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ADJOINT MAPS BETWEEN IMPLICATIVE SEMILATTICES AND CONTINUITY OF LOCALIC MAPS

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ABSTRACT: We study residuated homomorphisms (r–morphisms) and their adjoints, the so-called localizations (or l–morphisms), between implicative semilattices, because these objects may be characterized as semilattices whose unary meet operations have adjoints. Since left resp. right adjoint maps are the residuated resp. residual maps (having the property that preimages of principal downsets resp. upsets are again such), one may not only regard the l–morphisms as abstract continuous maps in a pointfree framework (as familiar in the complete case), but also characterize them by concrete closure-theoretical continuity properties. These concepts apply to locales (frames, complete Heyting lattices) and provide generalizations of continuous and open maps between spaces to an algebraic (not necessarily complete) pointfree setting.

KEYWORDS: Adjoint map, Complement, Implicative semilattice, Localic map, Nuclear range, Sublocale.

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1. Introduction

A basic tool in order theory with countless applications in other fields of mathematics is provided by adjoint pairs of maps between partially ordered sets (posets): given posets A, B and maps $h\colon A \longrightarrow B$ and $f\colon B \longrightarrow A$ related by the equivalence

$$ha \leq b \Leftrightarrow a \leq fb,$$

f is the (*right* or *upper*) *adjoint* of h, and h the *coadjoint* (*left* or *lower adjoint*) of f (we omit parentheses if maps are applied to elements). Either partner of an adjunction is uniquely determined by the other. The letter h has been chosen because in our investigations h often will be an algebraic homomorphism, whereas f will represent certain continuous functions. In

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Section 5, g will stand for the *left* (french: *gauche*) adjoint of h, provided it exists. In other contexts, it is more common to denote a *left* adjoint by f and its (*right*) adjoint by g; accordingly, in the categorical version of adjointness [1, 21], left adjoint functors generalize *free functors*, and their adjoints *grounding functors*. Some authors use the opposite order $\geq$, so that an upper adjoint g stands on the *left* side of the inequality $gb \geq a$ [18].

The adjoining equivalence allows to shift parts of one side of an inequality to the other side in a very convenient way. It is well known and easy to see that a map is coadjoint (left adjoint) iff it is *residuated*, i.e., preimages of principal downsets

$$\downarrow c = \{b \in B \mid b \leq c\}$$

are again principal downsets, and a map is (right) adjoint iff it is *residual*, i.e., preimages of principal upsets

$$\uparrow c = \{a \in A \mid c \leq a\}$$

are again principal upsets. Residuated maps preserve all existing joins, and residual maps all existing meets. In particular, both kinds of maps are isotone (order-preserving). Moreover, a map between complete lattices is residuated (resp. residual) iff it preserves arbitrary joins (resp. meets). The most important fact in the theory of (Galois) adjunctions is that any adjoint pair of maps induces mutually inverse isomorphisms between their ranges. Note also that one partner of an adjunction is idempotent iff so is the other, that it is injective iff the partner is surjective, and that passing to adjoints inverts composition; see [8, 13, 18].

A basic instance of an adjunction is obtained as follows. Every map $f\colon S \longrightarrow T$ between sets induces an adjoint pair of maps between the power sets $\mathcal{P}S$ and $\mathcal{P}T$: the image map $f_{\rightarrow}$ with $f_{\rightarrow}U = fU = f[U] = \{fu \mid u \in U\}$ and the preimage map $f^{\leftarrow}$ with $f^{\leftarrow}V = f^{-1}[V] = \{s \in S \mid fs \in V\}$. They are related by the equivalence

$$f_{\rightarrow}U \subseteq V \;\Leftrightarrow\; U \subseteq f^{\leftarrow}V.$$

Topologies are prototypes of so-called *frames* or *locales* [21, 36], that is, complete lattices in which binary meets distribute over arbitrary joins. The category Frm of frames has as morphisms the *frame homomorphisms*, that is, maps which preserve arbitrary joins and finite meets, while the opposite category Loc of locales has the same objects but as morphisms the adjoints of frame homomorphisms, so-called *locale morphisms* or *localic maps*. Denoting for any topological space T its topology, regarded as a frame resp. locale, by $\mathcal{O}T$, we obtain for every continuous map $f\colon S \longrightarrow T$ between topological spaces an adjoint pair of maps

$$\mathcal{O}_{\leftarrow} f\colon \mathcal{O}T \longrightarrow \mathcal{O}S, \quad V \longmapsto f^{\leftarrow}V,$$

$$\mathcal{O}^{\rightarrow} f\colon \mathcal{O}S \longrightarrow \mathcal{O}T, \quad U \longmapsto (f_{\rightarrow}(U'))^{-\prime},$$

where $^-$ denotes closure and $'$ set-theoretical complement; the preimage map is now the left (i.e. lower !) adjoint partner:

$$\mathcal{O}_{\leftarrow} f\, V \subseteq U \;\Leftrightarrow\; U' \subseteq f^{\leftarrow}V' \;\Leftrightarrow\; (f_{\rightarrow}(U'))^{-} \subseteq V' \;\Leftrightarrow\; V \subseteq \mathcal{O}^{\rightarrow} f\, U$$

(working with closed sets would be more natural, as demonstrated in [11]). $\mathcal{O}^{\rightarrow}$ is an adjoint functor from the category Top of topological spaces to the category Loc, and $\mathcal{O}_{\leftarrow}$ is a contravariant functor from Top to Frm.

$$\begin{tikzcd} & \mathsf{Top} \arrow[dl, "\mathcal{O}_{\leftarrow}"'] \arrow[dr, "\mathcal{O}^{\rightarrow}"] & \\ \mathsf{Frm} \arrow[rr, leftrightarrow, "op"] & & \mathsf{Loc} \end{tikzcd}$$

This may be regarded as the "starting point of pointfree topology" (cf. Johnstone [22]). Comprehensive references to themes of pointfree topology are Dowker and Papert [10], Isbell [20], Johnstone [21, 22, 23], Picado, Pultr and Tozzi [35, 36, 38], and Simmons [40, 41, 42].

For the important special case where S is a subspace of a space T and e is the inclusion map from S into T, the above construction yields an adjoint pair of maps

$$\mathcal{O}_{\leftarrow} e\colon \mathcal{O}T \longrightarrow \mathcal{O}S,\; V \longmapsto S \cap V,$$

$$\mathcal{O}^{\rightarrow} e\colon \mathcal{O}S \longrightarrow \mathcal{O}T,\; U \longmapsto S \rightarrow U,$$

where $S \rightarrow U = (S \smallsetminus U)^{-\prime}$ (closure and complement in T). By the machinery of Galois adjunctions, the induced topology $\mathcal{O}S = \{S \cap V \mid V \in \mathcal{O}T\}$ is isomorphic to

$$\mathcal{O}_T S = S^{\sim} = \{S \rightarrow U \mid U \in \mathcal{O}S\},$$

which is a meet–closed and left $\rightarrow$–closed subset, that is, a *sublocale* of $\mathcal{O}T$ [35, 36]. In this sense, one may say that sublocales represent subspaces; however, the map $\mathcal{O}_T$ from $\mathcal{P}T$ to the coframe of all sublocales is neither one-to-one nor onto in general. For topological characterizations of those spaces for which $\mathcal{O}_T$ is injective, surjective or bijective, respectively, see [36] and [40].

In view of the connections between spaces and locales, categorically inspired authors refer to localic maps (or to the opposite arrows of frame homomorphisms [21]) as "continuous maps". This raises the question of whether that kind of maps may be characterized by certain concrete continuity properties in the closure-theoretical sense (preimages of closed subobjects are closed, and the formation of preimages commutes with complementation). We shall give an affirmative answer to that question; the explicit characterization of localic maps in terms related to continuity is, however, a bit delicate: the complements of closed sublocales have to be formed in the lattice of all sublocales and not set-theoretically as in classical topology.

Motivated by the previous observations, we shall study, more generally, implicative semilattices, that is, meet-semilattices with top elements in which the unary meet operations $\lambda_a = a \wedge -$ have adjoints $\alpha_a = a \rightarrow -$. The frames resp. locales are just the complete implicative semilattices, and the frame homomorphisms are nothing but the residuated semilattice homomorphisms preserving the top elements. Our arguments are often shorter than those found in the literature for the case where joins exist, and nevertheless provide proper extensions to the setting of semilattices; new ideas are required when certain joins or meets are not available.

