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ON LAX EPIMORPHISMS AND THE ASSOCIATED FACTORIZATION

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Abstract: We study lax epimorphisms in 2-categories, with special attention to Cat and $\mathcal{V}$-Cat. We show that any 2-category with convenient colimits has an orthogonal *LaxEpi*-factorization system, and we give a concrete description of this factorization in Cat.

Keywords: fully faithful morphisms, factorization systems, 2-categories, enriched categories, weighted limits.
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Introduction

A morphism $e : A \to B$ in a category $\mathbb{A}$ is an epimorphism if, for every object C, the map $\mathbb{A}(e,C) : \mathbb{A}(B,C) \to \mathbb{A}(A,C)$ is injective; looking at the hom-sets as discrete categories, this means that the functor $\mathbb{A}(e,C)$ is fully faithful. *Lax epimorphisms* (also called *co-fully-faithful* morphisms) are a 2-dimensional version of epimorphisms; in a 2-category they are precisely the 1-cells e making $\mathbb{A}(e,C)$ fully faithful for all C.

One of the most known (orthogonal) factorization systems in the category of small categories and functors is the comprehensive factorization system of Street and Walters [20]. Another known factorization system consists of bijective-on-objects functors in the left-hand side and fully faithful functors in the right. Indeed in both cases we have an orthogonal factorization system in the 2-category Cat in the sense of Definition 2.1. This means that with the usual notion in ordinary categories we have a 2-dimensional aspect of the diagonal fill-in property. Here we show that Cat has also an orthogonal $(\mathcal{E},\mathcal{M})$-factorization system where $\mathcal{E}$ is the class of all lax epimorphisms, and present a concrete description of it, making use of a characterization of the lax epimorphic functors given in [1] (Theorem 3.5).

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Moreover, any 2-category has an orthogonal $(LaxEpi, LaxStrongMono)$-factorization system provided that it has 2-colimits and is almost cowellpowered with respect to lax epimorphisms (Theorem 2.16). Here to be *almost cowellpowered* with respect to a class $\mathcal{E}$ of morphisms means that, for every morphism f, the category of all factorizations $d \cdot e$ of f with $e \in \mathcal{E}$ has a weakly terminal set. A key property is the fact that lax epimorphisms are closed under 2-colimits (Theorem 2.10).

We dedicate the last section to the study of lax epimorphisms in the 2-category $\mathcal{V}$-Cat for $\mathcal{V}$ a complete symmetric monoidal closed category. In this context, it is natural to consider a variation of the notion of lax epimorphism: We say that a $\mathcal{V}$-functor $J : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{V}$-lax epimorphism if the $\mathcal{V}$-functor $\mathcal{V}\text{-Cat}[J,\mathcal{C}] : \mathcal{V}\text{-Cat}[\mathcal{B},\mathcal{C}] \to \mathcal{V}\text{-Cat}[\mathcal{A},\mathcal{C}]$ is $\mathcal{V}$-fully faithful for all small $\mathcal{V}$-categories $\mathcal{C}$. Assuming that $\mathcal{V}$ is also cocomplete, Theorem 4.6 gives several characterizations of the lax epimorphisms in the 2-category $\mathcal{V}$-Cat. In particular, we show that they are precisely the $\mathcal{V}$-lax epimorphisms, and also precisely those $\mathcal{V}$-functors for which there is an isomorphism $\mathrm{Lan}_J\mathcal{B}(B,J-) \cong \mathcal{B}(B,-)$ ($\mathcal{V}$-natural in $B \in \mathcal{B}^{\mathrm{op}}$). Moreover, $\mathcal{V}$-lax epimorphisms are equivalently defined if above we replace all small $\mathcal{V}$-categories $\mathcal{C}$ by all possibly large $\mathcal{V}$-categories $\mathcal{C}$, or by just the category $\mathcal{V}$. This last characterization as well as Theorem 4.9, which characterizes $\mathcal{V}$-lax epimorphisms as absolutely $\mathcal{V}$-(co)dense $\mathcal{V}$-functors, have been proved before for $\mathcal{V} = \mathsf{Set}$ in [1].

For the basic theory on 2-categories we refer to [15] and [16]. For a detail account on 2-dimensional (co)limits, see [14]; here we use the notation $\lim(W,F)$ for the limit of $F : \mathbb{A} \to \mathbb{B}$ weighted ("indexed" in Kelly's language) by $W : \mathbb{A} \to \mathsf{Cat}$. Concerning enriched categories, we refer to [13].

1. Lax epimorphisms in 2-categories

In this section we present some basic properties and examples on lax epimorphisms. We end up by showing that, under reasonable conditions, for 2-categories $\mathbb{S}$ and $\mathbb{B}$, every lax epimorphism of the 2-category $2\text{-Cat}[\mathbb{S},\mathbb{B}]$ is pointwise. Pointwise lax epimorphisms will have a role in the main result of Section 2.

Definition 1.1. A *lax epimorphism* in a 2-category $\mathbb{A}$ is a 1-cell $f : A \to B$ for which all the hom-functors

$$\mathbb{A}(f,C) : \mathbb{A}(B,C) \to \mathbb{A}(A,C)$$

(with $C \in \mathbb{A}$) are fully faithful.

Remark 1.2 (Duality and Coduality)**.** The notion of lax epimorphism is dual to the the notion of *fully faithful morphism* (in a 2-category). That is the reason why lax epimorphisms are also called *co-fully-faithful morphisms*. Indeed, the notion fully faithful morphism in the 2-category of small categories Cat coincides with the notion of fully faithful functor, since a functor $P : \mathbb{A} \to \mathbb{B}$ is fully faithful if and only if the functor

$$\mathsf{Cat}(\mathbb{C}, P) : \mathsf{Cat}(\mathbb{C}, \mathbb{B}) \to \mathsf{Cat}(\mathbb{C}, \mathbb{A})$$

is fully faithful for all categories $\mathbb{C}$.

On the other hand, the notion of lax epimorphism is *self-codual*. Namely, a morphism $p : A \to B$ is a lax epimorphism in $\mathbb{A}$ if and only if the corresponding morphism in $\mathbb{A}^{\mathrm{co}}$ (the 2-category obtained after inverting the directions of the 2-cells in $\mathbb{A}$) is a lax epimorphism.

Remark 1.3. Lax epimorphisms are closed for isomorphism classes. That is to say, if $f \cong g$ and g is a lax epimorphism, then so is f. Moreover, we have that lax epimorphisms are closed under composition and are right-cancellable: for composable morphisms r and s, if r and sr are lax epimorphisms, so is s.

Examples 1.4.
(1) In a locally discrete 2-category, lax epimorphisms are just epimorphisms, since fully faithful functors between discrete categories are injective functions on the objects. But, in general, the class of lax epimorphisms and the one of epimorphisms are different and no one contains the other (see [1]).
(2) Coequifiers are lax epimorphisms. The property of being a lax epimorphism encompasses the two-dimensional aspect of the universal property of a coequifier (see [14, pag. 309]). But, as observed in [1], coequalizers in Cat are not necessarily lax epimorphisms.
(3) Any equivalence is a lax epimorphism. Recall that a morphism $g : A \to B$ is an equivalence if there is $f : B \to A$ with $gf \cong 1_B$ and $fg \cong 1_A$. This is equivalent to the existence of an ajunction between f and g with both unit and counit being invertible, and it is also well known that it is equivalent to the existence of an adjunction $(\varepsilon, \eta) : f \dashv g$ together with both f and g fully faithful. Dually, $g : A \to B$ is an equivalence if and only if there is an adjunction $(\varepsilon, \eta) : f \dashv g$ with both f and g being lax epimorphisms. Moreover, given

an adjunction $(\varepsilon, \eta) : f \dashv g : A \to B$ in a 2-category $\mathbb{A}$, the morphism g is a lax epimorphism if and only if f is fully faithful, if and only if η is invertible (see [16, Lemma 2.1]).

(4) In a locally thin 2-category (i.e., with the hom-categories being preordered sets), the lax epimorphisms are the order-epimorphisms, i.e., morphisms f for which $g \cdot f \leq h \cdot f$ implies $g \leq h$; and coinserters are lax epimorphisms – this immediately follows from the definition of coinserter (see, for instance, [14, pag. 307]).

However, coinserters are not lax epimorphisms in general; we indicate a simple counter-example in the 2-category Cat of small categories.* Let $\mathcal{A}$ be the discrete category with a unique object A, $\mathcal{B}$ the discrete category with two objects, FA and GA, and $F, G : \mathcal{A} \to \mathcal{B}$ the functors defined according to the name of the objects of $\mathcal{B}$. The coinserter of F and G is an inclusion $P : \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ has the same objects as B and a unique non trivial morphism, $\alpha_A : FA \to GA$. More precisely, the coinserter is given by the pair (P, α). (For a description of coinserters in Cat, see [5], Example 6.5.) But P is not a lax epimorphism. Indeed, let $J, K : \mathcal{C} \to \mathcal{D}$ be two functors, where the category $\mathcal{D}$ consists of four objects and six non trivial morphisms as in the diagram below, with $K\alpha_A \cdot \gamma_{FA} = r \neq s = \gamma_{GA} \cdot J\alpha_A$:

$$\begin{tikzcd}
JFA \arrow[r, "\gamma_{FA}"] \arrow[d, "J\alpha_A"'] \arrow[dr, bend left=10, "r"] \arrow[dr, bend right=10, "s"'] \arrow[dr, phantom, "\neq"] & KFA \arrow[d, "K\alpha_A"] \\
JGA \arrow[r, "\gamma_{GA}"'] & KGA
\end{tikzcd}$$

Then, we have a natural transformation $\gamma : JP \to KP$ which cannot be expressed as $\gamma = \overline{\gamma} * \mathrm{id}_P$ for any $\overline{\gamma} : J \Rightarrow K$.

