95th birthday

ABSTRACT: This paper considers the compact hedgehog as a frame presented by generators and relations, based on the presentation of the frame of extended real numbers. The main focus will be on the point-free version of continuous and semi-continuous functions that arise from it, and their application in characterizations of variants of collectionwise normality. The variants to be considered are defined by selections of adequate families of sublocales and their characterizations depend on lattice-theoretic properties of the selected families. This way insertion and extension results for semicontinuous and continuous functions with values in the compact hedgehog frame are generalized and unified.

KEYWORDS: Hedgehog space, compact hedgehog, frame, locale, zero sublocale, z- and z_κ^c-embedded sublocales, collectionwise normality, total collectionwise normality, compact hedgehog-valued frame homomorphism, insertion results, sublocale selection.

2020 MATHEMATICS SUBJECT CLASSIFICATION: 06D22, 18F70, 54D15.

1. Introduction

Hedgehog spaces are an interesting source of examples in point-set topology. They constitute, in fact, three different classes of spaces [1]. Here we are interested in the so-called *compact* hedgehog spaces, from a point-free viewpoint.

As the name suggests, the hedgehog can be described as a set of spines identified at a single point. Given a cardinal κ and a set I of cardinality κ, the *hedgehog with κ spines* $J(\kappa)$ is the disjoint union $\bigcup_{i\in I}\left(\overline{\mathbb{R}}\times\{i\}\right)$ of κ copies (the spines) of the extended real line identified at $-\infty$:

The *compact hedgehog space* $\Lambda J(\kappa)$ is the hedgehog endowed with the (compact) topology generated by the subbasis

$$\{\,(r,\text{—})_i \mid r\in\mathbb{Q}, i\in I\,\}\cup\{\,(\text{—},r)_i \mid r\in\mathbb{Q}, i\in I\,\}$$

where $(r,\text{—})_i := (r,+\infty]\times\{i\}$ and $(\text{—},r)_i := J(\kappa)\setminus([r,+\infty]\times\{i\})$.

Received July 26, 2021.

$$J(\kappa) = \{-\infty\} \cup \bigcup_{i\in I}((-\infty, +\infty] \times \{i\}).$$

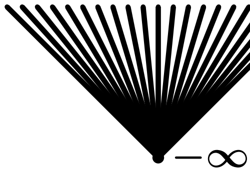

FIGURE 1. The hedgehog.

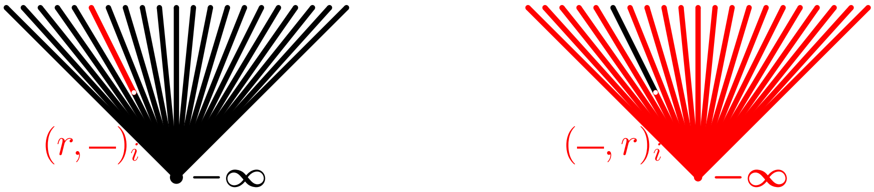

FIGURE 2. Subbasic opens of the compact topology.

An important feature of point-free topology is the algebraic nature of the category of frames that enables presentations of frames by generators and relations. One can present a frame by generators and relations by specifying a set G of generators and a set $R \subseteq G \times G$ of relations (written in a language with symbols for finite meets and arbitrary joins). Then there exists a frame $\mathsf{Frm}\,\langle G \mid R\rangle$ such that for any frame L, the set of frame homomorphisms $\mathsf{Frm}\,\langle G \mid R\rangle \to L$ is in bijective correspondence with functions $f\colon G \to L$ satisfying $f(a) = f(b)$ for all $(a, b) \in R$. For an account of how to construct frames from generators and relations, see [24].

Our aim with the present paper is to study the compact topology of the hedgehog via frame presentations by generators and relations (starting just from the rationals, independently of any notion of real number). The main focus will be on the point-free version of continuous and semicontinuous functions, with values in the compact hedgehog, that arise from it, and their relation with variants and generalizations of normality and collectionwise normality. The variants to be considered are defined by selections of adequate families of sublocales. These selections are fixed by a convenient general format that allows to treat the results in a unified manner and to identify the conditions for them by lattice-theoretic properties of the given selection. Examples of such variants are normality, mild normality, F-property, or total κ-collectionwise normality, and their duals (like e.g. extremal disconnectedness, extremal δ-disconnectedness and Oz-property).

In detail, this will proceed as follows. We begin in Section 2 with a brief account of the necessary background and terminology. Next, in Section 3, the core of the paper, we introduce the frame $\mathfrak{L}(cJ(\kappa))$ of the compact hedgehog with κ spines and present its main properties. We end the section by proving that the spatial spectrum of $\mathfrak{L}(cJ(\kappa))$ is precisely the compact hedgehog.

Section 4 takes up lower and upper semicontinuous functions with values in the compact hedgehog and Section 5 discusses variants of collectionwise normality for frames such as κ-collectionwise normality and total κ-collectionwise normality (for a cardinal κ). Section 6 is devoted to the study of (disjoint) continuous extensions (it provides in particular a Tietze-type theorem for totally κ-collectionwise normal frames) and in Section 7 we present the corresponding Katětov-Tong-type insertion results.

In the last four sections we use the sublocale selections of [20] to get a general view over these notions and results with the aim of identifying the conditions on the sublocale selection under which the results hold. Section 8 presents some general Katětov-Tong-type insertion results, Section 9 extends the notions of zero and z-embedded sublocales, Section 10 covers variants of total κ-collectionwise normality, and finally Section 11 provides some general versions of Tietze-type theorems (for some variants of normal frames and frames where certain classes of sublocales are z_κ^c-embedded).

2. Preliminaries

Our general reference for topology is [13]. Our terminology regarding the categories of frames and locales will be that of [25]. The Heyting operator in a locale L, right adjoint to the meet operator, will be denoted by $\to$; for each $a \in L$, $a^* = a \to 0$ is the pseudocomplement of a. An $a \in L$ is *regular* if $a = b^*$ for some b (in other words, $a^{**} = a$). Furthermore, an element b is rather below a (written $b \prec a$) if $b^* \vee a = 1$.

A *point* of a locale L is an element $p \neq 1$ in L such that

$$x \wedge y \leq p \;\Rightarrow\; x \leq p \text{ or } y \leq p.$$

The *spectrum* of L is the topological space

$$\Sigma(L) = \Big(\{p \mid p \in L, p \text{ prime}\}, \{\Sigma_a \mid a \in L\}\Big)$$

where $\Sigma_a = \{p \mid a \not\leq p\}$ (it is easy to see that $\Sigma_0 = \emptyset$, $\Sigma_1 = \{\text{primes of } L\}$, $\Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b$ and $\Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i}$).

A *sublocale* of a locale L is a subset $S \subseteq L$ closed under arbitrary meets such that

$$\forall a \in L \ \ \forall s \in S \ \ (a \to s \in S).$$

These are precisely the subsets of L for which the embedding $j_S \colon S \hookrightarrow L$ is a morphism of locales. We shall denote by ν_S the left adjoint of j_S, defined by $\nu_S(a) = \bigwedge\{s \in S \mid s \geq a\}$.

The system $\mathcal{S}(L)$ of all sublocales of L, partially ordered by inclusion, is a coframe [25, Thm. III.3.2.1], that is, its dual lattice is a frame. Infima and suprema are given by

$$\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i, \quad \bigvee_{i\in I} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i\}.$$

The least element is the sublocale $\mathsf{O} = \{1\}$ and the greatest element is the entire locale L.

Since $\mathcal{S}(L)$ is a coframe, there is a co-Heyting operator which gives the *difference* $S \smallsetminus T$ of two sublocales $S, T \in \mathcal{S}(L)$. This operator is characterized by the equivalence $S \smallsetminus T \subseteq R$ if and only if $S \subseteq T \vee R$ for $R, S, T \in \mathcal{S}(L)$.

For any $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = {\uparrow} a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L, respectively (that we shall denote simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion). For each $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathcal{S}(L)$ and satisfy the identities

$$\bigcap_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \tag{2.1}$$

$$\bigvee_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b).$$

For any sublocale S of L, the closed (resp. open) sublocales $\mathfrak{c}_S(a)$ (resp. $\mathfrak{o}_S(a)$) of S are precisely the intersections $\mathfrak{c}(a) \cap S$ (resp. $\mathfrak{o}(a) \cap S$) and we have, for any $a \in L$, $\mathfrak{c}(a) \cap S = \mathfrak{c}_S(j_S^*(a))$ and $\mathfrak{o}(a) \cap S = \mathfrak{o}_S(j_S^*(a))$.

A closed sublocale $\mathfrak{c}(a)$ will be called *regular* (resp. *δ-regular*) if a is regular (resp. δ-regular, that is, $a = \bigvee_{n=0}^{\infty} a_n$ with $a_n \prec a$).

Recall the frame of reals $\mathfrak{L}(\mathbb{R})$ from [3]. Here we define it, equivalently, as the frame presented by generators $(r, \text{—})$ and $(\text{—}, r)$ for all rationals r, and relations

(r1) $(p, \text{—}) \wedge (\text{—}, q) = 0$ if $q \leq p$,
(r2) $(p, \text{—}) \vee (\text{—}, q) = 1$ if $p < q$,

(r3) $(p,\text{—}) = \bigvee_{r>p}(r,\text{—})$,
(r4) $(\text{—},q) = \bigvee_{s<q}(\text{—},s)$,
(r5) $\bigvee_{p\in\mathbb{Q}}(p,\text{—}) = 1$,
(r6) $\bigvee_{q\in\mathbb{Q}}(\text{—},q) = 1$.

By dropping relations (r5) and (r6) one has the *frame of extended reals* $\mathfrak{L}(\overline{\mathbb{R}})$ ([5]).

A *continuous real-valued function* [3] (resp. *extended continuous real-valued function* [5]) on a frame L is a frame homomorphism $h\colon \mathfrak{L}(\mathbb{R}) \to L$ (resp. $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$). We denote by $\mathsf{C}(L)$ and $\overline{\mathsf{C}}(L)$, respectively, the collections of all continuous (resp. extended continuous) real-valued functions on L. $\mathsf{C}(L)$ and $\overline{\mathsf{C}}(L)$ are partially ordered by $f \le g$ iff $f(r,\text{—}) \le g(r,\text{—})$ for all $r \in \mathbb{Q}$ (or, equivalently, $g(\text{—},r) \le f(\text{—},r)$ for all $r \in \mathbb{Q}$).

An $a \in L$ is said to be a *cozero element* if $a = f\left((\text{—},0) \vee (0,\text{—})\right)$ for some $f \in \mathsf{C}(L)$. Equivalently, a is a cozero element iff $a = f(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))$ for some $f \in \overline{\mathsf{C}}(L)$. As usual, $\operatorname{Coz} L$ will denote the σ-frame of all cozero elements of L. Following the terminology used by several authors (e.g. [11, 23]), we say that a *zero sublocale* is a closed sublocale $\mathfrak{c}(a)$ with $a \in \operatorname{Coz} L$. The class of zero sublocales is denoted by $\mathsf{Z}(L)$.

There is a useful way of specifying (extended) continuous real-valued functions with the help of scales ([16, Section 4]). An *extended scale* in L is a map $\sigma\colon \mathbb{Q} \to L$ such that $\sigma(p) \vee \sigma(q)^* = 1$ whenever $p < q$. An extended scale is a *scale* if $\bigvee_{p\in\mathbb{Q}}\sigma(p) = 1 = \bigvee_{p\in\mathbb{Q}}\sigma(p)^*$. For each extended scale σ in L, the formulas

$$h(p,\text{—}) = \bigvee_{r>p}\sigma(r) \quad \text{and} \quad h(\text{—},q) = \bigvee_{r<q}\sigma(r)^*, \qquad (\text{for all } p,q \in \mathbb{Q}), \tag{2.2}$$

determine an $h \in \overline{\mathsf{C}}(L)$ ([5, Lem. 1]); h is in $\mathsf{C}(L)$ iff σ is a scale. If $h_1, h_2 \in \overline{\mathsf{C}}(L)$ are generated by extended scales σ_1 and σ_2 respectively, then one has

$$h_1 \le h_2 \text{ iff } \sigma_1(r) \le \sigma_2(s), \qquad \text{for all } r > s. \tag{2.3}$$

Let $\mathcal{S}(L)^{op} = (\mathcal{S}(L), \le)$, with $\le \equiv \supseteq$, be the dual lattice of $\mathcal{S}(L)$. Meets and joins in $\mathcal{S}(L)^{op}$ are given by respectively $\bigsqcap_{i\in I} S_i = \bigvee_{i\in I} S_i$ and $\bigsqcup_{i\in I} S_i = \bigcap_{i\in I} S_i$. Let $\overline{\mathsf{F}}(L) = \overline{\mathsf{C}}(\mathcal{S}(L)^{op})$: the elements of $\overline{\mathsf{F}}(L)$ are the *extended real functions* on L [5]. By the identities in (2.1), the set $\mathfrak{c}L$ of all closed sublocales of L is a subframe of $\mathcal{S}(L)^{op}$ isomorphic to the given L. Hence the ℓ-ring

$\overline{\mathsf{F}}(L)$ is an extension of $\overline{\mathsf{C}}(L)$, partially ordered by

$$f \leq g \;\text{ iff }\; f(\text{—}, r) \subseteq g(\text{—}, r) \;\text{ iff }\; g(r, \text{—}) \subseteq f(r, \text{—}) \quad (\text{for all } r \in \mathbb{Q}). \tag{2.4}$$

An $f \in \overline{\mathsf{F}}(L)$ is *lower semicontinuous* (resp. *upper semicontinuous*) if $f(r,\text{—})$ (resp. $f(\text{—},r)$) is closed for every $r \in \mathbb{Q}$. These classes are denote by $\overline{\mathsf{LSC}}(L)$ and $\overline{\mathsf{USC}}(L)$. By the isomorphism $L \simeq \mathfrak{c}L$, $\overline{\mathsf{C}}(L)$ can be identified as the set of all $f \in \overline{\mathsf{F}}(L)$ such that $f(p,\text{—})$ and $f(\text{—},p)$ are closed (for any rational p), that is, $\overline{\mathsf{C}}(L) = \overline{\mathsf{LSC}}(L) \cap \overline{\mathsf{USC}}(L)$.

3. The compact hedgehog frame

The usual topology on the (extended) reals can be naturally introduced in two completely different ways:

- It is the metric topology induced by the euclidean metric.
- It is the Lawson topology induced by the linear order.

The first approach is probably the best known. In this case the topology, being a metric topology, is generated by the basis of all open balls, i.e. the open intervals $\langle a, b\rangle$ with $a < b$ in $\mathbb{R}$ (or just with $a < b$ in $\mathbb{Q}$).

The second approach is of particular interest when one is interested in notions like lower and upper semicontinuity. In this case one first generates two topologies:

(1) The *Scott topology*, that is, the smallest topology in which the sets $\mathord{\uparrow\!\!\!\uparrow} a = \{ x \in \mathbb{R} \mid a < x \}$ are *open* for all a in $\mathbb{R}$ (or, equivalently, with a just in $\mathbb{Q}$).

(2) The *lower topology*, that is, the smallest topology in which the principal filters $\uparrow a = \{ x \in \mathbb{R} \mid a \leq x \}$ are *closed* for all a in $\mathbb{R}$ (or, equivalently, with $a \in \mathbb{Q}$).

Then the usual euclidean topology is the Lawson topology, that is, the common refinement of the Scott and the lower topologies ([14, Chapter III]).

The metric topology on $J(\kappa)$ is precisely the cardinal generalization of the metric topology on the unit real interval. Pointfreely, it is described by the *frame of the metric hedgehog with κ spines* [19], the frame $\mathfrak{L}(J(\kappa))$ generated by abstract symbols $(r,\text{—})_i$ and $(\text{—},r)$, $r \in \mathbb{Q}$ and $i \in I$, subject to the following relations:

(h0) $(r,\text{—})_i \wedge (s,\text{—})_j = 0$ whenever $i \neq j$.

(h1) $(r,\text{—})_i \wedge (\text{—},s) = 0$ whenever $r \geq s$ and $i \in I$.

(h2) $\bigvee_{i\in I} (r_i,\text{—})_i \vee (\text{—},s) = 1$ whenever $r_i < s$ for every $i \in I$.

(h3) $(r,\text{—})_i = \bigvee_{s>r}(s,\text{—})_i$ for every $r \in \mathbb{Q}$ and $i \in I$.
(h4) $(\text{—},r) = \bigvee_{s<r}(\text{—},s)$ for every $r \in \mathbb{Q}$.

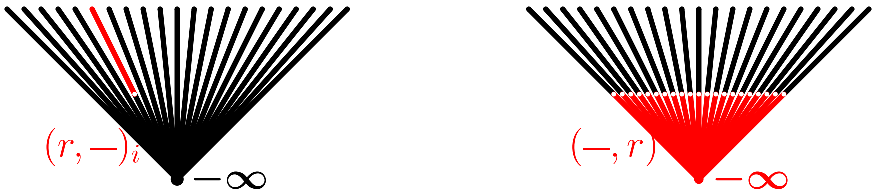

FIGURE 3. The metric hedgehog generators.