If we wish to have Frm resp. Loc as *full subcategories* of two respective dual categories whose objects are implicative semilattices, we have to consider as morphisms not the usual *implicative* homomorphisms (which preserve finite meets and the binary residuation $\rightarrow$) but the residuated top-preserving semilattice homomorphisms, briefly referred to as *r–morphisms*, and in the opposite direction their adjoints, the so-called *localizations* (Bezhanishvili and Ghilardi [6]) or *l–morphisms*. Thus,

r–morphisms have *right* adjoints and preserve finite meets,
l–morphisms have *left* adjoints that preserve finite meets,

and the respective categories are duals of each other via Galois adjunction. For continuous maps f between spaces, $\mathcal{O}_{\leftarrow} f$ is an r–morphism and $\mathcal{O}^{\rightarrow} f$ its adjoint l–morphism.

Under the point of view we adopt in the present paper, it is reasonable to regard implicative semilattices as algebras $(A, \wedge, \top, \alpha)$, where α is the family of all *unary residuations* or *relative pseudocomplementations* α_a $(a \in A)$. Here we leave the classical area of varieties, because the signature depends on A, and the subalgebras are those subsemilattices which are closed under each α_a; we call them *l–ideals*. Now, all unary meet operations become r–morphisms, and the image (but not the preimage) of an l–ideal under an l–morphism is always an l–ideal. Those subsets for which the inclusion map is an r– resp. l–morphism will be referred to as *r–* resp. *l–domains*. In the complete case of frames/locales, the r–domains are the *subframes*, whereas the l–domains are the *sublocales*. In our general setting, they are still nothing but the ranges of nuclei, that is, closure operations preserving finite meets.

The idea to characterize algebraic homomorphisms and their categorical duals by continuity properties is central in the development of general Stone duality [14], and also in the present context morphisms receive a concrete topological flavor: regarding principal upsets as basic closed sets renders adjoint maps "basic continuous": preimages of basic closed sets are basic closed. More precisely, we justify the term "continuous" for locale morphisms by showing that the l–morphisms between implicative semilattices are characterized by the following continuity condition: the preimage of the zero ideal (the least basic closed set) is zero, the preimage of any basic closed set is basic closed, and its complement in the lattice of l–ideals is contained in the preimage of the complement – a triviality in the counterpart of *set-theoretical* complements, but an unavoidable additional condition in the lattice-theoretical setting.

In the complete case of locales, the prefix "basic" is omitted, since the basic closed sets then form a closure system, that is, a collection of sets closed under arbitrary intersections (with $\bigcap \emptyset$ being the entire ground set). The closed sublocales and their lattice-theoretical complements, the open sublocales, represent (via $\mathcal{O}_T$) closed resp. open subspaces. In a suitable categorical framework, they also correspond to Isbell's abstract *open* resp. *closed parts* [20]. However, the analogy to the topological situation should not be overinterpreted. If for an l–morphism $f\colon B \longrightarrow A$ and an l–domain C of A there is a greatest l–domain D of B contained in $f^{\leftarrow}C$ then D is called the *localic preimage* of C. Such localic preimages exist in the complete case of locales, but not in general. Four questions arise immediately:

(1) Is any adjoint map whose preimages of opens are open an l–morphism?

(2) Is any adjoint map with open localic preimages of opens an l–morphism?

(3) Are the set-theoretical preimages of opens under any l–morphism open?

(4) Are the localic preimages of opens under any l–morphism open?

Only (4) has a positive response, while the other three questions have to be answered in the negative, even in the complete case:

In boolean locales (and only in these), all sublocales are open and closed [21, 38]. Hence, every residual map has here the property that preimages of open resp. closed sublocales are again open resp. closed. But not all residuated maps preserve binary meets, and consequently, their adjoints need not be localic. On the other hand, on a chain every residuated map trivially preserves binary meets; hence, all top-preserving residuated maps between complete chains are frame homomorphisms (r–morphisms), and all residual maps f satisfying $fx = \top \Rightarrow x = \top$ are localic (l–morphisms). But there are l–morphisms on four-element chains such that not all preimages of open sublocales are open: take $f\top = \top$ and $fn = n+1$ otherwise.

Concerning (basic) open morphisms and quasi-open morphisms (having the property that the image of any basic open set is contained in a least basic open set of the codomain), we extend the Joyal–Tierney Theorem [25] about open localic maps to the non-complete situation, by establishing a dual isomorphism between the category of implicative semilattices with basic open l–morphisms as morphisms and the category having the same objects but as morphisms those implicative maps which are biadjoint, that is, both adjoint and coadjoint; in fact, these are just the coadjoints of the basic open l–morphisms. More generally, via Galois adjunction, arbitrary biadjoint maps correspond to the quasi-open l–morphisms. Similar phenomena have been observed in other contexts (cf. Erné [12], Hofmann and Mislove [19]).

Finally, in order to bring together all pieces of the puzzle, we introduce a category of *basic zero-dimensional spaces*, similar to categories considered in [11] and [14]. The objects are closure spaces with a distributive closure system containing a specified meet-base of complemented members. All the categories discussed here, and many more, like that of T_D-spaces (Aull and Thron [2]), front spaces (Skula [43]), and of course, Stone spaces and the more general zero-dimensional spaces (Johnstone [21]), are embedded in the category of basic zero-dimensional spaces. Hence, that category might be an interesting subject of future research.

2. Closure operations, closure ranges and adjunctions

Let A be a partially ordered set (poset), $\leq$ its order relation, and A^{op} the dual poset. We write $b = a \vee c$ if b is the least upper bound (*supremum*, *join*), and dually $b = a \wedge c$ if b is the greatest lower bound (*infimum*, *meet*) of $\{a, c\}$. This convention also applies when not all binary suprema resp. infima exist (that is, not only in lattices but in arbitrary posets). Recall that completeness is a self-dual property: all subsets have joins iff all subsets have meets. A least element (*bottom*) of A is denoted by $\perp$ or $\perp A$, and a greatest element (*top*) by $\top$ or $\top A$. If $a \vee c = \top$ and $a \wedge c = \perp$ then c is a *complement* of a.

A (*unary*) *operation* on A is a map from A into A. Any set of operations on A is ordered pointwise. A *closure operation* or *hull operation* is an isotone (order-preserving), inflationary (extensive) and idempotent operation. The dual notion is *coclosure* or *kernel operation.* Closure operations j may be characterized by the single equivalence

$$x \leq jy \Leftrightarrow jx \leq jy.$$

We call a subset C of A a *closure range* if for each $a \in A$ there is a least $c \in C$ with $a \leq c$. Other names for such subsets are *closure system*, *partial ordinal* [3], or *relatively meet–closed set* [17, 44]. Indeed, any closure range is closed under all existing meets, and the closure ranges of a complete lattice are exactly its meet–closed subsets. We reserve the term "closure system" for sets that are closed under intersections and consequently complete lattices with respect to the inclusion order. Recall that a closure system is *topological* if it is closed under finite unions, and *algebraic* if it is closed under directed unions. The term "closure range" is justified by the following fact [3, 13, 32]:

Proposition 2.1. *Sending each closure operation to its range, one obtains a dual isomorphism between the pointwise ordered set of all closure operations on A and the set of all closure ranges in A, ordered by inclusion.*

Every map $h\colon A \longrightarrow B$ naturally factors into its surjective corestriction $h_0\colon A \longrightarrow hA$ and the inclusion map $h^0\colon hA \longrightarrow B$. Note that h is a closure operation iff h^0 is adjoint to h_0, and h is a homomorphism iff h^0 and h_0 are homomorphisms. For easy reference, we record the main connections between closure operations and adjoint maps (Blyth and Janowitz [8], Erné [13, 15]).

Proposition 2.2. *Any residuated map $h\colon A \longrightarrow B$ with range D and its adjoint $f\colon B \longrightarrow A$ with range C satisfy the equations $f = fhf$ and $h = hfh$. Hence, C is the range of the closure operation $g = fh$, D is the range of the coclosure operation $k = hf$, and $i = h_0|_C\colon C \longrightarrow D$ is an isomorphism with $h = k^0 i g_0$. This provides a factorization of h into a surjective, a bijective and an injective residuated map. A dual factorization $f = g^0 i^{-1} k_0$ into residual maps holds in the opposite direction.*

Proposition 2.3. *Let $h\colon A \longrightarrow B$ be a residuated map and $f\colon B \longrightarrow A$ its adjoint. For any closure operation j on B with range C, the "image" fjh is a closure operation on A with range fC. Hence, residual maps send closure ranges to closure ranges.*

3. Morphisms between implicative semilattices

By a semilattice we always mean a $\wedge$-semilattice with top element. As morphisms between semilattices we take residuated maps that preserve finite meets, called *r–morphisms*; their adjoints are referred to as *localizations* or *l–morphisms.* Note that if the codomain of an injective r– or l–morphism is complete then so is the domain, and in the opposite direction, if the domain of a surjective r– or l–morphism is complete then so is the codomain.

