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AN OPTIMAL LIOUVILLE THEOREM FOR THE POROUS MEDIUM EQUATION

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ABSTRACT: Under a sharp asymptotic growth condition at infinity, we prove a Liouville type theorem for the inhomogeneous porous medium equation, provided it stays universally close to the heat equation. Additionally, for the homogeneous equation in a strip, we show that for the conclusion to hold, it is enough to assume the sharp asymptotic growth at infinity only in the space variable. The results are optimal, meaning that the growth condition at infinity cannot be weakened.

KEYWORDS: Porous medium equation, Liouville theorem, intrinsic scaling, degenerate parabolic equations.
AMS SUBJECT CLASSIFICATION (2000): 35K55, 35K65, 35B53.

1. Introduction

As it is well known, bounded harmonic function in the whole space $\mathbb{R}^n$ must be a constant (Liouville theorem). This is true for parabolic equations as well, namely, in $\mathbb{R}^n \times \mathbb{R}_-$ bounded solutions of the heat equation are constant, [12]. However, there is a key difference between elliptic and parabolic equations. For instance, entire harmonic functions, which are bounded only from below (or above), are constant, [5], but one sided bound is not enough to make the same conclusion for entire caloric functions, as shows the example of the function $u(x,t) = e^{x+t}$, which solves the heat equation in $\mathbb{R} \times \mathbb{R}$, is bounded from below, but obviously is not a constant. Nevertheless, the absence of the two sided bound can be compensated by a growth condition, as established by Bernstein. He argued that in the plane entire solutions of uniformly elliptic equations, growing sublinearly at infinity, must be constant, [5]. A result of similar spirit was established for entire solutions of the heat equation by Hirschman in [6]. Recent advances in geometric analysis, [9], allowed to

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obtain Liouville type results for degenerate elliptic and parabolic equations. More precisely, in [10] the authors show that weak solutions of

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \;\; p > 2$$

in $\mathbb{R}^n \times \mathbb{R}$ are constant, provided their growth at infinity is controlled in an intrinsic manner. Its proof makes use of sharp Hölder regularity for weak solutions under an appropriate intrinsic scaling.

In this note we prove a Liouville type theorem for weak solutions of the inhomogeneous porous medium equation

$$u_t - \operatorname{div}(u^\gamma \nabla u) = f, \tag{1.1}$$

in $\mathbb{R}^n \times \mathbb{R}$, where $f \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ and for a small slow diffusion parameter $\gamma \geq 0$. More precisely, we show that if an entire weak solution of (1.1) vanishes at a point and

$$u(x,t) = o\left(|x|^\alpha + |t|^{\frac{\alpha}{2}}\right), \quad \text{as} \quad |x| + |t| \to \infty, \tag{1.2}$$

where $\alpha = 2(\gamma+1)^{-1}$, then it must vanish everywhere (Theorem 3.1). Moreover, our result is optimal, meaning that the growth condition at infinity cannot be weakened, i.e., if (1.2) holds for $\alpha = 2(\gamma+1)^{-1} + \delta$, $\delta > 0$, then the conclusion of the theorem fails, as shows the example of the function $u(x,t) = |x|^{\frac{2}{\gamma+1}}$, which vanishes at zero and obviously is not identically zero. Observe that the growth condition at infinity is intrinsically related to the local regularity of solutions. In fact, using local $C^{1,\beta}$ estimates, in [7] the authors proved that entire stationary p-harmonic functions, which grow at infinity as $|x|^{1+\beta}$, are linear (see also [10]). It is noteworthy, that even in uniformly elliptic case, one sided bound is not enough to guarantee a Liouville type result for the non-homogeneous equation, as shows the example of the non-constant function $u(x) = |x|^2 \geq 0$, which satisfies $\Delta u = 2n$ in $\mathbb{R}^n$. Furthermore, for the homogeneous equation in a strip, we show that for the conclusion to hold, it is enough to assume the sharp asymptotic growth at infinity only in the space variable (Theorem 3.2). Our result extends similar results for the heat equation, [6, 12], and adjusts the result from [4, Proposition 16.2]. To the best of authors' knowledge, there are only a few Liouville type results for the porous medium equation, [2, 8], and unlike those, the degree of homogeneity in (1.1) does not depend on the right hand side. The main ingredient in our proof is the higher regularity estimate for these type of equations, obtained recently in [1].

2. Preliminaries

One may note that the classical porous medium equation has the coefficient $(\gamma+1)$ in the divergence term of (1.1), however, it does not effect the proof of the arguments, therefore, to keep things simple, we consider the equation (1.1).

Next, we introduce notations and recall a regularity result for future reference. We start by defining the intrinsic cylinder by

$$G_r(x,t) := B_r(x) \times \left(-\frac{1}{2}r^{\frac{2}{\gamma+1}} + t,\, \frac{1}{2}r^{\frac{2}{\gamma+1}} + t\right), \quad r > 0$$

and the intrinsic norm by

$$\|(x,t)\| := |x| + |t|^{\frac{\gamma+1}{2}}.$$

When x and t are zero, we will omit them, i.e., $G_r = G_r(0,0)$.

