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$\mathcal{O}$-OPERATORS ON LIE ∞-ALGEBRAS WITH RESPECT TO LIE ∞-ACTIONS

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ABSTRACT: We define $\mathcal{O}$-operators on a Lie ∞-algebra E with respect to an action of E on another Lie ∞-algebra and we characterize them as Maurer-Cartan elements of a certain Lie ∞-algebra obtained by Voronov's higher derived brackets construction. The Lie ∞-algebra that controls the deformation of $\mathcal{O}$-operators with respect to a fixed action is determined.

KEYWORDS: Lie ∞-algebra, $\mathcal{O}$-operator, Maurer Cartan element.
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Introduction

The first instance of Rota-Baxter operator appeared in the context of associative algebras in 1960, in a paper by Baxter [1], as a tool to study fluctuation theory in probability. Since then, these operators were widely used in many branches of mathematics and mathematical physics.

Almost forty years later, Kupershmidt [4] introduced $\mathcal{O}$-operators on Lie algebras as a kind of generalization of classical r-matrices, thus opening a broad application of $\mathcal{O}$-operators to integrable systems. Given a Lie algebra $(E, [\cdot,\cdot])$ and a representation Φ of E on a vector space V, an $\mathcal{O}$-operator on E with respect to Φ is a linear map $T : V \to E$ such that $[T(x), T(y)] = T\big(\Phi(T(x))(y) - \Phi(T(y))(x)\big)$. When Φ is the adjoint representation of E, T is a Rota-Baxter operator (of weight zero). $\mathcal{O}$-operators are also called relative Rota-Baxter operators or generalized Rota-Baxter operators.

In recent years Rota-Baxter and $\mathcal{O}$-operators, in different algebraic and geometric settings, have deserved a great interest by mathematical and physical communities.

In [9], a homotopy version of $\mathcal{O}$-operators on symmetric graded Lie algebras was introduced. This was the first step towards the definition of a $\mathcal{O}$-operator on a Lie ∞-algebra with respect to a representation on a graded vector space that was given in [6]. The current paper also deals with $\mathcal{O}$-operators on Lie

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∞-algebras, but with a different approach which uses Lie ∞-actions instead of representations of Lie ∞-algebras. Our definition is therefore different from the one given in [6] but there is a relationship between them.

There are two equivalent definitions of Lie ∞-algebra structure on a graded vector space E, both given by collections of n-ary brackets which are either symmetric or skew-symmetric, depending on the definition we are considering, and must satisfy a kind of generalized Jacobi identities. One goes from one to the other by shifting the degree of E and applying a *décalage* isomorphism. We use the definition in its symmetric version, where the brackets have degree $+1$. Equivalently, this structure can be defined by a degree $+1$ coderivation M_E of $\bar{S}(E)$, the reduced symmetric algebra of E, such that the commutator $[M_E, M_E]_c$ vanishes.

Representations of Lie ∞-algebras on graded vector spaces were introduced in [7]. In [6], the authors consider a representation Φ of a Lie ∞-algebra E on a graded vector space V and define an $\mathcal{O}$-operator (homotopy relative Rota-Baxter operator) on E with respect to Φ as a degree zero element T of $\mathrm{Hom}(\bar{S}(V), E)$ satisfying a family of suitable identities. Inspired by the notion of an action of a Lie ∞-algebra on a graded manifold [8], we define an action of a Lie ∞-algebra (E, M_E) on a Lie ∞-algebra (V, M_V) as a Lie ∞-morphism Φ between E and $\mathrm{Coder}(\bar{S}(V))[1]$, the symmetric DGLA of coderivations of $\bar{S}(V)$. An $\mathcal{O}$-operator on E with respect to the action Φ is a comorphism between $\bar{S}(V)$ and $\bar{S}(E)$ that intertwines the coderivation M_E and a degree $+1$ coderivation of $\bar{S}(V)$ built from M_V and Φ, which turns out to be a Lie ∞-algebra structure on V too.

As we said before, the two $\mathcal{O}$-operator definitions, ours and the one in [6], are different. However, since there is a close connection between Lie ∞-actions and representations of Lie ∞-algebras, the two definitions can be related. On the one hand, any representation of (E, M_E) on a complex (V, d) can be seen as a Lie ∞-action of (E, M_E) on (V, D), with D the coderivation given by the differential d, and for this very "simple" Lie ∞-algebra structure on V our $\mathcal{O}$-operator definition recovers the one given in [6]. On the other hand, any action Φ of (E, M_E) on (V, M_V) yields a representation ρ on the graded vector space $\bar{S}(V)$ and an $\mathcal{O}$-operator with respect to the action Φ is not the same as an $\mathcal{O}$-operator with respect to the representation ρ. However, there is a way to relate the two concepts.

A well-known Voronov's construction [10] defines a Lie ∞-algebra structure on an abelian Lie subalgebra $\mathfrak{h}$ of $\mathrm{Coder}(\bar{S}(E \oplus V))$ and we show that $\mathcal{O}$-operators with respect to the action Φ are Maurer-Cartan elements of $\mathfrak{h}$.

In general, deformations of structures and morphisms are governed by DGLA's or, more generally, by Lie ∞-algebras. We do not intend to deeply study deformations of $\mathcal{O}$-operators on Lie ∞-algebras with respect to Lie ∞-actions. Still, we prove that deformations of an $\mathcal{O}$-operator are controlled by the twisting of a Lie ∞-algebra, constructed out of a graded Lie subalgebra of $\mathrm{Coder}(\bar{S}(E \oplus V))$.

The paper is organized in four sections. In Section 1 we collect some basic results on graded vector spaces, graded symmetric algebras and Lie ∞-algebras that will be needed along the paper. In Section 2, after recalling the definition of a representation of a Lie ∞-algebra on a complex (V, d) [7], we introduce the notion of action of a Lie ∞-algebra on another Lie ∞-algebra (Lie ∞-action) and we prove that a Lie ∞-action of E on V induces a Lie ∞-algebra structure on $E \oplus V$. We pay special attention to the adjoint action of a Lie ∞-algebra. In Section 3 we introduce the main notion of the paper – $\mathcal{O}$-operator on a Lie ∞-algebra E with respect to an action of E on another Lie ∞-algebra, and we give the explicit relation between these operators and $\mathcal{O}$-operators on E with respect to a representation on a graded vector space introduced in [6]. Given an $\mathcal{O}$-operator T on E with respect to a Lie ∞-action Φ on V, we show that V inherits a new Lie ∞-algebra structure given by a degree $+1$ coderivation which is the sum of the initial one on V with a degree $+1$ coderivation obtained out of Φ and T. We prove that symmetric and invertible comorphisms $T : \bar{S}(E^*) \to S(E)$ are $\mathcal{O}$-operators with respect to the coadjoint action if and only if a certain element of $\bar{S}(E^*)$, which is defined using the inverse of T, is a cocycle for the Lie ∞-algebra cohomology of E. Section 3 ends with the characterization of $\mathcal{O}$-operators as Maurer-Cartan elements of a Lie ∞-algebra obtained by Voronov's higher derived brackets construction. The main result in Section 4 shows that Maurer-Cartan elements of a graded Lie subalgebra of $\mathrm{Coder}(\bar{S}(E \oplus V))$ encode a Lie ∞-algebra on E and an action of E on V. Moreover, we obtain the Lie ∞-algebra that controls the deformation of $\mathcal{O}$-operators with respect to a fixed action.

1. Lie ∞-algebras

We begin by reviewing some concepts about graded vector spaces, graded symmetric algebras and Lie ∞-algebras.

1.1. Graded vector spaces and graded symmetric algebras. We will work with $\mathbb{Z}$-graded vector spaces with finite dimension over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $E = \oplus_{i\in\mathbb{Z}} E_i$ be a finite dimensional graded vector space. We call E_i the homogeneous component of E of degree i. An element x of E_i is said to be homogeneous with degree $|x| = i$. For each $k \in \mathbb{Z}$, one may shift all the degrees by k and obtain a new grading on E. This new graded vector space is denoted by $E[k]$ and is defined by $E[k]_i = E_{i+k}$.

A morphism $\Phi : E \to V$ between two graded vector spaces is a degree preserving linear map, i.e. a collection of linear maps $\Phi_i : E_i \to V_i$, $i \in \mathbb{Z}$. We call $\Phi : E \to V$ a (homogeneous) morphism of degree k, for some $k \in \mathbb{Z}$, and we write $|\Phi| = k$, if it is a morphism between E and $V[k]$. This way we have a natural grading in the vector space of linear maps between graded vector spaces:

$$\mathrm{Hom}(E, V) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(E, V).$$

In particular, $\mathrm{Hom}(E, E) = \mathrm{End}(E) = \oplus_{i\in\mathbb{Z}} \mathrm{End}_i(E)$.

The dual E^* of E is naturally a graded vector space whose component of degree i is, for all $i \in \mathbb{Z}$, the dual $(E_{-i})^*$ of E_{-i}. In equation: $(E^*)_i = (E_{-i})^*$.

Given two graded vector spaces E and V, their direct sum $E \oplus V$ is a vector space with grading

$$(E \oplus V)_i = E_i \oplus V_i$$

and their usual tensor product comes equipped with the grading

$$(E \otimes V)_i = \oplus_{j+k=i} E_j \otimes V_k.$$

We will adopt the Koszul sign convention, for homogeneous linear maps $f : E \to V$ and $g : F \to W$ the tensor product $f \otimes g : E \otimes F \to V \otimes W$ is the morphism of degree $|f| + |g|$ given by

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y),$$

for all homogeneous $x \in E$ and $y \in F$.

For each $k \in \mathbb{N}_0$, let $T^k(E) = \otimes^k E$, with $T^0(E) = \mathbb{K}$, and let $T(E) = \oplus_k T^k(E)$ be the tensor algebra over E. The **graded symmetric algebra**

over E is the quotient

$$S(E) = T(E)/\left\langle x \otimes y - (-1)^{|x||y|} y \otimes x \right\rangle.$$

The symmetric algebra $S(E) = \oplus_{k\geq 0} S^k(E)$ is a graded commutative algebra, whose product we denote by $\odot$. For $x = x_1 \odot \ldots \odot x_k \in S^k(E)$, we set $|x| = \sum_{i=1}^k |x_i|$.

For $n \geq 1$, let S_n be the permutation group of order n. For any homogeneous elements $x_1, \ldots, x_n \in E$ and $\sigma \in S_n$, the Koszul sign is the element in $\{-1, 1\}$ defined by

$$x_{\sigma(1)} \odot \ldots \odot x_{\sigma(n)} = \epsilon(\sigma)\, x_1 \odot \ldots \odot x_n.$$

As usual, writing $\epsilon(\sigma)$ is an abuse of notation because the Koszul sign also depends on the x_i.

An element σ of S_n is called an $(i, n-i)$-unshuffle if $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$. The set of $(i, n-i)$-unshuffles is denoted by $Sh(i, n-i)$. Similarly, $Sh(k_1, \ldots, k_j)$ is the set of $(k_1, \ldots, k_j)$-unshuffles, i.e., elements of S_n with $k_1 + \ldots + k_j = n$ such that the order is preserved within each block of length k_i, $1 \leq i \leq j$.

The reduced symmetric algebra $\bar{S}(E) = \oplus_{k\geq 1} S^k(E)$ has a natural coassociative and cocommutative coalgebra structure given by the coproduct $\Delta : \bar{S}(E) \to \bar{S}(E) \otimes \bar{S}(E)$,

$$\Delta(x) = 0, \;\; x \in E;$$

$$\Delta(x_1 \odot \ldots \odot x_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in Sh(i,n-i)} \epsilon(\sigma) \big(x_{\sigma(1)} \odot \ldots \odot x_{\sigma(i)}\big) \otimes \big(x_{\sigma(i+1)} \odot \ldots \odot x_{\sigma(n)}\big),$$

for $x_1, \ldots, x_n \in E$.