An *implicative semilattice* [6, 7, 33] (or *Brouwerian semilattice* [26, 27]) is a semilattice whose unary operations $\lambda_a = a \wedge -$ have adjoints $\alpha_a = a \!\rightarrow\! -$:

$$a \wedge x \leq y \Leftrightarrow x \leq a \!\rightarrow\! y.$$

As announced in the introduction, we regard implicative semilattices as algebras

$$(A, \wedge, \top, \alpha)$$

with the family $\alpha = (\alpha_a\colon a \in A)$ of unary residuations, so that each λ_a is an r–morphism, and each α_a is an l–morphism. Observe that an r–morphism between implicative semilattices preserves not only existing joins but also complements.

Deviating from [39], we reserve the terms *Heyting semilattice* and *Heyting lattice* for bounded implicative semilattices resp. lattices (having a least element $\perp$). In the lattice case,

$$(A, \vee, \wedge, \top, \perp, \rightarrow)$$

is a *Heyting algebra.* All these algebraic structures are equationally definable (see, e.g., Esakia [17] or Köhler [26, 27]). In Heyting semilattices, the element $\neg\, a = a \!\rightarrow\! \perp$, also denoted by $a^{\perp}$ or a^{*}, is the *pseudocomplement* or *negation* of a. From now on,

A denotes an implicative semilattice with top element $\top$.

By an *interior operation* we mean a kernel operation preserving finite meets. On the other hand, a *nucleus* (see, for example, Bezhanishvili and Ghilardi [6], and for the complete case, Banaschewski [4], Johnstone [21], Simmons [42]) is a closure operation j preserving finite meets; instead of the latter condition, it suffices to postulate the seemingly weaker but equivalent inequality

$$x \wedge jy \leq j(x \wedge y).$$

There is a description of nuclei on implicative semilattices by one equation, due to Macnab [30], who calls nuclei on Heyting algebras *modal operators*:

$$x \to jy = jx \to jy.$$

Notice that every nucleus j fulfils the inequality

$$j(x \to y) \leq jx \to jy$$

but equality need not hold, that is, j need not be *implicative* (preserve the *formal implication* $\to$). An inner characterization of the ranges of nuclei is provided by the next definition: a *nuclear range* (*modal subalgebra* in [30], *strong ideal* in [39]) is a closure range C that is left $\to$–closed, or *l–closed*, i.e. closed under the *unary* operations α_a, which means that $a \to c \in C$ for all $a \in A$ and $c \in C$. As a closure range contains all existing meets of subsets, every nuclear range is an *l–ideal*, that is, an l–closed subsemilattice (*total subalgebra* in [27], *ideal* in [39]). By definition, the l–ideals are the left ideals with respect to the operation $\to$; all order-theoretical *filters* (*dual ideals*), i.e. nonempty $\wedge$–closed upsets, are l–ideals, but not conversely.

We denote by $\mathcal{T}A$ the algebraic closure system of all l–ideals (total subalgebras), by $\mathcal{S}lA$ its $\vee$-subsemilattice of those l–ideals that are closure ranges, and by $\mathcal{N}A$ the same set, but ordered by *dual* inclusion. Notice that $\mathcal{S}lA$ need not be a closure system if A is not complete. The *zero ideal* $\mathsf{0} = \{\top\}$ is the *least* element of $\mathcal{S}lA$ but the *greatest* element of $\mathcal{N}A$. The subsequent description of the members of $\mathcal{S}lA$ resp. $\mathcal{N}A$ is familiar in the more restricted theory of frames and locales, where they are known as *sublocales* [21, 36]. The case of Heyting algebras, due to Macnab [5, 29, 30], extends without any alteration to implicative semilattices.

Proposition 3.1. *Sending each nucleus to its range yields an isomorphism between the semilattice* $\mathrm{N}A$ *of all nuclei and the semilattice* $\mathcal{N}A$ *of all nuclear ranges. Hence, these are not only the ranges of nuclei but also the l–domains, that is, those subsets for which the inclusion map into* A *is an l–morphism.*

Analogously, by an *r–domain* we mean a subset for which the inclusion map is an r–morphism. In light of our general remarks on adjunctions in Section 2, and observing that composites of maps preserve finite meets if the factors do, we draw the following conclusions:

Theorem 3.1. *Let $h\colon A \longrightarrow B$ be an r–morphism between implicative semilattices with range C and $f\colon B \longrightarrow A$ its adjoint l–morphism with range D. Then $g = fh$ is a nucleus, $k = hf$ is an interior operation, and h has a unique factorization $h = k^0 i\, g_0$, where*

$g_0\colon A \longrightarrow C$ is the corestriction of a nucleus (an extremal r–epimorphism),

$i\colon C \longrightarrow D$ is an isomorphism (an extremal r–epi- and -monomorphism),

$k^0\colon D \longrightarrow B$ is the inclusion of an r–domain (extremal r–monomorphism).

A dual extremal epi-mono-factorization into l–morphisms holds for f.

Corollary 3.1. *The surjective corestrictions of nuclei are representative for the equivalence classes of surjective r–morphisms, and the inclusion maps of l–domains for the equivalence classes of injective l–morphisms.*

Corollary 3.2. *The poset of r–morphisms between implicative semilattices A and B is dual to the poset of l–morphisms from B to A, and isomorphic to the poset of isomorphisms between l–domains of A and r–domains of B.*

Proposition 3.2. *For an l–morphism $f\colon B \longrightarrow A$ adjoint to an r–morphism $h\colon A \longrightarrow B$, and a nucleus j on B, the "image" fjh is a nucleus on A whose range is the image of the range jB under f. Hence, l–morphisms map l–domains (that is, nuclear ranges) to l–domains.*

In the complete case of frames resp. locales, the r–domains are just the subframes, and on the other hand, the l–domains are just the sublocales. Categorically thinking people mean by a sublocale an extremal l–monomorphism between locales or an extremal r–epimorphism between frames [20, 21, 38]. In view of Corollary 3.1 all three interpretations are well compatible.

Let us recall a few facts concerning $\mathcal{T}A$ and $\mathcal{S}lA$ (cf. [21, 30, 36] for the case of frames, where $\mathcal{S}lA$ is a closure system). Binary joins in $\mathcal{T}A$ and $\mathcal{S}lA$ are given by

$$C \vee D = \{x \wedge y \mid x \in C,\, y \in D\}.$$

The next proposition from [16] generalizes results in [27] and [39].

Proposition 3.3. *For l–ideals C and D_i $(i \in I)$ of A, the distributive law*

$$C \vee \bigcap_{i\in I} D_i = \bigcap_{i\in I}(C \vee D_i)$$

holds whenever I is finite or C is a nuclear range. In particular,

(1) *$\mathcal{T}A$ is an algebraic frame,*

(2) *$\mathcal{S}lA$ is a coframe whenever it is a closure system.*
Hence, in the latter case, the isomorphic lattices $\mathrm{N}A$ *and* $\mathcal{N}A$ *are frames.*

For each $a \in A$, the adjoint map $\alpha_a = a \to -$ is known to be a nucleus (see, e.g., Macnab [30]), and its range is

$$\begin{aligned} \mathfrak{a}a &= \{\, a \to x \mid x \in A\} \\ &= \{x \in A \mid x = a \to x\} \\ &= \{x \in A \mid (a \to x) \to x = \top\}. \end{aligned}$$

The following facts are from [16] (cf. [39], and [36] for the case of locales):

Proposition 3.4. *The map $\mathfrak{a}_A = \mathfrak{a}$ is an embedding of A in $\mathcal{S}lA$; it preserves finite meets* (*though $\mathcal{S}lA$ need not be a $\wedge$-semilattice*) *and all existing joins.*

There is also a canonical embedding $\mathfrak{c} = \mathfrak{c}_A$ of A in $(\mathcal{T}A)^{op}$, sending a to $\mathfrak{c}a$, the principal upset $\uparrow a$, which is always an l–ideal but need not be nuclear unless A is a lattice, in which case $\gamma_a = a \vee -$ is the associated nucleus. We record a result that is known for Heyting algebras [30] and frames [36, 39]; it extends, by a different argument given in [16], replacing $a \vee x$ with $(a \to x) \to x$, to implicative semilattices.