We can also consider an extension on $J(\kappa)$ of the Lawson topology. For that, introduce first the following (partial) order on $J(\kappa)$:

$$(t,i) \leq (s,j) \;\equiv\; t = -\infty \quad \text{or} \quad i = j,\ t \leq s.$$

The poset $(J(\kappa), \leq)$ is evidently a cardinal generalization of $(\overline{\mathbb{R}}, \leq)$, being $(J(1), \leq)$ precisely $(\overline{\mathbb{R}}, \leq)$. In general, for an arbitrary cardinal κ, it fails to be a complete lattice (but it is still a bounded complete domain [14]). We can still generate two topologies:

(1) The *Scott topology*, that is, the smallest topology in which the sets $\twoheaduparrow(r,i) = \{\,(t,j) \in J(\kappa) \mid (r,i) \ll (t,j)\,\} = (r,+\infty] \times \{i\}$ are *open* for all $r \in \mathbb{Q}$ and $i \in I$.
(2) The *lower topology*, that is, the smallest topology in which the principal filters $\uparrow(r,i) = \{\,(t,j) \in J(\kappa) \mid (r,i) \leq (t,j)\,\} = [r,+\infty] \times \{i\}$ are *closed* for all $r \in \mathbb{Q}$ and $i \in I$.

The *Lawson topology* is the common refinement of the Scott and the lower topologies. This is a compact Hausdorff topology on $J(\kappa)$ (see [14, Exercise III-3.2 and Theorem III-5.8]) that yields a separable metrizable space if and only if $\kappa \leq \aleph_0$ (see [14, Corollary III-4.6.] and [1, Properties 6.7 (6)–(7)]).

We recall from [17] that a function f defined on a topological space X with values in the hedgehog $J(\kappa)$ is said to be *lower semicontinuous* if it is continuous with respect to the Scott topology, i.e. $f^{-1}((r,+\infty] \times \{i\})$ is open in X for every $r \in \mathbb{Q}$ and $i \in I$ (this notion should not be confused with the one of Blair and Swardson [8]); similarly, it is *upper semicontinuous* if it is continuous with respect to the lower topology, i.e. $f^{-1}(J(\kappa) \setminus ([r,+\infty] \times \{i\}))$ is open in X for every $r \in \mathbb{Q}$ and $i \in I$. It is said to be *continuous* if it is continuous with respect to the Lawson topology, i.e. if it is both lower and upper semicontinuous.

Frame presentation and fundamental properties. Pointfreely, the compact hedgehog is described by the *frame of the compact hedgehog with κ spines*, that is, the frame $\mathfrak{L}(cJ(\kappa))$ presented by generators $(r,\text{—})_i$ and $(\text{—},r)_i$, $r\in\mathbb{Q}$ and $i\in I$, subject to the following relations (cf. Figure 2):

(ch0) $(r,\text{—})_i \wedge (s,\text{—})_j = 0$ whenever $i\neq j$.
(ch1) $(r,\text{—})_i \wedge (\text{—},s)_i = 0$ whenever $r\geq s$ for every $i\in I$.
(ch2) $(r,\text{—})_i \vee (\text{—},s)_i = 1$ whenever $r<s$ for every $i\in I$.
(ch3) $(r,\text{—})_i = \bigvee_{s>r}(s,\text{—})_i$ for every $r\in\mathbb{Q}$ and $i\in I$.
(ch4) $(\text{—},r)_i = \bigvee_{s<r}(\text{—},s)_i$ for every $r\in\mathbb{Q}$ and $i\in I$.

Remark 3.1. Note that there is an alternative presentation for the frame $\mathfrak{L}(cJ(\kappa))$: take the subframe $\mathfrak{L}_c(J(\kappa))$ of $\mathfrak{L}(J(\kappa))$ generated by the elements

$$(r,\text{—})_i \quad\text{and}\quad (r,\text{—})_i{}^* = \bigvee_{j\neq i}(r-1,\text{—})_j \vee (\text{—},r), \qquad r\in\mathbb{Q},\ i\in I.$$

It is a straightforward (but tedious) exercise to check that $\mathfrak{L}_c(J(\kappa)) \cong \mathfrak{L}(cJ(\kappa))$.

Proposition 3.2. *$\mathfrak{L}_c(J(\kappa))$ is a proper subframe of $\mathfrak{L}(J(\kappa))$ if and only if κ is infinite.*

Proof: If κ is finite, then $\bigwedge_{i\in I}(r,\text{—})_i{}^* = \big(\bigwedge_{i\in I}\bigvee_{j\neq i}(r-1,\text{—})_j\big)\vee(\text{—},r) = (\text{—},r)$, hence $\mathfrak{L}_c(J(\kappa)) = \mathfrak{L}(J(\kappa))$.

Otherwise, if κ is infinite, then the frame $\mathfrak{L}(J(\kappa))$ is not compact (this is a consequence of the defining relation (h2), see [19, Remarks 3.1]) But, as we shall see in 3.3 below, $\mathfrak{L}(cJ(\kappa))$, and hence $\mathfrak{L}_c(J(\kappa))$, is a compact frame. ■

For each $i\in I$, the map $\sigma_i\colon\mathbb{Q}\to\mathfrak{L}(cJ(\kappa))$ given by $\sigma_i(r)=(r,\text{—})_i$ is an extended scale in $\mathfrak{L}(cJ(\kappa))$. Hence (recall (2.2)) the formulas

$$\pi_i(p,\text{—}) = \bigvee_{s>p}(s,\text{—})_i = (p,\text{—})_i \quad\text{and}$$
$$\pi_i(\text{—},q) = \bigvee_{s<q}(s,\text{—})_i{}^* = \bigvee_{s<q}(\text{—},s)_i = (\text{—},q)_i$$

determine an extended continuous real function

$$\pi_i\colon\mathfrak{L}(\overline{\mathbb{R}})\to\mathfrak{L}(cJ(\kappa)).$$

We shall refer to π_i as the *i-th projection.*

Theorem 3.3. *$\mathfrak{L}(cJ(\kappa))$ is a compact regular frame.*

Proof: Consider the unique frame homomorphism f, given by coproduct universal property, for which the following diagram commutes:

$$\begin{tikzcd} \mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "\iota_i"] \arrow[dr, "\pi_i"'] & \bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}) \arrow[d, dashed, "f"] \\ & \mathfrak{L}(cJ(\kappa)) \end{tikzcd}$$

Let

$$a=\bigvee_{i\neq j}\iota_i\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big)\wedge\iota_j\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big)\in\bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}).$$

By (ch0) we have

$$f(a)=\bigvee_{i\neq j}f\Big(\iota_i\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big)\Big)\wedge f\Big(\iota_j\Big(\bigvee_{s\in\mathbb{Q}}(s,\text{—})\Big)\Big)=\bigvee_{i\neq j}\bigvee_{r\in\mathbb{Q}}\bigvee_{s\in\mathbb{Q}}(r,\text{—})_i\wedge(s,\text{—})_j=0.$$

Moreover, $f(a\vee\iota_i(r,\text{—}))=f(\iota_i(r,\text{—}))=\pi_i(r,\text{—})=(r,\text{—})_i$ and $f(a\vee\iota_i(\text{—},s))=f(\iota_i(\text{—},s))=\pi_i(\text{—},s)=(\text{—},s)_i$ for every $i\in I$ and $r,s\in\mathbb{Q}$. Hence the map $k\colon\mathfrak{c}(a)\to\mathfrak{L}(cJ(\kappa))$ given by $k(x)=f(x)$ for each $x\in\mathfrak{c}(a)$ is a surjective frame homomorphism making the following triangle

$$\begin{tikzcd} \bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "f"] \arrow[d, "(-)\vee a"'] & \mathfrak{L}(cJ(\kappa)) \\ \mathfrak{c}(a) \arrow[ur, "k"'] & \end{tikzcd}$$

commute. On the other hand, the assignments

$$(\text{—},r)_i\longmapsto\iota_i(r,\text{—})\vee a\quad\text{ and }\quad(s,\text{—})_i\longmapsto\iota_i(\text{—},s)\vee a$$

for each $r,s\in\mathbb{Q}$ and $i\in I$ determine a frame homomorphism

$$g\colon\mathfrak{L}(cJ(\kappa))\to\mathfrak{c}(a)$$

(the fact that they turn the relations (ch0)–(ch4) into identities in $\mathfrak{c}(a)$ follows easily from the fact that each ι_i is a frame homomorphism and the fact that the relations (r1)–(r4) are satisfied in $\mathfrak{L}(\overline{\mathbb{R}})$). Thus g is the unique frame homomorphism that makes the triangle

$$\begin{tikzcd} \bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "f"] \arrow[d, "(-)\vee a"'] & \mathfrak{L}(cJ(\kappa)) \arrow[dl, "g"] \\ \mathfrak{c}(a) & \end{tikzcd}$$

commutative (the fact that it commutes obviously follows from the fact that the coproduct injections are jointly epic). Consequently, $\mathfrak{L}(cJ(\kappa))$ and $\mathfrak{c}(a)$ are isomorphic frames, and the latter is regular and compact because it is a closed sublocale of a regular and compact locale (by Tychonoff's Theorem for locales [25]). ■

Remark 3.4. Since Tychonoff's Theorem for locales and compactness of $\mathfrak{L}(\overline{\mathbb{R}})$ are constructively valid, the proof above is also constructively valid provided the index set I has decidable equality (i.e., for all $i, j \in I$, one has either $i = j$ or $i \neq j$), a condition that was already assumed in the relation (ch0).

Proposition 3.5. *$\mathfrak{L}(cJ(\kappa))$ is metrizable if and only if $\kappa \leq \aleph_0$.*

Proof: The coproduct of countably many metrizable frames is metrizable by virtue of [22, p. 31]. Hence, if $\kappa \leq \aleph_0$, then, for $|I| = \kappa$, $\bigoplus_{i\in I} \mathfrak{L}(\overline{\mathbb{R}})$ is metrizable, and so is any of its quotients, thus $\mathfrak{L}(cJ(\kappa))$ is metrizable.

Conversely, if $\mathfrak{L}(cJ(\kappa))$ is metrizable, and since it is also compact, then it must have a countable $\bigvee$-basis ([6, 4.3]). Let $B = \{b_n\}_{n\in\mathbb{N}}$ be such a basis. Then, for each $i \in I$, there is some $n_i \in \mathbb{N}$ such that $0 \neq b_{n_i} \leq (0, \text{—})_i$. Consequently, $\{b_{n_i}\}_{i\in I}$ is a pairwise disjoint family of nonzero elements contained in B, hence $\kappa = |I| \leq \aleph_0$. ■

Hence, by [10, Prop. 3], we have:

Corollary 3.6. *For $\kappa \leq \aleph_0$, any regular subframe of $\mathfrak{L}(cJ(\kappa))$ is metrizable.*

Further properties. It is a straightforward exercise to check that for each $i \in I$ the assignments from $\mathfrak{L}(cJ(\kappa))$ into $\mathfrak{L}(\overline{\mathbb{R}})$ given by

$$(r, \text{—})_j \longmapsto \begin{cases} (r, \text{—}), & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \quad \text{and} \quad (\text{—}, r)_j \longmapsto \begin{cases} (\text{—}, r), & \text{if } j = i, \\ 1, & \text{if } j \neq i, \end{cases}$$

turn the defining relations (ch0)–(ch4) into identities in $\mathfrak{L}(\overline{\mathbb{R}})$ and thus determine a surjective frame homomorphism $h_i \colon \mathfrak{L}(cJ(\kappa)) \to \mathfrak{L}(\overline{\mathbb{R}})$ (such that $h_i \circ \pi_i$ is the identity in $\mathfrak{L}(\overline{\mathbb{R}})$):

(1) $(r, \text{—})_i \wedge (s, \text{—})_j = 0$ if and only if $i \neq j$.
(2) $(r, \text{—})_i \wedge (\text{—}, s)_i = 0$ if and only if $r \geq s$.
(3) $(r, \text{—})_i \vee (\text{—}, s)_j = 1$ if and only if $r < s$ and $i = j$.
(4) $(\text{—}, r)_i \vee (\text{—}, s)_j = 1$ if and only if $i \neq j$.
(First note that $(\text{—}, r)_i \vee (\text{—}, s)_i = (\text{—}, r \vee s)_i \neq 1$, by (ch4). On the other hand, if $i \neq j$ then $(\text{—}, r)_i \vee (\text{—}, s)_j \geq (\text{—}, r)_i \vee (r - 1, \text{—})_i = 1$.)

(5) $\bigvee_{i\in I}\bigvee_{r\in\mathbb{Q}}(r,\text{—})_i\neq 1$.
(Indeed $\bigvee_{i\in I}\bigvee_{r\in\mathbb{Q}}(r,\text{—})_i=1$ would imply $\bigvee_{r\in\mathbb{Q}}(r,\text{—})=1$ in $\mathfrak{L}(\overline{\mathbb{R}})$, a contradiction.)
(6) For each $i\in I$, $\bigvee_{r\in\mathbb{Q}}(\text{—},r)_i\neq 1$.
($\bigvee_{r\in\mathbb{Q}}(\text{—},r)_i=1$ would imply $\bigvee_{r\in\mathbb{Q}}(\text{—},r)=1$ in $\mathfrak{L}(\overline{\mathbb{R}})$.)

Lemma 3.7. *All the following elements of $\mathfrak{L}(cJ(\kappa))$ are prime (hence maximal):*
(a) $\bigvee_{r>t}(r,\text{—})_i\vee\bigvee_{r<t}(\text{—},r)_i$ *for all* $t\in\mathbb{R}$ *and* $i\in I$.
(b) $\bigvee_{i\in I}\bigvee_{r\in\mathbb{Q}}(r,\text{—})_i$.
(c) $\bigvee_{r\in\mathbb{Q}}(\text{—},r)_i$ *for all* $i\in I$.

Proof: First note that since $\mathfrak{L}(cJ(\kappa))$ is a regular frame, any prime element is maximal.

We only show the case for (b), the others may be proved similarly.

By property (5) above, the element $p=\bigvee_{r\in\mathbb{Q}}\bigvee_{i\in I}(r,\text{—})_i$ is not the top element. Clearly, p is prime iff the map

$$h\colon \mathfrak{L}(cJ(\kappa))\to\{\,0<1\,\},$$

given by $h(x)=0$ if $x\leq p$ and $h(x)=1$ otherwise, is a frame homomorphism. For that it suffices to show that the assignments $h(r,\text{—})_i=0$ iff $(r,\text{—})_i\leq p$ and $h(\text{—},r)_i=0$ iff $(\text{—},r)_i\leq p$ send the defining relations into identities. But $(r,\text{—})_i\leq p$ for any $r\in\mathbb{Q}$ and $i\in I$. Hence $h(r,\text{—})_i=0$ for all $r\in\mathbb{Q}$ and $i\in I$. Moreover, $(\text{—},r)_i\leq p$ together with (ch2) would imply $p=1$, hence $h(\text{—},r)_i=1$ for all $r\in\mathbb{Q}$ and $i\in I$. Now it is clear that h turns relations (ch0)-(ch4) into identities in the two-element frame $\{\,0<1\,\}$. ■

Next we compute the spectrum of $\mathfrak{L}(cJ(\kappa))$.

Lemma 3.8. *For each point $p\in\Sigma(\mathfrak{L}(cJ(\kappa)))$ let*

$$\alpha(p)=\bigvee\Big\{\,r\in\mathbb{Q}\mid\bigvee_{i\in I}(r,\text{—})_i\not\leq p\,\Big\}\in\overline{\mathbb{R}}.$$

We have:
(1) $\alpha(p)=-\infty$ *if and only if* $p=\bigvee_{r\in\mathbb{Q}}\bigvee_{i\in I}(r,\text{—})_i$.
(2) *If $\alpha(p)\neq-\infty$, then there is a unique $i_p\in I$ such that $(r,\text{—})_{i_p}\not\leq p$ for some $r\in\mathbb{Q}$.*
(3) *If $\alpha(p)\neq-\infty$, then $\alpha(p)=\bigwedge\{\,s\in\mathbb{Q}\mid(\text{—},s)_{i_p}\not\leq p\,\}$.*
(4) *If $\alpha(p)\in\mathbb{R}$, then $p=\big(\bigvee_{r>\alpha(p)}(r,\text{—})_{i_p}\big)\vee\big(\bigvee_{s<\alpha(p)}(\text{—},s)_{i_p}\big)$.*

(5) *If* $\alpha(p)=+\infty$, *then* $p=\bigvee_{r\in\mathbb{Q}}(\text{—},r)_{i_p}$.