(5) In the 2-category Pos of posets, monotone functions and pointwise order between maps, lax epimorphisms coincide with epimorphisms, and also with coinserters of some pair of morphisms (see [4, Lemma 3.6]).

(6) In Preord, lax epimorphisms need not to be epimorphisms: they are just the monotone maps $f : A \to B$ such that every $b \in B$ is isomorphic to $f(a)$ for some a.

*This rectifies [1, Example 2.1.1].

Moreover, coinserters are strictly contained in lax epimorphisms, they are precisely the monotone bijections. Indeed, given $f,g : A \to B$, let $\bar{B}$ be the underlying set of B with the preorder given by the reflexive and transitive closure of $\leq_B \cup \leq'$, where $\leq_B$ is the order in B and $y \leq' z$ whenever there is some $x \in A$ with $y \leq f(x)$ and $g(x) \leq z$; the coinserter is the identity map from B to $\bar{B}$. Conversely, if $h : B \to C$ is a monotone bijection, it is the coinserter of the projections $\pi_1, \pi_2 : P \to B$, where P is the comma object of h along itself.

Observe that the functor $P : \mathcal{B} \to \mathcal{C}$ of Example (4) is indeed a morphism of the full 2-subcategory Preord of Cat; it is a lax epimorphism in Preord but not in Cat.

(7) Let Grp be the 2-category of groups, homomorphisms, and 2-cells from f to g in Grp(A,B) given by those elements α of B with $f(x) \circ \alpha = \alpha \circ g(x)$, for all $x \in A$ (where $\circ$ denotes the group multiplication). The horizontal composition of $\alpha : f \to g$ with $\beta : h \to k : B \to C$ is given by $\beta * \alpha = h(\alpha) \circ \beta = \beta \circ k(\alpha)$; and the unit on an arrow $f : A \to B$ is simply the neutral element of B (see [6])[†].

The lax epimorphisms of Grp are precisely the regular epimorphisms, that is, surjective homomorphisms. Indeed, given a surjective homomorphism $f : A \to B$, homomorphisms $g, h : B \to C$ and an element $\gamma \in C$, the equalities $g(f(x)) \circ \gamma = \gamma \circ (h(f(x))$ for all $x \in A$ imply $g(y) \circ \gamma = \gamma \circ h(y)$ for all $y \in B$, showing that f is a lax epimorphism. Conversely, given a lax epimorphism $f : A \to B$, consider its $(RegEpi, Mono)$-factorization in Grp:

$$A \xrightarrow{q} M \xrightarrow{m} B\ .$$

Since q and qm, so is m, by Remark 1.3. We show that then m is an isomorphism. In Grp, monomorphisms are regular (see [2]); let $g, h : B \to C$ be a pair whose equalizer is the inclusion $m : M \hookrightarrow B$, that is, $M = \{y \in B \mid g(y) = h(y)\}$. Denoting the neutral element of C by e, we have a 2-cell $e : gm \to hm$. Since m is a lax epimorphism, there is a unique $\alpha : g \to h$ with $\alpha * e = e$. But $\alpha * e = g(e) \circ \alpha = \alpha \circ h(e) = \alpha$; hence $\alpha = e$, that is, $g(y) \circ e = e \circ h(y)$ for all $y \in B$. Thus, $B = M$ and m is the identity morphism.

[†]This 2-category is the full subcategory of 2-Cat of all groupoids with just one object.

Remark 1.5. In [1], lax epimorphisms were characterized in the 2-category Cat of small categories, functors and natural transformations: given a functor $F : A \to B$ and a morphism $g : b \to c$ in B, let $g//F$ denote the category whose objects are triples (h,a,k) such that the composition

$$b \xrightarrow{h} Fa \xrightarrow{k} c$$

is equal to g, and whose morphisms $f : (h,a,k) \to (h',a',k')$ are those $f : a \to a'$ of A with $Fa \cdot h = h'$ and $k' \cdot Fa = k$. Then:

Theorem 1.6. [1] *A functor $F : A \to B$ is a lax epimorphism in* Cat *if and only if, for every morphism g of B, the category $g//F$ is connected.*

Remark 1.7. Recall that a 2-functor $G : \mathbb{A} \to \mathbb{B}$ is *locally fully faithful* if, for any $A,B \in \mathbb{A}$, the functor $G_{A,B} : \mathbb{A}(A,B) \to \mathbb{B}(G(A),G(B))$ is fully faithful.

It is natural to consider lax epimorphisms in the context of 2-adjunctions or biadjunctions. Let $(\varepsilon,\eta) : F \dashv G : \mathbb{A} \to \mathbb{B}$ be a 2-adjunction (respectively, biadjunction). In this case, we have that, for any $A,B \in \mathbb{A}$,

$$\begin{tikzcd}
\mathbb{A}(A,B) \arrow[r, "G_{A,B}"] \arrow[dr, "\mathbb{A}(\varepsilon_A, B)"'] & \mathbb{B}(G(A),G(B)) \arrow[d, "\chi_{G(A),B}"] \\
 & \mathbb{A}(FG(A),B)
\end{tikzcd} \tag{1.0.1}$$

commutes (respectively, commutes up to an invertible natural transformation), in which

$$\begin{array}{rll} \chi_{G(A),B} : & \mathbb{B}(G(A),G(B)) & \to \mathbb{A}(FG(A),B) \\ & h & \mapsto \varepsilon_B \circ F(h) \end{array}$$

is the invertible functor (respectively, equivalence) of the 2-adjunction (biadjunction).

In the situation above, since isomorphisms (respectively, equivalences) are fully faithul and fully faithful functors are left-cancellable (see Remark 1.3), we have that $G_{A,B} : \mathbb{A}(A,B) \to \mathbb{B}(G(A),G(B))$ is fully faithful if, and only if, $\mathbb{A}(\varepsilon_A,B)$ is fully faithful. Therefore, the 2-functor $G : \mathbb{A} \to \mathbb{B}$ is locally fully faithful if and only if ε_C is a lax epimorphism for every $C \in \mathbb{A}$.

Remarks 1.8. It is known that in a 2-category with cotensor products, fully faithful morphisms are those $p : A \to B$ such that the comma object of p along itself is isomorphic to the cotensor product $2 \pitchfork A$. Dually, assuming the existence of tensor products, a morphism $p : A \to B$ is a lax epimorphism if and only if

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "p"'] \arrow[dr, phantom, "\xRightarrow{\alpha * \mathrm{id}_p}"] & B \arrow[d, "\nu_1"] \\ B \arrow[r, "\nu_0"'] & 2 \otimes B \end{tikzcd}$$

is an opcomma object, in which

$$\begin{tikzcd} B \arrow[r, bend left, "\nu_1"] \arrow[r, bend right, "\nu_0"'] \arrow[r, phantom, "\Uparrow\alpha"] & 2 \otimes B \end{tikzcd}$$

is the tensor product.

Remarks 1.9 (Preservation and reflection of lax epimorphisms)**.** Since, in the presence of tensor products, lax epimorphisms are characterized by opcomma objects as above, we conclude that:

Lemma 1.10. *Let $F : \mathbb{B} \to \mathbb{A}$ be a 2-functor.*

1. *Assuming that $\mathbb{B}$ has tensor products, if F preserves opcomma objects and tensor products, then F preserves lax epimorphisms.*
2. *Assuming that $\mathbb{A}$ has tensor products, if G creates opcomma objects and tensor products, then G reflects lax epimorphisms.*

Moreover, we also have that:

Lemma 1.11. *Let $F \dashv G$ be a 2-adjunction.*

(1) The 2-functor $F : \mathbb{B} \to \mathbb{A}$ preserves lax epimorphisms.
(2) If G is essentially surjective, then F reflects lax epimorphisms.

Proof: For any object W of $\mathbb{A}$ and any morphism $p : A \to B$ of $\mathbb{B}$, we have that

$$\begin{tikzcd} \mathbb{A}(F(B),W) \arrow[r, "\mathbb{A}(F(p),W)"] \arrow[d, leftrightarrow, "\chi_{A,W}"', "\cong"] & \mathbb{A}(F(A),W) \arrow[d, leftrightarrow, "\cong"', "\chi_{B,W}"] \\ \mathbb{B}(B,G(W)) & \mathbb{B}(A,G(W)) \arrow[l, "\mathbb{B}(p,G(W))"] \end{tikzcd} \tag{1.0.2}$$

commutes.