We will mainly use Sweedler notation: given $x \in \bar{S}(E)$,

$$\Delta^{(1)}(x) = \Delta(x) = x_{(1)} \otimes x_{(2)},$$

and the coassociativity yields

$$\Delta^{(n)}(x) = (\mathrm{id} \otimes \Delta^{(n-1)})\Delta(x) = x_{(1)} \otimes \ldots \otimes x_{(n+1)}, \quad n \geq 2.$$

Notice that

$$\Delta^{(n)}(x) = 0, \quad x \in S^{\leq n}(E).$$

The cocommutativity of the coproduct is expressed, for homogeneous elements of $\bar{S}(E)$, as

$$x_{(1)} \otimes x_{(2)} = (-1)^{|x_{(1)}||x_{(2)}|} x_{(2)} \otimes x_{(1)}.$$

Let V be another graded vector space. A linear map $f : \bar{S}(E) \to V$ is given by a collection of maps $f_k : S^k(E) \to V$, $k \geq 1$, and is usually denoted by $f = \sum_k f_k$.

Remark 1.1. Every linear map $f : S^k(E) \to V$ corresponds to a graded symmetric k-linear map $f \in \mathrm{Hom}(\otimes^k E, V)$ through the quotient map $p_k : \otimes^k E \to S^k(E)$ i.e., $f \equiv f \circ p_k$. In the sequel, we shall often write

$$f(x_1 \odot \ldots \odot x_k) = f(x_1, \ldots, x_k), \quad x_i \in E.$$

A **coalgebra morphism** (or **comorphism**) between the coalgebras $(\bar{S}(E), \Delta_E)$ and $(\bar{S}(V), \Delta_V)$ is a morphism $F : \bar{S}(E) \to \bar{S}(V)$ of graded vector spaces such that

$$(F \otimes F) \circ \Delta_E = \Delta_V \circ F.$$

There is a one-to-one correspondence between coalgebra morphisms $F : \bar{S}(E) \to \bar{S}(V)$ and degree preserving linear maps $f : \bar{S}(E) \to V$. Each f determines F by

$$F(x) = \sum_{k \geq 1} \frac{1}{k!} f(x_{(1)}) \odot \ldots \odot f(x_{(k)}),\ x \in \bar{S}(E),$$

and $f = p_V \circ F$, with $p_V : \bar{S}(V) \to V$ the projection map.

A degree k **coderivation** of $\bar{S}(E)$, for some $k \in \mathbb{Z}$, is a linear map $Q : \bar{S}(E) \to \bar{S}(E)$ of degree k such that

$$\Delta \circ Q = (Q \otimes \mathrm{id} + \mathrm{id} \otimes Q) \circ \Delta.$$

We also have a one to one correspondence between coderivations of $\bar{S}(E)$ and linear maps $q = \sum_i q_i : \bar{S}(E) \to E$:

Proposition 1.2. *Let E be a graded vector space and $p_E : \bar{S}(E) \to E$ the projection map. For every linear map $q = \sum_i q_i : \bar{S}(E) \to E$, the linear map $Q : \bar{S}(E) \to \bar{S}(E)$ given by*

$$Q(x_1 \odot \ldots \odot x_n) = \sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n-i)} \epsilon(\sigma) q_i \left(x_{\sigma(1)}, \ldots, x_{\sigma(i)} \right) \odot x_{\sigma(i+1)} \odot \ldots \odot x_{\sigma(n)}, \tag{1}$$

is the unique coderivation of $\bar{S}(E)$ *such that* $p_E \circ Q = q$.

In Sweedler notation, Equation (1) is written as:

$$Q(x) = q(x_{(1)}) \odot x_{(2)} + q(x), \;\; x \in \bar{S}(E).$$

When E is a finite dimensional graded vector space, we may identify $S(E^*)$ with $(SE)^*$. Koszul sign conventions yield, for each homogeneous elements $f, g \in E^*$,

$$(f \odot g)(x \odot y) = (-1)^{|x||g|} f(x)\, g(y) + f(y)\, g(x), \quad x, y \in E.$$

1.2. Lie ∞-algebras. We briefly recall the definition of Lie ∞-algebra [5], some basic examples and related concepts.

We will consider the symmetric approach to Lie ∞-algebras.

Definition 1.3. A **symmetric Lie ∞-algebra** (or a **Lie**[1] **∞-algebra**) is a graded vector space $E = \oplus_{i\in\mathbb{Z}} E_i$ together with a family of degree $+1$ linear maps $l_k : S^k(E) \to E$, $k \geq 1$, satisfying

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh(i,j-1)} \epsilon(\sigma)\, l_j \left(l_i \left(x_{\sigma(1)}, \ldots, x_{\sigma(i)} \right), x_{\sigma(i+1)} \ldots, x_{\sigma(n)} \right) = 0, \qquad (2)$$

for all $n \in \mathbb{N}$ and all homogeneous elements $x_1, \ldots, x_n \in E$.

The *décalage* isomorphism [10] establishes a one to one correspondence between skew-symmetric Lie ∞-algebra structures $\{l'_k\}_{k\in\mathbb{N}}$ on E and symmetric Lie ∞-algebra structures $\{l_k\}_{k\in\mathbb{N}}$ on $E[1]$:

$$l_k(x_1, \ldots, x_k) = (-1)^{(k-1)|x_1|+(k-2)|x_2|+\ldots+|x_{k-1}|} l'_k(x_1, \ldots, x_k).$$

In the sequel, we frequently write Lie ∞-algebra, omitting the term symmetric.

Example 1.4 (Symmetric graded Lie algebra)**.** A symmetric graded Lie algebra is a symmetric Lie ∞-algebra $E = \oplus_{i\in\mathbb{Z}} E_i$ such that $l_n = 0$ for $n \neq 2$. Then the degree 0 bilinear map on $E[-1]$ defined by

$$[\![x, y]\!] := (-1)^i l_2(x, y), \text{ for all } x \in E_i, y \in E_j, \qquad (3)$$

is a graded Lie bracket. In particular, if $E = E_{-1}$ is concentrated on degree -1, we get a Lie algebra structure.

Example 1.5 (Symmetric DGLA algebra)**.** A symmetric differential graded Lie algebra (DGLA) is a symmetric Lie ∞-algebra $E=\oplus_{i\in\mathbb{Z}}E_i$ such that $l_n=0$ for $n\neq 1$ and $n\neq 2$.

Then, from (2), we have that $d:=l_1$ is a degree $+1$ linear map $d:E\to E$ squaring zero and satisfies the following compatibility condition with the bracket $[\cdot,\cdot]:=l_2(\cdot,\cdot)$:

$$\begin{cases} d\,[x,y]+[d(x),y]+(-1)^{|x|}\,[x,d(y)]=0,\\ [[x,y]\,,z]+(-1)^{|y||z|}\,[[x,z]\,,y]+(-1)^{|x|}\,[x,[y,z]]=0, \end{cases}$$

Applying the *décalage* isomorphism, $(E[-1],d,[\![\cdot,\cdot]\!])$ is a (skew-symmetric) DGLA, with $[\![\cdot,\cdot]\!]$ given by (3).

Example 1.6. Let $(E=\oplus_{i\in\mathbb{Z}}E_i,\mathrm{d})$ be a cochain complex. Then $\mathrm{End}(E)[1]$ has a natural symmetric DGLA structure with $l_1=\partial$ and $l_2=[\cdot,\cdot]$ given by:

$$\begin{cases} \partial\phi=-\mathrm{d}\circ\phi+(-1)^{|\phi|+1}\phi\circ\mathrm{d},\\ [\phi,\psi]=(-1)^{|\phi|+1}\left(\phi\circ\psi-(-1)^{(|\phi|+1)(|\psi|+1)}\psi\circ\phi\right), \end{cases}$$

for ϕ,ψ homogeneous elements of $\mathrm{End}(E)[1]$. In other words, $\partial\phi=-[\mathrm{d},\phi]_c$ and $[\phi,\psi]=(-1)^{\deg(\phi)}[\phi,\psi]_c$, with $[\cdot,\cdot]_c$ the graded commutator on $\mathrm{End}(E)$ and $\deg(\phi)$ the degree of ϕ in $\mathrm{End}(E)$.

The symmetric Lie bracket $[\cdot,\cdot]$ on $\mathrm{End}(\bar{S}(E))[1]$ preserves $\mathrm{Coder}(\bar{S}(E))[1]$, the space of coderivations of $\bar{S}(E)$, so that $(\mathrm{Coder}(\bar{S}(E))[1],\partial,[\cdot,\cdot])$ is a symmetric DGLA.

The isomorphism between $\mathrm{Hom}(\bar{S}(E),E)$ and $\mathrm{Coder}(\bar{S}(E))$ given by Proposition 1.2, induces a Lie bracket on $\mathrm{Hom}(\bar{S}(E),E)$ known as the Richardson-Nijenhuis bracket:

$$[f,g]_{_{RN}}(x)=f(G(x))-(-1)^{|f||g|}g(F(x)),\quad x\in\bar{S}(E),$$

for each $f,g\in\mathrm{Hom}(\bar{S}(E),E)$, where F and G denote the coderivations defined by f and g, respectively. In other words, $[F,G]_c$ is the (unique) coderivation of $\bar{S}(E)$ determined by $[f,g]_{_{RN}}\in\mathrm{Hom}(\bar{S}(E),E)$.

Elements $l:=\sum_k l_k$ of $\mathrm{Hom}(\bar{S}(E),E)$ satisfying $[l,l]_{_{RN}}=0$ define a Lie ∞-algebra structure on E. This way we have an alternative definition of Lie ∞-algebra [5]:

Proposition 1.7. *A Lie ∞-algebra is a graded vector space E equipped with a degree $+1$ coderivation M_E of $\bar{S}(E)$ such that*

$$[M_E,M_E]_c=2M_E^2=0.$$

The dual of the coderivation M_E yields a differential d_* on $\bar{S}(E^*)$. The **cohomology of the Lie ∞-algebra** $\big(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}}\big)$ is the cohomology defined by the differential d_*.

A **Maurer-Cartan element** of a Lie ∞-algebra $(E, \{l_k\}_{k\in\mathbb{N}})$ is a degree zero element z of E such that

$$\sum_{k\geq 1} \frac{1}{k!}\, l_k(z, \dots, z) = 0. \tag{4}$$

The set of Maurer-Cartan elements of E is denoted by $\mathrm{MC}(E)$. Let z be a Maurer-Cartan element of $(E, \{l_k\}_{k\in\mathbb{N}})$ and set, for $k \geq 1$,

$$l_k^z(x_1, \dots, x_k) := \sum_{i\geq 0} \frac{1}{i!}\, l_{k+i}(z, \dots, z, x_1, \dots, x_k). \tag{5}$$

Then, $(E, \{l_k^z\}_{k\in\mathbb{N}})$ is a Lie ∞-algebra, called the *twisting of* E by z [3]. For filtered, or even weakly filtered Lie ∞-algebras, the convergence of the infinite sums defining Maurer-Cartan elements and twisted Lie ∞-algebras (Equations (4) and (5)) is guaranteed (see [3, 2, 6]).

For a symmetric graded Lie algebra (E, l_2), the twisting by $z \in \mathrm{MC}(E)$ is the symmetric DGLA $(E, l_1^z = l_2(z, \cdot), l_2^z = l_2)$.

1.3. Lie ∞-morphisms. A morphism of Lie ∞-algebras is a morphism between symmetric coalgebras that is compatible with the Lie ∞-structures.