Proof: (1) Clearly $\alpha(p)=-\infty$ iff $\bigvee_{r\in\mathbb{Q}}\bigvee_{i\in I}(r,\text{—})_i\le p$. The conclusion follows from Lem. 3.7 (b).
(2) The existence is obvious from the definition of $\alpha(p)$. For uniqueness, assume that there are distinct $i_p,j_p\in I$ such that $(r,\text{—})_{i_p}\not\le p$ and $(s,\text{—})_{j_p}\not\le p$. Then $(r,\text{—})_{i_p}\wedge(s,\text{—})_{j_p}\not\le p$ since p is prime, which contradicts (ch0).
(3) Let $r\in\mathbb{Q}$ such that $\bigvee_{i\in I}(r,\text{—})_i\not\le p$. Then there is an $i\in I$ satisfying $(r,\text{—})_i\not\le p$. By uniqueness of i_p, $i=i_p$. Let $s\in\mathbb{Q}$ such that $(\text{—},s)_{i_p}\not\le p$. Then $r\le s$ (otherwise, by (ch0), $s<r$ would imply $0=(r,\text{—})_{i_p}\wedge(\text{—},s)_{i_p}\not\le p$). Hence $\alpha(p)\le\bigwedge\{\,s\in\mathbb{Q}\mid(\text{—},s)_{i_p}\not\le p\,\}$. The inequality cannot be strict, otherwise there would exist $r_1,s_1\in\mathbb{Q}$ such that

$$\alpha(p)<r_1<s_1<\bigwedge\{\,s\in\mathbb{Q}\mid(\text{—},s)_{i_p}\not\le p\,\},$$

and then $(r_1,\text{—})_{i_p}\le p$ and $(\text{—},s_1)_{i_p}\le p$, a contradiction (since $1=(r_1,\text{—})_{i_p}\vee(\text{—},s_1)_{i_p}$ by (ch2)).
(4) Suppose $\alpha(p)\in\mathbb{R}$. By Lem. 3.7(a), it is enough to show that for all $r>\alpha(p)$ and all $s<\alpha(p)$ one has $(r,\text{—})_{i_p}\le p$ and $(\text{—},s)_{i_p}\le p$. Now, the former inequality follows from the definition of $\alpha(p)$ while the latter follows from (3).
(5) It follows from (3) that $\bigvee_{r\in\mathbb{Q}}(\text{—},r)_{i_p}\le p$. The equality follows then from Lem. 3.7 (c). ∎

Proposition 3.9. *The spectrum of* $\mathfrak{L}(cJ(\kappa))$ *is homeomorphic to the compact hedgehog space* $\Lambda J(\kappa)$.

Proof: Consider the map $\pi\colon\Sigma(\mathfrak{L}(cJ(\kappa)))\longrightarrow\Lambda J(\kappa)$ given by

$$\pi(p)=\begin{cases}(\alpha(p),i_p) & \text{if } \alpha(p)\neq-\infty;\\ -\infty & \text{otherwise.}\end{cases}$$

It readily follows from Lem. 3.8 (assertions 1, 4 and 5) that π is one-to-one. Let us show that π is also onto.

By 3.8 (1), $\pi\big(\bigvee_{r\in\mathbb{Q}}\bigvee_{i\in I}(r,\text{—})_i\big)=-\infty$, and, by 3.8 (5), $\pi\big(\bigvee_{r\in\mathbb{Q}}(\text{—},r)_i\big)=(+\infty,i)$. For each $t\in\mathbb{R}$ and $i\in I$ set

$$p_{(t,i)}=\Big(\bigvee_{r>t}(r,\text{—})_i\Big)\vee\Big(\bigvee_{r<t}(\text{—},r)_i\Big).$$

It is straightforward to check that $\bigvee_{j\in I}(s,\text{—})_j\le p_{(t,i)}$ if and only if $s\ge t$. Hence $\alpha(p_{(t,i)})=\bigvee\{\,s\mid s<t\,\}=t$. Moreover, if we select $s<t$, then

we have $(s,—)_i \not\leq p_{(t,i)}$ (as otherwise $p_{(t,i)} = (s,—)_i \vee p_{(t,i)} = 1$ by (ch2), contradicting maximality). Therefore $i_{p_{(t,i)}} = i$ and so $\pi(p_{(t,i)}) = (t,i)$.

Furthermore, π is lower semicontinuous, since

$$\pi^{-1}((r,+\infty] \times \{i\}) = \{\, p \in \Sigma(\mathfrak{L}(cJ(\kappa))) \mid (r,—)_i \not\leq p \,\} = \Sigma_{(r,—)_i}$$

is open for every $r \in \mathbb{Q}$ and $i \in I$, and upper semicontinuous, since

$$\pi^{-1}(J(\kappa) \setminus ([r,+\infty] \times \{i\})) = \{\, p \in \Sigma(\mathfrak{L}(cJ(\kappa))) \mid (—,r)_i \not\leq p \,\} = \Sigma_{(—,r)_i}$$

is open for every $r \in \mathbb{Q}$ and $i \in I$. Hence π is continuous.

Finally, let us prove that π is an open map. Note that, since $\mathfrak{L}(cJ(\kappa)))$ is generated by $\{(r,—)_i, (—,r)_i \mid r \in \mathbb{Q}, i \in I\}$ and π is a bijection, it suffices to show that the sets $\pi(\Sigma_{(r,—)_i})$ and $\pi(\Sigma_{(—,r)_i})$ are open for every $r \in \mathbb{Q}$ and $i \in I$. We have

$$\pi(\Sigma_{(r,—)_i}) = \{\, \pi(p) \mid (r,—)_i \not\leq p \,\} = \{\, (t,i) \mid t > r \,\} = (r,1] \times \{i\} \qquad \text{and}$$
$$\pi(\Sigma_{(—,r)_i}) = \{\, \pi(p) \mid (—,r)_i \not\leq p \,\} = J(\kappa) \setminus ([r,1] \times \{i\}).$$

Hence π is a homeomorphism. ■

4. Semicontinuities

We introduce now the following classes of frame homomorphisms:
A *compact hedgehog-valued*

– *function* on L is a frame homomorphism $f \colon \mathfrak{L}(cJ(\kappa)) \to \mathcal{S}(L)^{op}$,
– *lower semicontinuous function* on L is a frame homomorphism $f \colon \mathfrak{L}(cJ(\kappa)) \to \mathcal{S}(L)^{op}$ such that $f((r,—)_i)$ is closed for every $r \in \mathbb{Q}$ and $i \in I$,
– *upper semicontinuous function* on L is a frame homomorphism $f \colon \mathfrak{L}(cJ(\kappa)) \to \mathcal{S}(L)^{op}$ such that $f((—,r)_i)$ is closed for every $r \in \mathbb{Q}$ and $i \in I$,
– *continuous function* on L is a frame homomorphism $f \colon \mathfrak{L}(cJ(\kappa)) \to \mathcal{S}(L)^{op}$ such that $f((—,r)_i)$ and $f((r,—)_i)$ are closed for every $r \in \mathbb{Q}$ and $i \in I$.

The corresponding classes of compact hedgehog-valued functions will be denoted by, respectively,

$$\mathsf{F}_\kappa(L),\ \mathsf{LSC}_\kappa(L),\ \mathsf{USC}_\kappa(L),\ \mathsf{C}_\kappa(L).$$

Note that $\mathsf{C}_\kappa(L) = \mathsf{LSC}_\kappa(L) \cap \mathsf{USC}_\kappa(L)$.

The following lemma is an immediate consequence of the definition of the i-th projection π_i.

Lemma 4.1. *Let $f \in \mathsf{F}_\kappa(L)$. Then:*
(1) *$f \in \mathsf{LSC}_\kappa(L)$ if and only if $f \circ \pi_i \in \overline{\mathsf{LSC}}(L)$ for all $i \in I$.*
(2) *$f \in \mathsf{USC}_\kappa(L)$ if and only if $f \circ \pi_i \in \overline{\mathsf{USC}}(L)$ for all $i \in I$.*
(3) *$f \in \mathsf{C}_\kappa(L)$ if and only if $f \circ \pi_i \in \overline{\mathsf{C}}(L)$ for all $i \in I$.*

Let

$$\mathcal{H} = \left\{ h_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)^{op} \right\}_{i\in I} \subseteq \overline{\mathsf{F}}(L)$$

be a family of extended real-valued functions on L and $S_i = h_i(\bigvee_{r\in\mathbb{Q}}(r,\text{—})$. We say that $\mathcal{H}$ is *disjoint* if $S_i \sqcap S_j = 0$ (i.e. $S_i \vee S_j = L$) for every $i \neq j$.

Proposition 4.2. *If $\mathcal{H} = \{ h_i \mid i \in I \}$ is a disjoint κ-family of extended real-valued functions on L, then there is a unique $f \in \mathsf{F}_\kappa(L)$ such that $f \circ \pi_i = h_i$ for all $i \in I$. Conversely, given $f \in \mathsf{F}_\kappa(L)$, the κ-family $\{ f \circ \pi_i \}_{i\in I}$ is disjoint.*

Proof: *Uniqueness.* If $f \circ \pi_i = h_i$ for all i, then $f((r,\text{—})_i) = f(\pi_i(r,\text{—})) = h_i(r,\text{—})$ and $f(\text{—},r)_i = f(\pi_i(\text{—},r)) = h_i(\text{—},r)$, and thus f is uniquely determined.

Existence. Define $f \colon \mathfrak{L}(cJ(\kappa)) \to \mathcal{S}(L)^{op}$ by the assignments $f(\text{—},r)_i = h_i(\text{—},r)$ and $f((r,\text{—})_i) = h_i(r,\text{—})$. Let us confirm that it turns the relations (ch0)–(ch4) into identities in $\mathcal{S}(L)^{op}$:

(ch0) Let $i \neq j$. Then $f((r,\text{—})_i) \sqcap f((s,\text{—})_j) = h_i(r,\text{—}) \sqcap h_j(s,\text{—}) \leq S_i \sqcap S_j = 0$.
(ch1) Let $r \geq s$. Then $f((r,\text{—})_i) \sqcap f((\text{—},s)_i) = h_i(r,\text{—}) \sqcap h_i(\text{—},s) = h_i((r,\text{—}) \wedge (\text{—},s)) = h_i(0) = 0$.
(ch2) Let $r < s$. Then $f((r,\text{—})_i) \sqcup f((\text{—},s)_i) = h_i(r,\text{—}) \sqcup h_i(\text{—},s) = h_i((r,\text{—}) \vee (\text{—},s)) = h_i(1) = 1$.
(ch3) $f((r,\text{—})_i) = h_i(r,\text{—}) = h_i\left(\bigvee_{s>r}(s,\text{—})\right) = \bigsqcup_{s>r} f((s,\text{—})_i)$.
(ch4) Similar to (ch3).

Trivially, $f \circ \pi_i = h_i$ for all $i \in I$. The converse statement is an easy consequence of (ch0) and the frame distributive law. ■

In the following, $f_{\mathcal{H}}$ is always the compact hedgehog-valued function provided by 4.2. We conclude with some immediate corollaries of 4.1 and 4.2.

Corollary 4.3. *Let $\mathcal{H} \subseteq \overline{\mathsf{F}}(L)$ be a disjoint κ-family. Then:*
(1) *$f_{\mathcal{H}} \in \mathsf{LSC}_\kappa(L)$ if and only if $f \in \overline{\mathsf{LSC}}(L)$ for all $f \in \mathcal{H}$.*

(2) *$f_{\mathcal{H}} \in \mathsf{USC}_\kappa(L)$ if and only if $f \in \overline{\mathsf{USC}}(L)$ for all $f \in \mathcal{H}$.*
(3) *$f_{\mathcal{H}} \in \mathsf{C}_\kappa(L)$ if and only if $f \in \overline{\mathsf{C}}(L)$ for all $f \in \mathcal{H}$.*

Corollary 4.4. *Let $\{a_i\}_{i\in I}$ be a disjoint κ-family of elements of L. Then a_i is a cozero element for every $i \in I$ if and only if there is an $f \in \mathsf{C}_\kappa(L)$ such that $a_i = \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i)$ for all $i \in I$.*

Given a complemented sublocale S of a locale L, the *extended characteristic function* $\chi_S \in \overline{\mathsf{F}}(L)$ (see [5, Example 2]) is defined by

$$\chi_S(r,\text{—}) = S^* \quad \text{and} \quad \chi_S(\text{—},r) = S, \quad r \in \mathbb{Q}.$$

Obviously, $\chi_S \in \overline{\mathsf{LSC}}(L)$ (resp. $\chi_S \in \overline{\mathsf{USC}}(L)$) if and only if S is an open (resp. closed) sublocale.

Remark 4.5. A κ-family $\mathcal{C} = \{S_i\}_{i\in I}$ of complemented sublocales of L is pairwise disjoint in $\mathcal{S}(L)$ if and only if the corresponding κ-family $\{\,\chi_{S_i}\,\}_{i\in I}$ of extended characteristic functions is disjoint. Hence, by Prop. 4.2, it induces an $f \in \mathsf{F}_\kappa(L)$ such that $f \circ \pi_i = \chi_{S_i}$ for all $i \in I$. This f will be denoted by $\chi_{\mathcal{C}}$ and we shall refer to it as the *characteristic function* of the family $\mathcal{C}$.

Finally, from Cor. 4.3 we obtain:

Corollary 4.6. *Let $\mathcal{C} = \{S_i\}_{i\in I}$ be a pairwise disjoint κ-family of complemented sublocales of L. Then:*
(1) *$\chi_{\mathcal{C}} \in \mathsf{LSC}_\kappa(L)$ if and only if each S_i is open.*
(2) *$\chi_{\mathcal{C}} \in \mathsf{USC}_\kappa(L)$ if and only if each S_i is closed.*
(3) *$\chi_{\mathcal{C}} \in \mathsf{C}_\kappa(L)$ if and only if each S_i is clopen.*

5. Variants of collectionwise normality

A family $\{a_i\}_{i\in I}$ of elements of L is said to be *disjoint* if $a_i \wedge a_j = 0$ for every $i \neq j$. It is *discrete* (resp. *co-discrete*) if there is a cover C of L such that for any $c \in C$, $c \wedge a_i = 0$ (resp. $c \le a_i$) for all i with at most one exception. Note, in particular, that any discrete family is clearly disjoint, and that a pair $\{a, b\}$ is co-discrete if and only if $a \vee b = 1$. Trivially, if a finite $\{a_1, a_2, \ldots, a_n\}$ is co-discrete then $a_1 \vee a_2 \vee \cdots \vee a_n = 1$, but not conversely for $n \ge 3$.

Recall from [19] (see also [27] for more information) that a frame is *κ-collectionwise normal* if for every co-discrete κ-family $\{a_i\}_{i\in I}$, there is a discrete $\{b_i\}_{i\in I}$ with $b_i \vee a_i = 1$ for all $i \in I$. We take this opportunity to rectify a slip in [19]: we can replace discrete families by disjoint families in the definition of *κ-collectionwise normal* frames, as we shall show in 5.2 below.

Lemma 5.1. (Cf. [28, Lem. 1.13]) *For any co-discrete $\{a_i\}_{i\in I} \subseteq L$ and $b \in L$, $b \vee \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(b \vee a_i)$.*

Proposition 5.2. *A frame L is κ-collectionwise normal if and only if for any co-discrete $\{a_i\}_{i\in I}$ with $|I| \le \kappa$, there is a disjoint $\{b_i\}_{i\in I}$ such that $a_i \vee b_i = 1$ for all $i \in I$.*

Proof: The implication '$\Rightarrow$' is obvious since any discrete family is disjoint.

Conversely, let $\{a_i\}_{i\in I}$ be a co-discrete family. Then there is a disjoint family $\{b_i\}_{i\in I}$ such that $b_i \vee a_i = 1$ for all $i \in I$. Set

$$D := \{\, a \in L \mid a \wedge b_i \neq 0 \text{ for at most one } i \,\}$$

and $\overline{d} := \bigvee D$. Clearly $b_i \in D$, and hence $b_i \le \overline{d}$, for each i. Then, by the previous lemma, $\overline{d} \vee \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(\overline{d} \vee a_i) \ge \bigwedge_{i\in I}(b_i \vee a_i) = 1$. Moreover, since κ-collectionwise normality implies normality [19], there are $u, v \in L$ such that $u \vee \bigwedge_{i\in I} a_i = 1 = v \vee \overline{d}$ and $u \wedge v = 0$. The family

$$\{u_i := b_i \wedge u\}_{i\in I}$$

is then the required discrete system. Indeed, $C := D \cup \{v\}$ is a cover of L (since $\bigvee C = \overline{d} \vee v = 1$), each $c \in C$ meets at most one u_i (since $u_i \wedge v \le u \wedge v = 0$) and $u_i \vee a_i = (b_i \vee a_i) \wedge (u \vee a_i) = u \vee a_i \ge u \vee \bigwedge_{i\in I} a_i = 1$ for every i. ■

Next, we define a sublocale S of L to be *z^c_κ-embedded* in L if for every disjoint κ-family $\{a_i\}_{i\in I}$ of cozero elements of S, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $\nu_S(b_i) = a_i$ (that is, such that $\mathfrak{o}_S(a_i) = S \cap \mathfrak{o}(b_i)$) for every $i \in I$.

Remark 5.3. In view of [7, Thm. 3.8] (see also [19]), we shall say that a sublocale S of L is *z_κ-embedded* in L if for every $f\colon \mathfrak{L}(J(\kappa)) \to S$, there is a $g\colon \mathfrak{L}(J(\kappa)) \to L$ such that $\nu_S(\bigvee_{r\in\mathbb{Q}} g(r,\text{—})_i) = \bigvee_{r\in\mathbb{Q}} f(r,\text{—})_i$ for every $i \in I$. We shall see in Lem. 6.2 below that the notion above of z^c_κ-embedding is precisely what we get by replacing the (metric) hedgehog frame with the *compact* hedgehog frame in the notion of z_κ-embedding: a sublocale S is z^c_κ-embedded in L if and only if for every $f\colon \mathfrak{L}(cJ(\kappa)) \to S$, there is a $g\colon \mathfrak{L}(J(\kappa)) \to L$ such that $\nu_S(\bigvee_{r\in\mathbb{Q}} g(r,\text{—})_i) = \bigvee_{r\in\mathbb{Q}} f(r,\text{—})_i$ for every $i \in I$. This is the point-free counterpart of the notion of κ-total z-embedding from [18]. Since these two cardinal generalizations of z-embeddedness are generally unrelated, we prefer to use the term 'z^c_κ-embedding'.