(1) If $p : A \to B$ is a lax epimorphism in $\mathbb{B}$, for any $W \in \mathbb{A}$, we have that $\mathbb{B}(p, G(W))$ is fully faithful and, hence, by the commutativity of (1.0.2), $\mathbb{A}(F(p), W)$ is fully faithful.

(2) Assuming that G is essentially surjective, if $F(p) : F(A) \to F(B)$ is a lax epimorphism in $\mathbb{A}$, then, for any $Z \in \mathbb{B}$, there is $W \in \mathbb{A}$ such that $G(W) \cong Z$. Moreover, we have that $\mathbb{A}(F(p), W)$ is fully faithful and, hence, $\mathbb{B}(p, G(W))$ is fully faithful by the commutativity of (1.0.2). This implies that $\mathbb{B}(p, Z)$ is fully faithful for any $Z \in \mathbb{B}$. ■

Definition 1.12. A 2-natural transformation $\lambda : F \to G : \mathbb{S} \to \mathbb{B}$ is:

1. a *pointwise lax epimorphism* if, for any $C \in \mathbb{S}$, the morphism

$$\lambda_C : F(C) \to G(C)$$

   is a lax epimorphism in $\mathbb{B}$;
2. a *lax epimorphism* if λ is a lax epimorphism in the 2-category of 2-Cat$[\mathbb{S}, \mathbb{B}]$ of 2-functors, 2-natural transformations and modifications.

Proposition 1.13. *Let $\lambda : F \to G : \mathbb{S} \to \mathbb{B}$ be a 2-natural transformation. If λ is a pointwise lax epimorphism then it is a lax epimorphism in the 2-category* 2-Cat$[\mathbb{S}, \mathbb{B}]$.

Proof: Let $\lambda : F \to G : \mathbb{A} \to \mathbb{B}$ be a 2-natural transformation with each $\lambda_A : FA \to GA$ a lax epimorphism in $\mathbb{B}$. Let $\alpha, \beta : G \to H : \mathbb{A} \to \mathbb{B}$ be two 2-natural transformations, and let $\Theta : \alpha * \lambda \rightsquigarrow \beta * \lambda$ be a modification. In particular, we have 2-cells in $\mathbb{B}$ indexed by $A \in \mathbb{A}$:

$$\begin{tikzcd} & GA \arrow[dr, "\alpha_A"] & \\ FA \arrow[ur, "\lambda_A"] \arrow[dr, "\lambda_A"'] \arrow[rr, phantom, "\Downarrow\Theta_A"] & & HA \\ & GA \arrow[ur, "\beta_A"'] & \end{tikzcd}$$

This gives rise to unique 2-cells

$$\begin{tikzcd} GA \arrow[rr, bend left, "\alpha_A"] \arrow[rr, bend right, "\beta_A"'] \arrow[rr, phantom, "\Downarrow\Phi_A"] & & HA \end{tikzcd}$$

with $\Phi_A * \lambda_A = \Theta_A$. The uniqueness of $\Phi = (\Phi_A)_{A\in\mathbb{A}}$ is clear. It is straightforward to see that Φ is indeed a modification. ∎

However, not every lax epimorphism 2-natural transformation is a pointwise lax epimorphism. In fact, this is known to be true for epimorphisms and, as observed in (1) of Examples 1.4, lax epimorphisms in locally discrete 2-categories are the same as epimorphisms.

More precisely, consider the locally discrete 2-category $\mathbb{S}$ generated by

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

with the equation $fh = gh$. The pair (h, f) gives an epimorphism in 2-Cat$[2, \mathbb{S}]$ but h clearly is not an epimorphism in $\mathbb{S}$. Since $\mathbb{S}$ and 2-Cat$[2, \mathbb{S}]$ are locally discrete, this proves that (h, f) gives a 2-natural transformation which is a lax epimorphism but not a pointwise lax epimorphism.

Yet, it follows from Lemma 1.11 that the converse holds for many interesting cases. More precisely:

Theorem 1.14. *Let $\mathbb{B}$ be a 2-category with cotensor products. Then, a 2-natural transformation $\lambda : F \to G : \mathbb{S} \to \mathbb{B}$ is a lax epimorphism if and only if it is a pointwise lax epimorphism.*

Proof: By Proposition 1.13, every pointwise lax epimorphism is an epimorphism. We prove the converse below.

Let 1 be the terminal category with only the object 0. For each $s \in \mathbb{S}$, we denote by $\overline{s} : 1 \to \mathbb{S}$ the functor defined by s. For each $\overline{B} : 1 \to \mathbb{B}$, we have the pointwise right Kan extension (see [8, Theorem I.4.2]) given by

$$\mathrm{Ran}_{\overline{s}}\overline{B}(a) = \lim\left(\mathbb{S}(a, \overline{s}-), \overline{B}\right) \cong \mathbb{S}(a, s) \pitchfork \left(\overline{B}0\right).$$

We conclude, then, that, for any $s \in \mathbb{S}$, we have the 2-adjunction

$$2\text{-}\mathsf{Cat}[\overline{s}, \mathbb{B}] \dashv \mathrm{Ran}_{\overline{s}}.$$

Therefore, by Lemma 1.11, assuming that $\lambda : F \to G : \mathbb{S} \to \mathbb{B}$ is a lax epimorphism in 2-$\mathsf{Cat}[\mathbb{S}, \mathbb{B}]$, we have that, for every $s \in \mathbb{S}$,

$$2\text{-}\mathsf{Cat}[\overline{s}, \mathbb{B}](\lambda) = \lambda * \mathrm{id}_{\overline{s}} = \lambda_s$$

is a lax epimorphism in $\mathbb{B}$. ∎

2. The orthogonal *LaxEpi*-factorization system

Factorization systems in categories have largely shown their importance, taking the attention of many authors since the pioneering work exposed in [10]. (For a comprehensive account of the origins of the study of categorical factorization techniques see [21].) When the category has appropriate colimits, we get one of the most common orthogonal factorization systems, the $(Epi, StrongMono)$ one. Since lax epimorphisms look an adequate 2-version of epimorphisms, it is natural to ask for a factorization system involving them. In this section, we will obtain an orthogonal $(LaxEpi, LaxStrongMono)$-factorization system in 2-categories. In the next section we give a description of this orthogonal factorization system in Cat.

The notion of orthogonal factorization system in 2-categories generalizes the ordinary one by incorporating the two-dimensional aspect in the diagonal fill-in property. Here we use a strict version of the orthogonal factorization systems studied in [9] (see Remark 2.2):

Definition 2.1. In the 2-category $\mathbb{A}$, let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms closed under composition and containing the isomorphisms. The pair $(\mathcal{E}, \mathcal{M})$ forms an *orthogonal factorization system* provided that:

(i) Every morphism f of $\mathbb{A}$ factors as a composition $f = me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

(ii) For every $A \xrightarrow{e} B$ in $\mathcal{E}$ and $C \xrightarrow{m} D$ in $\mathcal{M}$, the square

$$\begin{tikzcd} \mathbb{A}(B,C) \arrow[r, "\mathbb{A}(B,m)"] \arrow[d, "\mathbb{A}(e,C)"'] & \mathbb{A}(B,D) \arrow[d, "\mathbb{A}(e,D)"] \\ \mathbb{A}(A,C) \arrow[r, "\mathbb{A}(A,m)"] & \mathbb{A}(A,D) \end{tikzcd}$$

is a pullback in Cat.

Remark 2.2. In [9], Dupont and Vitale studied orthogonal factorization systems in 2-categories in a non-strict sense. Thus, in (i) of Definition 2.1 the factorization holds up to equivalence, and in (ii), instead of a pullback, we have a bi-pullback.

Remark 2.3. The one-dimensional aspect of (ii) asserts, for each pair of morphisms $f : A \to C$ and $g : B \to D$ with $mf = ge$, the existence of a unique $t : B \to C$ with $te = f$ and $mt = g$. The two-dimensional aspect of (ii) means that, whenever, with the above equalities, we have $t'e = f'$ and $mt' = g'$, and 2-cells $\alpha : f \to f'$ and $\beta : g \to g'$ such that $m * \alpha = \beta * e$,

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[dl, dashed, bend right, "t"'] \arrow[dl, dashed, bend left, "t'"] \arrow[dl, phantom, "\overset{\theta}{\Rightarrow}"] \arrow[d, bend right, "g'"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\beta}{\Leftarrow}"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \tag{2.0.1}$$

then there is a unique 2-cell $\theta : t \to t'$ with $\theta * e = \alpha$ and $m * \theta = \beta$.

If $\mathcal{E}$ is made of lax epimorphisms, the two-dimensional aspect comes for free. Indeed, for $\alpha : f = te \Rightarrow t'e = f'$, there is a unique $\theta : t \Rightarrow t'$ with $\theta * e = \alpha$; and, since $\beta * e = m * \alpha = m * \theta * e$, we have $\beta = m * \theta$.

Definition 2.4. A 1-cell $m : C \to D$ is said to be a *lax strong monomorphism* if it has the diagonal fill-in property with respect to lax epimorphisms; that is, for every commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dotted, "t"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \tag{2.0.2}$$

with e a lax epimorphism, there is a unique $t : B \to C$ such that $te = f$ and $mt = g$.