Definition 1.8. Let $(E, \{l_k\}_{k\in\mathbb{N}})$ and $(V, \{m_k\}_{k\in\mathbb{N}})$ be Lie ∞-algebras. A **Lie ∞-morphism** $\Phi : E \to V$ is given by a collection of degree zero linear maps:

$$\Phi_k : S^k(E) \to V, \quad k \geq 1,$$

such that, for each $n \geq 1$,

$$\sum_{\substack{k+l=n\\ \sigma\in Sh(k,l)\\ l\geq 0,\, k\geq 1}} \varepsilon(\sigma)\Phi_{1+l}\Big(l_k(x_{\sigma(1)},\dots,x_{\sigma(k)}),x_{\sigma(k+1)},\dots,x_{\sigma(n)}\Big) = \tag{6}$$

$$= \sum_{\substack{k_1+\ldots+k_j=n\\ \sigma\in Sh(k_1,\ldots,k_j)}} \frac{\varepsilon(\sigma)}{j!} m_j\Big(\Phi_{k_1}(x_{\sigma(1)},\dots x_{\sigma(k_1)}),\Phi_{k_2}(x_{\sigma(k_1+1)},\dots x_{\sigma(k_1+k_2)}),\dots,$$

$$\Phi_{k_j}(x_{\sigma(k_1+\ldots+k_{j-1}+1)},\dots,x_{\sigma(n)})\Big),$$

If $\Phi_k=0$ for $k\neq 1$, then Φ is called a **strict Lie ∞-morphism.**

A **curved Lie ∞-morphism** $E\to V$ is a degree zero linear map $\Phi : S(E)\to V$ satisfying, for $n\geq 0$, an adapted version of (6) where the indexes $k_1,\dots,k_j$ on the right hand side of the equation run from 0 to n. The zero component $\Phi_0:\mathbb{R}\to V_0$ gives rise to an element $\Phi_0(1)\in V_0$, which by abuse of notation we denote by Φ_0. The curved adaptation of (6), for $n=0$, then reads $0=\sum_{j\geq 1}\frac{1}{j!}m_j(\Phi_0,\dots,\Phi_0)$. In other words, Φ_0 is a Maurer Cartan element of V [8].

Considering the coalgebra morphism $\Phi:\bar{S}(E)\to\bar{S}(V)$ defined by the collection of degree zero linear maps

$$\Phi_k : S^k(E)\to V,\quad k\geq 1,$$

we see that Equation (6) is equivalent to Φ preserving the Lie ∞-algebra structures:

$$\Phi\circ M_E = M_V\circ\Phi.$$

2. Representations of Lie ∞-algebras

A complex (V,d) induces a natural symmetric DGLA structure in $\mathrm{End}(V)[1]$, see Example 1.6.

Definition 2.1. A **representation** of a Lie ∞-algebra $(E,\{l_k\}_{k\in\mathbb{N}})$ on a complex (V,d) is a Lie ∞-morphism

$$\Phi : (E,\{l_k\}_{k\in\mathbb{N}})\to(\mathrm{End}(V)[1],\partial,[\cdot,\cdot]),$$

i.e., $\Phi \circ M_E = M_{\mathrm{End}(V)[1]} \circ \Phi$, where M_E is the coderivation determined by $\sum_k l_k$ and $M_{\mathrm{End}(V)[1]}$ is the coderivation determined by $\partial + [\cdot,\cdot]$.

Equivalently, a representation of E is defined by a collection of degree $+1$ maps

$$\Phi_k : S^k(E) \to \mathrm{End}(V), \quad k \geq 1,$$

such that, for each $n \geq 1$,

$$\sum_{\substack{i=1 \\ \sigma \in Sh(i,n-i)}}^{n} \varepsilon(\sigma) \Phi_{n-i+1}\left(l_i\left(x_{\sigma(1)}, \ldots, x_{\sigma(i)}\right), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\right) \tag{7}$$

$$= \partial \Phi_n(x_1, \ldots, x_n) + \frac{1}{2} \sum_{\substack{j=1 \\ \sigma \in Sh(j,n-j)}}^{n-1} \varepsilon(\sigma) \big[\Phi_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), \Phi_{n-j}(x_{\sigma(j+1)}, \ldots, x_{\sigma(n)})\big].$$

Remark 2.2. A representation on a complex (V, d) can be seen as a curved Lie ∞-morphism $\Phi : E \to \mathrm{End}(V)[1]$, with $\Phi = \sum_{k \geq 0} \Phi_k$ and $\Phi_0 = \mathrm{d}$. In fact, the first term on the right hand-side of Equations (7) is given by

$$\partial \Phi_n(x_1, \ldots, x_n) = [\Phi_0, \Phi_n(x_1, \ldots, x_n)],$$

and we have a curved Lie ∞-morphism

$$\Phi : (E, \{l_k\}_{k \in \mathbb{N}}) \to (\mathrm{End}(V)[1], [\cdot,\cdot])$$

between the Lie ∞-algebra E and the symmetric graded Lie algebra $(\mathrm{End}(V)[1], [\cdot,\cdot])$ (see [8], Lemma 2.5). This is why sometimes a representation of a Lie ∞-algebra E on a complex (V, d) is called a representation on the graded vector space V (compatible with the differential d of V).

Any representation $\Phi : E \to \mathrm{End}(V)[1]$ of a Lie ∞-algebra E on a complex (V, d) has a dual one. Let

$$^* : \mathrm{End}(V) \to \mathrm{End}(V^*)$$

be the Lie ∞-morphism given by

$$\langle f^*(\alpha), v \rangle = -(-1)^{|\alpha||f|} \langle \alpha, f(v) \rangle, \quad f \in \mathrm{End}(V), \alpha \in V^*, v \in V. \tag{8}$$

The **dual representation** ${}^*\Phi : E \to \mathrm{End}(V^*)[1]$ is obtained by composition of Φ with this Lie ∞-morphism. It is a representation on the complex (V^*, d^*)

and is given by

$$\langle {}^*\Phi(e)(\alpha), v\rangle = -(-1)^{(|e|+1)|\alpha|}\langle \alpha, \Phi(e)(v)\rangle\,,\ e\in \bar{S}(E),\ \alpha\in V^*,\ v\in V. \quad (9)$$

Remark 2.3. Given a representation $\Phi : E \to \mathrm{End}(V)[1]$ on a complex (V, d), defined by the collection of degree $+1$ linear maps $\Phi_k : S^k(E) \to \mathrm{End}(V)$, $k \geq 1$, one may consider the collection of degree $+1$ maps $\phi_k : S^k(E)\otimes V \to V$, $k \geq 0$, where $\phi_0 = \mathrm{d} : V \to V$ and $\phi_k(x, v) = (\Phi_k(x))(v)$, $k \geq 1$.

The embedding $\bar{S}(E) \oplus \big(S(E)\otimes V\big) \hookrightarrow \bar{S}(E\oplus V)$, provides a collection of maps

$$\tilde{\Phi}_k : S^k(E\oplus V) \to E\oplus V, \ \ k\geq 1,$$

given by

$$\begin{aligned}&\tilde{\Phi}_k\left((x_1, v_1), \ldots, (x_k, v_k)\right)\\ &= \left(l_k(x_1,\ldots,x_k), \sum_{i=1}^{k}(-1)^{|x_i|(|x_{i+1}|+\ldots+|x_k|)}\phi_{k-1}(x_1,\ldots,\widehat{x_i},\ldots,x_k,v_i)\right),\end{aligned}$$

and we may express Equations (7) as

$$\tilde{\Phi}_\bullet\left(\tilde{\Phi}_\bullet(x_{(1)})\odot x_{(2)}\right) + \tilde{\Phi}_1\tilde{\Phi}_\bullet(x) = 0, \quad x\in \bar{S}(E\oplus V). \quad (10)$$

Equation (10) means that $\tilde{\Phi}$ equips $E\oplus V$ with a Lie ∞-algebra structure.

Now suppose the graded vector space V has a Lie ∞-algebra structure $\{m_k\}_{k\in\mathbb{N}}$ given by a coderivation M_V of $\bar{S}(V)$. By the construction in Example 1.6, the coderivation M_V of $\bar{S}(V)$ defines a symmetric DGLA structure in $\mathrm{Coder}(\bar{S}(V))[1]$:

$$\partial_{M_V} Q = -M_V\circ Q + (-1)^{\deg(Q)} Q\circ M_V,$$

$$[Q, P] = (-1)^{\deg(Q)}\Big(Q\circ P - (-1)^{\deg(Q)\deg(P)} P\circ Q\Big),$$

where $\deg(Q)$ and $\deg(P)$ are the degrees of Q and P in $\mathrm{Coder}\,\bar{S}(V)$.

Generalizing the notion of an action of a graded Lie algebra on another graded Lie algebra, we have the following definition of an action of a Lie ∞-algebra on another Lie ∞-algebra:

Definition 2.4. An **action of the Lie ∞-algebra** $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ on the Lie ∞-algebra $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$, or a **Lie ∞-action** of E on V, is a Lie ∞-morphism

$$\Phi : (E, \{l_k\}_{k\in\mathbb{Z}}) \to (\mathrm{Coder}(\bar{S}(V))[1], \partial_{M_V}, [\cdot,\cdot]).$$

Remark 2.5. Being a Lie ∞-morphism, an action

$$\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$$

is univocally defined by a collection of degree +1 linear maps

$$\Phi_k : S^k(E) \to \mathrm{Coder}\,(\bar{S}(V)), \quad k \geq 1.$$

By the isomorphism provided in Proposition 1.2, and since each $\Phi_k(x)$, $x \in S^k(E)$, is a coderivation of $\bar{S}(V)$, we see that an action is completely defined by a collection of linear maps

$$\Phi_{k,i} : S^k(E) \otimes S^i(V) \to V, \quad i, k \geq 1. \tag{11}$$

We will denote the coderivation $\Phi_k(x)$ simply by Φ_x.

Remark 2.6. If we define $\Phi_0 := M_V$, then an action is equivalent to a curved Lie ∞-morphism between E and the graded Lie algebra $\mathrm{Coder}(\bar{S}(V))$ (compatible with the Lie ∞-structure in V) [8]. In this case, $\Phi = \sum_{k\geq 0} \Phi_k$ is called a **curved Lie ∞-action**.

There is a close relationship between representations and actions on Lie ∞-algebras.

First notice that each linear map $\ell : V \to V$ induces a (co)derivation of $\bar{S}(V)$. Hence we may see $\mathrm{End}(V)[1]$ as a Lie ∞-subalgebra of $\mathrm{Coder}(\bar{S}(V))[1]$. Therefore, given a representation $\Phi : E \to \mathrm{End}(V)[1]$ of the Lie ∞-algebra E on the complex (V, d), we have a natural action of E on the Lie ∞-algebra (V, M_V), where M_V is the coderivation defined by the map $\mathrm{d} : V \to V$. In this case, we say **the action is induced by a representation.**

Moreover, for each action $\Phi : E \to \mathrm{Coder}\,(\bar{S}(V))[1]$ of E on the Lie ∞-algebra $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$, we have a representation of E on V given by the collection of maps $\Phi_{k,1} : S^k(E) \otimes V \to V$, $k \geq 1$, or equivalently, $\Phi_{k,1} \equiv \rho_k : S^k(E) \to \mathrm{End}(V)$, $k \geq 1$. The morphism $\rho = \sum_k \rho_k$ is a representation of the Lie ∞-algebra E on the complex $(V, \mathrm{d} = m_1)$, called the **linear representation defined by** Φ.

Finally one should notice that, given a Lie ∞-algebra (V, M_V), the graded vector space $\mathrm{Coder}(\bar{S}(V))[1]$ is a Lie ∞-subalgebra of $\mathrm{End}(\bar{S}(V))[1]$. Therefore, any action $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ of the Lie ∞-algebra E on (V, M_V) yields a representation of E on the graded vector space $\bar{S}(V)$. We call it the **representation induced by the action** Φ. The coderivation M_V defines a (co)derivation of $\bar{S}(\bar{S}(V))$ and the representation is compatible with this (co)derivation.

Remark 2.7. In [8], the authors define an action of a finite dimensional Lie ∞-algebra E on a graded manifold $\mathscr{M}$ as a Lie ∞-morphism $\Phi : E \to \mathfrak{X}(\mathscr{M})[1]$. As the authors point out, when $\mathscr{M}$ is the graded manifold defined by a finite dimensional Lie ∞-algebra, we have an action of a Lie ∞-algebra on another Lie ∞-algebra. The definition presented here is a particular case of theirs because we are only considering coderivations of $\bar{S}(V)$, i.e. coderivations of $S(V)$ vanishing on the field $S^0(E)$. This restrictive case reduces to the usually Lie algebra action on another Lie algebra (and its semi-direct product) while the definition given in [8], gives rise to general Lie algebra extensions. For our purpose, this definition is more adequate.

Next, with the identification $S^n(E \oplus V) \simeq \oplus_{k=0}^{n} S^{n-k}(E) \otimes S^k(V)$, we see that the action Φ determines a coderivation of $\bar{S}(E \oplus V)$. Together with M_E and M_V we have a Lie ∞-algebra structure on $E \oplus V$. Next proposition can be deduced from [8].