Clearly, z^c_1-embedding coincides with the usual notion of z-embedding (cf. [2]). Moreover, z^c_2-embedding is also equivalent to z-embedding by [12, Prop. 3.3].

Furthermore, a locale will be said to be *totally κ-collectionwise normal* if every closed sublocale is z^c_κ-embedded. A locale is *totally collectionwise normal* if it is totally κ-collectionwise normal for every cardinal κ.

We shall need the following localic version of the pasting lemma:

Proposition 5.4. (cf. Prop. 4.4 in [26] or Prop. 7.2 in [19]) *Let L and M be locales, $a_1, a_2 \in M$ and $h_i\colon L \to \mathfrak{c}(a_i)$ $(i = 1, 2)$ frame homomorphisms such that $h_1(x) \vee a_2 = h_2(x) \vee a_1$ for every $x \in L$. Then the map $h\colon L \to \mathfrak{c}(a_1 \wedge a_2)$ given by $h(x) = h_1(x) \wedge h_2(x)$ is a frame homomorphism such that the triangle*

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[dr, "h_i"'] & \mathfrak{c}(a_1 \wedge a_2) \arrow[d, "\nu_{\mathfrak{c}(a_i)}"] \\ & \mathfrak{c}(a_i) \end{tikzcd}$$

commutes for $i = 1, 2$.

Proposition 5.5. *Every totally κ-collectionwise normal frame is κ-collectionwise normal.*

Proof: Let $\{a_i\}_{i\in I}$ be a co-discrete κ-family in a frame L. Fix some $i \in I$ and consider constant extended real valued functions $h_1^{(i)}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a_i)$ and $h_2^{(i)}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(\bigwedge_{j\neq i} a_j)$, given by

$$h_1^{(i)}(r,\text{—}) = 1, \quad h_1^{(i)}(\text{—},r) = a_i \quad \text{and} \quad h_2^{(i)}(r,\text{—}) = \bigwedge_{j\neq i} a_j, \quad h_2^{(i)}(\text{—},r) = 1.$$

One has $a_i \vee \bigwedge_{j\neq i} a_j = \bigwedge_{j\neq i}(a_i \vee a_j) = 1$ (the first equality follows from 5.1 and the obvious fact that any subfamily of a co-discrete family is co-discrete, whereas the second equality holds because the family $\{\mathfrak{c}(a_i)\}_{i\in I}$ is pairwise disjoint whenever $\{a_i\}_{i\in I}$ is co-discrete). Then, by Prop. 5.4, there is a frame homomorphism

$$h^{(i)}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a_i) \vee \mathfrak{c}(\bigwedge_{j\neq i} a_j) = \mathfrak{c}(\bigwedge_{j\in I} a_j)$$

given by $h^{(i)}(x) = h_1^{(i)}(x) \wedge h_2^{(i)}(x)$. But

$$\bigvee_{r\in\mathbb{Q}} h^{(i)}(r,\text{—}) = \bigwedge_{j\neq i} a_j,$$

thus $\bigwedge_{j\neq i} a_j$ is a cozero element in $\mathfrak{c}(\bigwedge_{j\in I} a_j)$. Hence the family $\{\bigwedge_{j\neq i} a_j\}_{i\in I}$ is a disjoint family of cozero elements in the closed sublocale $\mathfrak{c}(\bigwedge_{j\in I} a_j)$. Finally, by assumption, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $b_i \vee \bigwedge_{j\in I} a_j = \bigwedge_{j\neq i} a_j$; in particular,

$$b_i \vee a_i = b_i \vee \big(\bigwedge_{j\in I} a_j\big) \vee a_i = \big(\bigwedge_{j\neq i} a_j\big) \vee a_i = \bigwedge_{j\neq i} (a_j \vee a_i) = 1. \qquad \blacksquare$$

Finally we recall again that κ-collectionwise normality implies normality [19] to conclude:

Corollary 5.6. *Every totally κ-collectionwise normal frame is normal.*

6. Disjoint extensions

Let $S \subseteq L$ be a sublocale and $f \in \mathsf{C}_\kappa(S)$. An $\overline{f} \in \mathsf{C}_\kappa(L)$ is said to be a (*continuous*) *extension* of f to L if $\nu_S \circ \overline{f} = f$.

We say that a disjoint κ-family $\mathcal{H}_S \subseteq \overline{\mathsf{C}}(S)$ *can be disjointly extended* to L if there is a disjoint κ-family

$$\mathcal{H} = \big\{\, \overline{f} \mid f \in \mathcal{H}_S \,\big\} \subseteq \overline{\mathsf{C}}(L)$$

in which each $\overline{f}$ is an extension of f. Further, a locale L will be said to have the *κ-disjoint extension property* if for each $a \in L$ every disjoint κ-family $\mathcal{H}_{\mathfrak{c}(a)} \subseteq \overline{\mathsf{C}}(\mathfrak{c}(a))$ can be disjointly extended to L. The following result can be easily inferred from Lem. 4.1 (3) and Prop. 4.2.

Proposition 6.1. *The following are equivalent for a locale L.*

(i) *L has the κ-disjoint extension property.*
(ii) *For each $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

We can now characterize z_κ^c-embeddedness as a property about appropriate compact hedgehog-valued functions.

Lemma 6.2. *The following are equivalent for a sublocale $S \subseteq L$.*

(i) *S is z_κ^c-embedded in L.*
(ii) *For each $f \in \mathsf{C}_\kappa(S)$, there is a $g \in \mathsf{C}_\kappa(L)$ such that $\nu_S\big(\bigvee_{r\in\mathbb{Q}} g((r,\text{—})_i)\big) = \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i)$ for every $i \in I$.*

Proof: (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}_\kappa(S)$. For each $i \in I$ set

$$a_i := \bigvee_{r\in\mathbb{Q}} (f \circ \pi_i)(r, \text{—}) = \bigvee_{r\in\mathbb{Q}} f((r, \text{—})_i).$$

$\{a_i\}_{i\in I}$ is a disjoint κ-family of cozero elements of S. Then, by assumption, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $\nu_S(b_i) = a_i$ for every $i \in I$. Applying Cor. 4.4 to $\{b_i\}_{i\in I}$ we get a $g \in \mathsf{C}_\kappa(L)$ such that $b_i = \bigvee_{r\in\mathbb{Q}} g((r, \text{—})_i)$. Finally,

$$\nu_S\big(\bigvee_{r\in\mathbb{Q}} g((r, \text{—})_i)\big) = \nu_S(b_i) = \bigvee_{r\in\mathbb{Q}} f((r, \text{—})_i)$$

for every $i \in I$.

(ii) $\Longrightarrow$ (i): Let $\{a_i\}_{i\in I}$ be a disjoint κ-family of cozero elements of S, and take the $f \in \mathsf{C}_\kappa(S)$, provided by Cor. 4.4, that satisfies $a_i = \bigvee_{r\in\mathbb{Q}} f((r, \text{—})_i)$ for all $i \in I$. By hypothesis, there is a $g \in \mathsf{C}_\kappa(L)$ such that $\nu_S\left(\bigvee_{r\in\mathbb{Q}} g\left((r, \text{—})_i\right)\right) = a_i$ for all $i \in I$. Set $b_i := \bigvee_{r\in\mathbb{Q}} g((r, \text{—})_i)$ for each $i \in I$. Clearly, $\{b_i\}_{i\in I}$ is the claimed disjoint family. ■

Theorem 6.3 (Tietze-type theorem for total κ-collectionwise normality)**.** *The following are equivalent for a locale L.*

(i) *L is totally κ-collectionwise normal.*

(ii) *For each $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Proof: (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ and set $a_i := \bigvee_{r\in\mathbb{Q}} f\left((r, \text{—})_i\right)$. By the previous lemma, there is a $g \in \mathsf{C}_\kappa(L)$ such that $a \vee b_i = a_i$ for every $i \in I$, where $b_i := \bigvee_{r\in\mathbb{Q}} g\left((r, \text{—})_i\right)$. For each $i \in I$, consider

$$h_1^{(i)} = f \circ \pi_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a) \quad \text{and} \quad h_2^{(i)} = \mathbf{0} \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(b_i)$$

(the latter defined by $h_2^{(i)}(r, \text{—}) = b_i$ and $h_2^{(i)}(\text{—}, r) = 1$ for every $r \in \mathbb{Q}$). Let us show that $h_1^{(i)}(x) \vee b_i = h_2^{(i)}(x) \vee a$ for every $x \in \mathfrak{L}(\overline{\mathbb{R}})$ by checking it for the generators of $\mathfrak{L}(\overline{\mathbb{R}})$. For each $r \in \mathbb{Q}$ we have:

$$a \vee b_i \le h_1^{(i)}(r, \text{—}) \vee b_i = f((r, \text{—})_i) \vee b_i \le a_i \vee b_i = a \vee b_i.$$

Hence $h_1^{(i)}(r, \text{—}) \vee b_i = a \vee b_i = h_2^{(i)}(r, \text{—}) \vee a$. On the other hand, pick some rational $t < r$ and conclude that

$$\begin{aligned} h_1^{(i)}(\text{—}, r) \vee b_i = (h^{(i)}(\text{—}, r) \vee a) \vee b_i = f((\text{—}, r)_i) \vee (a \vee b_i) = f((\text{—}, r)_i) \vee a_i \\ \ge f((\text{—}, r)_i) \vee a_i \ge f((\text{—}, r)_i) \vee f\left((t, \text{—})_i\right) = 1. \end{aligned}$$

Hence $h_1^{(i)}(-,r) \vee b_i = 1 = h_2^{(i)}(-,r) \vee a$.

Consequently, by Prop. 5.4, there exists a frame homomorphism

$$h_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a \wedge b_i),$$

such that $\nu_1 \circ h_i = h_1^{(i)}$ and $\nu_2 \circ h_i = h_2^{(i)}$, where $\nu_1 \colon \mathfrak{c}(a \wedge b_i) \to \mathfrak{c}(a)$ and $\nu_2 \colon \mathfrak{c}(a \wedge b_i) \to \mathfrak{c}(b_i)$ are the associated surjections. Since L is normal (by Cor. 5.6), the standard point-free version of Tietze's extension theorem yields a frame homomorphism $g_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $\nu \circ g_i = h_i$, where $\nu \colon L \to \mathfrak{c}(a \wedge b_i)$ is the corresponding surjection. Observe that the family $\{g_i\}_{i\in I}$ is disjoint since

$$g_i(\bigvee_{r\in\mathbb{Q}} (r,-)) \leq h_i(\bigvee_{r\in\mathbb{Q}} (r,-)) \leq h_2^{(i)}(\bigvee_{r\in\mathbb{Q}} (r,-)) = b_i$$

and $\{b_i\}_{i\in I}$ is disjoint. Hence, consider the hedgehog-valued function $h \colon \mathfrak{L}(cJ(\kappa)) \to L$ defined by $h \circ \pi_i = g_i$ for all $i \in I$ (provided by Prop. 4.2), which is continuous by Cor. 4.3. This is the claimed extension. Indeed, denote by $\nu_{\mathfrak{c}(a)} \colon L \to \mathfrak{c}(a)$ the surjection associated to $\mathfrak{c}(a)$. Then

$$\nu_{\mathfrak{c}(a)} \circ h \circ \pi_i = \nu_{\mathfrak{c}(a)} \circ g_i = \nu_1 \circ \nu \circ g_i = \nu_1 \circ h_i = h_1^{(i)} = f \circ \pi_i,$$

and thus $\nu_{\mathfrak{c}(a)} \circ h = f$ follows from the uniqueness of Prop. 4.2.

(ii) $\Longrightarrow$ (i) is an immediate consequence of the implication (ii) $\Longrightarrow$ (i) in the previous lemma. ∎

We end up this section with the investigation of the case $\kappa = \aleph_0$.

Proposition 6.4. *A sublocale $S \subseteq L$ is $z^c_{\aleph_0}$-embedded if and only if it is z-embedded.*

Proof: The "only if" part is obvious. Conversely, assume that S is z-embedded and let $\{a_n\}_{n\in\mathbb{N}}$ be a countable disjoint family of cozero elements of S. For each $n \in \mathbb{N}$, let b_n be the join in S of the family $\{\, a_m \mid m \neq n \,\}$. Note that $\{\, a_n, b_n \,\}$ is a disjoint pair of cozero elements of S (since a countable join of cozero elements is again a cozero). Then, by [12, Prop. 3.3], there is a disjoint pair $\{\, c_n, d_n \,\}$ of cozero elements of L such that $\nu_S(c_n) = a_n$ and $\nu_S(d_n) = b_n$. Take

$$u_1 = c_1 \quad \text{and} \quad u_n = c_n \wedge d_1 \wedge \cdots \wedge d_{n-1} \quad (n > 1).$$

$\{u_n\}_{n\in\mathbb{N}}$ is the required disjoint family of cozero elements of L that extends $\{a_n\}_{n\in\mathbb{N}}$. Indeed, each u_n is a cozero element (because cozero elements are closed under finite meets); for disjointness, let $n < m$ and observe that $u_n \wedge u_m \leq c_n \wedge d_n = 0$. Finally, $\nu_S(u_1) = \nu_S(c_1) = a_1$ and, for $n > 1$,

$$\nu_S(u_n) = \nu_S(c_n) \wedge \nu_S(d_1) \wedge \cdots \wedge \nu_S(d_{n-1}) = a_n \wedge b_1 \wedge \cdots \wedge b_{n-1}.$$

Note that $a_n \leq b_m$ for $m = 1, \ldots, n-1$, hence $\nu_S(u_n) = a_n$ as claimed. ■

Corollary 6.5. *For $1 \leq \kappa \leq \aleph_0$, total κ-collectionwise normality is equivalent to normality.*

Corollary 6.6. *The following are equivalent for a locale L.*

(i) *L is normal.*
(ii) *For each $a \in L$, every $f \in \mathsf{C}_{\aleph_0}(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_{\aleph_0}(L)$.*

7. Insertion results

Recall the partial order in $\overline{\mathsf{F}}(L)$ (2.4). We may extend it to $\mathsf{F}_\kappa(L)$ by defining, for any $f, g \in \mathsf{F}_\kappa(L)$,

$$f \leq g \;\equiv\; f \circ \pi_i \leq g \circ \pi_i \text{ for every } i \in I.$$

Inspired by [18], we obtain a first Katětov-Tong-type insertion result for $\mathsf{F}_\kappa(L)$ that characterizes normality.

Theorem 7.1. *The following are equivalent for a locale L.*

(i) *L is normal.*
(ii) *For every $\kappa \geq 1$, and every $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*
(iii) *There is a $\kappa \geq 1$ such that for every $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Proof: (i) $\Longrightarrow$ (ii): Let $\kappa \geq 1$, $|I| = \kappa$, $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ with $f \leq g$ and $i \in I$. Then $f \circ \pi_i \leq g \circ \pi_i$ in $\overline{\mathsf{F}}(L)$, and $f \circ \pi_i \in \overline{\mathsf{USC}}(L)$ and $g \circ \pi_i \in \overline{\mathsf{LSC}}(L)$ (by Cor. 4.1). By the standard point-free version of Katětov-Tong insertion theorem [16], there is an $h_i \in \overline{\mathsf{C}}(L)$ such that $f \circ \pi_i \leq h_i \leq g \circ \pi_i$. Since $\{g \circ \pi_i\}_{i \in I}$ is a disjoint family, then so is $\{h_i\}_{i \in I}$. Let $h \in \mathsf{C}_\kappa(L)$ be the function defined by $h \circ \pi_i = h_i$ for all $i \in I$ (provided by Prop. 4.2 and Cor. 4.3). Obviously, $f \leq h \leq g$.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): Let $|I| = \kappa$ and fix some $i_0 \in I$. Let $a, b \in L$ such that $a \vee b = 1$.