In other words, taking into account Remark 2.3, $m : C \to D$ is a lax strong monomorphism if for every lax epimorphism e, the morphisms e and m fulfil condition (ii) of Definition 2.1.

Remark 2.5. It is obvious that lax strong monomorphisms are closed under composition and left-cancellable; moreover, their intersection with lax epimorphisms are isomorphisms.

Proposition 2.6. *In a 2-category:*

(i) Every inserter is a lax strong monomorphism.
(ii) In the presence of coequifiers, every lax strong monomorphism is faithful, i.e., a morphism m such that $\mathbb{A}(X,m)$ is faithful for all X.

Proof: (i) For the commutative square (2.0.2) above let e be a lax epimorphism and let the diagram

$$\begin{tikzcd} & D \arrow[dr, "r"] & \\ C \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & E \\ & D \arrow[ur, "s"'] & \end{tikzcd} \tag{2.0.3}$$

be an inserter. Since e is a lax epimorphism, there is a unique $\beta : rg \Rightarrow sg$ with $\alpha * f = \beta * e$. This implies the existence of a unique $t : B \to C$ such that $mt = g$ and $\alpha * t = \beta$. Then we have $\alpha*(te) = \beta*e = \alpha*f$ and $m(te) = ge = mf$. Hence, by the universality of (m,α), we conclude that $te = f$. And t is unique: if $mt = mt'$ and $te = t'e$, then we have $\alpha * t * e = \alpha * t' * e$, which implies $\alpha * t = \alpha * t'$; this together with $mt = mt'$ shows that $t = t'$.

(ii) Given a lax strong monomorphism $m : A \to B$ and two 2-cells $\alpha, \beta : r \to s : X \to A$ with $m * \alpha = m * \beta$, let $e : A \to C$ be the coequifier of the 2-cells. Then m factors through e. Since, by 1.4(2), e is a lax epimorphism, using the diagonal fill-in property, there is some $t : C \to A$ with $te = 1_A$. Then $\alpha = \beta$. ■

Examples 2.7. (1) In Pos and Preord the converse of 2.6(i) also holds. In Pos lax strong monomorphisms are just order-embeddings‡ and order-embeddings coincide with inserters ([4, Lemma 3.3]).

‡A morphism $m : X \to Y$ in Pos or Preord is an *order-embedding* if m is injective and $m(x) \le m(y) \Leftrightarrow x \le y$.

Also in Preord lax strong monomorphisms coincide with inserters. It is easily seen that lax strong monomorphisms are precisely the order-embeddings $m : X \to Y$ with $m[X]$ closed in Y under isomorphic elements. Let $m : X \to Y$ be a lax strong monomorphism. Let Z be obtained from Y just replacing every element $y \in Y \setminus m[X]$ by two unrelated elements $(y, 1)$ and $(y, 2)$, and let the maps $f_1, f_2 : Y \to Z$ be equal on $m[X]$ and $f_i(y) = (y, i)$, $i = 1, 2$, for the other cases. Endowing Z with the least preorder which makes f_1 and f_2 monotone, we see that m is the inserter of f_1 and f_2.

(2) But, in general, the converse of 2.6(i) is false. Just consider an ordinary category (i.e. a locally discrete 2-category) where strong monomorphisms and regular monomorphisms do not coincide. For that, it suffices that regular monomorphisms are not closed under composition.

Remark 2.8. In contrast to 2.6, neither equifiers nor equalizers are, in general, lax strong monomorphisms. Consider the following equivalence of categories, where only the non trivial morphisms are indicated:

$$A = \boxed{a} \xrightarrow{E} \boxed{a \underset{f^{-1}}{\overset{f}{\rightleftarrows}} b} = B$$

The functor E is a lax epimorphism (see Example 1.4(3)), but not a lax strong monomorphism, since there is no $T : B \to A$ making the following two triangles

$$\begin{tikzcd} A \arrow[r, "E"] \arrow[d, equal] & B \arrow[d, equal] \arrow[dl, "T"] \\ A \arrow[r, "E"'] & B \end{tikzcd}$$

commutative. But E is both an equifier and an equalizer. To see that it is an equalizer consider the pair $F, \mathrm{id}_B : B \to B$, where F takes all objects on a

and all morphisms on 1_a. To see that it is an equafier consider the category

$$C = \boxed{\begin{tikzcd} Ra \arrow[r, bend left, "Rf"] \arrow[d, "\alpha_a=\beta_a"'] & Rb \arrow[l, bend left, "Rf^{-1}"] \arrow[d, bend right, "\beta_b"'] \arrow[d, bend left, "\alpha_b"] \\ Sa \arrow[r, bend left, "Sf"] & Sb \arrow[l, bend left, "Sf^{-1}"] \end{tikzcd}} \tag{2.0.4}$$

and 2-cells $\alpha, \beta : R \to S : B \to C$ given in the obvious way.

A key property in the sequel is the closedness of lax epimorphisms under colimits, in the sense of 2.9 below. The closedness of classes of morphisms under limits in ordinary categories was studied in [11].

Definition 2.9. Let $\mathcal{E}$ be a class of morphisms in a 2-category $\mathbb{B}$. We say that $\mathcal{E}$ *is closed under (2-dimensional) colimits* in $\mathbb{A}$ if, for every small 2-category $\mathbb{S}$, every weight $W : \mathbb{S}^{\mathrm{op}} \to \mathrm{Cat}$ and every 2-natural transformation

$$\lambda : D \to D' : \mathbb{S} \to \mathbb{B},$$

the induced morphism

$$\mathrm{colim}\,(W, \lambda) : \mathrm{colim}(W, D) \to \mathrm{colim}(W, D')$$

is a morphism in the class $\mathcal{E}$ whenever, for any $C \in \mathbb{S}$, λ_C is a morphism in $\mathcal{E}$.

Theorem 2.10. *Lax epimorphisms are closed under (2-dimensional) colimits.*

Proof: In fact, if the 2-natural transformation $\lambda : D \to D' : \mathbb{S} \to \mathbb{B}$ is a pointwise lax epimorphism, then, for any $A \in \mathbb{B}$, the 2-natural transformation

$$\mathbb{B}(\lambda, A) : \mathbb{B}(D'-, A) \to \mathbb{B}(D-, A),$$

pointwise defined by $\mathbb{B}(\lambda, A)_C = \mathbb{B}(\lambda_C, A)$, is pointwise fully faithful. Hence it is fully faithful in the 2-category $\mathrm{Cat}[\mathbb{S}, \mathbb{B}]$ (dual of Proposition 1.13). Therefore, for any weight $W : \mathbb{S}^{\mathrm{op}} \to \mathrm{Cat}$ and $X \in \mathbb{B}$,

$$\mathbb{B}(\mathrm{colim}\,(W, \lambda), X) \cong 2\text{-}\mathrm{Cat}\,[\mathbb{S}, \mathbb{B}]\,(W, \mathbb{B}(\lambda, A))$$

is fully faithful. This proves that $\mathrm{colim}\,(W, \lambda)$ is a lax epimorphism in $\mathbb{B}$. $\blacksquare$

Remark 2.11. As shown in [7], for any orthogonal $(\mathcal{E}, \mathcal{M})$-factorization system in an ordinary category, $\mathcal{E}$ and $\mathcal{M}$ are closed under, respectively, colimits and limits.

In Cat, lax epimorphisms are not closed under (2-dimensional) limits, fully faithful functors are not closed under (2-dimensional) colimits, and, moreover, equivalences are neither closed under limits nor colimits.

Indeed, consider the category $\nabla 2$ with two objects and one isomorphism between them. Let d^0 and d^1 be the two possible inclusions $1 \to \nabla 2$ of the terminal category in $\nabla 2$. There is only one 2-natural transformation ι between the diagram

$$1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} \nabla 2$$

and the terminal diagram $1 \rightrightarrows 1$.

Clearly, ι is a pointwise equivalence (and, hence, a pointwise lax epimorphism and fully faithful functor). However, the induced functor between the equalizers and the coequalizers are respectively

$$\bar{\iota} : \emptyset \to 1 \qquad \text{and} \qquad \underline{\iota} : \Sigma\mathbb{Z} \to 1$$

in which $\Sigma\mathbb{Z}$ is just the group $(\mathbb{Z}, +, 0)$ seen as a category with only one object. The functor $\bar{\iota}$ is not a lax epimorphism, while $\underline{\iota}$ is not fully faithful. Hence, both are not equivalences.

Therefore, equivalences may not be the left or the right class of a (strict) orthogonal factorization system in a 2-category with reasonable (co)limits.

Remark 2.12. The closedness under colimits has several nice consequences, we indicate three of them, which are going to be useful in the proof of Theorem 2.16 below (cf. [11]).