Proposition 2.8. *Let* $(E, M_E \equiv \{l_k\}_{k\in\mathbb{N}})$ *and* $(V, M_V \equiv \{m_k\}_{k\in\mathbb{N}})$ *be Lie* ∞*-algebras. An action*

$$\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$$

defines a Lie ∞*-algebra structure in* $E \oplus V$*.*

Proof: We consider the brackets $\{\mathfrak{l}_n\}_{n\in\mathbb{N}}$ on $E \oplus V$ given by:

$$\begin{aligned}
&\mathfrak{l}_n(x_1, \dots, x_n) = l_n(x_1, \dots, x_n), \quad x_i \in E\\
&\mathfrak{l}_n(v_1, \dots, v_n) = m_n(v_1, \dots, v_n), \quad v_i \in V\\
&\mathfrak{l}_{k+n}(x_1, \dots, x_k, v_1, \dots, v_n) = \Phi_{k,n}(x_1, \dots, x_k, v_1, \dots, v_n),
\end{aligned}$$

with $\Phi_{k,n} : S^k(E) \otimes S^n(V) \to V$ the collection of linear maps defining Φ (see Remark 2.5).

The collection of linear maps $\Phi_{k,n}$ defines a coderivation of $\bar{S}(E \oplus V)$,

$$\Upsilon : \bar{S}(E \oplus V) \to \big(\bar{S}(E) \otimes \bar{S}(V)\big) \oplus \bar{S}(V) \subset \bar{S}(E \oplus V)$$

related to the action Φ by

$$\Upsilon(x \otimes v) = \Phi_x(v), \quad x \in E,\, v \in \bar{S}(V)$$

and

$$\Upsilon(x \otimes v) = \Phi_x(v) + (-1)^{|x_{(1)}|} x_{(1)} \otimes \Phi_{x_{(2)}}(v), \quad x \in S^{\geq 2}(E),\, v \in \bar{S}(V).$$

The degree $+1$ coderivation of $\bar{S}(E\oplus V)$ determined by $\{\mathfrak{l}_n\}_{n\in\mathbb{N}}$ is

$$M_{E\oplus V}=M_E+\Upsilon+M_V.$$

Let us prove that $M^2_{E\oplus V}=0$. For $x\in\bar{S}(E)$ and $v\in\bar{S}(V)$,

$$M^2_{E\oplus V}(x)=M^2_E(x)=0 \quad\text{and}\quad M^2_{E\oplus V}(v)=M^2_V(v)=0$$

while, for mixed terms, we have

$$M_{E\oplus V}(x\otimes v)=M_E(x)\otimes v+(-1)^{|x|}x\otimes M_V(v)+(-1)^{|x_{(1)}|}x_{(1)}\otimes\Phi_{x_{(2)}}(v)+\Phi_x(v)$$

and

$$\begin{aligned}\mathfrak{l}(M_{E\oplus V}(x\otimes v))=&(\Phi_{M_E(x)})_\bullet(v)+(-1)^{|x|}(\Phi_x)_\bullet(M_V(v))\\&+(-1)^{|x_{(1)}|}(\Phi_{x_{(1)}})_\bullet(\Phi_{x_{(2)}}(v))+m_\bullet(\Phi_x(v)).\end{aligned}$$

Since Φ is a Lie ∞-morphism, we have

$$\Phi_{M_E(x)}=-M_V\circ\Phi_x-(-1)^{|x|}\Phi_x\circ M_V+\frac{1}{2}\left[\Phi_{x_{(1)}},\Phi_{x_{(2)}}\right],$$

which implies $M^2_{E\oplus V}=0$. ■

The Lie ∞-algebra structure in $E\oplus V$ presented in Remark 2.3, is a particular case of Proposition 2.8, with $M_V=\mathrm{d}$.

Adjoint representation and adjoint action. An important example of a representation is given by a Lie ∞-algebra structure.

Let $\big(E,M_E\equiv\{l_k\}_{k\in\mathbb{N}}\big)$ be a Lie ∞-algebra; thus (E,l_1) is a complex. The collection of degree $+1$ maps

$$\begin{array}{rrcl}\mathrm{ad}_k: & S^k(E) & \to & \mathrm{End}(E)\\ & x_1\odot\ldots\odot x_k & \mapsto & \mathrm{ad}_{x_1\odot\cdots\odot x_k}:=l_{k+1}(x_1,\ldots,x_k,\ -\ )\end{array},\quad k\geq 1,$$

satisfies Equations (7). (Note that Equations (7) are equivalent to Equations (2)). So, this collection of maps defines a representation $\mathrm{ad}=\sum_k\mathrm{ad}_k$ of the Lie ∞-algebra E on (E,l_1).

Definition 2.9. The representation ad is called the **adjoint representation** of the Lie ∞-algebra $\big(E,M_E\equiv\{l_k\}_{k\in\mathbb{N}}\big)$.

Moreover, notice that for each $x\in S^i(E)$, $i\geq 1$, we may consider the degree $|x|+1$ coderivation ad^D_x of $\bar{S}(E)$ defined by the family of linear maps

$$\begin{array}{rrcl}(\mathrm{ad}_x)_k: & S^k(E) & \to & E\\ & e & \mapsto & l_{i+k}(x,e),\quad k\geq 1.\end{array}$$

So, we have a collection of degree +1 linear maps

$$\begin{array}{rccl} \mathrm{ad}_i : & S^i(E) & \to & \mathrm{Coder}(\bar{S}(E)) \\ & x & \mapsto & \mathrm{ad}^D_x \end{array}, \quad i \geq 1, \tag{12}$$

and we set $\mathbf{ad} = \sum_i \mathrm{ad}_i$.

Proposition 2.10. *The collection of degree* $+1$ *linear maps given by* (12) *defines a Lie* ∞*-morphism*

$$\mathbf{ad} : (E, \{l_k\}_{k\in\mathbb{N}}) \to \big(\mathrm{Coder}\,\bar{S}(E)[1], \partial_{M_E}, [\cdot,\cdot]\big)$$

from the Lie ∞*-algebra* E *to the symmetric DGLA* $\mathrm{Coder}\,\bar{S}(E)[1]$.

Proof: For each $x \in S^i(E)$, let $\mathrm{ad}_x = \sum_k (\mathrm{ad}_x)_k$ and set $l = \sum_k l_k$.

If $x \in \oplus_{i\geq 2} S^i(E)$ and $e \in \bar{S}(E)$, we have

$$\begin{aligned} M_E(x \odot e) &= M_E(x) \odot e + (-1)^{|x|} x \odot M_E(e) + (-1)^{|e||x_{(2)}|} l(x_{(1)}, e) \odot x_{(2)} \\ &\quad + l(x, e_{(1)}) \odot e_{(2)} + (-1)^{|e_{(1)}||x_{(2)}|} l(x_{(1)}, e_{(1)}) \odot x_{(2)} \odot e_{(2)} + l(x, e) \end{aligned}$$

and so,

$$\begin{aligned} &\mathrm{ad}_x(M_E(e)) = l(x, M_E(e)) \\ &= (-1)^{|x|} \underbrace{l(M_E(x \odot e))}_{=0 \text{ by } (2)} - (-1)^{|x|} l(M_E(x), e) - (-1)^{|x|} l(\mathrm{ad}^D_x(e)) \\ &\quad - (-1)^{|x_{(1)}| + |x_{(1)}||x_{(2)}|} l(x_{(2)}, \mathrm{ad}^D_{x_{(1)}}(e)) \\ &= \big(-(-1)^{|x|} \mathrm{ad}_{M_E(x)} - (-1)^{|x|} l \circ \mathrm{ad}^D_x - (-1)^{|x_{(2)}|} \mathrm{ad}_{x_{(1)}} \circ \mathrm{ad}^D_{x_{(2)}} \big)(e), \end{aligned}$$

which is equivalent to

$$\mathrm{ad}_{M_E(x)} = -l \circ \mathrm{ad}^D_x - (-1)^{|x|} \mathrm{ad}_x \circ M_E - (-1)^{|x_{(2)}|} \mathrm{ad}_{x_{(1)}} \circ \mathrm{ad}^D_{x_{(2)}}$$

or to

$$\mathrm{ad}_{M_E(x)} = -[l, \mathrm{ad}_x]_{RN} - \frac{1}{2}(-1)^{|x_{(1)}|} [\mathrm{ad}_{x_{(1)}}, \mathrm{ad}_{x_{(2)}}]_{RN}. \tag{13}$$

Note that the coderivation defined by the second member of (13) is

$$[M_E, \mathrm{ad}^D_x] + \frac{1}{2}[\mathrm{ad}^D_{x_{(1)}}, \mathrm{ad}^D_{x_{(2)}}] = \partial_{M_E}(\mathrm{ad}^D_x) + \frac{1}{2}[\mathrm{ad}^D_{x_{(1)}}, \mathrm{ad}^D_{x_{(2)}}].$$

If $x \in E$, a similar computation gives

$$\mathrm{ad}_{l_1(x)} = -l \circ \mathrm{ad}^D_x - (-1)^{|x|} \mathrm{ad}_x \circ M_E = -[l, \mathrm{ad}_x]_{RN}. \tag{14}$$

Equations (13) and (14) mean that the map $\mathbf{ad} : E \to \operatorname{Coder} \bar{S}(E)[1]$ is a Lie ∞-morphism. ∎

Definition 2.11. The linear map $\mathbf{ad} : E \to \operatorname{Coder}(\bar{S}(E))[1]$ is an action of the Lie ∞-algebra E on itself, called the **adjoint action of** E.

3. $\mathscr{O}$-operators on a Lie ∞-algebra

In this section we define $\mathscr{O}$-operators on a Lie ∞-algebra E with respect to an action of E on a Lie ∞-algebra V. This is the main notion of the paper.

3.1. $\mathscr{O}$-operators with respect to a Lie ∞-action. Let $(E, M_E \equiv \{l_k\}_{k\geq 1})$ and $(V, M_V \equiv \{m_k\}_{k\geq 1})$ be Lie ∞-algebras and $\Phi : E \to \operatorname{Coder}(\bar{S}(V))[1]$ a Lie ∞-action of E on V. Remember we are using Sweedler's notation: for each $v \in \bar{S}(V)$,

$$\Delta(v) = v_{(1)} \otimes v_{(2)}$$

and

$$\Delta^2(v) = (\mathrm{id} \otimes \Delta)\Delta(v) = (\Delta \otimes \mathrm{id})\Delta(v) = v_{(1)} \otimes v_{(2)} \otimes v_{(3)}.$$

Each degree zero linear map $T : \bar{S}(V) \to \bar{S}(E)$ defines a degree $+1$ linear map $\Phi^T : \bar{S}(V) \to \bar{S}(V)$ given by

$$\begin{aligned} &\Phi^T(v) = 0, \quad v \in V, \\ &\Phi^T(v) = \Phi_{T(v_{(1)})}\, v_{(2)}, \quad v \in S^{\geq 2}(V). \end{aligned}$$

Lemma 3.1. *The linear map $\Phi^T : \bar{S}(V) \to \bar{S}(V)$ is a degree $+1$ coderivation of $\bar{S}(V)$ and is defined by the collection of linear maps $\sum \Phi_{\bullet,\bullet}(T \otimes \mathrm{id})\Delta$.*

Proof: For the linear map $\Phi^T : \bar{S}(V) \to \bar{S}(V)$ to be a coderivation it must satisfy:

$$\Delta\Phi^T(v) = \left(\Phi^T \otimes \mathrm{id} + \mathrm{id} \otimes \Phi^T\right)\Delta(v), \quad v \in \bar{S}(V).$$

This equation is trivially satisfied for $v \in V$.

For each $v = v_1 \odot v_2 \in S^2(V)$ we have $\Phi^T(v) \in V$ and consequently, $\Delta\Phi^T(v) = 0$. On the other hand, since $\Phi^T{}_{|V} = 0$, we see that

$$\left(\Phi^T \otimes \mathrm{id} + \mathrm{id} \otimes \Phi^T\right)\Delta(v) = 0$$

and the equation is satisfied in $S^2(V)$.