Solutions of (1.1) are understood in the weak sense (for the precise definition see, for example, [4, 11]). Our result makes use of a higher regularity result obtained for solutions of the porous medium equation near the heat equation. Since the improved regularity results in [1] are local, with a slight adaptation, one can extend those to two-sided cylinders defined above. Thus, we use the following refined regularity result from [1, Theorem 2].

Theorem 2.1. *If $u \geq 0$ is a weak solution of*

$$u_t - \operatorname{div}(u^\gamma \nabla u) = f \quad in \quad G_1, \tag{2.1}$$

with $f \in L^\infty(G_1)$, then there exist $\varepsilon > 0$ and $C > 0$, depending only on $\|f\|_\infty$ and n, such that for $\gamma \in (0,\varepsilon)$ and $(x_0,t_0) \in \partial\{u>0\} \cap G_{1/8}$, in $G_{1/10}(x_0,t_0)$, one has

$$u(x,t) \leq C\|u\|_\infty \|(x,t)\|^{\frac{2}{\gamma+1}}.$$

We finish this section with the following result from [4, Proposition 16.2], which will later be used to prove that for the homogeneous equation in a strip, for the Liouville type result to hold, it is enough to assume the sharp asymptotic growth at infinity only in the space variable.

Proposition 2.1. *If $u \geq 0$ is a weak solution of*

$$u_t - \operatorname{div}(u^\gamma \nabla u) = 0$$

in a strip $S_T := \mathbb{R}^n \times (-\infty, T)$, $T \in \mathbb{R}$, and $\inf_{S_T} u = 0$, then

$$\lim_{t \to -\infty} u(x,t) = 0, \quad for\ all \quad x \in \mathbb{R}^n.$$

3. Liouville type theorems

In this section we prove that the only non-negative entire weak solution of the inhomogeneous porous medium type equation that vanishes at a point and has a certain intrinsic growth at infinity, is zero, provided the equation is universally close to the heat equation. Our result is optimal, i.e., the growth condition at infinity can not be weakened. Furthermore, for the homogeneous equation in a strip, we show that for the conclusion to hold the asymptotic growth at infinity only in space variable is enough. Thus, we assume that $\gamma \geq 0$ is small enough, so we are in the regularity regime of Theorem 2.1. Note that allowing the right hand side to be any bounded function covers not only the homogeneous equation, but also several interesting classes, including equations with obstacle type non-homogeneity (see [3]), such as

$$u_t - \operatorname{div}(u^\gamma \nabla u) = \chi_{\{u>0\}},$$

where $\chi_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$.

Theorem 3.1. *Let $\gamma \geq 0$ be as in Theorem 2.1, $f \in L^\infty(\mathbb{R}^n \times \mathbb{R})$, and $u \geq 0$ be an entire solution of*

$$u_t - \operatorname{div}(u^\gamma \nabla u) = f.$$

If $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and

$$u(x,t) = o\left(\|(x,t)\|^{\frac{2}{\gamma+1}}\right), \quad \text{as} \quad \|(x,t)\| \to \infty, \tag{3.1}$$

then $u \equiv 0$.

Proof: Since the function $v(x,t) := u(x - x_0, t - t_0)$ solves the same equation and satisfies $v(x_0, t_0) = 0$, without loss of generality, we assume that $x_0 = 0$ and $t_0 = 0$.

Set

$$\sigma := \frac{2}{\gamma + 1}$$

and for $k \in \mathbb{N}$ define

$$u_k(x,t) := \frac{u(kx, k^\sigma t)}{k^\sigma}.$$

We divide the proof into two steps.

Step 1. We claim that in $G_{1/10}$ there holds

$$\|u_k\|_\infty \to 0, \quad \text{as} \quad k \to \infty. \tag{3.2}$$

Indeed, direct computation shows

$$(u_k)_t - \operatorname{div}(u_k^\gamma \nabla u_k) = u_t - \gamma k^{2(1-\sigma)-\sigma(\gamma-1)} u^{\gamma-1}|\nabla u|^2 - k^{2-\sigma-\gamma\sigma} u^\gamma \Delta u,$$

but

$$2(1-\sigma)-\sigma(\gamma-1) = 0 \quad \text{and} \quad 2-\sigma-\gamma\sigma = 0,$$

therefore,

$$(u_k)_t - \operatorname{div}(u_k^\gamma \nabla u_k) = u_t - \gamma u^{\gamma-1}|\nabla u|^2 - u^\gamma \Delta u = u_t - \operatorname{div}(u^\gamma \nabla u) = f(kx, k^\sigma t).$$