Proposition 3.5. *For each $a \in A$, the l–ideal $\mathfrak{c}a$ is the complement and so the pseudocomplement of the nuclear range $\mathfrak{a}a$ in $\mathcal{T}A$, hence also the complement in $\mathcal{S}lA$ and in $\mathcal{N}A$ if A is a lattice. The embedding $\mathfrak{c}_A$ of A in $(\mathcal{T}A)^{op}$ resp. in $\mathcal{N}A$ preserves finite meets and existing joins. If A is a Heyting lattice then $\mathfrak{c}_A : A \longrightarrow \mathcal{N}A$ is an r–morphism whose adjoint sends $C \in \mathcal{N}A$ to $\bot C$.*

In view of Proposition 3.5 and the resemblance to the situation of topological spaces, the sets $\mathfrak{a}a$ are called *basic open*, and the sets $\mathfrak{c}a$ *basic closed* ($\mathfrak{apertus}$ = latin for *open*, $\mathfrak{clusus}$ = latin for *closed*). Of course, the complete case is more intuitive: here, the basic closed sets are merely called *closed*, since they form a closure system, and their lattice complements *open*, though the system of all open sets need *not* be closed under unions. However, via $\mathcal{O}_T$, the open sublocales represent the open subspaces, and the closed sublocales the closed subspaces of a topological space T. Observe that even for frames A, the lattice-theoretical complements in $\mathcal{T}A$, in $\mathcal{S}lA$ and in its dual $\mathcal{N}A$ differ from the set-theoretical complements; see [36] for details.

4. l–morphisms as continuous maps

We now turn to a more thorough investigation of r–morphisms and their adjoints, the l–morphisms.

Proposition 4.1. *If a map $f\colon B \longrightarrow A$ between implicative semilattices is adjoint to $h\colon A \longrightarrow B$ then the following conditions on an element $a \in A$ are equivalent:*

(a) $h(a \wedge c) = ha \wedge hc$ *for all* $c \in A$.

(b) $f(ha \to b) = a \to fb$ *for all* $b \in B$.

(c) $f(\mathfrak{a}ha \cap \mathfrak{a}b) = \mathfrak{a}a \cap f\mathfrak{a}b$ *for all* $b \in B$.

(d) $f\mathfrak{a}ha \subseteq \mathfrak{a}a$, *that is,* $\mathfrak{a}ha \subseteq f^{\leftarrow}\mathfrak{a}a$.

Proof: (a) $\Rightarrow$ (b) follows from the equivalences

$c \le f(ha \to b) \;\Leftrightarrow\; hc \le ha \to b \;\Leftrightarrow\; ha \wedge hc = h(a \wedge c) \le b \;\Leftrightarrow\; a \wedge c \le fb$
$\;\Leftrightarrow\; c \le a \to fb.$

(b) $\Rightarrow$ (c): The inclusion $f(\mathfrak{a}ha \cap \mathfrak{a}b) \subseteq \mathfrak{a}a \cap f\mathfrak{a}b$ is clear from (b). Conversely, for any $d = a \to d = f(b \to c) \in \mathfrak{a}a \cap f\mathfrak{a}b$ we have

$d = a \to f(b \to c) = f(ha \to (b \to c)) = f((ha \wedge b) \to c).$

By Proposition 3.4, $(ha \wedge b) \to c \in \mathfrak{a}(ha \wedge b) = \mathfrak{a}ha \cap \mathfrak{a}b$, hence $d \in f(\mathfrak{a}ha \cap \mathfrak{a}b)$.
(c) $\Rightarrow$ (d): $f\mathfrak{a}ha = f(\mathfrak{a}ha \cap \mathfrak{a}\top) = \mathfrak{a}a \cap f\mathfrak{a}\top \subseteq \mathfrak{a}a$.
(d) $\Rightarrow$ (a): Given $c \in A$, put $b = h(a \wedge c)$. By (d), $f(ha \to b) = a \to d$ for some $d \in A$. Then $a \wedge c \le fb$ and $b \le ha \to b$, hence $fb \le f(ha \to b) = a \to d$, $a \wedge c \le a \wedge fb \le d$, $c \le a \to d = f(ha \to b)$ and so $hc \le ha \to b$, $ha \wedge hc \le b = h(a \wedge c) \le ha \wedge hc$, since h is istone. ■

Proposition 4.2. *The image of an l–ideal (l–domain) under an l–morphism f is an l–ideal (l–domain). For injective f the preimage of an l–ideal is an l–ideal.*
For surjective f, a set is basic closed iff its preimage under f is basic closed.

Proof: Let $f\colon B \longrightarrow A$ be an l–morphism with coadjoint h, and let D be an l–ideal of B. The image fD is a subsemilattice of A (as f preserves meets). For $a \in A$ and $b \in D$, we get $a \to fb = f(ha \to b) \in fD$ by Proposition 4.1. Thus, fD is an l–ideal. By Propositions 2.3 and 3.2, fD is a closure range resp. l–domain if D is one.

If f is injective then h is surjective. For each l–ideal C of A, the preimage $f^{\leftarrow}C$ is an l–ideal, being a subsemilattice such that for $b, c \in B$ with $fb \in C$ there is an $a \in A$ with $c = ha$, hence $f(c \to b) = f(ha \to b) = a \to fb \in C$ and $c \to b \in f^{\leftarrow}C$.

Now, suppose f is surjective and $f^{\leftarrow}C$ is basic closed, say $f^{\leftarrow}C = \uparrow b$. Then, for $a = fb$ we get $fha = a \in C$, hence $b \leq ha$, and then

$$a \leq x \Rightarrow b \leq hx \Rightarrow hx \in f^{\leftarrow}C \Rightarrow x = fhx \in C \Rightarrow a = fb \leq fhx = x.$$

Thus, $C = \uparrow a$ is basic closed. ■

The following example demonstrates that the preimage of an l–domain under an injective l–morphism need not be an l–domain.

Example 4.1. Like every bounded chain, the rational chain

$$A = \{\pm\tfrac{1}{n} \mid n \in \mathbb{N}\}$$

(with $\mathbb{N}$ the chain of positive integers) is a Heyting lattice. It is easy to see that in the semilattice $\mathcal{N}A$ (the dual of $\mathcal{S}lA$), the subset $\{B, C\}$ with

$$B = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{-\tfrac{1}{2n-1} \mid n \in \mathbb{N}\} \ \text{ and } \ C = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{-\tfrac{1}{2n} \mid n \in \mathbb{N}\}$$

has no join, and the complement of C has neither in $\mathcal{S}lA$ nor in $\mathcal{N}A$ a pseudocomplement; details are given in [16]. Define maps f and g on A by

$$f(\tfrac{1}{n}) = g(\tfrac{1}{n}) = \tfrac{1}{n},\ f(-\tfrac{1}{n}) = g(-\tfrac{1}{2n-1}) = g(-\tfrac{1}{2n}) = -\tfrac{1}{2n}.$$

Both f and g are l–morphisms with range C, but the preimage of the l–domain B is the filter $F = \{\tfrac{1}{n} \mid n \in \mathbb{N}\}$, which is not an l–domain. While f is injective but not a nucleus, g is not injective but a nucleus.

In analogy to semilinear maps between vector spaces, modules and algebras, we call a map $f\colon B \longrightarrow A$ *semilinear* with respect to binary operations on A and B, both denoted by $*$, if it has a coadjoint $h\colon A \longrightarrow B$ satisfying the *Frobenius identity* (cf. [9, 28, 31, 35])

$$a * fb = f(ha * b).$$

In that case, we also say f is $*$*–semilinear*. From Proposition 4.1, we deduce one algebraic and one closure-theoretical characterization of the adjoints of residuated $\wedge$-homomorphisms (which need not preserve the top elements).