Now let $v \in S^{\geq 3}(V)$, then

$$\begin{aligned}\Delta\Phi^T(v) &= \Delta\Phi_{T(v_{(1)})}v_{(2)}\\ &= \Big(\Phi_{T(v_{(1)})}\otimes \mathrm{id} + \mathrm{id}\otimes \Phi_{T(v_{(1)})}\Big)\,\Delta(v_{(2)}).\end{aligned}$$

The coassociativity of Δ ensures that

$$\begin{aligned}\Delta\Phi^T(v) &= \Phi_{T(v_{(1)})}v_{(2)}\otimes v_{(3)} + (-1)^{(|v_{(1)}|+1)|v_{(2)}|}v_{(2)}\otimes \Phi_{T(v_{(1)})}v_{(3)}\\ &= \Phi_{T(v_{(1)})}v_{(2)}\otimes v_{(3)} + (-1)^{|v_{(1)}|}v_{(1)}\otimes \Phi_{T(v_{(2)})}v_{(3)}\\ &= \left(\Phi^T\otimes \mathrm{id} + \mathrm{id}\otimes\Phi^T\right)\Delta(v).\end{aligned}$$

■

Definition 3.2. Let $(E, M_E \equiv \{l_k\}_{k\geq 1})$ and $(V, M_V \equiv \{m_k\}_{k\geq 1})$ be Lie ∞-algebras and $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ an action. An **$\mathcal{O}$-operator** on E with respect to the action Φ is a (degree 0) morphism of coalgebras $T : \bar{S}(V) \to \bar{S}(E)$ such that

$$M_E \circ T = T \circ \left(\Phi^T + M_V\right). \tag{15}$$

Definition 3.3. A **Rota Baxter operator (of weight 1)** on a Lie ∞-algebra $(E, M_E \equiv \{l_k\}_{k\geq 1})$ is an $\mathcal{O}$-operator with respect to the adjoint action.

An $\mathcal{O}$-operator $T : \bar{S}(V) \to \bar{S}(E)$ with respect to an action $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ of $(E, M_E \equiv \{l_k\}_{k\geq 1})$ on $(V, M_V \equiv \{m_k\}_{k\geq 1})$ is defined by a linear map $t = \sum_i t_i : \bar{S}(V) \to E$ satisfying:

(i) $l_1(t_1(v)) = t_1(m_1(v)), \quad v \in V$

(ii) $l(T(v)) = t\left(\Phi_{T(v_{(1)})}v_{(2)} + m(v_{(1)})\odot v_{(2)}\right), \quad v \in \oplus_{i\geq 2}S^i(V)$.

In particular, the $\mathcal{O}$-operator T is a comorphism i.e., for each $v \in S^n(E)$, $n \geq 1$,

$$T(v) = \sum_{k_1+\ldots+k_r=n} \frac{1}{r!}\, t_{k_1}(v_{(k_1)})\odot\ldots\odot t_{k_r}(v_{(k_r)}),$$

so, detailing (ii) for $v = v_1\odot v_2$, we get

$$\begin{aligned}l_1\Big(t_2(v_1,\, v_2)\Big) &+ l_2\Big(t_1(v_1),\, t_1(v_2)\Big)\\ &= t_1\left(\Phi_{t_1(v_1)}v_2 + (-1)^{|v_1|\,|v_2|}\Phi_{t_1(v_2)}v_1 + m_2(v_1,v_2)\right)\\ &\quad + t_2\Big(m_1(v_1), v_2\Big) + (-1)^{|v_1|}t_2\Big(v_1, m_1(v_2)\Big).\end{aligned}$$

Generally, for every $v = v_1 \odot \ldots \odot v_n \in S^n(V)$, $n \geq 3$, we have

$$\sum_{\substack{k_1+\ldots+k_i=n \\ \sigma\in Sh(k_1,\ldots,k_i)}} \frac{\epsilon(\sigma)}{i!} l_i\Big(t_{k_1}(v_{\sigma(1)},\ldots,v_{\sigma(k_1)}),\ldots,t_{k_i}(v_{\sigma(k_1+\ldots+k_{i-1}+1)},\ldots,v_{\sigma(n)})\Big)$$

$$= \sum_{\substack{k_1+\ldots+k_{i+2}=n \\ \sigma\in Sh(k_1,\ldots,k_{i+2})}} \frac{\epsilon(\sigma)}{i!}$$

$$t_{1+k_{i+2}}\Big(\Phi_{i,k_{i+1}}\Big(t_{k_1}(v_{\sigma(1)}\ldots,v_{\sigma(k_1)})\odot\ldots\odot t_{k_i}(v_{\sigma(k_1+\ldots+k_{i-1}+1)},\ldots,v_{\sigma(k_1+\ldots+k_i)}),$$

$$v_{\sigma(k_1+\ldots+k_i+1)}\odot\cdots\odot v_{\sigma(k_1+\ldots+k_{i+1})}\Big), v_{\sigma(k_1+\ldots+k_{i+1}+1)}\odot\cdots\odot v_{\sigma(n)}\Big)$$

$$+\sum_{i=1}^{n}\sum_{\sigma\in Sh(i,n-i)} \epsilon(\sigma)\, t_{n-i+1}\left(m_i(v_{\sigma(1)},\ldots,v_{\sigma(i)}),v_{\sigma(i+1)},\ldots,v_{\sigma(n)}\right).$$

Remark 3.4. When $M_V = 0$ we are considering V simply as a graded vector space, with no Lie ∞-algebra attached and an $\mathcal{O}$-operator must satisfy

$$M_E \circ T = T \circ \Phi^T.$$

In this case, the terms of above equations involving the brackets m_i on V vanish.

Remark 3.5. When $(E, [\cdot,\cdot]_E)$ and $(V, [\cdot,\cdot]_V)$ are Lie algebras, for degree reasons, a morphism $T = t_1$ must be a strict morphism. Moreover, our definition coincides with the usual definition of $\mathcal{O}$-operator (of weight 1) between Lie algebras [4] :

$$[t_1(v), t_1(w)]_E = t_1\left(\Phi_{t_1(v)}w - \Phi_{t_1(w)}v + [v,w]_V\right), \quad v, w \in V.$$

Remark 3.6. When (V, d) is just a complex and the action $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ is induced by a representation $\rho : E \to \mathrm{End}(V)[1]$ we have that $\Phi(x)$ is the (co)derivation defined by $\rho(x)$. In this case, $\mathcal{O}$-operators with respect to Φ coincide with $\mathcal{O}$-operators with respect to ρ (or relative Rota Baxter operators) given in [6].

In [6] the authors define $\mathcal{O}$-operators with respect to representations of Lie ∞-algebras. Any action induces a representation and $\mathcal{O}$-operators with respect to an action are related with $\mathcal{O}$-operators of with respect to the induced representation. We shall see that this relation is given by the comorphism

$$I = \sum_{n\geq 1} \mathrm{i}_n : \bar{S}(V) \to \bar{S}(\bar{S}(V)),$$

defined by the family of inclusion maps $\mathrm{i}_n : S^n(V) \hookrightarrow \bar{S}(V)$, $n \geq 1$.

Notice that any coderivation D of $\bar{S}(V)$ induces a (co)derivation D^{d} of $\bar{S}(\bar{S}(V))$. The comorphism I preserves these coderivations:

Lemma 3.7. *Let V be a graded vector space and D a coderivation of $\bar{S}(V)$. The map $I : \bar{S}(V) \to \bar{S}(\bar{S}(V))$ satisfies*

$$I \circ D = D^{\mathrm{d}} \circ I.$$

Proof: We will denote by $\cdot$ the symmetric product in $\bar{S}(\bar{S}(V))$, to distinguish from the symmetric product $\odot$ in $\bar{S}(V)$.

Let $v \in S^n(V)$, $n \geq 1$, and denote by $\{m_k\}_{k\geq 1}$ the family of linear maps defining the coderivation D. For $v \in V$, we immediately have $D^{\mathrm{d}} \circ I(v) = D \circ I(v) = I \circ D(v)$. For $v \in S^{\geq 2}(V)$ we have

$$\begin{aligned}
D^{\mathrm{d}} \circ I(v) &= D^{\mathrm{d}}\left(\sum_{k=1}^{n} \frac{1}{k!} v_{(1)} \cdot \ldots \cdot v_{(k)}\right) \\
&= \sum_{k=1}^{n} \frac{1}{k!}\Big(D(v_{(1)}) \cdot v_{(2)} \cdot \ldots \cdot v_{(k)} \\
&\qquad + \ldots + (-1)^{|D|(|v_{(1)}|+\ldots+|v_{(k-1)}|)} v_{(1)} \cdot \ldots \cdot v_{(k-1)} \cdot D(v_{(k)})\Big) \\
&= \sum_{k=1}^{n} \frac{1}{(k-1)!} D(v_{(1)}) \cdot v_{(2)} \cdot \ldots \cdot v_{(k)} \\
&= D(v_{(1)}) \cdot I(v_{(2)}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
I \circ D(v) &= I(m_{\bullet}(v_{(1)}) \odot v_{(2)}) \\
&= m_{\bullet}(v_{(1)}) \cdot I(v_{(2)}) + (m_{\bullet}(v_{(1)}) \odot v_{(2)}) \cdot I(v_{(3)}) \\
&= D(v_{(1)}) \cdot I(v_{(2)})
\end{aligned}$$

and the result follows. ∎

Remark 3.8. In particular, if D defines a Lie ∞-algebra structure on V, then D^{d} defines a Lie ∞-algebra structure on $\bar{S}(V)$ and I is a Lie ∞-morphism.

Proposition 3.9. *Let $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ be an action of the Lie ∞-algebra $\big(E, M_E \equiv \{l_k\}_{k\geq 1}\big)$ on the Lie ∞-algebra $\big(V, M_V \equiv \{m_k\}_{k\geq 1}\big)$ and $\tilde{T} : \bar{S}(\bar{S}(V)) \to E$ be an $\mathscr{O}$-operator with respect to the induced representation $\rho : E \to \mathrm{End}(\bar{S}(V))[1]$. Then $T = \tilde{T} \circ I$ is an $\mathscr{O}$-operator with respect to the action Φ.*

Proof: For each $x \in \bar{S}(E)$, let us denote by

$$\Phi_x^{\mathrm{d}} := \Phi(x)^{\mathrm{d}} = \rho(x)^{\mathrm{d}},$$

the (co)derivation of $\bar{S}(\bar{S}(V))$ defined by $\rho(x)$.

Let $\tilde{T}$ be an $\mathscr{O}$-operator with respect to the induced representation. This means that

$$M_E \circ \tilde{T}(w) = \tilde{T}\Big(\Phi^{\mathrm{d}}_{\tilde{T}(w_{(1)})} w_{(2)} + {M_V}^{\mathrm{d}}(w)\Big), \quad w \in \bar{S}(\bar{S}(V)).$$

Then, for each $w = I(v)$, $v \in \bar{S}(V)$, we have:

$$M_E \circ \tilde{T}(I(v)) = \tilde{T}\Big(\Phi^{\mathrm{d}}_{\tilde{T}(I(v)_{(1)})} I(v)_{(2)} + {M_V}^{\mathrm{d}} \circ I(v)\Big).$$

Using the fact that I is a comorphism and Lemma 3.7, we rewrite last equation as

$$\begin{aligned} M_E \circ T(v) &= \tilde{T}\Big(\Phi^{\mathrm{d}}_{\tilde{T}(I(v_{(1)}))} I(v_{(2)}) + {M_V}^{\mathrm{d}} \circ I(v)\Big) \\ &= \tilde{T}\Big(I \circ \Phi_{T(v_{(1)})} v_{(2)} + I \circ M_V(v)\Big) \\ &= T\Big(\Phi_{T(v_{(1)})} v_{(2)} + M_V(v)\Big). \end{aligned}$$

Taking into account this equation and that T is a comorphism, because is the composition of two comorphisms, the result follows. ■

Proposition 3.10. *Let T be an $\mathscr{O}$-operator on $\big(E, M_E \equiv \{l_k\}_{k\geq 1}\big)$ with respect to a Lie ∞-action $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ on $\big(V, M_V \equiv \{m_k\}_{k\geq 1}\big)$. Then, V has a new Lie ∞-algebra structure*

$$M_{V^T} = \Phi^T + M_V$$

and $T : (V, M_{V^T}) \to (E, M_E)$ is a Lie ∞-morphism.