(1) Let the two squares in the following picture be pushouts:

$$\begin{tikzcd}
\bullet \arrow[rrr, equal] \arrow[ddd, "g"'] \arrow[dr, dotted, "\mathrm{id}"] & & & \bullet \arrow[ddd, "g"] \arrow[dl, dotted, "f"'] \\
& \bullet \arrow[r, "f"] \arrow[d, "g"'] & \bullet \arrow[d, "g'"] & \\
& \bullet \arrow[r, "f'"'] & \bullet & \\
\bullet \arrow[rrr, equal] \arrow[ur, dotted, "\mathrm{id}"] & & & \bullet \arrow[ul, dashed, "f'"']
\end{tikzcd}$$

Then, the dotted arrows form a 2-natural transformation between the corresponding origin diagrams, and the dashed arrow is the unique one induced by the universality of the inner square. From Theorem 2.10, if f is a lax epimorphism, so is f'. In conclusion, lax epimorphisms are stable under pushouts.

(2) Analogously, we see that the cointersection $e : A \to E$ of a family $e_i : A \to E_i$ of lax epimorphisms is a lax epimorphism.

$$\begin{tikzcd}
& & A \arrow[dll, equal] \arrow[dl, equal] \arrow[dd, dashed, "e"] \\
A \arrow[r, equal] \arrow[d, dotted, "\mathrm{id}"'] & A \arrow[d, dotted, "e_i"] & \\
A \arrow[r, "e_i"] \arrow[rr, "e"'] & E_i \arrow[r] & E
\end{tikzcd}$$

(3) Moreover, the closedness under colimits ensures that, given a family of morphisms $f_i : B \to C$ equalized by a lax epimorphism e, i.e., $f_i e = f_j e$ for all f_i and f_j of the family, the multiple coequalizer of all f_i, if it exists, is also a lax epimorphism.

$$\begin{tikzcd}
E \arrow[r, "f_i e"] \arrow[d, dotted, "e"'] & B \arrow[r, equal] \arrow[d, dotted, "\mathrm{id}"] & B \arrow[d, dashed, "c"] \\
A \arrow[r, "f_i"'] & B \arrow[r, "c"'] & C
\end{tikzcd}$$

Remark 2.13. Many of everyday categories are cowellpowered, that is, the family of epimorphisms with a same domain is essentially small. By contrast, in the "mother" of all 2-categories, Cat, the class of lax epimorphisms is not cowellpowered: For every cardinal n, let A_n denote the category whose objects are a_i, $i \in n$, and whose morphisms are $f_{ij} : a_i \to a_j$ with $f_{jk} f_{ij} = f_{ik}$ and $f_{ii} = 1_{a_i}$ for $i, j, k \in n$. Every inclusion functor $E_n : A_0 \to A_n$, being an equivalence, is a lax epimorphism, but the family of all these E_n is a proper class. Moreover, the family E_n, $n \in$ Card, fails to have a cointersection in Cat. However, Cat is almost cowellpowered in the sense of Definition 2.14 as shown in the next section.

Definition 2.14. Let $\mathcal{E}$ be a class of 1-cells in a 2-category $\mathbb{A}$. Given a morphism $f : A \to B$, denote by $\mathcal{E}|f$ the category whose objects are factorizations $A \xrightarrow{d} D \xrightarrow{p} B$ of f with $d \in \mathcal{E}$, and whose morphisms $u : (d, D, p) \to (e, E, m)$ are 1-cells $u : D \to E$ with $ud = e$ and $md = p$. We say that $\mathbb{A}$ *is almost cowellpowered with respect to* $\mathcal{E}$, if $\mathcal{E}|f$ has a weakly terminal set for every morphism f.

Remark 2.15. Clearly, a (2-)category with an orthogonal $(\mathcal{E}, \mathcal{M})$-factorization system is almost cowellpowered with respect to $\mathcal{E}$: the $(\mathcal{E}, \mathcal{M})$-factorization of $f : A \to B$ is indeed a terminal object of $\mathcal{E}|f$.

The closedness of lax epimorphisms under colimits allows to obtain the following:

Theorem 2.16. *Let a 2-category $\mathbb{A}$ have conical colimits and be almost cowellpowered with respect to lax epimorphisms. Then $\mathbb{A}$ has an orthogonal $(LaxEpi, LaxStrongMono)$-factorization system.*

Proof: Let $\mathcal{E}$ be the class of lax epimorphisms in $\mathbb{A}$. Given a morphism $f : A \to B$, let $\{(e_i, E_i, m_i) | i \in I\}$ be a weakly terminal object of the category $\mathcal{E}|f$; that is, for every factorization $A \xrightarrow{d} D \xrightarrow{p} B$ of f with $d \in \mathcal{E}$ there is some i and some morphism $u : (d, D, p) \to (e_i, E_i, m_i)$. Take the cointersection $e : A \to E$ of all $e_i : A \to E_i$. By Remark 2.12(2), the morphism e belongs to $\mathcal{E}$;

$$\begin{tikzcd}
A \arrow[r, "e_i"] \arrow[dr, "e"'] & E_i \arrow[r, "m_i"] \arrow[d, "t_i"] & B \\
& E \arrow[ur, "m"'] &
\end{tikzcd}$$

moreover, the cointersection gives rise to a unique $m : E \to B$ with $me = f$. Thus, (e, E, m) is clearly a weakly terminal object of $\mathcal{E}|f$.

Consider all $s : E \to E$ forming a morphism $s : (e, E, m) \to (e, E, m)$ in $\mathcal{E}|f$. Let $c : E \to C$ be the multiple coequalizer of the family of all these morphisms $s : E \to E$. By Remark 2.12(3), c is a lax epimorphism. Since 1_E is one of those morphisms s, and $ms = m$ for all them, the universality of c gives a unique $n : C \to B$ with $nc = m$. It is easy to see that

$$c : (e, E, m) \to (ce, C, n)$$

is also the coequalizer in $\mathcal{E}|f$ of all the above morphisms s. Hence, (ce, C, n) is a terminal object of $\mathcal{E}|f$ (cf. [19], Ch.V, Sec.6).

We show that $n : C \to B$ is a lax strong monomorphism. In the following diagram, let the outer square be commutative with $q \in \mathcal{E}$; form the pushout $(\bar{q}, \bar{r})$ of q along r, and let w be the unique morphism with $w\bar{q} = n$ and $w\bar{r} = s$:

$$\begin{tikzcd}
P \arrow[rr, "q"] \arrow[dd, "r"'] & & Q \arrow[dl, "\bar{r}"'] \arrow[dd, "s"] \\
& R \arrow[dr, "w"] & \\
C \arrow[ur, "\bar{q}"] \arrow[rr, "n"'] & & B
\end{tikzcd} \tag{2.0.5}$$

The closedness under colimits of lax epimorphisms ensures that $\bar{q}$ is a lax epimorphism (Remark 2.12(1)), so $(\bar{q}ce, R, w) \in \mathcal{E}|f$. Since (ce, C, n) is terminal, there is a unique $u : R \to C$ forming a morphism in $\mathcal{E}|f$ from $(\bar{q}ce, R, w)$ to (ce, C, n), and it makes $u\bar{q} : C \to C$ an endomorphism on (ce, C, n), then $u\bar{q} = 1_C$. The morphism $t = u\bar{r}$ fulfils the equalities $tq = r$ and $nt = s$. Moreover t is unique; indeed, if t' is another morphism fulfilling the same equalities, let k be the coequalizer of t and t' and let $p : K \to B$ be such that $pk = n$. Again by Remark 2.12, (kce, K, p) belongs to $\mathcal{E}|f$. Arguing as before for $\bar{q}$, we conclude that k is a split monomorphism, then $t = t'$.

Taking into account Remark 2.3, we conclude that we have indeed an orthogonal factorization system in the 2-category $\mathbb{A}$. ■

Remark 2.17. In [9], an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ which, as the $(LaxEpi, LaxStrongMono)$ one, has $\mathcal{E}$ made of lax epimorphisms and $\mathcal{M}$ made of faithful morphisms is said to be *(1,2)-proper*.

Examples 2.18. Some of the well-known orthogonal factorization systems in ordinary categories are indeed of the $(LaxEpi, LaxStrongMono)$ type for convenient 2-cells. This is the case in the 2-categories Pos and Grp. In Pos it is the usual orthogonal *(Surjections, Order-embeddings)*-factorization system. Analogously for the category Top of topological spaces and continuous maps, with 2-cells given by the pointwise specialization order, we obtain *(Surjections, Embeddings)*. For the 2-category Grp, the $(LaxEpi, LaxStrongMono)$ factorization is precisely the $(RegEpi, Mono)$ one.

Recall that, for every category with an orthogonal factorization system $(\mathcal{E},\mathcal{M})$, we have that $\mathcal{M} = \mathcal{E}^{\downarrow}$, i.e., $\mathcal{M}$ consists of all morphisms m fulfilling the diagonal fill-in property as in (2.0.2) of Definition 2.4. From the proof of Theorem 2.16, Remark 2.11 and Remark 2.15, it immediately follows that, more generally, we have the following:

Theorem 2.19. *Let $\mathcal{E}$ be a class of morphisms closed under composition and containing isomorphisms in a cocomplete category $\mathcal{A}$. Then, $(\mathcal{E},\mathcal{E}^{\downarrow})$ forms an orthogonal factorization system if and only if $\mathcal{A}$ is almost cowellpowered with respect to $\mathcal{E}$ and $\mathcal{E}$ is closed under colimits.*

3. The Lax Epi-factorization in Cat

In this section we describe the orthogonal $(LaxEpi, LaxStrongMono)$-factorization system in the 2-category Cat of small categories, functores and natural transformations. All we do applies also to the bigger universe CAT of possibly large categories.