Theorem 4.1. *A map $f\colon B \longrightarrow A$ between implicative semilattices is $\to$–semilinear, or equivalently, adjoint to a $\wedge$-homomorphism, iff each basic closed set has a basic closed preimage whose complement in $\mathcal{T}B$ is a subset of (but not necessarily equal to) the preimage of the complement in $\mathcal{T}A$.*

Proof: If $f\colon B \longrightarrow A$ is adjoint to a $\wedge$-homomorphism $h\colon A \longrightarrow B$ then preimages of basic closed sets $\mathfrak{c}a = \uparrow a$ are basic closed: $f^{\leftarrow}\mathfrak{c}a = \mathfrak{c}ha$. By Proposition 3.5, $\mathfrak{a}a$ is the complement $\neg\,\mathfrak{c}a$ of $\mathfrak{c}a$ in $\mathcal{T}A$, and by Proposition 4.1, we have

$$\neg\, f^{\leftarrow}\mathfrak{c}a = \neg\,\mathfrak{c}ha = \mathfrak{a}ha \subseteq f^{\leftarrow}\mathfrak{a}a = f^{\leftarrow}\neg\,\mathfrak{c}a.$$

Conversely, assume that $f^{\leftarrow}\mathfrak{c}a$ is basic closed and $\neg\, f^{\leftarrow}\mathfrak{c}a \subseteq f^{\leftarrow}\neg\,\mathfrak{c}a$ for all $a \in A$. The first condition just expresses that f is adjoint to a map $h\colon A \longrightarrow B$ with $f^{\leftarrow}\mathfrak{c}a = \mathfrak{c}ha$. From the second condition, it follows as above that

$$\mathfrak{a}ha = \neg\, f^{\leftarrow}\mathfrak{c}a \subseteq f^{\leftarrow}\neg\,\mathfrak{c}a = f^{\leftarrow}\mathfrak{a}a.$$

Thus, by Proposition 4.1, f is $\rightarrow$-semilinear, or equivalently, adjoint to a $\wedge$-homomorphism. ∎

A map f between topped posets is called *codense* if $fb = \top$ implies $b = \top$. If f is adjoint to h then preservation of top elements by h is equivalent to codensity of f:

$$h\top = \top \;\Leftrightarrow\; (fb = \top \Rightarrow b = \top) \;\Leftrightarrow\; f^{\leftarrow}\mathsf{0} = \mathsf{0}.$$

Note that a map $f\colon B \longrightarrow A$ is an l–morphism iff it is codense and there exists a map $h\colon A \longrightarrow B$ satisfying the Frobenius identity for $\rightarrow$, because that entails

$$ha \leq b \;\Leftrightarrow\; ha \rightarrow b = \top \;\Leftrightarrow\; f(ha \rightarrow b) = \top \;\Leftrightarrow\; a \rightarrow fb = \top \;\Leftrightarrow\; a \leq fb,$$

so that h is necessarily the coadjoint of f.

If for an l–morphism $f\colon A \longrightarrow B$ and some $C \in \mathcal{S}lA$ there is a greatest $D \in \mathcal{S}lB$ contained in the preimage $f^{\leftarrow}C$ then this D is called the *localic preimage* of C and denoted by $f_{\leftarrow}C$. In the complete case, one finds the following result in [36] (where $f[D]$ stands for fD and $f_{-1}[C]$ for $f_{\leftarrow}C$):

Proposition 4.3. *Let $f\colon B \longrightarrow A$ be a localic map between locales. For each $D \in \mathcal{S}lB$ the image $f_{\rightarrow}D$ belongs to $\mathcal{S}lA$, for each $C \in \mathcal{S}lA$ the localic preimage $f_{\leftarrow}C$ exists, and this provides an adjunction between $\mathcal{S}lB$ and $\mathcal{S}lA$:*

$$f_{\rightarrow}D \subseteq C \;\Leftrightarrow\; D \subseteq f_{\leftarrow}C.$$

For categorically versed readers: the l–inclusion map of the localic preimage under a localic map f is the pullback of the l–inclusion map along f [37], and one defines localic preimages of extremal l–monomorphisms, regarded as sublocales, by taking pullbacks [20, 21, 38]. Non-complete situations are less comfortable, as Example 4.1 shows: there is no greatest l-domain contained in the preimage of the l–domain B under the injective l–morphism f.

We come to the main characterization of l–morphisms in closed and open terms:

Theorem 4.2. *For a map $f\colon B \longrightarrow A$ between implicative semilattices, the following conditions are equivalent:*

(a) *f is an l–morphism.*
(b) *f is codense and $\to$-semilinear.*
(c) *$f^{\leftarrow}\mathsf{0} = \mathsf{0}$, preimages of basic closed sets are basic closed and have complements in $\mathcal{T}B$ contained in the preimages of the complements in $\mathcal{T}A$.*
(d) *f is isotone with $f_{\leftarrow}\mathfrak{a} = \mathfrak{a}h$ and $f_{\leftarrow}\mathfrak{c} = \mathfrak{c}h$ for some map $h\colon A \longrightarrow B$.*
(e) *f is isotone, localic preimages of basic open sets exist, are basic open, and their complements are the preimages of the complements in $\mathcal{T}A$.*

Proof: Theorem 4.1 assures the equivalence of (a), (b) and (c).
(b) $\Rightarrow$ (d): By (b) $\Rightarrow$ (a), f has a coadjoint h satisfying the equation $\mathfrak{c}h = f_{\leftarrow}\mathfrak{c}$ and is therefore isotone. By (b) $\Rightarrow$ (d) in Proposition 4.1, $\mathfrak{a}ha \subseteq f^{\leftarrow}\mathfrak{a}a$. On the other hand, if D is any l–ideal of B with $D \subseteq f^{\leftarrow}\mathfrak{a}a$, then for $d \in D$ and $b = (ha \to d) \to d$, we have $ha \leq b \in D$ and so $fb \in fD \subseteq \mathfrak{a}a$; hence,

$$fb = a \to fb = f(ha \to b) = f\top = \top,$$

and

$$(ha \to d) \to d = b = \top$$

by codensity, whence $d \in \mathfrak{a}ha$. Thus, $D \subseteq \mathfrak{a}ha$. This proves the equation $\mathfrak{a}ha = f_{\leftarrow}\mathfrak{a}a$.
(d) $\Rightarrow$ (c): For isotone f, the identity $\mathfrak{c}h = f_{\leftarrow}\mathfrak{c}$ implies that f is adjoint to h:

$$ha \leq b \;\Leftrightarrow\; \mathfrak{c}b \subseteq \mathfrak{c}ha = f_{\leftarrow}\mathfrak{c}a \;\Leftrightarrow\; f_{\rightarrow}\mathfrak{c}b \subseteq \mathfrak{c}a \;\Leftrightarrow\; fb \in \mathfrak{c}a \;\Leftrightarrow\; a \leq fb.$$

Further, $f^{\leftarrow}\mathsf{0} = \mathsf{0}$, as

$$h\top \in \mathfrak{c}h\top = \neg\,\mathfrak{a}h\top = \neg\, f_{\leftarrow}\mathfrak{a}\top = \neg\, f_{\leftarrow}A = \neg\, B = \{\top\}.$$

(d) $\Leftrightarrow$ (e) is straightforward, using the fact that $\mathfrak{a}$ and $\mathfrak{c}$ are embeddings. ■

In condition (e) one may add that localic preimages of basic closed sets are basic closed to make the condition more "symmetric". An obvious question is whether condition (d) is tantamount to the weaker condition

(d') $f_{\leftarrow}\mathfrak{a} = \mathfrak{a}h$ and $f_{\leftarrow}\mathfrak{c} = \mathfrak{c}h$ for some map $h\colon A \longrightarrow B$.

It is true that any such map h has to be isotone, since

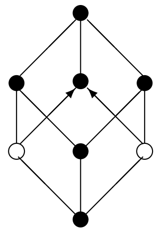
$$a \le b \Rightarrow \mathfrak{c}b \subseteq \mathfrak{c}a \Rightarrow f_{\leftarrow}\mathfrak{c}b \subseteq f_{\leftarrow}\mathfrak{c}a \Rightarrow \mathfrak{c}hb \subseteq \mathfrak{c}ha \Rightarrow ha \le hb,$$

and that h has to commute with all existing complements:

$$\mathfrak{c}h\neg a = f_{\leftarrow}\mathfrak{c}\neg a = f_{\leftarrow}\mathfrak{a}a = \mathfrak{a}ha = \mathfrak{c}\neg ha, \text{ whence } h\neg a = \neg ha.$$

However, condition (d') does not imply that f is isotone, not even if h is the identity map on an eight-element boolean algebra B:

Example 4.2.