Proof: By Lemma 3.1 we know Φ^T is a degree $+1$ coderivation of $\bar{S}(V)$ hence so is M_{V^T}.

Since Φ is an action, so that $\Phi\circ M_E = M_{\mathrm{Coder}(\bar{S}(V))[1]}\circ\Phi$, and T is a comorphism, we have, for each $v\in\bar{S}(V)$,

$$\begin{aligned}\Phi_{M_E T(v_{(1)})}v_{(2)} = & -M_V\Phi_{T(v_{(1)})}v_{(2)} - (-1)^{|v_{(1)}|}\Phi_{T(v_{(1)})}M_V(v_{(2)}) \\ & +(-1)^{|v_{(1)}|+1}\Phi_{T(v_{(1)})}\Phi_{T(v_{(2)})}v_{(3)}.\end{aligned} \tag{16}$$

On the other hand, T is an $\mathcal{O}$-operator:

$$M_E\circ T(v) = T\circ\Phi_{T(v_{(1)})}v_{(2)} + T\circ M_V(v)$$

and this yields

$$\Phi_{M_E T(v_{(1)})}v_{(2)} = \Phi_{T\Phi_{T(v_{(1)})}v_{(2)}}v_{(3)} + \Phi_{TM_V(v_{(1)})}v_{(2)}. \tag{17}$$

Moreover, due to the fact that both Φ^T and M_V are coderivations and $M_V^2=0$, we have

$$\begin{aligned}M_{V^T}^2(v) &= (\Phi^T)^2(v) + \Phi^T\circ M_V(v) + M_V\circ\Phi^T(v)\\ &= \Phi_{T(\Phi_{T(v_{(1)})}v_{(2)})}v_{(3)} + (-1)^{|v_{(1)}|}\Phi_{T(v_{(1)})}\Phi_{T(v_{(2)})}v_{(3)}\\ &\quad + \Phi_{TM_V(v_{(1)})}v_{(2)} + (-1)^{|v_{(1)}|}\Phi_{T(v_{(1)})}M_V(v_{(2)}) + M_V(\Phi_{T(v_{(1)})}v_{(2)}).\end{aligned}$$

Taking into account Equations (16) and (17) we conclude $M_{V^T}^2=0$. Therefore, M_{V^T} defines a Lie ∞-algebra structure on V and Equation (15) means that $T:\bar{S}(V)\to\bar{S}(E)$ is a Lie ∞-morphism between the Lie ∞-algebras (V, M_{V^T}) and (E, M_E). ■

The brackets of the Lie ∞-algebra structure on V defined by the coderivation M_{V^T} are given by

$$m_1^T(v) = m_1(v)$$

and, for $n\geq 2$,

$$\begin{aligned}m_n^T(v_1,\dots,v_n) &= m_n(v_1,\dots v_n)\\ &+ \sum_{\substack{k_1+\ldots+k_i=j\\ 1\leq j\leq n-1}}\ \sum_{\sigma\in Sh(k_1,\dots,k_i,n-j)}\epsilon(\sigma)\,\frac{1}{n!}\,\Phi_{i,n-j}\big(t_{k_1}(v_{\sigma(1)},\dots,v_{\sigma(k_1)})\\ &\qquad\odot\ldots\odot t_{k_i}(v_{\sigma(k_1+\dots+k_{i-1}+1)},\dots,v_{\sigma(j)}), v_{\sigma(j+1)}\odot\cdots\odot v_{\sigma(n)}\big)\,,\end{aligned}$$

with $\Phi_{i,n-j}$, $i\geq 1$, the linear maps determined by the action Φ (see (11)).

$\mathcal{O}$-operators for the coadjoint representation. Let $(E, M_E \equiv \{l_k\}_{k\geq 1})$ be a finite dimensional Lie ∞-algebra. Next, we consider the dual of the adjoint representation of E (see (9)), called the coadjoint representation.

Definition 3.11. The **coadjoint representation** of E, $\mathrm{ad}^* : E \to \mathrm{End}(E^*)[1]$, is defined by

$$\langle \mathrm{ad}^*_x(\alpha), v\rangle = -(-1)^{|\alpha|(|x|+1)} \langle \alpha, \mathrm{ad}_x v\rangle , \quad v \in E,\, x \in \bar{S}(E),\, \alpha \in E^*.$$

Notice that E^* is equipped with the differential l_1^* (see (8)).

An $\mathcal{O}$-operator on E with respect to the coadjoint representation $\mathrm{ad}^* : E \to \mathrm{End}\,(E^*)[1]$ is a coalgebra morphism $T : \bar{S}(E^*) \to \bar{S}(E)$ given by a collection of maps $t = \sum_i t_i : \bar{S}(E^*) \to E$ satisfying

$$\begin{aligned} l(T(\alpha)) &= \sum_{\substack{1\leq i\leq n-1\\ \sigma\in Sh(i,n-i)}} \varepsilon(\sigma)\, t_{n-i+1}(\mathrm{ad}^*_{T(\alpha_{\sigma(1)}\odot\ldots\odot\alpha_{\sigma(i)})}\, \alpha_{\sigma(i+1)}, \alpha_{\sigma(i+2)}, \ldots, \alpha_{\sigma(n)}) \\ &\quad + \sum_{i=1}^{n} (-1)^{|\alpha_1|+\cdots+|\alpha_{i-1}|} t_n(\alpha_1, \ldots, l_1^*\alpha_i, \ldots, \alpha_n), \end{aligned} \tag{18}$$

for all $\alpha = \alpha_1 \odot \ldots \odot \alpha_n \in S^n(E^*)$, $n \geq 1$.

We say that T is **symmetric** if

$$\langle \beta, t_n(\alpha_1, \ldots, \alpha_n)\rangle = (-1)^{|\alpha||\beta|+|\alpha_n|(|\alpha_1|+\ldots+|\alpha_{n-1}|)} \langle \alpha_n, t_n(\alpha_1, \ldots, \alpha_{n-1}, \beta)\rangle ,$$

for all $\alpha_1, \ldots, \alpha_n, \beta \in E^*$ and $n \geq 1$.

When T is invertible, its inverse $T^{-1} : \bar{S}(E) \to \bar{S}(E^*)$, given by $t^{-1} = \sum_n t_n^{-1}$, is also symmetric:

$$\left\langle t_n^{-1}(x_1, \ldots, x_n), y\right\rangle = (-1)^{|y||x_n|} \left\langle t_n^{-1}(x_1, \ldots, x_{n-1}, y), x_n\right\rangle ,$$

for every $x_1, \ldots x_n, y \in E$, $n \geq 1$.

One should notice that t_n^{-1} is **not** the inverse map of t_n. It simply denotes the n-component of the inverse T^{-1} of T.

For each $n \geq 1$, let $\omega^{(n)} \in \otimes^n E^*$ be defined by $\omega^{(1)} = 0$ and

$$\left\langle \omega^{(n)}, x_1 \otimes \ldots \otimes x_n \right\rangle = \left\langle t_{n-1}^{-1}(x_1, \ldots, x_{n-1}), x_n\right\rangle , \quad x_1, \ldots, x_n \in E.$$

The symmetry of T^{-1} guarantees that $\omega = \sum_{n\geq 1} \omega^{(n)}$ is an element of $\bar{S}(E^*)$.

Proposition 3.12. *Let $T : \bar{S}(E^*) \to \bar{S}(E)$ be an invertible symmetric comorphism. The linear map T is an $\mathcal{O}$-operator with respect to the coadjoint action if and only if $\omega \in \oplus_{n\geq 2} S^n(E^*)$, given by*

$$\langle \omega, x_1 \odot \ldots \odot x_{k+1} \rangle = \left\langle t_k^{-1}(x_1, \ldots, x_k), x_{k+1} \right\rangle, \quad x_1, \ldots, x_{k+1} \in E,\ k \geq 1,$$

is a cocycle for the Lie ∞-algebra cohomology.

Proof: When T is invertible, Equation (18) is equivalent to equations

$$t_1^{-1} l_1(x) = l_1^* t_1^{-1}(x), \quad x \in E,$$

and

$$t^{-1} M_E(x) = \mathrm{ad}^*_{x_{(1)}} t^{-1}(x_{(2)}) + l_1^* t_n^{-1}(x), \quad x \in S^n(E), n \geq 2.$$

Let $x = x_1 \odot \ldots \odot x_n \in S^n(E)$, $n \geq 1$, and $y \in E$, such that $|y| = |x| + 1$. We have:

$$\begin{aligned}
\langle \omega, M_E(x \odot y) \rangle &= \left\langle t^{-1}(M_E(x)), y \right\rangle + (-1)^{|x|} \left\langle t^{-1}(x_1, \ldots, x_n), l_1(y) \right\rangle \\
&\quad + (-1)^{|x_{(1)}|} \left\langle t^{-1}(x_{(1)}), \mathrm{ad}_{x_{(2)}} y \right\rangle \\
&= \left\langle t^{-1}(M_E(x)), y \right\rangle - \left\langle l_1^* t^{-1}(x), y \right\rangle - \left\langle \mathrm{ad}^*_{x_{(1)}} t^{-1}(x_{(2)}), y \right\rangle
\end{aligned}$$

and the result follows. ∎

3.2. $\mathcal{O}$-operators as Maurer-Cartan elements. Let $(E, M_E \equiv \{l_k\}_{k\geq 1})$ and $(V, M_V \equiv \{m_k\}_{k\geq 1})$ be Lie ∞-algebras.

The graded vector space of linear maps between $\bar{S}(V)$ and E will be denoted by $\mathfrak{h} := \mathrm{Hom}(\bar{S}(V), E)$. It can be identified with the space of coalgebra morphisms between $\bar{S}(V)$ and $\bar{S}(E)$. On the other hand, since

$$S^n(E \oplus V) \simeq \oplus_{k=0}^n \left(S^{n-k}(E) \otimes S^k(V) \right), \quad n \geq 1,$$

the space $\mathfrak{h}$ can be seen as a subspace of $\mathrm{Coder}(\bar{S}(E \oplus V))$, the space of coderivations of $\bar{S}(E \oplus V)$. Its elements define coderivations that only act on elements of $\bar{S}(V)$, they are $S(E)$-linear.

The space $S(E \oplus V)$ has a natural $S(E)$-bimodule structure. With the above identification we have:

$$e \cdot (x \otimes v) = (e \odot x) \otimes v = (-1)^{|e|(|x|+|v|)} (x \otimes v) \cdot e,$$

for $e \in S(E)$, $x \otimes v \in S(E \oplus V) \simeq S(E) \otimes S(V)$.

Let $t : \bar{S}(V) \to E$ be an element of $\mathfrak{h}$ defined by the collection of maps $t_k : S^k(V) \to E$, $k \geq 1$. Let us denote by $T : \bar{S}(V) \to \bar{S}(E)$ the coalgebra morphism and by $\mathfrak{t}$ the coderivation of $\bar{S}(E \oplus V)$ defined by t. Notice that

$$\mathfrak{t}(v) = t_1(v), \quad v \in V$$

and

$$\mathfrak{t}(v) = t(v_{(1)}) \otimes v_{(2)} + t(v), \quad v \in S^{\geq 2}(V).$$

and also, for $x \in \bar{S}(E)$,

$$\mathfrak{t}(x \otimes v) = (-1)^{|x||t|} x \cdot \mathfrak{t}(v), \quad v \in \bar{S}(V).$$

Proposition 3.13. *The space $\mathfrak{h}$ is an abelian Lie subalgebra of* $\mathrm{Coder}(\bar{S}(E \oplus V))$.

Proof: Let $t = \sum_i t_i : \bar{S}(V) \to E$ and $w = \sum_i w_i : \bar{S}(V) \to E$ be elements of $\mathfrak{h}$. Denote by $\mathfrak{t}$ and $\mathfrak{w}$ the coderivations of $\bar{S}(E \oplus V)$ defined by t and w, respectively.