Since u is an entire solution of (1.1), then u_k is a solution of

$$(u_k)_t - \operatorname{div}(u_k^\gamma \nabla u_k) = g_k$$

in G_1, where $g_k(x,t) = f(kx, k^\sigma t)$ is bounded independent of k, as $f \in L^\infty(\mathbb{R}^n \times \mathbb{R})$. Let $(x_k, t_k) \in \overline{G}_{1/10}$ be a point where u_k attains its supremum, i.e.,

$$\sup_{\overline{G}_{1/10}} u_k = u_k(x_k, t_k).$$

Note that

$$u_k(0,0) = 0, \ \ \forall k \in \mathbb{N},$$

and at least for $k \in \mathbb{N}$ big enough, $(x_k, t_k) \neq (0,0)$, since otherwise $u_k \equiv 0$ in $G_{1/10}$ and hence, $u \equiv 0$ in $G_{k/10}$, for any $k \geq k_0$, and there is nothing to prove. As

$$\|(kx_k, k^\sigma t_k)\|^\sigma = \left(|kx_k| + |k^\sigma t_k|^{\frac{1}{\sigma}}\right)^\sigma \leq (2k)^\sigma,$$

then

$$\|u_k\|_\infty = \sup_{\overline{G}_{1/10}} u_k = 2^\sigma \frac{u(kx_k, k^\sigma t_k)}{(2k)^\sigma} \leq 2^\sigma \frac{u(kx_k, k^\sigma t_k)}{\|(kx_k, k^\sigma t_k)\|^\sigma}. \tag{3.3}$$

If $\|(kx_k, k^\sigma t_k)\| \to \infty$, as $k \to \infty$, then (3.1) and (3.3) imply (3.2). Additionally, (3.1) and (3.3) in $G_{1/10}$ guarantee that $\|u_k\|_\infty$ is bounded independent of k. Applying Theorem 2.1 to u_k, one has

$$u_k(x_k, t_k) \leq C\|u_k\|_\infty \|(x_k, t_k)\|^\sigma,$$

where $C > 0$ is a constant depending only on $\|f\|_\infty$ and n. Therefore, if $\|(kx_k, k^\sigma t_k)\|$ remains bounded, as $k \to \infty$, then $u(kx_k, k^\sigma t_k)$ also remains bounded, since

$$\begin{aligned} u(kx_k, k^\sigma t_k) &= k^\sigma u_k(x_k, t_k) \\ &\leq k^\sigma C\|u_k\|_\infty \|(x_k, t_k)\|^\sigma \\ &= C\|u_k\|_\infty \|(kx_k, k^\sigma t_k)\|^\sigma. \end{aligned}$$

This gives

$$u_k(x_k, t_k) \to 0, \quad as \quad k \to \infty,$$

and hence (3.2) holds.

Step 2. Suppose there exists a point $(y, \tau) \in \mathbb{R}^n \times \mathbb{R}$ such that $u(y, \tau) > 0$. From Theorem 2.1 in $G_{1/10}$ we have

$$u_k(x, t) \leq C\|u_k\|_\infty \|(x, t)\|^\sigma,$$

which, combined with (3.2), gives

$$\sup_{G_{1/10}} \frac{u_k(x, t)}{\|(x, t)\|^\sigma} < \frac{u(y, \tau)}{2\|(y, \tau)\|^\sigma}. \tag{3.4}$$

Choosing $k \in \mathbb{N}$ large enough, one can guarantee that $(y, \tau) \in G_{k/10}$. Using (3.2) and (3.4), we then obtain

$$\begin{aligned} \frac{u(y, \tau)}{\|(y, \tau)\|^\sigma} &\leq \sup_{G_{k/10}} \frac{u(x, t)}{\|(x, t)\|^\sigma} \\ &= \sup_{G_{1/10}} \frac{u_k(x, t)}{\|(x, t)\|^\sigma} \\ &< \frac{u(y, \tau)}{2\|(y, \tau)\|^\sigma}, \end{aligned}$$

which is a contradiction. ■

As a consequence of Theorem 3.1, we obtain the next result, which reveals that for the conclusion to hold for the homogeneous equation in a strip, it is enough to assume the sharp asymptotic growth only in the space variable.

Theorem 3.2. *Let $\gamma \geq 0$ be as in Theorem 2.1, and $u \geq 0$ be a solution of*

$$u_t - \operatorname{div}(u^\gamma \nabla u) = 0$$

in a strip $\mathbb{R}^n \times (-\infty, T)$, $T \in \mathbb{R}$. If $u(x_0, t_0) = 0$ for some (x_0, t_0) in the strip, and

$$u(x, t) = o\left(|x|^{\frac{2}{\gamma+1}}\right), \quad as \quad |x| \to \infty, \tag{3.5}$$

then $u \equiv 0$.

Proof: As remarked earlier, since the improved regularity result, namely, Theorem 2.1, is local, it and therefore, Theorem 3.1, remains true also in the one-sided strip. On the other hand, for all $x \in \mathbb{R}^n$, Proposition 2.1 guarantees

$$\lim_{t \to -\infty} u(x,t) = 0.$$

The latter, combined with (3.5), implies (3.1) and therefore, $u \equiv 0$. ∎

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