The sketched map f is extensive and idempotent but not isotone, and satisfies

$$f_{\leftarrow}\mathfrak{c}b = \mathfrak{c}b = \mathfrak{a}\neg b = f_{\leftarrow}\mathfrak{a}\neg b \text{ for all } b \in B.$$

This example also shows that in condition (c) of Theorem 4.2 it does not suffice to postulate localic preimages of basic closed sets to be basic closed. But isotone maps may also be characterized by a continuity condition, namely with respect to the topologies formed by all unions of basic closed sets.

Let us summarize the main conclusions for the case of locales (where the l–domains are the sublocales) and stress the analogy but also the differences to the classical case of topological spaces. Applying Theorem 3.2 to the complete case shows the localic maps in a very pleasing light, namely as a natural analogue of the topologically continuous functions. In accordance with [36, Ch. III–4] we have for any localic map that

the set-theoretic image of a sublocale is always a sublocale,
the set-theoretic preimage of a closed sublocale is a closed sublocale,
the set-theoretic preimage of an open sublocale need not be a sublocale,
the localic preimage of an open sublocale is an open sublocale,
the localic preimage map is adjoint to the image map.

Recall that a map between locales for which preimages of closed resp. open sets are closed resp. open need not be localic. Open sublocales are complementary to closed sublocales (in the lattice of all sublocales). Now, the characterization of localic maps in "terms of continuity" reads as follows:

Corollary 4.1. *A function f between the underlying sets of locales B and A is a localic map from B to A iff the preimage of zero is zero and for all closed C in A, $f^{\leftarrow}C$ is closed in B and satisfies $\neg f^{\leftarrow}C \subseteq f^{\leftarrow}\neg C$.*

Here $\neg$ denotes the complement in the coframe of sublocales. The inclusion for the set-theoretic preimage in the last formula replaces equality for the localic preimage, as the standard set-theoretic preimage $f^{\leftarrow}\neg C$ need not be a sublocale; the displayed inclusion avoids any reference to localic preimages.

Boolean lattices may be characterized in terms of basic closed resp. open subsets (see [30], and for the case of locales, [21] and [38]).

Proposition 4.4. *A Heyting lattice A is a boolean lattice iff the l–domains are the basic closed sets, or equivalently, the l–domains are the basic open sets.*

Let $\mathcal{B}A$ denote the boolean sublattice of $(\mathcal{T}A)^{op}$ generated by the basic closed resp. basic open sets. Without proof we note:

Proposition 4.5. *For any Heyting lattice A, $\mathcal{B}A$ is its free boolean extension. The r–embedding $\mathfrak{c}\colon A \longrightarrow \mathcal{B}A$ has the adjoint $\mathfrak{l}\colon \mathcal{B}A \longrightarrow A$, $C \longmapsto \perp C$.*

This provides a categorical description of basic closed sets in Heyting lattices:

Theorem 4.3. *For a Heyting lattice A and $C \subseteq A$ the following are equivalent:*

(a) *C is a basic closed set.*
(b) *All preimages of C under l–morphisms are l–domains.*
(c) *The preimage of C under the l–morphism $\mathfrak{l}\colon \mathcal{B}A \longrightarrow A$ is an l–domain.*
(d) *$f^{\leftarrow}C \in SlB$ for some l–morphism f from a boolean lattice B onto A.*
(e) *$f^{\leftarrow}C$ is basic closed for some surjective l–morphism $f\colon B \longrightarrow A$.*

Proof: (a) ⇒ (b): Theorem 4.2.
(b) ⇒ (c) ⇒ (d): Proposition 4.5.
(d) ⇒ (e): Proposition 4.4.
(e) ⇒ (a): Proposition 4.2 (last sentence). ∎

Corollary 4.2. *A subset of a locale A is closed iff all its preimages under localic maps are sublocales iff its preimage under $\mathfrak{l}\colon \mathcal{B}A \longrightarrow A$ is a sublocale.*

By Theorem 4.2, the r–morphisms h between implicative semilattices with adjoints f are those isotone maps for which this diagram commutes:

$$
\begin{tikzcd}
 & A \arrow[rrr, "h"'] \arrow[rrr, dotted, leftarrow, bend left, "f"] \arrow[ddl, "\mathfrak{a}_A"'] \arrow[dr, "\mathfrak{c}_A"] & & & B \arrow[ddl, "\mathfrak{a}_B"'] \arrow[dr, "\mathfrak{c}_B"] & \\
 & & \mathfrak{c}A \arrow[rrr, "f_{\leftarrow}"] \arrow[dll, leftrightarrow, "\neg"] \arrow[dd, hook] & & & \mathfrak{c}B \arrow[dll, leftrightarrow, "\neg"] \arrow[dd, hook] \\
\mathfrak{a}A \arrow[rrr, "f_{\leftarrow}"] \arrow[dd, hook] & & & \mathfrak{a}B \arrow[dd, hook] & & \\
 & & \mathcal{B}A^{op} \arrow[rrr, "f_{\leftarrow}"] \arrow[dll, leftrightarrow, "\neg"] & & & \mathcal{B}B^{op} \arrow[dll, leftrightarrow, "\neg"] \\
\mathcal{B}A \arrow[rrr, "f_{\leftarrow}"] & & & \mathcal{B}B & &
\end{tikzcd}
$$

5. Biadjoint morphisms

By a *biadjoint map* between posets we mean one that is both adjoint and coadjoint. A biadjoint map preserves not only all existing joins, but also all existing meets. Hence, a biadjoint map between semilattices is certainly an r–morphism; and a map between complete lattices is biadjoint iff it preserves arbitrary joins and meets, in other words, it is a complete homomorphism.

Let us consider some further Frobenius identities:

Proposition 5.1. *For a biadjoint map $h\colon A \longrightarrow B$ between implicative semilattices with adjoint f and coadjoint g, the following conditions are equivalent:*

(a) *For all $a \in A$ and $c \in A$, $h(a \to c) = ha \to hc$.*
(b) *For all $a \in A$ and $b \in B$, $g(ha \wedge b) = a \wedge gb$.*
(c) *For all $c \in A$ and $b \in B$, $f(b \to hc) = gb \to c$.*

Proof: The claim is immediate from the following chains of equivalences:

$$b \leq h(a \to c) \;\Leftrightarrow\; gb \leq a \to c \;\Leftrightarrow\; a \wedge gb \leq c \;\Leftrightarrow\; a \leq gb \to c$$
$$b \leq ha \to hc \;\Leftrightarrow\; ha \leq b \to hc \;\Leftrightarrow\; g(ha \wedge b) \leq c \;\Leftrightarrow\; a \leq f(b \to hc). \quad \blacksquare$$

A map between (topological or closure) spaces is called *quasi-open* if for each open set in the domain there is a least open set containing its image (Erné [12], Hofmann and Mislove [19]). It is easy to see that via the functor $\mathcal{O}_{\leftarrow}$ the quasi-open continuous maps correspond to the complete homomorphisms or, equivalently, biadjoint maps between the associated lattices of open sets. In full analogy, we call an l–morphism $f\colon B \longrightarrow A$ *quasi-open* iff for each basic open set U in B there is a least basic open set in A containing the image of U; and we say f is *basic open* if images of basic open sets are again such.

Theorem 5.1. *An r–morphism is (bi)adjoint iff its adjoint is quasi-open. Hence, via Galois adjunction, the category of implicative semilattices and biadjoint maps is dual to the category with the same objects and quasi-open l–morphisms.*

Proof: For any r–morphism $h\colon A \longrightarrow B$ with adjoint $f\colon B \longrightarrow A$, we have

$$f\mathfrak{a}b \subseteq \mathfrak{a}a \;\Leftrightarrow\; \mathfrak{a}b \subseteq f_{\leftarrow}\mathfrak{a}a \;\Leftrightarrow\; \mathfrak{a}b \subseteq \mathfrak{a}ha \;\Leftrightarrow\; b \leq ha.$$

Thus, if h has a coadjoint g then the equivalence $gb \leq a \;\Leftrightarrow\; b \leq ha$ yields $f\mathfrak{a}b \subseteq \mathfrak{a}a \;\Leftrightarrow\; \mathfrak{a}gb \subseteq \mathfrak{a}a$, whence $\mathfrak{a}gb$ is the least basic open set containing $f\mathfrak{a}b$. Conversely, if such an $\mathfrak{a}gb$ exists for each $b \in B$, then we obtain

$$gb \leq a \;\Leftrightarrow\; \mathfrak{a}gb \subseteq \mathfrak{a}a \;\Leftrightarrow\; f\mathfrak{a}b \subseteq \mathfrak{a}a \;\Leftrightarrow\; b \leq ha,$$

i.e., g is coadjoint to h. ∎

We are ready for a generalized version of the Joyal–Tierney Theorem [25] about open localic maps between frames/locales (cf. [36] for the sublocale version):