Let $v \in \bar{S}(V)$. The Lie bracket of $\mathfrak{t}$ and $\mathfrak{w}$ is given by:

$$\begin{aligned}
&[\mathfrak{t}, \mathfrak{w}]_c(v) = \mathfrak{t} \circ (w(v_{(1)}) \otimes v_{(2)}) - (-1)^{|t||w|} \mathfrak{w} \circ (t(v_{(1)}) \otimes v_{(2)}) \\
&= (-1)^{|t|(|w|+|v_{(1)}|)} w(v_{(1)}) \cdot \mathfrak{t}(v_{(2)}) - (-1)^{|t||w|}(-1)^{|w|(|t|+|v_{(1)}|)} t(v_{(1)}) \cdot \mathfrak{w}(v_{(2)}) \\
&= \Big((-1)^{|t|(|w|+|v_{(1)}|)} w(v_{(1)}) \cdot t(v_{(2)}) \\
&\qquad - (-1)^{|t||w|}(-1)^{|w|(|t|+|v_{(1)}|)} t(v_{(1)}) \cdot w(v_{(2)}) \Big) \otimes v_{(3)} \\
&\quad + (-1)^{|t|(|w|+|v_{(1)}|)} w(v_{(1)}) \cdot t(v_{(2)}) - (-1)^{|t||w|}(-1)^{|w|(|t|+|v_{(1)}|)} t(v_{(1)}) \cdot w(v_{(2)}) \\
&= \Big((-1)^{|t|(|w|+|v_{(1)}|)} w(v_{(1)}) \odot t(v_{(2)}) \\
&\qquad - (-1)^{|t|(|w|+|v_{(2)}|)+|v_{(1)}||v_{(2)}|} w(v_{(2)}) \odot t(v_{(1)}) \Big) \otimes v_{(3)} \\
&\quad + (-1)^{|t|(|w|+|v_{(1)}|)} w(v_{(1)}) \odot t(v_{(2)}) \\
&\quad - (-1)^{|t|(|w|+|v_{(2)}|)+|v_{(1)}||v_{(2)}|} w(v_{(2)}) \odot t(v_{(1)}),
\end{aligned}$$

where we used the fact that $\mathfrak{t}$ and $\mathfrak{w}$ are $\bar{S}(E)$-linear. Because of cocommutativity of the coproduct, the last expression vanishes. ■

Now, let $\Phi : E \to \mathrm{Coder}(\bar{S}(V))[1]$ be an action of the Lie ∞-algebra E on the Lie ∞-algebra V. By Proposition 2.8, Φ induces a coderivation Υ of $\bar{S}(E \oplus V)$ and $M_{E \oplus V} = M_E + \Upsilon + M_V$ is a Lie ∞-algebra structure on $E \oplus V$. Let $\mathscr{P} : \mathrm{Coder}(\bar{S}(E \oplus V)) \to \mathfrak{h}$ be the projection onto $\mathfrak{h}$.

Then we have:

Proposition 3.14. *The quadruple* $\big(\mathrm{Coder}(\bar{S}(E\oplus V)), \mathfrak{h}, \mathscr{P}, M_{E\oplus V}\big)$ *is a* V*-data and* $\mathfrak{h}$ *has a Lie* ∞*-algebra structure.*

Proof: We already know that $\mathrm{Coder}(\bar{S}(E\oplus V))$, equipped with the commutator, is a graded Lie algebra and $\mathfrak{h}$ is an abelian Lie subalgebra.

Let $p : \bar{S}(E\oplus V) \to E$ be the projection and $i : \bar{S}(V) \to \bar{S}(E\oplus V)$ the inclusion.

Notice that, for each $Q \in \mathrm{Coder}(\bar{S}(E\oplus V))$ we have $\mathscr{P}(Q) = p\circ Q\circ i$ so

$$\ker\mathscr{P} = \big\{Q \in \mathrm{Coder}(\bar{S}(E\oplus V)) : Q\circ i \text{ is a coderivation of } \bar{S}(V)\big\}$$

is clearly a Lie subalgebra of $\mathrm{Coder}(\bar{S}(E\oplus V))$:

$$\begin{aligned}\mathscr{P}([Q,P]_c) &= p\circ [Q,P]_c \circ i = p\circ QP\circ i - (-1)^{|Q||P|} p\circ PQ\circ i\\ &= p\circ Q\circ i\circ P\circ i - (-1)^{|Q||P|} p\circ P\circ i\circ Q\circ i = 0, \quad P, Q\in\ker\mathscr{P}.\end{aligned}$$

Moreover

$$M_{E\oplus V}\circ i = M_V, \text{ so } M_{E\oplus V} \in (\ker\mathscr{P})_1$$

and, since $M_{E\oplus V}$ defines a Lie ∞-structure in $E\oplus V$, we have:

$$[M_{E\oplus V}, M_{E\oplus V}]_c = 0.$$

Voronov's construction [10] guarantees that $\mathfrak{h}$ inherits a (symmetric) Lie ∞-structure given by:

$$\partial_k\big(t_1,\dots,t_k\big) = \mathscr{P}([[\dots[M_{E\oplus V}, t_1]_{_{RN}}\dots]_{_{RN}}, t_k]_{_{RN}}\big), \quad t_1,\dots,t_k\in\mathfrak{h},\ k\geq 1.$$

■

Remark 3.15. A similar proof as in [6] shows that, with the above structure, $\mathfrak{h}$ is a filtered Lie ∞-algebra.

Lemma 3.16. *Let* $p : \bar{S}(E\oplus V)\to E$ *be the projection map and* $\mathfrak{t}$ *a coderivation of* $\bar{S}(E\oplus V)$ *defined by a degree zero element* $t : \bar{S}(V)\to E$ *of* $\mathfrak{h}$*. For each* $v\in\bar{S}(V)$*,*

$$\partial_1 t(v) = l_1 t(v) - t\circ M_V(v)$$

and

$$\partial_k(t,\dots,t)(v) = l_k\big(t(v_{(1)}),\dots,t(v_{(k)})\big) - k\, t\left(\Phi_{t(v_{(1)})\odot\ldots\odot t(v_{(k-1)})} v_{(k)}\right),\ k\geq 2.$$

Proof: Notice that

$$p \circ \mathfrak{t} = t$$
$$p \circ \mathfrak{t}^k = 0, \quad k \geq 2.$$

Consequently, for $k = 1$ we have

$$\partial_1 t(v) = p \circ M_{E\oplus V} \circ \mathfrak{t}(v) - p \circ \mathfrak{t} \circ M_V(v) = l_1 t(v) - t \circ M_V(v), \;\; v \in \bar{S}(V)$$

and, for $k \geq 2$,

$$\begin{aligned}\partial_k(t, \dots, t) &= p \circ M_{E\oplus V} \circ \mathfrak{t}^k - k\, p \circ \mathfrak{t} \circ M_{E\oplus V} \circ \mathfrak{t}^{k-1} \\ &= l \circ \mathfrak{t}^k - k\, t \circ M_{E\oplus V} \circ \mathfrak{t}^{k-1}\end{aligned}$$

and the result follows. ■

Remark 3.17. Notice that $\partial_k(t, \dots, t)(v) = 0$, for $v \in S^{<k}(V)$, as a consequence of $\mathfrak{t}^k(v) = 0$ and $\Phi \circ \mathfrak{t}^{k-1}(v) \in \bar{S}(E)$.

Next proposition realizes $\mathcal{O}$-operators as Maurer-Cartan elements of this Lie ∞-algebra $\mathfrak{h}$.

Proposition 3.18. *$\mathcal{O}$-operators on E with respect to an action Φ are Maurer-Cartan elements of $\mathfrak{h}$.*

Proof: Let $t : \bar{S}(V) \to E$ be a degree 0 element of $\mathfrak{h}$ and $\mathfrak{t}$ the corresponding coderivation of $\bar{S}(E \oplus V)$. Maurer-Cartan equation yields

$$\partial_1 t + \frac{1}{2}\partial_2(t, t) + \dots + \frac{1}{k!}\partial_k(t, \dots, t) + \dots = 0.$$

Using Lemma 3.16 we have, for each $v \in S^k(V)$,

$$\begin{aligned}\partial_1 t(v) + &\frac{1}{2}\partial_2(t, t)(v) + \dots + \frac{1}{k!}\partial_k(t, \dots, t)(v) = \\ &= l_1 t(v) - t \circ M_V(v) \\ &+ \frac{1}{2} l_2\big(t(v_{(1)}), t(v_{(2)})\big) - t\left(\Phi_{t(v_{(1)})} v_{(2)}\right) + \dots + \\ &+ \frac{1}{k!} l_k\big(t(v_{(1)}), \dots, t(v_{(k)})\big) - \frac{1}{(k-1)!}\, t\left(\Phi_{t(v_{(1)})\odot \dots \odot t(v_{(k-1)})} v_{(k)}\right).\end{aligned}$$

Let $T : \bar{S}(V) \to \bar{S}(E)$ be the morphism of coalgebras defined by $t : \bar{S}(V) \to E$. Maurer-Cartan equation can be written as

$$l \circ T(v) - t \circ M_V(v) - t\, \Phi_{T(v_{(1)})} v_{(2)} = 0,$$

which is equivalent to T being an $\mathcal{O}$-operator (see Equation (15)). $\blacksquare$

4. Deformation of $\mathcal{O}$-operators

We prove that each Maurer-Cartan element of a special graded Lie subalgebra of $\mathrm{Coder}(\bar{S}(E \oplus V))$ encondes a Lie ∞-algebra structure on E and a curved Lie ∞-action of E on V. We study deformations of $\mathcal{O}$-operators.

4.1. Maurer-Cartan element of Coder $\bar{\mathbf{S}}(\mathbf{E} \oplus \mathbf{V})$. Let E and V be two graded vector spaces and consider the graded Lie algebra $\mathfrak{L} := (\mathrm{Coder}(\bar{S}(E \oplus V)), [\cdot,\cdot]_c)$. Since $\bar{S}(E \oplus V) \simeq \bar{S}(E) \oplus (\bar{S}(E) \otimes \bar{S}(V)) \oplus \bar{S}(V)$, the space $M := \mathrm{Coder}(\bar{S}(E))$ of coderivations of $\bar{S}(E)$ can be seen as a graded Lie subalgebra of $\mathfrak{L}$. Also, the space R of coderivations defined by linear maps of the space $\mathrm{Hom}((\bar{S}(E) \otimes \bar{S}(V)) \oplus \bar{S}(V), V)$ can be embedded in $\mathfrak{L}$. We will use the identifications $M \equiv \mathrm{Hom}(\bar{S}(E), E)$ and $R \equiv \mathrm{Hom}((\bar{S}(E) \otimes \bar{S}(V)) \oplus \bar{S}(V), V)$. Given $\rho \in R$, we will denote by ρ_0 the restriction of the linear map ρ to $\bar{S}(V)$ and by ρ_x the linear map obtained by restriction of ρ to $\{x\} \otimes \bar{S}(V)$, with $x \in \bar{S}(E)$. We set $\mathfrak{L}' := M \oplus R$.