Let us recall, by the way, two well-known orthogonal factorization systems $(\mathcal{E},\mathcal{M})$ in the category Cat:

(a) $\mathcal{E}$ consists of all functors bijective on objects and $\mathcal{M}$ consists of all fully faithful functors.
(b) $\mathcal{E}$ consists of all initial functors and $\mathcal{M}$ consists of all discrete opfibrations; analogously, for final functors and discrete fibrations [20].

It is easy to see that in both cases, (a) and (b), the system $(\mathcal{E},\mathcal{M})$ fulfils the two-dimensional aspect of the fill-in diagonal property, thus we have an orthogonal factorization system in the 2-category Cat as defined in 2.1.

We start by defining discrete splitting bifibrations. We will see that they are precisely the lax strong monomorphisms.

Notation 3.1. Recall from Remark 1.5 the definition of the category $g//P$ for a functor $P : A \to B$ and a morphism $g : b \to c$ of B. For every decomposition of g of the form $b \xrightarrow{r} Pe \xrightarrow{s} c$, we denote by $[(r,s)]$ the corresponding connected component. By composing a morphism $t : d \to b$ with $C = [(r,s)]$

we obtain $C \cdot t = [(rt, s)]$, a connected component of $tg/\!/P$. Analogously, for the composition on the right hand side: for $u : b \to c$, $u \cdot C = [(h, uk)]$.

Definition 3.2. Let $P : E \to B$ be a functor.

(a) A *P-split* consists of a factorization of an identity 1_b of the form

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[rr, bend right, "1_b"'] & Pe \arrow[r, "k"] & b \end{tikzcd}$$

with $[(1_{Pe}, hk)] = [(hk, 1_{Pe})]$.

(b) A *P-split diagram* is a rectangle

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[d, "g"'] & Pe \arrow[r, "k"] \arrow[d, phantom, "\wr"] & b \arrow[d, "g"] \\ c \arrow[r, "h'"'] & Pe' \arrow[r, "k'"'] & c \end{tikzcd} \tag{3.0.1}$$

where (h, k) and (h', k') are P-splits such that $[(h, gk)] = [(h'g, k')]$ in $g/\!/P$. The wavy line in the middle of the rectangle indicates the existence of an appropriate P-zig-zag between (h, gk) and $(h'g, k')$; that is, the existence of a finite number of morphisms h_i, k_i, f_i making the following diagram commutative:

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[d, equal] & Pe \arrow[r, "k"] \arrow[d, "Pf_0"] & b \arrow[d, "g"] \\ b \arrow[r, "h_1"] \arrow[d, equal] & Pe_1 \arrow[r, "k_1"] & c \arrow[d, equal] \\ b \arrow[r, "h_2"] \arrow[d, equal] & Pe_2 \arrow[r, "k_2"] \arrow[u, "Pf_1"'] \arrow[d, "Pf_2"] & c \arrow[d, equal] \\ b \arrow[r, "h_3"] \arrow[d, equal] & Pe_3 \arrow[r, "k_3"] \arrow[d, dotted, no head] & c \arrow[d, equal] \\ b \arrow[r, "h_n"] \arrow[d, "g"'] & Pe_n \arrow[r, "k_n"] & c \arrow[d, equal] \\ c \arrow[r, "h'"'] & Pe' \arrow[r, "k'"'] \arrow[u, "Pf_n"'] & c \end{tikzcd}$$

(c) The functor $P : E \to B$ is said to be a *discrete splitting bifibration* if, for every P-split diagram (3.0.1), there is a unique commutative rectangle in E of the form

$$\begin{tikzcd} b_0 \arrow[r, "h_0"] \arrow[d, "g_0"'] & e \arrow[r, "k_0"] & b_0 \arrow[d, "g_0"] \\ c \arrow[r, "h'_0"'] & e' \arrow[r, "k'_0"'] & c \end{tikzcd} \tag{3.0.2}$$

whose image by P is the outer rectangle of (3.0.1). (That is, $Px_0 = x$, for each letter x with x_0 appearing in (3.0.2).)

Remark 3.3. If P is a discrete splitting bifibration, then it is clear that, for every P-split of 1_b,

$$b \xrightarrow{h} Pe \xrightarrow{k} b$$

there are unique morphisms $h_0 : b_0 \to e$ and $k_0 : e \to b_0$ such that $Ph_0 = h$ and $Pk_0 = k$.

Proposition 3.4. *Every discrete splitting bifibration*

(1) is faithful,
(2) is conservative, and
(3) reflects identities.

Proof: Let $P : E \to B$ be a discrete splitting bifibration.

(1) For $a \underset{g}{\overset{f}{\rightrightarrows}} b$ with $Pf = Pg = x$, consider the following diagrams:

$$\begin{tikzcd} Pa \arrow[r, equal] \arrow[d, "x"'] & Pa \arrow[r, equal] \arrow[d, squiggly, "Pf"'] & Pa \arrow[d, "x"] \\ Pb \arrow[r, equal] & Pb \arrow[r, equal] & Pb \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, equal] \arrow[d, "f"'] & a \arrow[r, equal] & a \arrow[d, "f"] \\ b \arrow[r, equal] & b \arrow[r, equal] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, equal] \arrow[d, "g"'] & a \arrow[r, equal] & a \arrow[d, "g"] \\ b \arrow[r, equal] & b \arrow[r, equal] & b \end{tikzcd}$$

The first one is a P-split rectangle and it is the image by P of the two last ones. Then $f = g$.

(2) Let $f : a \to b$ be such that Pf is an isomorphism in B. Then we have a P-split diagram:

$$\begin{tikzcd} Pb \arrow[r, equal] \arrow[d, "(Pf)^{-1}"'] & Pb \arrow[r, equal] & Pb \arrow[d, "(Pf)^{-1}"] \\ Pa \arrow[r, equal] & Pa \arrow[r, equal] \arrow[u, squiggly, "Pf"'] & Pa \end{tikzcd}$$

Consequently, there is a unique $t_0 : b \to a$ with $Pt_0 = (Pf)^{-1}$. Since, by (1), P is faithful, t_0 is the inverse of f.

(3) Let $f : d \to e$ be such that $Pf = 1_x$. By (2), f is an isomorphism. Concerning the diagrams

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, equal] & Pe \arrow[r, equal] \arrow[d, squiggly, "P1_e"] & x \arrow[d, equal] \\ x \arrow[r, equal] & Pe \arrow[r, equal] & x \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, "f"] \arrow[d, "1_d"'] & e \arrow[r, "f^{-1}"] & d \arrow[d, "1_d"] \\ d \arrow[r, "f"] & e \arrow[r, "f^{-1}"] & d \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, "f"] \arrow[d, "f"'] & e \arrow[r, "f^{-1}"] & d \arrow[d, "f"] \\ e \arrow[r, "1_e"] & e \arrow[r, "1_e"] & e \end{tikzcd}$$

the first one is a P-split rectangle which is the image by P of the two rectangles on the right hand side. Consequently, $f = 1_d$.

■

Theorem 3.5. *For $\mathcal{E}$ the class of lax epimorphisms and $\mathcal{M}$ the class of discrete splitting bifibrations, $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorization system in* Cat *(and also in* CAT*).*

Proof: Along the proof we represent the categories by blackboard bold letters: $\mathbb{A}$, $\mathbb{B}$, etc.

(1) The factorization. Given a functor $F : \mathbb{A} \to \mathbb{B}$, we define the category $\mathbb{E}$ as follows:

ob $\mathbb{E}$: pairs (b, B) where $b \in \mathbb{B}$ and B is a connected component of the category $1_b//F$;

mor $\mathbb{E}$: all $(b, B) \xrightarrow{g} (c, C)$ with $g : b \to c$ a morphism of $\mathbb{B}$ and $g \cdot B = C \cdot g$, see Notation 3.1.

The identities and composition are obvious.

Let

$$\mathbb{E} \xrightarrow{P} \mathbb{B}$$

be the obvious projection, and define

$$\mathbb{A} \xrightarrow{E} \mathbb{E}$$

by $Ea = (Fa, C_a)$ where C_a is the connected component of $(1_{Fa}, 1_{Fa})$ in $1_{Fa}//F$, and $E(a \xrightarrow{f} a') = ((Fa, C_a) \xrightarrow{Ff} (Fa', C_{a'}))$. $\mathbb{E}$ is clearly well-defined and $F = P \cdot E$.