Theorem 5.2. *The basic open l–morphisms are precisely the adjoints of the implicative biadjoint maps between implicative semilattices. Hence, by virtue of Galois adjunction, the category of implicative semilattices and implicative biadjoint maps is dual to the category with the same objects and basic open l–morphisms.*

Proof: If f is a basic open l–morphism adjoint to h then, by Theorem 5.1, h has a coadjoint g such that $\mathfrak{a}gb = f\mathfrak{a}b$, and by Propositions 3.4 and 4.1,

$$\mathfrak{a}(a \wedge gb) = \mathfrak{a}a \cap \mathfrak{a}gb = \mathfrak{a}a \cap f\mathfrak{a}b = f(\mathfrak{a}ha \cap \mathfrak{a}b) = f\mathfrak{a}(ha \wedge b) = \mathfrak{a}g(ha \wedge b),$$

hence $g(ha \wedge b) = a \wedge gb$; so by (b) $\Rightarrow$ (a) in Proposition 5.1, h preserves $\to$. Conversely, assuming that h is coadjoint to f, adjoint to g, and preserves $\to$, we use (a) $\Rightarrow$ (c) in Proposition 5.1 twice to prove $f\mathfrak{a}b = \mathfrak{a}gb$, which will show that f is basic open. For $d = b \to d \in \mathfrak{a}b$ and $a = fd$ we get $ha \leq d$ and $gb \to a = f(b \to ha) \leq f(b \to d) = a \leq gb \to a$, hence $a = gb \to a \in \mathfrak{a}gb$. Thus, $f\mathfrak{a}b \subseteq \mathfrak{a}gb$. And each $gb \to c \in \mathfrak{a}gb$ is equal to $f(b \to hc) \in f\mathfrak{a}b$, whence $\mathfrak{a}gb \subseteq f\mathfrak{a}b$. ∎

6. Closure in Heyting lattices and interior in locales

The formation of closure and interior in topological spaces has strict analogues for frames/locales (but not for implicative semilattices, as certain completeness properties are required in order to guarantee the existence of the embedding $\mathfrak{c}$ of A in $\mathcal{N}A$ and of localic preimages; see Propositions 3.5 and 4.3). Some of the results below are folklore in pointfree topology; the formulation via concrete sublocale sets makes the involved concepts more handy.

Let A be a Heyting lattice. Recall that the embeddings $\mathfrak{a}\colon A \longrightarrow \mathcal{S}lA$ and $\mathfrak{c}\colon A \longrightarrow \mathcal{N}A$ preserve finite meets and all existing joins. By Proposition 3.5 and Theorem 3.1, applied to $h = \mathfrak{c}$ and its adjoint

$$\mathfrak{l}\colon \mathcal{N}A \longrightarrow A,\ C \longmapsto \perp C,$$

the dualized composite map

$$\mathfrak{cl}\colon \mathcal{S}lA \longrightarrow \mathcal{S}lA, \quad C \longmapsto \overline{C} = {\uparrow}{\perp}C$$

is a closure operation ($\mathfrak{cl}$!) preserving finite joins. On the other hand, for locales A, the embedding $\mathfrak{a}\colon A \longrightarrow \mathcal{S}lA$ has the adjoint

$$\mathfrak{u}\colon \mathcal{S}lA \longrightarrow A,\ C \longmapsto \bigvee\{a \mid \mathfrak{a}a \subseteq C\},$$

and one obtains an interior operation

$$\mathfrak{au}\colon \mathcal{S}lA \longrightarrow \mathcal{S}lA, \quad C \longmapsto C^\circ = \mathfrak{au}C.$$

Note that for the specific C in Example 4.1, neither $\mathfrak{u}C$ nor $\mathfrak{au}C$ exists.

Lemma 6.1. *If A is a locale and $C \in \mathcal{S}A$ has a complement $\neg\, C$ in $\mathcal{S}lA$ then*

$$(\neg\, C)^\circ = \neg\, \overline{C} \quad \textit{and} \quad \overline{\neg\, C} = \neg\, C^\circ.$$

Proof: The equation

$$\mathfrak{u}\neg\, C = \bigvee\{a \mid \mathfrak{a}a \subseteq \neg\, C\} = \bigvee\{a \mid C \subseteq \mathfrak{c}a\} = \perp C$$

gives $\mathfrak{au}(\neg\, C) = \mathfrak{al}C = \neg\, \mathfrak{cl}C$; replacing C with $\neg\, C$ gives $\mathfrak{cl}(\neg\, C) = \neg\, \mathfrak{au}C$. ∎

With respect to closure and interior, localic maps between locales behave quite similar to but not completely like continuous maps between spaces. Indeed, from the equivalence (a) ⇔ (d) ⇔ (e) in Theorem 4.2 one easily derives a further characterization of localic maps in terms of continuity:

Theorem 6.1. *An isotone map $f : B \longrightarrow A$ between locales is localic iff the localic preimages $f_{\leftarrow}C$ of all sublocales $C \in \mathcal{S}lA$ exist and satisfy*

$$f_{\leftarrow}C^{\circ} \subseteq (f_{\leftarrow}C)^{\circ}, \qquad \overline{f_{\leftarrow}C} \subseteq f_{\leftarrow}\overline{C},$$
$$f_{\leftarrow}\neg\, C^{\circ} = \neg\, f_{\leftarrow}C^{\circ}, \quad f_{\leftarrow}\neg\,\overline{C} = \neg\, f_{\leftarrow}\overline{C}.$$

Theorem 6.2. *A localic map $f : B \longrightarrow A$ between locales is open iff*

$$f_{\leftarrow}C^{\circ} = (f_{\leftarrow}C)^{\circ} \text{ for all } C \in \mathcal{S}lA.$$

Proof: For $b \in B$, $\mathfrak{a}b \subseteq f_{\leftarrow}C$ means $f\mathfrak{a}b \subseteq C$, which for open f entails $f\mathfrak{a}b \subseteq C^{\circ}$, that is, $\mathfrak{a}b \subseteq f^{\leftarrow}C^{\circ}$, whence $\mathfrak{a}b \subseteq f_{\leftarrow}C^{\circ}$, as $\mathfrak{a}b$ belongs to $\mathcal{S}lB$. In particular, for $b = \mathfrak{u}f_{\leftarrow}C$, this amounts to $(f_{\leftarrow}C)^{\circ} = \mathfrak{a}\mathfrak{u}f_{\leftarrow}C \subseteq f_{\leftarrow}C^{\circ}$.

Conversely, if $(f_{\leftarrow}C)^{\circ} \subseteq f_{\leftarrow}C^{\circ}$ for all $C \in \mathcal{S}lA$ then, since for each open sublocale $\mathfrak{a}b$ of B the image $f\mathfrak{a}b$ is a sublocale of A, we get

$$\mathfrak{a}b \subseteq (f_{\leftarrow}f\mathfrak{a}b)^{\circ} \subseteq f_{\leftarrow}(f\mathfrak{a}b)^{\circ} \subseteq f^{\leftarrow}(f\mathfrak{a}b)^{\circ}, \quad \text{hence} \quad f\mathfrak{a}b \subseteq (f\mathfrak{a}b)^{\circ},$$

i.e., $f\mathfrak{a}b$ is open. ■

The following characterization of boolean l–ideals is given in [16] (for the complete case see [36]):

Proposition 6.1. *For a subset B of an implicative semilattice A and an element $b \in A$, the following conditions are equivalent:*

(a) $B = \mathfrak{b}b := \{a \to b \mid a \in A\}$.
(b) *B is the least nuclear range in A containing b.*
(c) *B is an l–ideal of A and a boolean lattice with least element b.*

Using this fact, we prove:

Proposition 6.2. *If $f : B \longrightarrow A$ is a basic open l–morphism between Heyting lattices with coadjoint h then*

$$hC \subseteq f_{\leftarrow}C \subseteq h\overline{C} \subseteq {\uparrow} h\overline{C} = f_{\leftarrow}\overline{C} = \overline{f_{\leftarrow}C} \quad \text{for all } C \in \mathcal{S}lA.$$