Proposition 4.1. *The space $\mathfrak{L}'$ is a graded Lie subalgebra of $\mathfrak{L} = \mathrm{Coder}(\bar{S}(E \oplus V))$.*

Proof: Given $m \oplus \rho, m' \oplus \rho' \in \mathfrak{L}'$, let us see that

$$[m \oplus \rho, m' \oplus \rho']_{RN} = [m, m']_{RN} \oplus ([m, \rho']_{RN} + [\rho, m']_{RN} + [\rho, \rho']_{RN})$$

is an element of $\mathfrak{L}'$. It is obvious that $[m, m']_{RN} \in \mathrm{Hom}(\bar{S}(E), E)$. Consider m^D and ρ^D the coderivations of $\bar{S}(E \oplus V)$ defined by the morphisms m and ρ, respectively. For $x \in \bar{S}(E)$ and $v \in \bar{S}(V)$ we have,

$$\begin{aligned} [m, \rho']_{RN}(x) &= [m, \rho']_{RN}(v) = 0 \\ [m, \rho']_{RN}(x \otimes v) &= \left(m \circ \rho'^D - (-1)^{|m||\rho'|} \rho' \circ m^D\right)(x \odot v) \\ &= -(-1)^{|m||\rho'|} \rho'_{m^D(x)}(v) \in V \end{aligned}$$

and

$$
\begin{aligned}
&[\rho,\rho']_{RN}(x) = 0\\
&[\rho,\rho']_{RN}(v) = \rho\circ\rho'^D(v) - (-1)^{|\rho||\rho'|}\rho'\circ\rho^D(v) \in V\\
&[\rho,\rho']_{RN}(x\otimes v) = \underbrace{(-1)^{|x||\rho'|}\rho_x(\rho_0'^D(v)) + (-1)^{|x_{(1)}||\rho'|}\rho_{x_{(1)}}(\rho'^D_{x_{(2)}}(v)) + \rho_0(\rho'^D_x(v))}_{\in V}\\
&\quad - (-1)^{|\rho||\rho'|}\big(\underbrace{(-1)^{|x||\rho|}\rho'_x(\rho_0^D(v)) + (-1)^{|x_{(1)}||\rho|}\rho'_{x_{(1)}}(\rho^D_{x_{(2)}}(v)) + \rho'_0(\rho^D_x(v))}_{\in V}\big),
\end{aligned}
$$

which proves that $[m,\rho']_{RN} + [\rho,m']_{RN} + [\rho,\rho']_{RN} \in \mathrm{Hom}((\bar{S}(E)\otimes\bar{S}(V))\oplus \bar{S}(V),V)$. ■

Next theorem shows that an element $m\oplus\rho\in\mathfrak{L}'$ which is a Maurer-Cartan of $\mathfrak{L} = \mathrm{Coder}(\bar{S}(E\oplus V))$ encodes a Lie ∞-algebra structure on E and an action of E on the Lie ∞-algebra V.

Theorem 4.2. *Let E and V be two graded vector spaces and $m\oplus\rho\in\mathfrak{L}' = M\oplus R$. Then, $m\oplus\rho$ is a Maurer-Cartan element of $\mathfrak{L}'$ if and only if m^D defines a Lie ∞-structure on E and ρ is a curved Lie ∞-action of E on V .*

Proof: We have

$$
[m\oplus\rho, m\oplus\rho]_{RN} = 0 \Leftrightarrow \begin{cases} [m,m]_{RN} = 0\\ 2\,[m,\rho]_{RN} + [\rho,\rho]_{RN} = 0. \end{cases} \tag{19}
$$

Similar computations to those in the proof of Proposition 4.1 give, for all $v\in\bar{S}(V)$ and $x\in\bar{S}(E)$,

$$
\begin{aligned}
&\begin{cases} \left(2\,[m,\rho]_{RN} + [\rho,\rho]_{RN}\right)(v) = 0\\ \left(2\,[m,\rho]_{RN} + [\rho,\rho]_{RN}\right)(x\otimes v) = 0 \end{cases}\\
\Leftrightarrow &\begin{cases} \rho_0\circ\rho_0^D(v) = 0\\ \rho_{m^D(x)}(v) = \left(-[\rho_0,\rho_x]_{RN} - \frac{(-1)^{|x_{(1)}|}}{2}\left[\rho_{x_{(1)}},\rho_{x_{(2)}}\right]_{RN}\right)(v). \end{cases}
\end{aligned}
$$

Since $m\oplus\rho$ is a degree $+1$ element of $\mathfrak{L}'$, the right hand-side of (19) means that m^D defines a Lie ∞-algebra structure on E and $\rho = \sum_{k\geq 0}\rho_k$ is a curved Lie ∞-action of E on V. Notice that $\rho_0^D : \bar{S}(V)\to\bar{S}(V)$ equips V with a Lie ∞-structure.

Reciprocally, if (E, m^D) is a Lie ∞-algebra and ρ is a curved Lie ∞-action of E on V, the degree $+1$ element $m \oplus \rho$ of $\mathfrak{L}'$ is a Maurer-Cartan element of $\mathfrak{L}'$. ∎

Next proposition gives the Lie ∞-algebra that controls the deformations of the actions of E on V [3].

Proposition 4.3. *Let $m \oplus \rho$ be a Maurer-Cartan element of $\mathfrak{L}'$ and $m' \oplus \rho'$ a degree $+1$ element of $\mathfrak{L}'$. Then, $m \oplus \rho + m' \oplus \rho'$ is a Maurer-Cartan element of $\mathfrak{L}'$ if and only if $m' \oplus \rho'$ is a Maurer-Cartan element of ${\mathfrak{L}'}^{m \oplus \rho}$. Here, ${\mathfrak{L}'}^{m \oplus \rho}$ denotes the DGLA which is the twisting of $\mathfrak{L}'$ by $m \oplus \rho$.*

4.2. Deformation of $\mathscr{O}$-operators. Let $\mathfrak{h}$ be the abelian Lie subalgebra of $\mathfrak{L} = \mathrm{Coder}(\bar{S}(E \oplus V))$ considered in Proposition 3.13 and $\mathscr{P} : \mathfrak{L} \to \mathfrak{h}$ the projection onto $\mathfrak{h}$. Let $\Delta \in \mathfrak{L}'$ be a Maurer-Cartan element of $\mathfrak{L}$. Then, $(\mathfrak{L}, \mathfrak{h}, \mathscr{P}, \Delta)$ is a V-data and $\mathfrak{h}$ has a Lie ∞-structure given by the brackets:

$$\partial_k(a_1, \dots, a_k) = \mathscr{P}([[\dots[\Delta, a_1]_{_{RN}} \dots]_{RN}, a_k]_{RN}), \quad k \geq 1.$$

We denote by $\mathfrak{h}_\Delta$ the Lie ∞-algebra $\mathfrak{h}$ equipped with the above structure.

The V-data $(\mathfrak{L}, \mathfrak{h}, \mathscr{P}, \Delta)$ also provides a Lie ∞-algebra structure on $\mathfrak{L}[1] \oplus \mathfrak{h}$, that we denote by $(\mathfrak{L}[1] \oplus \mathfrak{h})_\Delta$, with brackets [10]:

$$\begin{cases} q_1^\Delta((x, a_1)) = (-\left[\Delta, x\right]_{_{RN}}, \mathscr{P}(x + \left[\Delta, a_1\right]_{_{RN}})) \\ q_2^\Delta(x, x') = (-1)^{deg(x)} \left[x, x'\right]_{_{RN}} \\ q_k^\Delta(x, a_1, \dots, a_{k-1}) = \mathscr{P}([\dots[[x, a_1]_{RN}, a_2]_{RN} \dots a_{k-1}]_{RN}), \quad k \geq 2, \\ q_k^\Delta(a_1, \dots, a_k) = \partial_k(a_1, \dots, a_k), \quad k \geq 1, \end{cases} \tag{20}$$

$x, x' \in \mathfrak{L}[1]$ and $a_1, \dots, a_{k-1} \in \mathfrak{h}$. Here, $deg(x)$ is the degree of x in $\mathfrak{L}$.

Moreover, since $\mathfrak{L}'$ is a Lie subalgebra of $\mathfrak{L}$ satisfying $[\Delta, \mathfrak{L}'] \subset \mathfrak{L}'$, the brackets $\left\{q_k^\Delta\right\}_{k \in \mathbb{N}}$ restricted to $\mathfrak{L}'[1] \oplus \mathfrak{h}$ define a Lie ∞-algebra structure on $\mathfrak{L}'[1] \oplus \mathfrak{h}$, that we denote by $(\mathfrak{L}'[1] \oplus \mathfrak{h})_\Delta$. Notice that the restrictions of the brackets $\left\{q_k^\Delta\right\}$ to $\mathfrak{L}'[1] \oplus \mathfrak{h}$ are given by the same expressions as in (20) except for $k = 1$:

$$q_1^\Delta((x, a_1)) = (-\left[\Delta, x\right]_{_{RN}}, \mathscr{P}(\left[\Delta, a_1\right]_{_{RN}})) = (-\left[\Delta, x\right]_{_{RN}}, \partial_1(a_1)),$$

because $\mathscr{P}(\mathfrak{L}') = 0$. Of course, $\mathfrak{h}_\Delta$ is a Lie ∞-subalgebra of $(\mathfrak{L}'[1] \oplus \mathfrak{h})_\Delta$.

Remark 4.4. The brackets (20) that define the Lie ∞-algebra structure of $(\mathfrak{L}[1]\oplus\mathfrak{h})_\Delta$ coincide with those of $\mathfrak{h}_\Delta$ for $x = x' = 0$. So, an easy computation yields

$$t \in \mathrm{MC}(\mathfrak{h}_\Delta) \;\Leftrightarrow\; (0,t) \in \mathrm{MC}(\mathfrak{L}'[1]\oplus\mathfrak{h})_\Delta.$$

Theorem 3 in [2] yields:

Proposition 4.5. *Consider the V-data $(\mathfrak{L},\mathfrak{h},\mathscr{P},\Delta)$, with $\Delta \in \mathrm{MC}(\mathfrak{L}')$ and let t be a degree zero element of $\mathfrak{h}$. Then,*

$$t \in \mathrm{MC}(\mathfrak{h}_\Delta) \;\Leftrightarrow\; (\Delta,t) \in \mathrm{MC}(\mathfrak{L}[1]\oplus\mathfrak{h})_\Delta.$$

Recall that, given an element $t \in \mathfrak{h} = \mathrm{Hom}(\bar{S}(V),E)$, the corresponding morphism of coalgebras $T : \bar{S}(V) \to \bar{S}(E)$ is an $\mathcal{O}$-operator if and only if t is a Maurer-Cartan element of $\mathfrak{h}_\Delta$ (Proposition 3.18). Moreover, given a Maurer-Cartan element $m\oplus\rho$ of $\mathfrak{L}'$, we know from Theorem 4.2 that (E,m^D) is a Lie ∞-algebra and ρ is a curved Lie ∞-action of E on V. So, an $\mathcal{O}$-operator can be seen as a Maurer-Cartan element of the Lie ∞-algebra $(\mathfrak{L}'[1]\oplus\mathfrak{h})_\Delta$:

Proposition 4.6. *Let E and V be two graded vector spaces. Consider a morphism of coalgebras $T : \bar{S}(V) \to \bar{S}(E)$ defined by $t \in \mathrm{Hom}(\bar{S}(V),E)$, and the V-data $(\mathfrak{L},\mathfrak{h},\mathscr{P},\Delta)$, with $\Delta := m\oplus\rho \in \mathrm{MC}(\mathfrak{L}')$. Then, T is an $\mathcal{O}$-operator on E with respect to the curved Lie ∞-action ρ if and only if (Δ,t) is a Maurer-Cartan element of $(\mathfrak{L}'[1]\oplus\mathfrak{h})_\Delta$.*

Corollary 4.7. *If T is an $\mathcal{O}$-operator on the Lie ∞-algebra (E,m^D) with respect to the curved Lie ∞-action ρ of E on V, then $((\mathfrak{L}'[1]\oplus\mathfrak{h})_{m\oplus\rho})^{(m\oplus\rho,t)}$ is a Lie ∞-algebra.*

As a consequence of Theorem 3 in [2], we obtain the Lie ∞-algebra that controls the deformation of $\mathcal{O}$-operators on E with respect to a fixed curved Lie ∞-action on V:

Corollary 4.8. *Let E and V be two graded vector spaces and consider the V-data $(\mathfrak{L},\mathfrak{h},\mathscr{P},\Delta := m\oplus\rho)$. Let T be an $\mathcal{O}$-operator on (E,m^D) with respect to the curved Lie ∞-action ρ and $T' : \bar{S}(V) \to \bar{S}(E)$ a (degree zero) morphism of coalgebras defined by $t' \in \mathrm{Hom}(\bar{S}(V),E)$. Then, $T+T'$ is an $\mathcal{O}$-operator on E with respect to the curved Lie ∞-action ρ if and only if (Δ,t') is a Maurer-Cartan element of $(\mathfrak{L}'[1]\oplus\mathfrak{h})_\Delta^{(\Delta,t)}$.*

Proof: Let $t \in \mathfrak{h}$ be the morphism defined by T. Then [2],

$$(\Delta,t+t') \in \mathrm{MC}(\mathfrak{L}'[1]\oplus\mathfrak{h})_\Delta \;\Leftrightarrow\; (\Delta,t') \in \mathrm{MC}(\mathfrak{L}'[1]\oplus\mathfrak{h})_\Delta^{(\Delta,t)}. \quad \blacksquare$$

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