(2) E is a lax epimorphism. Using the characterization of lax epimorphisms given in [1], see Remark 1.5, we need to show that, for every $(b, B) \xrightarrow{g} (d, D)$ in $\mathbb{E}$, the category $g//E$ is connected. Given two factorizations (u_i, Ea_i, v_i), $i = 1, 2$, of g in $\mathbb{E}$ as in the figure

$$\begin{tikzcd} & (Fa_1, C_{a_1}) \arrow[dr, "v_1"] & \\ (b, B) \arrow[ur, "u_1"] \arrow[rr, "g"] \arrow[dr, "u_2"'] & & (d, D) \\ & (Fa_2, C_{a_2}) \arrow[ur, "v_2"'] & \end{tikzcd} \tag{3.0.3}$$

by the definition of morphisms in $\mathbb{E}$, we have the following equalities of connected components in $g//F$ (see Notation 3.1): $g \cdot B = v_1 u_1 \cdot B = v_1 \cdot C_{a_1} \cdot u_1 = v_1 \cdot [(u_1, 1_{Fa_1})] = [(u_1, v_1)]$; and, analogously, $g \cdot B = [(u_2, v_2)]$, showing that $[(u_1, v_1)] = [(u_2, v_2)]$ in $g//F$; hence, $[(u_1, v_1)] = [(u_2, v_2)]$ also in $g//E$.

(3) P is a discrete splitting bifibration.

(3a) First observe that, given two factorizations in $\mathbb{B}$ of a same morphism g of the form

$$\begin{tikzcd} & P(e, E) \arrow[dr, "s"] & \\ b \arrow[ur, "r"] \arrow[dr, "r'"'] & & c \\ & P(e', E') \arrow[ur, "s'"'] & \end{tikzcd}$$

if $\big(r, (e, E), s\big)$ and $\big(r', (e', E'), s'\big)$ belong to the same connected component of $g//P$, then also $s \cdot E \cdot r = s' \cdot E' \cdot r'$ in $g//F$. Indeed, a P-zig-zag connecting these two factorizations, as illustrated in the left hand side diagram below gives rise to an F-zig-zag connecting $s \cdot E \cdot r$ to $s' \cdot E' \cdot r'$ in $g//F$, as indicated in the right hand side diagram, where $E = [(h,a,k)]$, $E' = [(h',a',k')]$ and $E_j = [(h_j,a_j,k_j)]$:

$$\begin{tikzcd}
 & P(e,E) \arrow[d, "Pf_1"] \arrow[dr, "s"] & \\
b \arrow[ur, "r"] \arrow[r] \arrow[dr] \arrow[ddr, "r'"'] & P(e_1,E_1) \arrow[r] & c \\
 & P(e_2,E_2) \arrow[u, "Pf_2"'] \arrow[ur] \arrow[d, dotted, no head] & \\
 & P(e',E') \arrow[uur, "s'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & e \arrow[r, "h"] \arrow[d, "f_1"'] & Fa \arrow[r, "k"] \arrow[d, rightsquigarrow, no head] & e \arrow[d, "f_1"'] \arrow[dr, "s"] & \\
b \arrow[ur, "r"] \arrow[r] \arrow[dr] \arrow[ddr] & e_1 \arrow[r, "h_1"] & Fa_1 \arrow[r, "k_1"] \arrow[d, rightsquigarrow, no head] & e_1 \arrow[r] & c \\
 & e_2 \arrow[u, "f_2"] \arrow[r, "h_2"] \arrow[d, dotted, no head] & Fa_2 \arrow[r, "k_2"] \arrow[d, dotted, no head] & e_2 \arrow[u, "f_2"] \arrow[ur] \arrow[d, dotted, no head] & \\
 & e' \arrow[r, "h'"'] & Fa' \arrow[r, "k'"'] & e' \arrow[uur, "s'"'] &
\end{tikzcd}$$

(3b) Let

$$\begin{tikzcd}
b \arrow[r, "u_1"] \arrow[d, "g"'] & P(d,D) \arrow[r, "v_1"] \arrow[d, rightsquigarrow, no head] & b \arrow[d, "g"] \\
c \arrow[r, "u_2"'] & P(e,E) \arrow[r, "v_2"'] & c
\end{tikzcd} \tag{3.0.4}$$

be a P-split diagram with $D = [(h_1,a_1,k_1)]$ and $E = [(h_2,a_2,k_2)]$. Let B and C be the connected components of $1_b//F$ and $1_c//F$ given, respectively, by

$$B = v_1 \cdot D \cdot u_1 = [(h_1u_1, a_1, v_1k_1)] \quad \text{and} \quad C = v_2 \cdot C \cdot u_2 = [(h_2u_2, a_2, v_2k_2)].$$

By (3a), since $[(1_d, u_1v_1)] = [(u_1v_1, 1_d)]$ in $u_1v_1//P$, we have that $u_1v_1 \cdot D = D \cdot u_1v_1$. Then $u_1B = u_1v_1Du_1 = Du_1v_1u_1 = Du_1$ and $Bv_1 = v_1Du_1v_1 = v_1u_1v_1D = v_1D$, showing that $(b,B) \xrightarrow{u_1} (d,D)$ and $(d,D) \xrightarrow{v_1} (b,B)$ are morphisms in $\mathbb{E}$. And B is unique, because, if B' is a connected component of $1_b//F$ such that $u_1B' = Du_1$ and $v_1D = B'v_1$, then $B' = v_1u_1B' = v_1Du_1 = B$. Analogously for $c \xrightarrow{u_2} P(e,E) \xrightarrow{v_2} c$.

It remains to show that $g : (b,B) \to (c,C)$ is a morphism of $\mathbb{E}$. By (3a), the P-split diagram (3.0.4) gives rise to the following F-split diagram:

$$\begin{tikzcd}
b \arrow[r, "u_1"] \arrow[d, "g"'] & d \arrow[r, "h_1"] & Fa_1 \arrow[r, "k_1"] \arrow[d, rightsquigarrow, no head] & d \arrow[r, "v_1"] & b \arrow[d, "g"] \\
c \arrow[r, "u_2"'] & e \arrow[r, "h_2"'] & Fa_2 \arrow[r, "k_2"'] & e \arrow[r, "v_2"'] & c
\end{tikzcd}$$

That is,

$$gv_1Du_1 = v_2Eu_2g \quad \text{in} \quad g/\!/F.$$

Hence, by definition of B and C,

$$gB = Cg,$$

showing that g is a morphism in $\mathbb{E}$. Since (b,B) and (c,C) are unique, g is clearly unique too. In conclusion, we have a unique diagram of morphisms of $\mathbb{E}$ of the form

$$\begin{tikzcd}
(b,B) \arrow[r, "u_1"] \arrow[d, "g"'] & (d,D) \arrow[r, "v_1"] & (b,B) \arrow[d, "g"] \\
(c,C) \arrow[r, "u_2"'] & (e,E) \arrow[r, "v_2"'] & (c,C)
\end{tikzcd}$$

whose image by P is the rectangle of (3.0.4).

(4) $(\mathcal{E},\mathcal{M})$ fulfils the diagonal fill-in property. Let

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "Q"] \arrow[d, "G"'] & \mathbb{B} \arrow[d, "H"] \\
\mathbb{C} \arrow[r, "M"'] & \mathbb{D}
\end{tikzcd}$$

be a commutative diagram where Q is a lax epimorphism and M is a discrete splitting bifibration.

(4a) We define $T : \mathbb{B} \to \mathbb{C}$ as follows:

Given $b \in \mathbb{B}$, since Q is a lax epimorphism, the category $b/\!/Q$ is connected. Let B be the unique connected component of $1_b/\!/Q$, and let (h,a,k) be a representative of B. It is a Q-split, since $(1_{Qa}, hk)$ and $(hk, 1_{Qa})$ belong to the same connected component of $hk/\!/Q$. Hence,

$$Hb \xrightarrow{Hh} MGa \xrightarrow{Hk} Hb$$

is an M-split in $\mathbb{D}$.

By Remark 3.3, since M is a discrete splitting bifibration, there are unique morphisms $h_0 : b_0 \to Ga$ and $k_0 : Ga \to b_0$ with $Mh_0 = Hh$ and $Mk_0 = Hk$. We put

$$Tb = b_0. \tag{3.0.5}$$

We show that b_0 does not depend on the representative of B. Indeed, for another representative (h',a',k'), we have a Q-split diagram as on the left hand side of (3.0.6); by applying H, we get the M-split diagram on the right hand side:

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[d, equal] & Qa \arrow[r, "k"] \arrow[d, squiggly, no head] & b \arrow[d, equal] \\ b \arrow[r, "h'"'] & Qa' \arrow[r, "k'"'] & b \end{tikzcd} \qquad \begin{tikzcd} Hb \arrow[r, "Hh"] \arrow[d, equal] & MGa \arrow[r, "Hk"] \arrow[d, squiggly, no head] & Hb \arrow[d, equal] \\ Hb \arrow[r, "Hh'"'] & MGa' \arrow[r, "Hk'"'] & Hb \end{tikzcd} \tag{3.0.6}$$

By hypothesis, there is a unique diagram

$$\begin{tikzcd} b_1 \arrow[r, "h_1"] \arrow[d, "s"'] & Ga \arrow[r, "k_1"] & b_1 \arrow[d, "s"] \\ b_2 \arrow[r, "h_2"'] & Ga' \arrow[r, "k_2"'] & b_2 \end{tikzcd}$$

whose image by M is the outside rectangle of the first diagram of (3.0.6). But M reflects identities, by Proposition 3.4. Then s is an identity and, taking into account the unicity of b_0 and k_0 above, it must be $b_1 = b_2 = b_0$ and $s = 1_{b_0}$.