Consider the pairwise disjoint κ-families $\mathcal{C} = \{S_i\}_{i\in I}$ and $\mathcal{D} = \{T_i\}_{i\in I}$ defined by $S_{i_0} = \mathfrak{c}(a)$, $T_{i_0} = \mathfrak{o}(b)$ and $S_i = \mathsf{O} = T_i$ for every $i \neq i_0$. By Cor. 4.6,

$$\chi_{\mathcal{C}} \in \mathsf{USC}_\kappa(L) \quad \text{and} \quad \chi_{\mathcal{D}} \in \mathsf{LSC}_\kappa(L).$$

Moreover, since $a \vee b = 1$ is equivalent to $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ in $\mathcal{S}(L)$, it follows that $\chi_{\mathcal{C}} \leq \chi_{\mathcal{D}}$. Hence, there is an $h \in \mathsf{C}_\kappa(L)$ such that $\chi_{\mathcal{C}} \leq h \leq \chi_{\mathcal{D}}$, from which it follows that $\chi_{\mathfrak{c}(a)} \leq h \circ \pi_{i_0} \leq \chi_{\mathfrak{o}(b)}$. The normality of L follows then from the standard point-free version of Urysohn's Lemma (as e.g. in the formulation of [16, Cor. 8.2]). ■

We end up this section with an insertion result that characterizes total collectionwise normality. To prove it we need the following property of total κ-collectionwise normality:

Lemma 7.2. *Total κ-collectionwise normality is hereditary with respect to closed sublocales.*

Proof: Let $S = \mathfrak{c}(a)$ be a closed sublocale of a totally κ-collectionwise normal locale L. Let $\mathfrak{c}_S(b)$ be a closed sublocale of S and let $\{a_i\}_{i\in I}$ be a disjoint κ-family of cozero elements of $\mathfrak{c}_S(b)$. Since $\mathfrak{c}_S(b) = \mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathfrak{c}(a \vee b)$, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $b_i \vee a \vee b = a_i$ for every $i \in I$. Then $\{b_i \vee a\}_{i\in I} \subseteq S$ is the desired disjoint family extending $\{a_i\}_{i\in I}$. ■

Theorem 7.3. *The following are equivalent for a locale L and a cardinal κ:*

(i) *L is totally κ-collectionwise normal.*
(ii) *For each $a \in L$ and every $f \in \mathsf{USC}_\kappa(\mathfrak{c}(a))$ and $g \in \mathsf{LSC}_\kappa(\mathfrak{c}(a))$ such that $f \leq g$, there exists an $\overline{h} \in \mathsf{C}_\kappa(L)$ such that $f \leq \nu_{\mathfrak{c}(a)} \circ \overline{h} \leq g$.*

Proof: (i) $\Longrightarrow$ (ii): Suppose L is totally κ-collectionwise normal and consider $a \in L$. By Cor. 5.6, L is normal, and so is $\mathfrak{c}(a)$ (because normality is a closed-hereditary property). Let $f \in \mathsf{USC}_\kappa(\mathfrak{c}(a))$ and $g \in \mathsf{LSC}_\kappa(\mathfrak{c}(a))$ such that $f \leq g$. By Thm. 7.1, there is an $h \in C_\kappa(\mathfrak{c}(a))$ such that $f \leq h \leq g$. Then, since $\mathfrak{c}(a)$ is also totally κ-collectionwise normal (by the lemma), the conclusion follows readily from Thm. 6.3.

(ii) $\Longrightarrow$ (i): Let $\mathfrak{c}(a)$ be a closed sublocale of L and let $h \in \mathsf{C}_\kappa(\mathfrak{c}(a))$. Applying the insertion condition to $h \leq h$ we get an $\overline{h} \in \mathsf{C}_\kappa(L)$ such that $h \leq \nu_{\mathfrak{c}(a)} \circ \overline{h} \leq h$. Of course, $h = \nu_{\mathfrak{c}(a)} \circ \overline{h}$ is an extension of h and the conclusion follows then from Thm. 6.3. ■

8. A general view: sublocale selections

For the rest of the article we want to treat several variants of normality and total κ-collectionwise normality at once. The following describes a convenient general setting for this.

An object function $\mathbb{F}$ on the category of frames will be called a *sublocale selection* if $\mathbb{F}(L)$ is a class of complemented sublocales of L. We shall denote by $\mathbb{F}^*$ the sublocale selection defined by

$$\mathbb{F}^*(L) = \{ S^* \mid S \in \mathbb{F}(L) \}.$$

We shall say that $\mathbb{F}$ is closed under (binary, countable, arbitrary) joins (resp. meets) if $\mathbb{F}(L)$ is closed under (binary, countable, arbitrary) joins (resp. meets), taken in $\mathcal{S}(L)$, for any L.

The standard examples of $\mathbb{F}$ are given by selecting closed sublocales, regular closed sublocales, zero sublocales and δ-regular closed sublocales. In the following, these will be denoted

$$\mathbb{F}_{\mathrm{c}},\ \mathbb{F}_{\mathrm{reg}},\ \mathbb{F}_{\mathrm{z}},\ \mathbb{F}_{\delta\mathrm{reg}}$$

respectively.

Now, for any sublocale selection $\mathbb{F}$, a locale L is called $\mathbb{F}$-normal if for any $S, T \in \mathbb{F}(L)$

$$S \cap T = \mathsf{O} \quad \Longrightarrow \quad \exists\ A, B \in \mathbb{F}(L)\colon\ S \cap A = \mathsf{O} = T \cap V,\ A \vee B = L.$$

Examples 8.1. (See [20]) For the selection $\mathbb{F} = \mathbb{F}_{\mathrm{c}}$, $\mathbb{F}$-normality is standard normality and $\mathbb{F}^*$-normality is just extremal disconnectedness, while for $\mathbb{F}_{\mathrm{reg}}$, $\mathbb{F}$-normality is *mild normality* and $\mathbb{F}^*$-normality is also extremal disconnectedness. For $\mathbb{F} = \mathbb{F}_{\mathrm{z}}$, $\mathbb{F}$-normality is a property inherent to any frame while $\mathbb{F}^*$-normality is the property of being an F-frame.

Furthermore, an $f \in \mathsf{F}_\kappa(L)$ will be called

- *lower* $\mathbb{F}$*-semicontinuous* if for every $p < q$ in $\mathbb{Q}$ and $i \in I$, there is an $F^i_{p,q} \in \mathbb{F}(L)$ such that $f(q,\text{—})_i \le F^i_{p,q} \le f(p,\text{—})_i$.
- *upper* $\mathbb{F}$*-semicontinuous* if for every $p < q$ in $\mathbb{Q}$ and $i \in I$, there is a $G^i_{p,q} \in \mathbb{F}(L)$ such that $f(\text{—},p)_i \le G^i_{p,q} \le f(\text{—},q)_i$.

Similarly (see [20]), an $f \in \overline{\mathsf{F}}(L)$ is called

- *lower* $\mathbb{F}$*-semicontinuous* if for every $p < q$ in $\mathbb{Q}$ there is an $F_{p,q} \in \mathbb{F}(L)$ such that $f(q,\text{—}) \le F_{p,q} \le f(p,\text{—})$.

– *upper* $\mathbb{F}$*-semicontinuous* if for every $p < q$ in $\mathbb{Q}$ there is a $G_{p,q} \in \mathbb{F}(L)$ such that $f(-,p) \le G_{p,q} \le f(-,q)$.

In both cases f is called $\mathbb{F}$*-continuous* if it is lower and upper $\mathbb{F}$-semicontinuous.

This defines the following subclasses of $\mathsf{F}_\kappa(L)$ and $\overline{\mathsf{F}}(L)$

$$\mathsf{LSC}^{\mathbb{F}}_\kappa(L), \quad \mathsf{USC}^{\mathbb{F}}_\kappa(L), \quad \mathsf{C}^{\mathbb{F}}_\kappa(L) = \mathsf{LSC}^{\mathbb{F}}_\kappa(L) \cap \mathsf{USC}^{\mathbb{F}}_\kappa(L),$$

$$\overline{\mathsf{LSC}}^{\mathbb{F}}(L), \quad \overline{\mathsf{USC}}^{\mathbb{F}}(L), \quad \overline{\mathsf{C}}^{\mathbb{F}}(L) = \overline{\mathsf{LSC}}^{\mathbb{F}}(L) \cap \overline{\mathsf{USC}}^{\mathbb{F}}(L).$$

For instance, for the selection $\mathbb{F}_{\mathrm{c}}$ above we obviously have

$$\mathsf{LSC}^{\mathbb{F}_{\mathrm{c}}}_\kappa(L) = \mathsf{LSC}_\kappa(L), \ \mathsf{USC}^{\mathbb{F}_{\mathrm{c}}}_\kappa(L) = \mathsf{USC}_\kappa(L) \ \text{ and } \ \mathsf{C}^{\mathbb{F}_{\mathrm{c}}}_\kappa(L) = \mathsf{C}_\kappa(L).$$

For $\mathbb{F}_{\mathrm{reg}}$ it yields the notions of lower and upper *normal-semicontinuities* and *continuity*; for $\mathbb{F}_{\mathrm{z}}$, it yields the notions of lower and upper *zero-semicontinuities* and *zero-continuity* (see [20] for details).

The following results are all very easy to prove.

Proposition 8.2. *Let* $f \in \mathsf{F}_\kappa(L)$*. Then:*

(1) $f \in \mathsf{LSC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *for all* $i \in I$*.*
(2) $f \in \mathsf{USC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *for all* $i \in I$*.*
(3) $f \in \mathsf{C}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *for all* $i \in I$*.*

Corollary 8.3. *Let* $f \in \mathsf{F}_\kappa(L)$*. Then:*

(1) $f \in \mathsf{LSC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \in \mathsf{USC}^{\mathbb{F}^*}_\kappa(L)$*.*
(2) $f \in \mathsf{C}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \in \mathsf{C}^{\mathbb{F}^*}_\kappa(L)$*.*

Corollary 8.4. *Let* $\mathcal{H} \subseteq \overline{\mathsf{F}}(L)$ *be a disjoint* κ*-family. Then:*

(1) $f_{\mathcal{H}} \in \mathsf{LSC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *for all* $f \in \mathcal{H}$*.*
(2) $f_{\mathcal{H}} \in \mathsf{USC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *for all* $f \in \mathcal{H}$*.*
(3) $f_{\mathcal{H}} \in \mathsf{C}^{\mathbb{F}}_\kappa(L)$ *if and only if* $f \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *for all* $f \in \mathcal{H}$*.*

Proposition 8.5. *Let* $\mathcal{C} = \{S_i\}_{i\in I}$ *be a pairwise disjoint* κ*-family of complemented sublocales of a locale* L*. Then:*

(1) $\chi_{\mathcal{C}} \in \mathsf{LSC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $S_i \in \mathbb{F}^*(L)$ *for all* $i \in I$*.*
(2) $\chi_{\mathcal{C}} \in \mathsf{USC}^{\mathbb{F}}_\kappa(L)$ *if and only if* $S_i \in \mathbb{F}(L)$ *for all* $i \in I$*.*
(3) $\chi_{\mathcal{C}} \in \mathsf{C}^{\mathbb{F}}_\kappa(L)$ *if and only if* $S_i \in \mathbb{F}(L) \cap \mathbb{F}^*(L)$ *for all* $i \in I$*.*

Recall from [20] the relations $\Subset_{\mathbb{F}}$ in $\mathcal{S}(L)^{op}$ (for any L):

$$S \Subset_{\mathbb{F}} T \ \equiv\ \exists U \in \mathbb{F}(L),\ \exists V \in \mathbb{F}^*(L)\colon S \leq V \leq U \leq T.$$

We shall say that a sublocale selection $\mathbb{F}$ is a *Katětov selection on L* if for $S, S', T, T' \in \mathcal{S}(L)$,

(K1) $S, S' \Subset_{\mathbb{F}} T \ \Rightarrow\ S \cap S' \Subset_{\mathbb{F}} T$, and

(K2) $S \Subset_{\mathbb{F}} T, T' \ \Rightarrow\ S \Subset_{\mathbb{F}} T \vee T'$.

$\mathbb{F}_{\rm c}$, $\mathbb{F}_{\rm z}$, $\mathbb{F}_{\delta\rm reg}$, $\mathbb{F}^*_{\rm c}$, $\mathbb{F}^*_{\rm z}$, $\mathbb{F}^*_{\delta\rm reg}$ and $\mathbb{F}^*_{\rm reg}$ are all Katětov selections on any locale while $\mathbb{F}_{\rm reg}$ is a Katětov selection on any mildly normal locale. We are now ready to prove a generalized insertion result for compact hedgehog-valued functions:

Theorem 8.6. *Let $\mathbb{F}$ be a sublocale selection. The following are equivalent for any locale L such that $\mathbb{F}$ is a Katětov selection on L and $L \in \mathbb{F}(L) \cap \mathbb{F}^*(L)$:*

(i) *L is $\mathbb{F}$-normal.*

(ii) *For every $\kappa \geq 1$, and every $f \in \mathsf{USC}^{\mathbb{F}}_{\kappa}(L)$ and $g \in \mathsf{LSC}^{\mathbb{F}}_{\kappa}(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}^{\mathbb{F}}_{\kappa}(L)$ such that $f \leq h \leq g$.*

(iii) *There is a $\kappa \geq 1$ such that for every $f \in \mathsf{USC}^{\mathbb{F}}_{\kappa}(L)$ and $g \in \mathsf{LSC}^{\mathbb{F}}_{\kappa}(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}^{\mathbb{F}}_{\kappa}(L)$ such that $f \leq h \leq g$.*

Proof: (i) $\Longrightarrow$ (ii): Let $\kappa \geq 1$, $|I| = \kappa$, $f \in \mathsf{USC}^{\mathbb{F}}_{\kappa}(L)$ and $g \in \mathsf{LSC}^{\mathbb{F}}_{\kappa}(L)$ with $f \leq g$ and $i \in I$. Then, by Cor. 8.2, $f \circ \pi_i \leq g \circ \pi_i$ in $\overline{\mathsf{F}}(L)$ with

$$f \circ \pi_i \in \overline{\mathsf{USC}}^{\mathbb{F}}(L) \quad \text{and} \quad g \circ \pi_i \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L).$$

By [20, Thm. 7.1], there is an $h_i \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ such that $f \circ \pi_i \leq h_i \leq g \circ \pi_i$. Since $\{g \circ \pi_i\}_{i \in I}$ is a disjoint family, then so is $\{h_i\}_{i \in I}$. Let $h \in \mathsf{C}^{\mathbb{F}}_{\kappa}(L)$ be the function given by Cor. 8.4 (defined by $h \circ \pi_i = h_i$ for every $i \in I$). It satisfies $f \leq h \leq g$, as claimed.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): Let $|I| = \kappa$ and fix some $i_0 \in I$. Let $S, T \in \mathbb{F}(L)$ such that $S \cap T = \mathsf{O}$. Define pairwise disjoint κ-families $\mathcal{C} = \{S_i\}_{i \in I}$ and $\mathcal{D} = \{T_i\}_{i \in I}$ by $S_{i_0} = S$, $T_{i_0} = T^*$ and $S_i = \mathsf{O} = T_i$ for $i \neq i_0$. By Prop. 8.5 (and the fact that O and L belong to $\mathbb{F}(L)$), one has $\chi_{\mathcal{C}} \in \mathsf{USC}^{\mathbb{F}}_{\kappa}(L)$ and $\chi_{\mathcal{D}} \in \mathsf{LSC}^{\mathbb{F}}_{\kappa}(L)$. Moreover, $\chi_{\mathcal{C}} \leq \chi_{\mathcal{D}}$, since $S \subseteq T^*$. Hence, there exists $h \in \mathsf{C}^{\mathbb{F}}_{\kappa}(L)$ such that $\chi_{\mathcal{C}} \leq h \leq \chi_{\mathcal{D}}$, from which it follows in particular that $\chi_S \leq h \circ \pi_{i_0} \leq \chi_{T^*}$. Let $A, B \in \mathbb{F}(L)$ such that

$$(h \circ \pi_{i_0})(1, \text{—}) \subseteq A \subseteq (h \circ \pi_{i_0})(2, \text{—}) \quad \text{and} \quad (h \circ \pi_{i_0})(\text{—}, 1) \subseteq B \subseteq (h \circ \pi_{i_0})(\text{—}, 0).$$

Then $A \vee B \supseteq (h \circ \pi_{i_0})((1,\text{—}) \wedge (\text{—},1)) = L$ and $S \cap A = \chi_S(\text{—},3) \cap A \subseteq (h \circ \pi_{i_0})((\text{—},3) \vee (2,\text{—})) = \mathsf{O}$. Similarly, $T \cap B = \mathsf{O}$. ∎

Note that the particular case $\mathbb{F} = \mathbb{F}_{\mathrm{c}}$ yields immediately Thm. 7.1. Similarly, for $\mathbb{F} = \mathbb{F}_{\mathrm{reg}}$ we obtain:

Corollary 8.7. *The following are equivalent for a locale L:*

(i) *L is mildly normal.*
(ii) *For every $\kappa \geq 1$, and every upper normal-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every lower normal-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*
(iii) *There is a $\kappa \geq 1$ such that for every upper normal-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and lower normal-semicontinuous $g \in \mathsf{F}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Proof: $\mathbb{F}_{\mathrm{reg}}$ is a Katětov selection on any mildly normal locale. And the proof of implication (iii) $\Longrightarrow$ (i) does not need $\mathbb{F}$ to be Katětov. ∎

There is yet another feature of this general setting, suggested by sublocale complementation in sublocale frames: each result applied to the 'dual' selection $\mathbb{F}^*$ provides an extra dual result. For example, for $\mathbb{F} = \mathbb{F}_{\mathrm{c}}^*$, a frame is $\mathbb{F}$-normal if and only if it is extremally disconnected, and lower (resp. upper) $\mathbb{F}$-semicontinuity is precisely upper (resp. lower) semicontinuity. Hence, it also follows readily from Thm. 8.6 that

Corollary 8.8. *The following are equivalent for a locale L:*

(i) *L is extremally disconnected.*
(ii) *For every $\kappa \geq 1$, and every $f \in \mathsf{LSC}_\kappa(L)$ and $g \in \mathsf{USC}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*
(iii) *There is a $\kappa \geq 1$ such that for every $f \in \mathsf{LSC}_\kappa(L)$ and $g \in \mathsf{USC}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

For $\mathbb{F} = \mathbb{F}_{\mathrm{z}}^*$, we obtain:

Corollary 8.9. *The following are equivalent for a locale L:*

(i) *L is an F-frame.*
(ii) *For every $\kappa \geq 1$, and every upper zero-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every lower zero-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists a zero-continuous $h \in \mathsf{F}_\kappa(L)$ such that $f \leq h \leq g$.*

(iii) *There is a* $\kappa \geq 1$ *such that for every upper zero-semicontinuous* $f \in \mathsf{F}_\kappa(L)$ *and every lower zero-semicontinuous* $g \in \mathsf{F}_\kappa(L)$ *satisfying* $f \leq g$*, there exists a zero-continuous* $h \in \mathsf{F}_\kappa(L)$ *such that* $f \leq h \leq g$*.*

9. Relative zero and z-embedded sublocales

Let $\mathbb{F}$ be a sublocale selection. A sublocale T of L will be called an $\mathbb{F}$*-zero sublocale of* L if there is some $\mathbb{F}$-continuous $f \in \overline{\mathsf{F}}(L)$ for which $T = f(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))$. We denote by $\mathbb{F}$-$\mathsf{Z}(L)$ the set of all $\mathbb{F}$-zero sublocales of L.