Proof: By Theorem 5.2, h is adjoint to a map g. For all $b \in B$ and $c \in C$, Proposition 5.1 yields $f(b \to hc) = gb \to c \in C$, that is,

$$\mathfrak{b}hc = \{b \to hc \mid b \in B\} \subseteq f^{\leftarrow}C.$$

From $hc \in \mathfrak{b}hc \in \mathcal{S}lB$ it follows that $hc \in f_{\leftarrow}C$, showing $hC \subseteq f_{\leftarrow}C$. Since f is adjoint to h, we get for $c = \bot C$:

$$f_{\leftarrow}\overline{C} = f_{\leftarrow}{\uparrow}c = f^{\leftarrow}{\uparrow}c = {\uparrow}hc = {\uparrow}h\overline{C}.$$

Proposition 6.1 assures $\overline{f_{\leftarrow}C} \subseteq f_{\leftarrow}\overline{C}$. Conversely, putting $b = \perp f_{\leftarrow}C$ and $a = gb$, we obtain $b \leq ha$ and

$$f_{\leftarrow}\mathfrak{c}a = \mathfrak{c}ha \subseteq \mathfrak{c}b = \overline{f_{\leftarrow}C}.$$

For $c \in C$, we have $hc \in f_{\leftarrow}C$, hence $b \leq hc$ and $a = gb \leq c$, that is, $c \in \mathfrak{c}a$. Thus, $C \subseteq \mathfrak{c}a = \overline{\mathfrak{c}a}$, $\overline{C} \subseteq \mathfrak{c}a$, and so

$$f_{\leftarrow}\overline{C} \subseteq f_{\leftarrow}\mathfrak{c}a \subseteq \overline{f_{\leftarrow}C}. \qquad \blacksquare$$

Summarizing the previous results, we arrive at the following closure-theoretical characterization of open localic maps:

Theorem 6.3. *An isotone map $f : B \longrightarrow A$ between locales is localic and open iff the localic preimages $f_{\leftarrow}C$ of all sublocales $C \in \mathcal{S}lA$ exist and satisfy*

$$\begin{aligned} f_{\leftarrow}C^{\circ} &= (f_{\leftarrow}C)^{\circ}, \qquad \overline{f_{\leftarrow}C} = f_{\leftarrow}\overline{C}, \\ f_{\leftarrow}\neg\, C^{\circ} &= \neg\, f_{\leftarrow}C^{\circ}, \quad f_{\leftarrow}\neg\,\overline{C} = \neg\, f_{\leftarrow}\overline{C}. \end{aligned}$$

In contrast to the situation with spaces, a localic map f satisfying

$$\overline{f_{\leftarrow}C} = f_{\leftarrow}\overline{C}$$

for all sublocales C need not be open, as the following reasoning shows:

Example 6.1. Consider a locale A and

$$B = \mathfrak{b}\perp = \{a \to \perp \mid a \in A\},$$

the smallest sublocale of A containing the bottom element $\perp$. This is the so-called *booleanization* of A, which is rarely open, whence the l–embedding $e\colon B \longrightarrow A$ is rarely an open map. For instance, if A is a chain and B is open then it has a closed complement $\mathfrak{c}a$, and a has to be the unique atom of A. Nevertheless, the embedding $e : B \longrightarrow A$ always satisfies the above closure equation, since every sublocale C of the boolean locale B is closed and thus

$$\overline{e_{\leftarrow}C} = e_{\leftarrow}C = B \cap C = B \cap \overline{C} = e_{\leftarrow}\overline{C}.$$

Indeed, any $a \to \perp \in \overline{C} = \uparrow\!\perp C$ must already be in C since $a \to \perp \geq \perp C$ implies $a \to \perp = a \to (a \wedge \perp C) = a \to \perp C \in C$.

For a thorough investigation of localic maps satisfying the above closure equation and related conditions (referred to as *hereditarily skeletal maps*) in a more categorical environment see Johnstone [24].

7. Basic zero-dimensional spaces

Our results suggest to consider so-called *basic zero-dimensional* (*closure*) *spaces*. These are triples $S = (X, \mathcal{C}, \mathcal{D})$ where

$\mathcal{D}$ is a closure system on X that is distributive as a lattice,

$\mathcal{C}$ is a subset of $\mathcal{D}$ with $X \in \mathcal{C}$, and each $C \in \mathcal{C}$ has a complement in $\mathcal{D}$,

$\mathcal{B} = \{B \vee \neg C \mid B, C \in \mathcal{C}\}$ is a meet-base of $\mathcal{D}$, which means that $\mathcal{D} = \{\bigcap \mathcal{X} \mid \mathcal{X} \subseteq \mathcal{B}\}$.

By distributivity, complements in $\mathcal{D}$ coincide with the pseudocomplements. We call the members of $\mathcal{C}$ *basic closed* and their complements *basic open*; but notice that $\mathcal{C}$ need not be a closure system. Putting

$$|S| = X,\ \mathcal{A}S = \{\neg C \mid C \in \mathcal{C}\},\ \mathcal{B}S = \mathcal{B},\ \mathcal{C}S = \mathcal{C},\ \mathcal{D}S = \mathcal{D},$$

we observe that $S^c = (|S|, \mathcal{A}S, \mathcal{D}S)$ is a basic zero-dimensional space, too, the *complementary space* of S; indeed, $S^{cc} = S$.

A *basic continuous map* between basic zero-dimensional spaces S and T is a map $f\colon |S| \longrightarrow |T|$ such that the preimage of $\bigcap \mathcal{D}T$ is $\bigcap \mathcal{D}S$, preimages of basic closed sets are basic closed, and their lattice complements in $\mathcal{D}S$ are contained in the preimages of the complements in $\mathcal{D}T$:

$$C \in \mathcal{C}T \text{ implies } f^{\leftarrow}C \in \mathcal{C}S \text{ and } \neg f^{\leftarrow}C \subseteq f^{\leftarrow}\neg C.$$

After having checked the composition law for basic continuous maps, one obtains a category $\mathsf{B0ds}$ of basic zero-dimensional spaces.

Here are a few prominent instances.

(1) Each T_D-closure space $(X, \mathcal{C})$ (in which $\overline{\{x\}} \setminus \{x\}$ is closed for all $x \in X$, see [11], and for the topological case [2, 36]) may be regarded as a basic zero-dimensional space $(X, \mathcal{C}, \mathcal{P}X)$; indeed, $\mathcal{C}$ is a meet-base for $\mathcal{P}X$ on account of the equation $X \setminus \{x\} = B \cup (X \setminus C)$, where $B = \overline{\{x\}} \setminus \{x\}$ and $C = \overline{\{x\}}$ are in $\mathcal{C}$. Then one checks that the category of T_D-closure spaces with the usual continuous maps is fully embedded in $\mathsf{B0ds}$.

(2) More generally, consider any closure space $(X, \mathcal{C})$ together with the topological closure system $\mathcal{D}$ consisting of all closed sets with respect to the topology generated by the differences $C \setminus D$ with $C, D \in \mathcal{C}$. The triple $(X, \mathcal{C}, \mathcal{D})$ is then a basic zero-dimensional space in which

complements are formed set-theoretically. In the case of a topological closure system $\mathcal{C}$, the topological space associated with $(X, \mathcal{D})$ is known as the *front space* of $(X, \mathcal{C})$ and its topology as the *Skula topology*; see [36, 43].

(3) Viewing each zero-dimensional topological space as a triple $(X, \mathcal{C}, \mathcal{D})$ where $\mathcal{D}$ is the system of closed sets and $\mathcal{C}$ consist of all clopen sets, one obtains another category fully embedded in $\mathsf{B0ds}$, namely that of zero-dimensional spaces and maps such that preimages of clopen sets are clopen – an important tool, e.g., in Stone duality. Recall that for boolean spaces (Stone spaces in [21]), that is, compact zero-dimensional Hausdorff spaces, the basic continuous maps are just the continuous ones.

(4) If A is an implicative semilattice with underlying set X then, for the closure system $\mathcal{M}A$ generated by $\mathcal{B}A$, the triple $(X, \mathfrak{c}A, \mathcal{M}A)$ is a basic zero-dimensional space. By Theorem 4.2, the l–morphisms are just the basic continuous maps between them. Thus, the category of implicative semilattices and l–morphisms is fully embedded in $\mathsf{B0ds}$. Specifically, in boolean lattices, the fact that all l–domains are basic open and closed considerably simplifies the situation: here, $\mathcal{M}A$ is merely the MacNeille completion of A. On the other hand, in the case of frames, $\mathcal{M}A$ coincides with $\mathcal{N}A$.

These examples may suffice for the moment to motivate future investigation of basic zero-dimensional spaces and suitable morphisms between them.

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