Remark 9.1. Since $\mathbb{F}$-continuity is a self-dual property, $\mathbb{F}$-zero sublocales and $\mathbb{F}^*$-zero sublocales are the same, i.e. $\mathbb{F}$-$\mathsf{Z}(L) = \mathbb{F}^*$-$\mathsf{Z}(L)$. For instance, for $\mathbb{F} = \mathbb{F}_{\mathrm{c}}$ or $\mathbb{F} = \mathbb{F}_{\mathrm{c}}^*$, the $\mathbb{F}$-zero sublocales are precisely the $\mathfrak{c}(a)$ with a being a cozero element of L, i.e. $\mathbb{F}$-$\mathsf{Z}(L) = \mathsf{Z}(L)$ (see [2]).

In the following, we say that a sublocale S of L is $\mathbb{F}$*-*z*-embedded* in L if for every $\mathbb{F}$-zero sublocale T of S there is an $\mathbb{F}$-zero sublocale T_S of L such that $T_S \cap S = T$.

Example 9.2. For $\mathbb{F} = \mathbb{F}_{\mathrm{c}}$ or $\mathbb{F} = \mathbb{F}_{\mathrm{c}}^*$, this is the standard notion of z-embeddedness of [21] (see also [2]).

Similarly, we say that a sublocale S of L is $\mathbb{F}$-z_κ^c*-embedded* in L if for every κ-family $\{S_i\}_{i\in I}$ which is disjoint in $\mathcal{S}(S)^{op}$ (i.e. $S_i \vee S_j = S$ whenever $i \neq j$) consisting of $\mathbb{F}$-zero sublocales of S, there is a disjoint family $\{T_i\}_{i\in I}$ of $\mathbb{F}$-zero sublocales of L such that $T_i \cap S = S_i$ for every $i \in I$.

Furthermore, we say that a frame L is an $\mathbb{F}$-z_κ^c *frame* if every $S \in \mathbb{F}$ is $\mathbb{F}$-z_κ^c-embedded.

Examples 9.3. (1) For $\mathbb{F} = \mathbb{F}_{\mathrm{c}}$ or $\mathbb{F} = \mathbb{F}_{\mathrm{c}}^*$, a sublocale S is $\mathbb{F}$-z_κ^c-embedded in L if and only if it is z_κ^c-embedded.

(2) Clearly, $\mathbb{F}_c$-z_κ^c frames are just totally κ-collectionwise normal frames. In particular, for $1 \leq \kappa \leq \aleph_0$, they are just the normal frames (by Cor. 6.5).

(3) For $\kappa = 1$, L is a $\mathbb{F}_c^*$-z_κ^c frame if and only if it is an Oz-frame ([4], cf. [21, 6.2.2]). It is well known that extremally disconnected (i.e. $\mathbb{F}_c^*$-normal) frames are Oz, but the reverse implication does not hold. Hence, the $\mathbb{F}$-z_κ^c property does not imply $\mathbb{F}$-normality in general.

As noted in [20], when $\mathbb{F}$ happens to be closed under countable meets, i.e., each $\mathbb{F}(L)$ is closed under countable intersections in $\mathcal{S}(L)$, $\mathbb{F}$-continuity has the following particularly simple description:

Lemma 9.4. *Assume $\mathbb{F}$ is closed under countable meets. Then:*
(1) $f \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *if and only if* $f(r,\text{—}) \in \mathbb{F}(L)$ *for all* $r \in \mathbb{Q}$.
(2) $f \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *if and only if* $f(\text{—},r) \in \mathbb{F}(L)$ *for all* $r \in \mathbb{Q}$.
(3) $f \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *if and only if* $f(r,\text{—}) \in \mathbb{F}(L)$ *and* $f(\text{—},r) \in \mathbb{F}(L)$ *for all* $r \in \mathbb{Q}$.

Proof: The implications $\Leftarrow$ always hold. Further, (3) is an obvious consequence of (1) and (2). Assume that $f \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ and consider $r \in \mathbb{Q}$. Then

$$f(r,\text{—}) = \bigsqcup_{s>r} f(s,\text{—}) \leq \bigsqcup_{s>r} F_{r,s} \leq f(r,\text{—})$$

for some $F_{r,s} \in \mathbb{F}(L)$ $(s > r)$. Hence $f(r,\text{—}) = \bigsqcup_{s>r} F_{r,s} = \bigcap_{s>r} F_{r,s} \in \mathbb{F}(L)$. The assertion in (2) may be proved in a similar way. ∎

Remark 9.5. If $\mathbb{F}$ is closed under countable meets, any $\mathbb{F}$-zero sublocale of L belongs to $\mathbb{F}(L)$. Indeed: such a sublocale is of the form $\bigcap_{r\in\mathbb{Q}} f(r,\text{—})$ for an $\mathbb{F}$-continuous function f, therefore it follows from the previous lemma that each $f(r,\text{—})$ belongs to $\mathbb{F}(L)$ and so does the countable join $\bigcap_{r\in\mathbb{Q}} f(r,\text{—})$.

In general, it may be hard to identify $\mathbb{F}$-zero sublocales of a locale L as they do not necessarily belong to $\mathbb{F}(L)$. However, if one assumes that $\mathbb{F}(L)$ (resp. $\mathbb{F}^*(L)$) is closed under countable joins and finite meets in $\mathcal{S}(L)^{op}$, then $\mathbb{F}(L)$ (resp. $\mathbb{F}^*(L)$) may be regarded as a sub-σ-frame of $\mathcal{S}(L)^{op}$ and hence as a σ-frame in its own right. Then, in this case, the theory of cozero elements in σ-frames from [9] is useful as we explain below.

The category of σ-frames is monadic over the category of sets. Hence, as in the category of frames, one has presentations by generators and relations. In this case, σ-frames may be presented by generators and relations involving only *countable* joins and *finite* meets. In particular, the frame $\mathfrak{L}(\overline{\mathbb{R}})$, can be regarded as a σ-frame generated by $(r,\text{—})$ and $(\text{—},s)$ for $r,s \in \mathbb{Q}$. Then, we have:

Proposition 9.6. *If $\mathbb{F}(L)$ is a sub-σ-frame of $\mathcal{S}(L)^{op}$, then a frame homomorphism $\mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)^{op}$ is $\mathbb{F}$-continuous if and only if it factors in the category of σ-frames through the inclusion $\iota\colon \mathbb{F}(L) \hookrightarrow \mathcal{S}(L)^{op}$.*

Proof: The "if" part is trivial. Conversely, given a frame homomorphism $\mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{S}(L)^{op}$ which is $\mathbb{F}$-continuous, we have by Lem. 9.4 (3) $f(r,\text{—}), f(\text{—},r) \in \mathbb{F}(L)$ for all $r \in \mathbb{Q}$. But then we may define a σ-frame homomorphism $g\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathbb{F}(L)$ given by $g(r,\text{—}) = f(r,\text{—})$ and $g(\text{—},s) = f(\text{—},s)$

for all $r, s \in \mathbb{Q}$. Of course, g sends relations into identities because so does f. Hence f factors as $\iota \circ g$. ∎

It follows readily that $\mathbb{F}$-zero sublocales of L are completely described by the cozero elements (in the sense of [9]) of the σ-frame $\mathbb{F}(L)$. Moreover, as we have already observed, since $\mathbb{F}$-continuity is a self-dual notion, so is the property of being an $\mathbb{F}$-zero sublocale.

Hence, assuming that

$$\mathbb{F}(L) \ (\textit{or } \mathbb{F}^*(L)) \textit{ is a sub-}\sigma\textit{-frame of } \mathcal{S}(L)^{op},$$

or, equivalently, that $\mathbb{F}$ (or $\mathbb{F}^*$) is closed under countable meets and finite joins, we get two lemmas. The first one follows immediately from the fact that cozero elements in a σ-frame form a sub-σ-frame [9, Cor. 1].

Lemma 9.7. *Let $\mathbb{F}$ be such that either $\mathbb{F}$ or $\mathbb{F}^*$ are closed under countable meets and finite joins. Then $\mathbb{F}$-$\mathsf{Z}(L) = \mathbb{F}^*$-$\mathsf{Z}(L)$ is a sub-σ-frame of $\mathcal{S}(L)^{op}$, i.e. countable meets and finite joins of $\mathbb{F}$-zero sublocales are $\mathbb{F}$-zero.*

The proof of the second lemma is based on [12, Prop. 3.3].

Lemma 9.8. *Let $\mathbb{F}$ be such that either $\mathbb{F}$ or $\mathbb{F}^*$ are closed under countable meets and finite joins and let U be an arbitrary sublocale of L. If S and T are $\mathbb{F}$-zero sublocales of L such that $U = (U \cap S) \vee (U \cap T)$, then there are $\mathbb{F}$-cozero sublocales S' and T' of L such that $S' \vee T' = L$, $S' \cap U = S \cap U$ and $T' \cap U = T \cap U$.*

Proof: Since $\mathbb{F}$-$\mathsf{Z}(L)$ is a regular sub-σ-frame of $\mathbb{F}(L)$ there are $\{S_n \prec S\}_{n\in\mathbb{N}}$, $\{T_n \prec T\}_{n\in\mathbb{N}} \subseteq \mathbb{F}$-$\mathsf{Z}(L)$ such that

$$S = \bigsqcup_{n\in\mathbb{N}} S_n \quad \text{and} \quad T = \bigsqcup_{n\in\mathbb{N}} T_n.$$

By substituting S_n by $S_1 \sqcup \cdots \sqcup S_n$ (that satisfies $S_1 \sqcup \cdots \sqcup S_n \prec S$ in $\mathbb{F}$-$\mathsf{Z}(L)$), and similarly for T, we may assume that $\{S_n\}_n$ and $\{T_n\}_n$ are increasing.

Since $S_n \prec S$ and $T_n \prec T$ in $\mathbb{F}$-$\mathsf{Z}(L)$, there are C_n, D_n in $\mathbb{F}$-$\mathsf{Z}(L)$ such that $S_n \sqcap C_n = L$, $S \sqcup C_n = \mathsf{O}$, $T_n \sqcap D_n = L$ and $T \sqcup D_n = \mathsf{O}$ for all $n \in \mathbb{N}$. Set

$$S' = \bigsqcup_{n\in\mathbb{N}} S_n \sqcap D_n \in \mathbb{F}\text{-}\mathsf{Z}(L) \quad \text{and} \quad T' = \bigsqcup_{n\in\mathbb{N}} T_n \sqcap C_n \in \mathbb{F}\text{-}\mathsf{Z}(L).$$

Clearly, $S' \sqcap T' = L$ (since $S_n \sqcap C_n = L$, $T_n \sqcap D_n = L$ and $\{S_n\}_n$ and $\{T_n\}_n$ are increasing). Now, observe that

$$T \sqcup S' = \bigsqcup_n (T \sqcup S_n) \sqcap (T \sqcup D_n) = \bigsqcup_n T \sqcup S_n = T \sqcup S.$$

Similarly, $T' \sqcup S = T \sqcup S$. Hence,

$$\begin{aligned} U \sqcup S' &= U \sqcup (S \sqcap T) \sqcup S' = U \sqcup [(S \sqcup S') \sqcap (T \sqcup S')] \\ &= U \sqcup [(S \sqcup S') \sqcap (T \sqcup S)] = U \sqcup S \sqcup (S' \sqcap T) \geq U \sqcup S \end{aligned}$$

while $U \sqcup S' \leq U \sqcup S$ is trivial from the definition of S'. Hence $U \sqcup S' = U \sqcup S$. The other identity follows by symmetry. ∎

10. Variants of total collectionwise normality

Despite the fact that total collectionwise normality implies normality (recall Cor. 5.6), we have also seen that the property of being $\mathbb{F}$-z_κ^c frame may be strictly weaker than $\mathbb{F}$-normality (cf. Remarks 9.3 (3)). In what follows, we study some conditions under which the $\mathbb{F}$-z_κ^c property implies $\mathbb{F}$-normality.

We shall say that a sublocale selection $\mathbb{F}$ is *hereditary* (resp. *weakly hereditary*) *on a locale* L if for each $S \in \mathbb{F}(L)$ the equality

$$\mathbb{F}(S) = \{ S \cap T \mid T \in \mathbb{F}(L) \}$$

(resp. the inclusion $\{ S \cap T \mid T \in \mathbb{F}(L) \} \subseteq \mathbb{F}(S)$) holds. Clearly, the standard examples $\mathbb{F}_{\mathrm{c}}$, $\mathbb{F}_{\mathrm{z}}$ and $\mathbb{F}_{\delta\mathrm{reg}}$ and their duals are weakly hereditary.

Note that, on the other hand, the selections $\mathbb{F}_{\mathrm{z}}$ and $\mathbb{F}_{\delta\mathrm{reg}}$ are not hereditary in general. For example, $\mathbb{F}_{\mathrm{z}}$ is hereditary on L if and only if L has the property that every zero sublocale is z-embedded. According to [15, 8.20 and 8.J (3)], the Tychonoff plank is a non-normal space whose zero-sets are all z-embedded; hence this property is strictly weaker than normality.

Problem 10.1. Characterize the locales in which every zero sublocale is z-embedded.

Problem 10.2. We also leave it as an open problem to characterize the class of locales in which $\mathbb{F}_{\delta\mathrm{reg}}$ is hereditary, that is, those locales L such that for every $a \in L$ and every δ-regular element b in $\mathfrak{c}(a)$, there is a δ-regular element c in L with $c \vee a = b$.

The following remark motivates the need of hereditary selections for dealing with extension results:

Remark 10.3. Assume L is a locale on which $\mathbb{F}$ is Katětov and closed under countable meets and finite joins (i.e. a *Tietze class* in [20]).

If $S \in \mathbb{F}(L)$ and $f \in \overline{\mathsf{F}}(S)$, the notion of $\mathbb{F}$-continuity relative to S from [20, Thm. 8.6] is more restrictive than the property of f being $\mathbb{F}$-continuous on

S in the sense of this paper. But if $\mathbb{F}$ is hereditary on L, then both notions of continuity coincide and therefore we may apply the relative version of Tietze's extension theorem [20, Thm. 8.6] in our context.

Theorem 10.4. *Let $\mathbb{F}$ be closed under countable meets and finite joins. If L is a $\mathbb{F}$-z_2^c frame and $\mathbb{F}$ is weakly hereditary on L then L is $\mathbb{F}$-normal.*

Proof: Let L be an $\mathbb{F}$-z_2^c frame and consider $S,T\in\mathbb{F}(L)$ such that $S\cap T=\mathsf{O}$. Then $S,T\in\mathbb{F}(S\vee T)$ (because $S=S\cap(S\vee T)$ and $T=T\cap(S\vee T)$ and $\mathbb{F}$ is weakly hereditary on L), hence $S=(S\vee T)\setminus T\in\mathbb{F}^*(S\vee T))$. Therefore, $\chi_S\in\overline{\mathsf{C}}^{\mathbb{F}}(S\vee T)$ by virtue of Prop. 8.5.

Observe that $\chi_S(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))=T$ and so T is an $\mathbb{F}$-zero sublocale of $S\vee T$. Exchanging the roles of S and T, we see that S is an $\mathbb{F}$-zero sublocale of $S\vee T$ as well. Now, $S\vee T\in\mathbb{F}(L)$ because $\mathbb{F}$ is closed under finite joins and therefore it is $\mathbb{F}$-z_2^c-embedded in L. Since $\{S,T\}$ is a disjoint family in $S\vee T$ consisting of $\mathbb{F}$-zero sublocales, there exist $\mathbb{F}$-zero sublocales A,B of L such that $A\vee B=L$, $A\cap(S\vee T)=T$ and $B\cap(S\vee T)=S$. Accordingly, $A\cap S=\mathsf{O}$ and $B\cap T=\mathsf{O}$. Finally, by Remark 9.5, A and B belong to $\mathbb{F}(L)$, and so L is $\mathbb{F}$-normal. ∎

In particular, this theorem ensures that $\mathbb{F}$-z_2^c frames are $\mathbb{F}$-normal for $\mathbb{F}=\mathbb{F}_c$, $\mathbb{F}_{\mathrm{z}}$ and $\mathbb{F}_{\delta\mathrm{reg}}$.

Remark 10.5. The assumption on $\mathbb{F}$ being closed under countable meets cannot be dropped. Indeed, for $\mathbb{F}=\mathbb{F}_{\mathrm{c}}^*$, the condition for being an $\mathbb{F}$-z_2^c frame is easily seen to be equivalent to the following statement:

> *for every open sublocale U of L and every cozero elements a, b of U such that $a\wedge b=0_U$, there exist cozero elements a', b' of L such that $a'\wedge b'=0$, $\nu_U(a')=a$ and $\nu_U(b')=b$.*

But according to Prop. 3.3 in [12] that is equivalent to L being an Oz-frame and we have already pointed out that the Oz property is strictly weaker than extremal disconnectedness (which is, in this case, $\mathbb{F}$-normality).

By 9.3 (3) and 10.5, $\mathbb{F}$-z_2^c frames and $\mathbb{F}$-z_1^c frames coincide for $\mathbb{F}=\mathbb{F}_{\mathrm{c}}^*$; the same happens with $\mathbb{F}=\mathbb{F}_{\mathrm{c}}$ (in this case, both notions are equivalent to normality). The following consequence of Lem. 9.8 shows that this holds more generally for any $\mathbb{F}$ such that either $\mathbb{F}$ or $\mathbb{F}^*$ is closed under countable meets and finite joins.

Corollary 10.6. *Let $\mathbb{F}$ be such that either $\mathbb{F}$ or $\mathbb{F}^*$ is closed under countable meets and finite joins. Then a frame is $\mathbb{F}$-z_2^c if and only if it is $\mathbb{F}$-z_1^c.*

Next, we consider the case $\kappa = \aleph_0$. In this case, we can also identify the $\mathbb{F}$-z_κ^c condition with the $\mathbb{F}$-z_2^c condition.

Proposition 10.7. *Let $\mathbb{F}$ be such that either $\mathbb{F}$ or $\mathbb{F}^*$ are closed under countable meets and finite joins. Then a frame is $\mathbb{F}$-$z_{\aleph_0}^c$ if and only if it is $\mathbb{F}$-z_2^c.*

Proof: The implication $\Rightarrow$ is trivial. Conversely, assume that L is an $\mathbb{F}$-z_2^c frame. Let $S \in \mathbb{F}(L)$ and pick a countable disjoint family $\{S_n\}_{n\in\mathbb{N}}$ of $\mathbb{F}$-zero sublocales of S. By Lem. 9.7, the sublocale

$$T_n = \bigsqcup_{m\neq n} S_m = \bigcap_{m\neq n} S_m \quad (n \in \mathbb{N})$$

is an $\mathbb{F}$-zero sublocale of S. Observe then that, for each $n \in \mathbb{N}$, $\{S_n, T_n\}$ is a disjoint pair of $\mathbb{F}$-zero sublocales of S. Since S is $\mathbb{F}$-z_2^c-embedded, there is a disjoint pair $\{S_n', T_n'\}$ of $\mathbb{F}$-zero sublocales of L such that $S \cap S_n' = S_n$ and $S \cap T_n' = T_n$. Finally, set

$$P_1 = S_1' \quad \text{and} \quad P_n = S_n' \vee T_1' \vee \cdots \vee T_{n-1}', \quad n > 1.$$

$\{P_n\}_{n\in\mathbb{N}}$ is the desired disjoint family of $\mathbb{F}$-zero sublocales of L extending $\{S_n\}_{n\in\mathbb{N}}$. Indeed, each P_n is an $\mathbb{F}$-zero sublocale since $\mathbb{F}$-zero sublocales are closed under finite joins by Lem. 9.7. For disjointness let $n < m$ and observe that $P_m \supseteq T_n'$ and $P_n \supseteq S_n'$, hence $P_n \cap P_m \supseteq S_n' \cap T_n' = L$. Finally, $S \cap P_1 = S_1$ and, for $n > 1$,

$$S \cap P_n = (S \cap S_n') \vee (S \cap T_1') \vee \cdots \vee (S \cap T_{n-1}') = S_n \vee T_1 \vee \cdots \vee T_{n-1}.$$

Note that $S_n \supseteq T_m$ for all $m = 1, \ldots, n-1$, hence $S \cap P_n = S_n$ as desired. ∎

Corollary 10.8. *Let $\mathbb{F}$ be closed under countable meets and finite joins. Then a locale on which $\mathbb{F}$ is hereditary and Katětov is $\mathbb{F}$-z_κ^c if and only if it is $\mathbb{F}$-normal, for any $1 \leq \kappa \leq \aleph_0$.*

Proof: Since $\mathbb{F}$-z_κ^c frames are $\mathbb{F}$-z_2^c frames for all $\kappa \geq 2$, necessity follows from Thm. 10.4. For sufficiency we first recall the fact that $\mathbb{F}$-normality implies $\mathbb{F}$-z_1^c follows from the relative version of the point-free Tietze's extension theorem from [20, Thm. 8.6]. The rest of the statement is a consequence of Cor. 10.6 and Prop. 10.7. ∎

11. General Tietze-type theorems

For convenience, we start this section with a particular case of the pasting lemma from [26] (cf. Prop. 5.4) that we shall need later.

Lemma 11.1. *Let L be a locale and $S, T \in \mathcal{S}(L)$. If $h_1 \in \overline{\mathsf{F}}(S)$ and $h_2 \in \overline{\mathsf{F}}(T)$ satisfy $h_1(x) \cap T = h_2(x) \cap S$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$, then there is an $h \in \overline{\mathsf{F}}(S \vee T)$ given by $h(x) = h_1(x) \vee h_2(x)$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$.*

Proof: One has $\mathcal{S}(S)^{op} = \mathfrak{c}(S)$ (where the latter closed sublocale is understood as a sublocale of $\mathcal{S}(L)^{op}$) and similarly $\mathcal{S}(T)^{op} = \mathfrak{c}(T)$. Hence, the statement follows at once from [26, Prop. 4.4]. ∎

We are now ready to prove two general extension results for $\mathbb{F}$-z_κ^c frames and $\mathbb{F}$-normal frames.

Theorem 11.2. *Let $\mathbb{F}$ be closed under countable meets and finite joins. The following are equivalent for a cardinal κ and a locale L on which $\mathbb{F}$ is hereditary and Katětov*:

(i) *L is an $\mathbb{F}$-z_κ^c frame.*
(ii) *For each $S \in \mathbb{F}(L)$, every $f \in \mathsf{C}_\kappa^{\mathbb{F}}(S)$ has an extension $\overline{f} \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$.*

Proof: (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}_\kappa^{\mathbb{F}}(S)$; for each $i \in I$ set $S_i = \bigvee_{r\in\mathbb{Q}}(f \circ \pi_i)(r, \text{—})$. Then $\{S_i\}_{i\in I}$ is a disjoint family of $\mathbb{F}$-zero sublocales of S. Since $S \in \mathbb{F}(L)$ is $\mathbb{F}$-z_κ^c-embedded, there is a disjoint family $\{T_i\}_{i\in I}$ of $\mathbb{F}$-zero sublocales of L such that $T_i \cap S = S_i$ for every $i \in I$.

For each $i \in I$, set $h_1^{(i)} := f \circ \pi_i \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ and consider the constant extended real valued function $h_2^{(i)} \in \overline{\mathsf{F}}(T_i)$ defined by

$$h_2^{(i)}(r, \text{—}) = T_i = 0_{\mathcal{S}(T_i)^{op}}, \quad \text{and} \quad h_2^{(i)}(\text{—}, r) = \mathsf{O} = 1_{\mathcal{S}(T_i)^{op}}$$

for every $r \in \mathbb{Q}$. Trivially, $h_2^{(i)} \in \overline{\mathsf{C}}^{\mathbb{F}}(T_i)$ (as $1_{\mathcal{S}(T_i)^{op}}, 0_{\mathcal{S}(T_i)^{op}} \in \mathbb{F}(T_i)$). Let us show that $h_1^{(i)}(x) \cap T_i = h_2^{(i)}(x) \cap S$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$ by showing it for the generators of $\mathfrak{L}(\overline{\mathbb{R}})$. For any $(r, \text{—})$ we have $S_i \subseteq (f \circ \pi_i)(r, \text{—}) \subseteq S$ and thus

$$S \cap T_i = S_i \cap T_i \subseteq (f \circ \pi_i)(r, \text{—}) \cap T_i \subseteq S \cap T_i.$$

Hence $h_1^{(i)}(r, \text{—}) \cap T_i = S \cap T_i = h_2^{(i)}(r, \text{—}) \cap S$. Further, for any $(\text{—}, r)$ select $t \in \mathbb{Q}$ such that $t < r$; we have

$$\begin{aligned}(f \circ \pi_i)(\text{—}, r) \cap T_i &= (f \circ \pi_i)(\text{—}, r) \cap S \cap T_i = (f \circ \pi_i)(\text{—}, r) \cap S_i \\ &\subseteq (f \circ \pi_i)(\text{—}, r) \cap (f \circ \pi_i)(t, \text{—}) = (f \circ \pi_i)((\text{—}, r) \vee (t, \text{—})) = \mathsf{O}.\end{aligned}$$

Hence $h_1^{(i)}(\text{—},r)\cap T_i = \mathsf{O} = h_2^{(i)}(\text{—},r)\cap S$. Since $\mathbb{F}$ is closed under binary meets and joins and $T_i, S\in\mathbb{F}(L)$, we may apply Lem. 11.1 to conclude that for each $i\in I$ there is a $h_i\in\overline{\mathsf{F}}(S\vee T_i)$ given by $h_i(x)=h_1^{(i)}(x)\vee h_2^{(i)}(x)$ for all $x\in\mathfrak{L}(\overline{\mathbb{R}})$. Now, because of the hereditary property and the fact that $\mathbb{F}$ is closed under finite joins, it follows easily that h_i is $\mathbb{F}$-continuous.

Since L is $\mathbb{F}$-normal (by 10.4) and $\mathbb{F}$ is a Katětov and hereditary selection on L closed under arbitrary meets and finite joins, we may apply the relative version of the point-free Tietze's extension theorem from [20, Thm. 8.6] and get frame homomorphisms $g_i\in\overline{\mathsf{C}}^{\mathbb{F}}(L)$ such that $g_i(x)\cap(S\vee T_i)=h_i(x)$ for all $x\in\mathfrak{L}(\overline{\mathbb{R}})$. Note that $\{g_i\}_{i\in I}$ is a disjoint family. Indeed, for each $i\neq j$,

$$\begin{aligned}g_i\big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\big)\vee g_j\big(\bigvee_{s\in\mathbb{Q}}(s,\text{—})\big) &\supseteq h_i\big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\big)\vee h_j\big(\bigvee_{s\in\mathbb{Q}}(s,\text{—})\big)\\ &\supseteq h_2^{(i)}\big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\big)\vee h_2^{(j)}\big(\bigvee_{s\in\mathbb{Q}}(s,\text{—})\big)=T_i\vee T_j=L.\end{aligned}$$

Now consider the hedgehog-valued function $h\colon\mathfrak{L}(cJ(\kappa))\to\mathcal{S}(L)^{op}$ defined by $h\circ\pi_i=g_i$ for all $i\in I$ (recall Prop. 4.2), which is $\mathbb{F}$-continuous by virtue of Cor. 8.4. We claim that h is the desired extension.

We need to check that $h(x)\cap S=f(x)$ for all $x\in\mathfrak{L}(cJ(\kappa))$ by showing it for the generators of $\mathfrak{L}(cJ(\kappa))$. For any $(r,\text{—})_i$ we have

$$\begin{aligned}S\cap g_i(r,\text{—}) &= S\cap(S\vee T_i)\cap g_i(r,\text{—})=S\cap h_i(r,\text{—})\\ &= h_1^{(i)}(r,\text{—})\vee(S\cap h_2^{(i)}(r,\text{—}))=(f\circ\pi_i)(r,\text{—})\vee(S\cap T_i)\\ &= (f\circ\pi_i)(r,\text{—})\vee S_i=(f\circ\pi_i)(r,\text{—}).\end{aligned}$$

Hence $h((r,\text{—})_i)\cap S=g_i(r,\text{—})\cap S=f((r,\text{—})_i)$. Furthermore, for any $(\text{—},r)_i$,

$$\begin{aligned}S\cap g_i(\text{—},r) &= S\cap(S\vee T_i)\cap g_i(\text{—},r)=S\cap h_i(\text{—},r)\\ &= h_1^{(i)}(\text{—},r)\vee(S\cap h_2^{(i)}(\text{—},r))=(f\circ\pi_i)(\text{—},r)\vee(S\cap\mathsf{O})\\ &= (f\circ\pi_i)(\text{—},r)\vee\mathsf{O}=(f\circ\pi_i)(\text{—},r).\end{aligned}$$

Hence $h((\text{—},r)_i)\cap S=g_i(\text{—},r)\cap S=f((\text{—},r)_i)$.

(ii) $\Longrightarrow$ (i): Let $S\in\mathbb{F}(L)$ and let $\{S_i\}_{i\in I}$ be a disjoint family of $\mathbb{F}$-zero sublocales of S. The latter means that for each $i\in I$ there is an $f_i\in\overline{\mathsf{C}}^{\mathbb{F}}(S)$ such that $f_i(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))=S_i$. Let $f\in\mathsf{F}_\kappa(S)$ be the unique frame homomorphism such that $f\circ\pi_i=f_i$ for all $i\in I$. By Cor. 8.4, $f\in\mathsf{C}_\kappa^{\mathbb{F}}(S)$. Then, by assumption, there is an $\mathbb{F}$-continuous extension $\overline{f}\in\mathsf{C}_\kappa^{\mathbb{F}}(L)$ of f.

Set $T_i = \overline{f}(\bigvee_{r\in\mathbb{Q}}(r,\text{—})_i)$ for each $i \in I$. It is clear that $\{T_i\}_{i\in I}$ is the desired disjoint family of $\mathbb{F}$-zero sublocales. ■

We now move to the extremally disconnected side of the parallel. We shall say that a sublocale selection $\mathbb{F}$ is *co-hereditary on a locale* L if for each $S \in \mathbb{F}(L)$ the equality

$$\mathbb{F}^*(S) = \{\, S \cap T \mid T \in \mathbb{F}^*(L)\,\}.$$

holds. $\mathbb{F}$ is *co-hereditary* if it is co-hereditary on any locale.

Lemma 11.3. *Each of the sublocale selections* $\mathbb{F} = \mathbb{F}^*_{\mathrm{c}}$, $\mathbb{F}^*_{\mathrm{z}}$, $\mathbb{F}^*_{\delta\mathrm{reg}}$ *is co-hereditary.*

Proof: $\mathbb{F}^*_{\mathrm{c}}$: It is well-known that closed sublocales of any sublocale S are of the form $T \cap S$ where T is closed in L.

$\mathbb{F}^*_{\mathrm{z}}$: amounts to showing that for every $a \in \operatorname{Coz} L$ and $b \in \operatorname{Coz} \mathfrak{o}(a)$, there exists $c \in \operatorname{Coz} L$ with $\nu_{\mathfrak{o}(a)}(c) = b$. But in this situation, one has $a \wedge b \in \operatorname{Coz} L$ (see for example [21, Cor. 5.6.2] or Cor. [12, 3.2.11]).

$\mathbb{F}^*_{\delta\mathrm{reg}}$: We have to show that for each δ-regular a in L and all δ-regular b in $\mathfrak{o}(a)$, there exists a δ-regular c in L with $\nu_{\mathfrak{o}(a)}(c) = b$. It is of course enough to show that $c := a \wedge b$ is δ-regular in L. By the isomorphism ${\downarrow}a \cong \mathfrak{o}(a)$, this is equivalent to show that if b is δ-regular in the frame ${\downarrow}a$, then it is δ-regular in L. Since a is δ-regular in L, one can write $a = \bigvee_n a_n$ where $a_n \prec a$ for all $n \in \mathbb{N}$ (i.e. for each n there is a c_n with $c_n \wedge a_n = 0$ and $c_n \vee a = 1$). Since $x \prec a$ and $y \prec a$ imply $x \vee y \prec a$, we may assume that $\{a_n\}_{n\in\mathbb{N}}$ is increasing. Moreover, b is δ-regular in ${\downarrow}a$, so one can write $b = \bigvee_n b_n$ where $\{b_n\}_{n\in\mathbb{N}}$ is increasing and for each $n \in \mathbb{N}$ there is a d_n with $d_n \wedge b_n = 0$ and $d_n \vee b = a$. Let $x_n = c_n \vee d_n$. Then

$$x_n \wedge (a_n \wedge b_n) = (c_n \wedge a_n \wedge b_n) \vee (d_n \wedge a_n \wedge b_n) = 0,$$

$$x_n \vee b = c_n \vee (d_n \vee b) = c_n \vee a = 1.$$

Finally, $b \le \bigvee_n a_n \wedge b_n$ because $b \le a$ and $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are increasing. Hence $b = \bigvee_n a_n \wedge b_n$ with $a_n \wedge b_n \prec b$ in L. ■

The following is the dual version of the relative Tietze's extension theorem which is missing in [20]:

Theorem 11.4. *Let* $\mathbb{F}$ *be such that* $\mathbb{F}^*$ *is closed under countable meets and finite joins. The following are equivalent for a locale* L *on which* $\mathbb{F}$ *is co-hereditary and Katětov*:

(i) *L is $\mathbb{F}$-normal.*

(ii) *For each $S \in \mathbb{F}(L)$, every $f \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ has an extension $\overline{f} \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$.*

Proof: (i) $\Longrightarrow$ (ii): Let $S \in \mathbb{F}(L)$, $f \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ and $r \in \mathbb{Q}$. Since f is $\mathbb{F}$-continuous and $\mathbb{F}^*$ is closed under countable meets, both $f(r,\text{—})$ and $f(\text{—},r)$ belong to $\mathbb{F}^*(S)$ (cf. Lem. 9.4). Since S is complemented then so are $f(r,\text{—})$ and $f(\text{—},r)$ and the maps $\sigma_1, \sigma_2 \colon \mathbb{Q} \to L$ given by

$$\sigma_1(r) = S^* \vee f(r,\text{—}) \quad \text{and} \quad \sigma_2(r) = S \cap f(\text{—},r)^*$$

are extended scales in $\mathcal{S}(L)^{op}$; denote by f_1 and f_2 the corresponding functions in $\overline{\mathsf{F}}(L)$.

Since $f(r,\text{—})$ and $f(\text{—},r)$ belong to $\mathbb{F}^*(S)$, by co-heredity there exist $U_r, V_r \in \mathbb{F}^*(L)$ such that $f(r,\text{—}) = U_r \cap S$ and $f(\text{—},r) = V_r \cap S$. Then one has (recall (2.2))

$$f_1(r,\text{—}) = \bigcap_{s>r} \sigma_1(s) = S^* \vee f(r,\text{—}) = S^* \vee (U_r \cap S) = S^* \vee U_r \in \mathbb{F}^*(L)$$

and

$$f_2(\text{—},r) = \bigcap_{s<r} \sigma_2(s)^* = S^* \vee f(\text{—},r) = S^* \vee (V_r \cap S) = S^* \vee V_r \in \mathbb{F}^*(L)$$

(as $\mathbb{F}^*(L)$ is closed under binary joins). It follows that f_1 is lower $\mathbb{F}^*$-continuous while f_2 is upper $\mathbb{F}^*$-continuous. This means that f_1 is upper $\mathbb{F}$-continuous and f_2 is lower $\mathbb{F}$-continuous. Moreover, by (2.3) we have $f_1 \leq f_2$ because $f(\text{—},s)^* \subseteq f(r,\text{—})$, and therefore $\sigma_1(r) \leq \sigma_2(s)$, for any $s < r$. Since $\mathbb{F}$ is Katětov on L, by [20, Thm. 7.1] there is an $\mathbb{F}$-continuous $h \in \overline{\mathsf{F}}(L)$ such that $f_1 \leq h \leq f_2$. Let $h_S \colon \mathcal{S}(L) \to \mathcal{S}(S)$ be the map $T \mapsto S \cap T$. One readily checks that

$$h_S(f_1(r,\text{—})) = f(r,\text{—}) \quad \text{and} \quad h_S(f_2(\text{—},s)) = f(\text{—},s)$$

and thus (recall the partial order (2.4)) $h_S \circ f_1 = f$ and $h_S \circ f_2 = f$. Finally note that

$$f = h_S \circ f_1 \leq h_S \circ h \leq h_S \circ f_2 = f.$$

It follows that $h_S \circ h = f$ and thus h is the desired $\mathbb{F}$-continuous extension of the given f.

(ii) $\Longrightarrow$ (i): Let $S, T \in \mathbb{F}(L)$ satisfy $S \cap T = \mathsf{O}$. Then $S \in \mathbb{F}(S \vee T) \cap \mathbb{F}^*(S \vee T)$. Indeed, $S \vee T \in \mathbb{F}(L)$ (because $\mathbb{F}^*$ is in particular closed under finite meets) and $T^* \in \mathbb{F}^*(L)$. Hence by co-heredity one has $S = T^* \cap (S \vee T) \in \mathbb{F}^*(S \vee T)$. Exchanging the roles of S and T we obtain $T \in \mathbb{F}^*(S \vee T)$ and hence $S =$

$(S \vee T) \setminus T \in \mathbb{F}(S \vee T)$. Then, by Prop. 8.5 one has $\chi_S \in \overline{\mathsf{C}}^{\mathbb{F}}(S \vee T)$. Since $S \vee T \in \mathbb{F}(L)$, there is an extension $f \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$. Choose $A, B \in \mathbb{F}(L)$ with

$$f(1,\text{–}) \subseteq A \subseteq f(2,\text{–}) \quad \text{and} \quad f(\text{–},1) \subseteq B \subseteq f(\text{–},0).$$

Then $A \vee B \supseteq f(1,\text{–}) \vee f(\text{–},1) = f((1,\text{–}) \wedge (\text{–},1)) = f(0) = L$ and $S \cap A = \chi_S(\text{–},3) \cap A \subseteq f(\text{–},3) \cap f(2,\text{–}) = \mathsf{O}$. Similarly, $T \cap B = \mathsf{O}$. ■

The cases $\mathbb{F} = \mathbb{F}_{\mathrm{c}}$, $\mathbb{F}_{\mathrm{z}}$, $\mathbb{F}_{\delta\mathrm{reg}}$ in Thm. 11.4 yield respectively the following corollaries:

Corollary 11.5. *Let L be a locale. The following are equivalent:*

(i) *L is extremally disconnected.*
(ii) *For each $a \in L$, every $f \in \overline{\mathsf{C}}(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \overline{\mathsf{C}}(L)$.*

Corollary 11.6. *Let L be a locale. The following are equivalent:*

(i) *L is an F-frame.*
(ii) *For each cozero element $a \in L$, every zero-continuous $f \in \overline{\mathsf{F}}(\mathfrak{o}(a))$ has a zero-continuous extension $\overline{f} \in \overline{\mathsf{F}}(L)$.*

Corollary 11.7. *Let L be a locale. The following are equivalent:*

(i) *L is extremally δ-disconnected.*
(ii) *For each δ-regular $a \in L$, every regular-continuous $f \in \overline{\mathsf{F}}(\mathfrak{o}(a))$ has a regular-continuous extension $\overline{f} \in \overline{\mathsf{F}}(L)$.*

We finally prove the cardinal generalization of Cor. 11.5. Following Blair [7], we shall say that L is an Oz^c_κ frame if it is an $\mathbb{F}^*_c$-z^c_κ frame, i.e. if for each $a \in L$ and every κ-family $\{a_i\}_{i\in I}$ of cozero elements of $\mathfrak{o}(a)$ satisfying $a_i \wedge a_j = a^*$ for all $i \neq j$, there is a κ-family $\{b_i\}_{i\in I}$ of disjoint cozero elements of L satisfying $\nu_{\mathfrak{o}(a)}(b_i) = a \to b_i = a_i$ for all $i \in I$. We note that every perfectly normal frame (i.e. one in which every element is a cozero) is automatically Oz^c_κ for any cardinal κ.

Theorem 11.8. *Let L be a locale and κ a cardinal. The following are equivalent:*

(i) *L is extremally disconnected and Oz^c_κ.*
(ii) *For each $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Proof: (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}_\kappa(\mathfrak{o}(a))$. For each $i \in I$, set

$$a_i = (f \circ \pi_i)\big(\bigvee_{r \in \mathbb{Q}} (r, \text{—})\big).$$

Then $\{a_i\}_{i \in I}$ is a κ-family consisting of cozero elements of $\mathfrak{o}(a)$ with $a_i \wedge a_j = a^*$ for all $i \neq j$. Since L is Oz^c_κ, there is a family $\{b_i\}_{i \in I}$ of disjoint cozero elements of L satisfying $a \to b_i = a_i$ for each $i \in I$. Hence $a_i \wedge a \wedge b_i^* = 0$.

For each $i \in I$, set $h_1^{(i)} := f \circ \pi_i \in \overline{\mathsf{C}}(\mathfrak{o}(a))$ and consider the constant extended real valued function $h_2^{(i)} \in \overline{\mathsf{C}}(\mathfrak{o}(b_i^*))$ defined by

$$h_2^{(i)}(r, \text{—}) = 0_{\mathfrak{o}(b_i^*)} = b_i^{**} \quad \text{and} \quad h_2^{(i)}(\text{—}, r) = 1$$

for every $r \in \mathbb{Q}$. Let us show that $h_1^{(i)}(x) \wedge a \wedge b_i^* = h_2^{(i)}(x) \wedge a \wedge b_i^*$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$ by showing it for the generators of $\mathfrak{L}(\overline{\mathbb{R}})$. For each $r \in \mathbb{Q}$ we have:

$$h_1^{(i)}(r, \text{—}) \wedge a \wedge b_i^* \leq a_i \wedge a \wedge b_i^* = 0 = h_2^{(i)}(r, \text{—}) \wedge a \wedge b_i^*.$$

Moreover, by (r2) one has $(f \circ \pi_i)(\text{—}, r) \vee a_i = 1$ and therefore $a \wedge b_i^* \leq (f \circ \pi_i)(\text{—}, r) \vee (a \wedge b_i^* \wedge a_i) = h_1^{(i)}(\text{—}, r)$. Hence

$$h_1^{(i)}(\text{—}, r) \wedge a \wedge b_i^* = a \wedge b_i^* = h_2^{(i)}(\text{—}, r) \wedge a \wedge b_i^*.$$

Consequently, by [26, 3.2, 3.3] there is an $h_i \in \overline{\mathsf{C}}(\mathfrak{o}(a \vee b_i^*))$ given by

$$h_i(x) = (h_1^{(i)}(x) \wedge a) \vee (h_2^{(i)}(x) \wedge b_i^*)$$

i.e. $h_i(r, \text{—}) = (f \circ \pi_i)(r, \text{—}) \wedge a$ and $h_i(\text{—}, r) = ((f \circ \pi_i)(\text{—}, r) \wedge a) \vee b_i^*$ for each $r \in \mathbb{Q}$ and which extends $h_1^{(i)}$ and $h_2^{(i)}$. By Cor. 11.5, for each $i \in I$ there is a $g_i \in \overline{\mathsf{C}}(L)$ which extends h_i (i.e. satisfying $\nu_{\mathfrak{o}(a \vee b_i^*)} \circ g_i = h_i$). Let us check that the family $\{g_i\}_{i \in I}$ is disjoint:

$$\begin{aligned} g_i\big(\bigvee_{r \in \mathbb{Q}} (r, \text{—})\big) \wedge g_j\big(\bigvee_{s \in \mathbb{Q}} (s, \text{—})\big) &\leq h_i\big(\bigvee_{r \in \mathbb{Q}} (r, \text{—})\big) \wedge h_j\big(\bigvee_{s \in \mathbb{Q}} (s, \text{—})\big) \\ &= (f \circ \pi_i)\big(\bigvee_{r \in \mathbb{Q}} (r, \text{—})\big) \wedge (f \circ \pi_j)\big(\bigvee_{s \in \mathbb{Q}} (s, \text{—})\big) \wedge a \\ &= a_i \wedge a_j \wedge a = a^* \wedge a = 0. \end{aligned}$$

For each $i \in I$, g_i extends h_i, and the latter extends $h_1^{(i)} = f \circ \pi_i$, hence g_i extends $f \circ \pi_i$. Therefore the function $\overline{f} \in \mathsf{C}_\kappa(L)$ given by $\overline{f} \circ \pi_i = g_i$ (cf. Cor. 4.3) extends f: indeed, note that $\nu_{\mathfrak{o}(a)} \circ \overline{f} \circ \pi_i = \nu_{\mathfrak{o}(a)} \circ g_i = f \circ \pi_i$ and use the uniqueness clause of 4.2.

(2) $\Longrightarrow$ (1): Extremal disconnectedness follows as in the proof of 11.4 and the fact that L is Oz^c_κ is as in the proof of 11.2. ■

We close with the following corollary which follows at once from 10.5, 10.6, 10.7 and the previous theorem:

Corollary 11.9. *Let L be a locale and $1 \leq \kappa \leq \aleph_0$ a cardinal. The following are equivalent*:

(i) *L is extremally disconnected.*
(ii) *For each $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Acknowledgements. This research was supported by the Basque Government (Grant IT974-16) and the Centre for Mathematics of the University of Coimbra (UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES). The first named author also acknowledges support from a predoctoral fellowship of the Basque Government (PRE-2018-1-0375).

References

[1] I. Arrieta Torres, *A Tale of Three Hedgehogs*, Bachelor Thesis, University of the Basque Country UPV/EHU, Bilbao, ArXiv:1711.08656 [math.GN], 2017.
[2] A.B. Avilez and J. Picado, *Continuous extensions of real functions on arbitrary sublocales and C-, C^*-, and z-embeddings*, J. Pure Appl. Algebra 225 (2021) art. n. 106702, 1–24.
[3] B. Banaschewski, *The real numbers in Pointfree Topology*, Textos de Matemática, Vol. 12, University of Coimbra, 1997.
[4] B. Banaschewski, T. Dube, C. Gilmour and J. Walters-Wayland, *Oz in pointfree topology*, Quaest. Math. **32** (2009) 215–227.
[5] B. Banaschewski, J. Gutiérrez García and J. Picado, *Extended real functions in pointfree topology*, J. Pure Appl. Algebra **216** (2012) 905–922.
[6] B. Banaschewski and A. Pultr, *A new look at pointfree metrization theorems*, Comment. Math. Univ. Carolinae **39** (1998) 167–175.
[7] R.L. Blair, *A cardinal generalization of z-embedding*, in: Rings of continuous functions, Lecture Notes in Pure and Applied Math., Marcel Dekker Inc., vol. 95, 1985, 7–66.
[8] R.L. Blair and M.A. Swardson, *Insertion and extension of hedgehog-valued functions*, Indian J. Math. **29** (1987) 229–250.
[9] J. Clarke and C. Gilmour, *Pseudocompact σ-frames*, Math. Slovaca **65** (2015) 289–300.
[10] T. Dube, *A short note on separable frames*, Comment. Math. Univ. Carolin. **37** (1996) 375–377.
[11] T. Dube and O. Ighedo, *More on locales in which every open sublocale is z-embedded*, Topology Appl. **201** (2016) 110–123.
[12] T. Dube and J. Walters-Wayland, *Coz-onto frame maps and some applications*, Appl. Categ. Structures **15** (2007) 119–133.
[13] R. Engelking, *General Topology*, Revised and completed ed.: Sigma Series in Pure Mathematics, Vol. 6, Heldermann Verlag, Berlin, 1989.

[14] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, *Continuous Lattices and Domains*, Encyclopedia of Mathematics and Its Applications, Vol. 93, Cambridge University Press, 2003.
[15] L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Van Nostrand, 1960.
[16] J. Gutiérrez García, T. Kubiak and J. Picado, *Localic real functions: a general setting*, J. Pure Appl. Algebra **213** (2009) 1064–1074.
[17] J. Gutiérrez García, T. Kubiak and M.A. de Prada Vicente, *Insertion of lattice-valued and hedgehog valued functions*, Topology Appl. **153** (2006) 1458–1475.
[18] J. Gutiérrez García, T. Kubiak and M.A. de Prada Vicente, *Controlling disjointness with a hedgehog*, Houst. J. Math. **35** (2009) 469–484.
[19] J. Gutiérrez García, I. Mozo Carollo, J. Picado and J. Walters-Wayland, *Hedgehog frames and a cardinal extension of normality*, J. Pure Appl. Algebra **223** (2019) 2345–2370.
[20] J. Gutiérrez García and J. Picado, *On the parallel between normality and extremal disconnectedness*, J. Pure Appl. Algebra **218** (2014) 784–803.
[21] J. Gutiérrez García, J. Picado and A. Pultr, *Notes on point-free real functions and sublocales*, in: Categorical Methods in Algebra and Topology, Textos de Matemática, DMUC, vol. 46, 2014, 167–200.
[22] J. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972) 5–32.
[23] P.B. Johnson, *κ-Lindelöf locales and their spatial parts*, Cahiers Topologie Géom. Différentielle Catég. **32** (1991) 297–313.
[24] P.T. Johnstone, *Stone Spaces*, Cambridge Studies in Advances Math. 3, Cambridge University Press, 1982.
[25] J. Picado and A. Pultr, *Frames and Locales: topology without points*, Frontiers in Mathematics **28**, Springer, Basel, 2012.
[26] J. Picado and A. Pultr, *Localic maps constructed from open and closed parts*, Categ. Gen. Algebraic Struct. Appl. **6** (2017) 21–35.
[27] J. Picado and A. Pultr, *Separation in point-free topology*, Birkhäuser/Springer, Cham, 2021.
[28] A. Pultr, *Remarks on metrizable locales*, in: Proc. of the 12th Winter School on Abstract Analysis, pp. 247–258, Rendiconti del Circolo Matematico di Palermo, 1984.

Igor Arrieta
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal, and Department of Mathematics, University of the Basque Country UPV/EHU, 48080 Bilbao, Spain
E-mail address: `igorarrieta@mat.uc.pt`

Javier Gutiérrez García
Department of Mathematics, University of the Basque Country UPV/EHU, 48080 Bilbao, Spain
E-mail address: `javier.gutierrezgarcia@ehu.eus`

Jorge Picado
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `picado@mat.uc.pt`