Let

$$b \xrightarrow{g} c$$

be a morphism in $\mathbb{B}$. Since Q is a lax epimorphism, there is some Q-split diagram of the form

$$\begin{tikzcd} b \arrow[r, "h_1"] \arrow[d, "g"'] & Qa_1 \arrow[r, "k_1"] \arrow[d, squiggly, no head] & b \arrow[d, "g"] \\ c \arrow[r, "h_2"] & Qa_2 \arrow[r, "k_2"] & c \end{tikzcd} .$$

By applying H to it, we obtain an M-split diagram:

$$\begin{tikzcd} Hb \arrow[r, "Hh_1"] \arrow[d, "Hg"'] & MGa_1 \arrow[r, "Hk_1"] \arrow[d, squiggly, no head] & Hb \arrow[d, "Hg"] \\ Hc \arrow[r, "Hh_2"] & MGa_2 \arrow[r, "Hk_2"] & Hc \end{tikzcd} . \tag{3.0.7}$$

By hypothesis, there are unique morphisms

$$\begin{array}{ccccc} b_0 & \xrightarrow{\hat{h}_1} & Ga_1 & \xrightarrow{\hat{k}_1} & b \\ {\scriptstyle g_0}\downarrow & & & & \downarrow{\scriptstyle g_0} \\ c_0 & \xrightarrow{\hat{h}_2} & Ga_2 & \xrightarrow{\hat{k}_2} & c_0 \end{array} \tag{3.0.8}$$

making the diagram commutative and whose image by M is the rectangle of (3.0.7). We put

$$Tg = g_0.$$

Again, by the unicity, we know that b_0 and c_0 do not depend on the representative of $1_b//Q$ and $1_c//Q$. And the unicity of g_0 follows then from the faithfulness of M (Proposition 3.4).

T is clearly a functor, the preservation of identities and composition being obvious.

(4b) We show that T satisfies the diagonal fill-in condition.

Given $b \in \mathrm{ob}\mathbb{B}$, $MTb = Mb_0 = Hb$, by construction, and, analogously, $MTg = Hg$, for each $g \in \mathrm{mor}\mathbb{B}$.

Given $f : a \to a'$ in $\mathbb{A}$, the M-split diagram

$$\begin{array}{ccccc} HQa = MGa & = & MGa & = & MGa \\ {\scriptstyle HQf=MGf}\downarrow & & \downarrow{\scriptstyle MGf} & & \downarrow{\scriptstyle MGf} \\ HQa' = MGa' & = & MGa' & = & MGa' \end{array}$$

ensures that $TQf = Gf$.

Finally, if $T' : \mathbb{B} \to \mathbb{C}$ is another functor such that $T'Q = G$ and $MT' = H$, we show that $T = T'$. Let $g : b \to d$ be a morphism of $\mathbb{B}$. The morphism $T(b \xrightarrow{g} d) = b_0 \xrightarrow{g_0} d_0$ is the unique one making part of a commutative rectangle as in (3.0.8) whose image by M is the rectangle of the M-split diagram (3.0.7). But the image by M of the rectangle

$$\begin{array}{ccccc} T'b & \xrightarrow{T'h} & Ga & \xrightarrow{T'k} & T'b \\ {\scriptstyle T'g}\downarrow & & & & \downarrow{\scriptstyle T'g} \\ T'd & \xrightarrow[T'h']{} & Ga' & \xrightarrow[T'k']{} & T'd \end{array}$$

gives also the M-split diagram (3.0.7). Then $T'g = g_0 = Tg$. ∎

Question 3.6. Inserters in Cat are discrete splitting bifibrations (by Proposition 2.6). We don't know if the converse is true or not.

4. Lax epimorphisms in the enriched context

In this section we study lax epimorphisms in the enriched setting.

Assumption 4.1. Along the section $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is a symmetric monoidal closed category with $\mathcal{V}_0$ complete.

We denote by $\mathcal{V}$-Cat the 2-category of *small* $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

Let $\mathcal{A}$ be a small $\mathcal{V}$-category, and $\mathcal{B}$ a (possibly large) $\mathcal{V}$-category. By abuse of language, we also denote by $\mathcal{V}\text{-Cat}(\mathcal{A},\mathcal{B})$ the category of $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{V}$-natural transformations between them. Moreover, in this setting, the designation $\mathcal{V}\text{-Cat}[\mathcal{A},\mathcal{B}]$ (or just $[\mathcal{A},\mathcal{B}]$) represents the $\mathcal{V}$-category of $\mathcal{V}$-functors; thus, for any pair of $\mathcal{V}$-functors $F, G : \mathcal{A} \to \mathcal{B}$, the hom-object $\mathcal{V}\text{-Cat}[\mathcal{A},\mathcal{B}](F,G)$ is given by the end

$$\int_{A\in\mathcal{A}} \mathcal{B}(FA, GA).$$

Recall that a $\mathcal{V}$-functor $P : \mathcal{A} \to \mathcal{B}$ is *$\mathcal{V}$-fully faithful* (called just *fully faithful* in [13]) if the map $P_{A,A'} : \mathcal{A}(A,A') \to \mathcal{B}(PA,PA')$ is an isomorphism in $\mathcal{V}_0$ for all $A, A' \in \mathcal{A}$.

Let $\mathcal{I}$ be the *unit $\mathcal{V}$-category* with one object 0 and $\mathcal{I}(0,0) = I$. Given a $\mathcal{V}$-functor $P : \mathcal{A} \to \mathcal{B}$, the underlying functor of P is denoted by $P_0 = \mathcal{V}\text{-Cat}(\mathcal{I}, P) : \mathcal{A}_0 \to \mathcal{B}_0$.

In general, we use the notations of [13]; concerning limits, we denote a weighted limit over a functor $F : \mathcal{D} \to \mathcal{C}$ with respect to a weight $W : \mathcal{D} \to \mathcal{V}$ by $\lim(W,F)$ (called indexed limit and designated by $\{W,F\}$ in [13]).

Lemma 4.2. *For a $\mathcal{V}$-functor $P : \mathcal{A} \to \mathcal{B}$, consider the following conditions.*

(a) P is $\mathcal{V}$-fully faithful.
(b) P_0 is fully faithful.
(c) The functor $\mathrm{Cat}(\mathcal{C}, P_0) : \mathrm{Cat}(\mathcal{C}, \mathcal{A}_0) \to \mathrm{Cat}(\mathcal{C}, \mathcal{B}_0)$ is fully faithful for every (ordinary) category $\mathcal{C}$.
(d) The functor $\mathcal{V}\text{-Cat}(\mathcal{C}, P) : \mathcal{V}\text{-Cat}(\mathcal{C}, \mathcal{A}) \to \mathcal{V}\text{-Cat}(\mathcal{C}, \mathcal{B})$ is fully faithful for every $\mathcal{V}$-category $\mathcal{C}$.
(e) The $\mathcal{V}$-functor $\mathcal{V}\text{-Cat}[\mathcal{C}, P] : \mathcal{V}\text{-Cat}[\mathcal{C}, \mathcal{A}] \to \mathcal{V}\text{-Cat}[\mathcal{C}, \mathcal{B}]$ is $\mathcal{V}$-fully faithful for every $\mathcal{V}$-category $\mathcal{C}$.

We have that

$$\text{(a)} \Longleftrightarrow \text{(e)} \Longrightarrow \text{(d)} \Longrightarrow \text{(c)} \Longleftrightarrow \text{(b)}$$

The five conditions are equivalent whenever (i) P has a left or right $\mathcal{V}$-adjoint, or (ii) $V = \mathcal{V}_0(I,-) : \mathcal{V}_0 \to \mathsf{Set}$ is conservative.

Proof: It is well-known that (a) $\Leftrightarrow$ (b) in case we have (i) or (ii) [13, 1.3 and 1.11].

(b) $\Leftrightarrow$ (c). It is just Remark 1.2.

(a) $\Rightarrow$ (d). Given two $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{A}$, and a $\mathcal{V}$-natural transformation $\beta : PF \to PG$, we want to show that there is a unique $\mathcal{V}$-natural transformation $\alpha : F \to G$ with $P\alpha = \beta$. Since P is $\mathcal{V}$-fully faithful, $P_{A,B}$ is a $\mathcal{V}_0$-isomorphism for all $A, B \in \mathcal{A}$. We just define $\alpha : F \to G$ with each component α_C given by

$$\alpha_C \equiv \left(I \xrightarrow{\beta_C} \mathcal{B}(PFC, PGC) \xrightarrow{(P_{FC,GC})^{-1}} \mathcal{A}(FC, GC)\right).$$

Clearly $\beta_C = P\alpha_C$ for each C, and α is unique. From the $\mathcal{V}$-naturality of β and the fact that P is a $\mathcal{V}$-functor, it immediatly follows that α is $\mathcal{V}$-natural.

(d) $\Rightarrow$ (b). It follows from the fact that $P_0 = \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{I}, P)$ by definition.

(e) $\Rightarrow$ (a). Recall that there is a bijection

$$\mathcal{A} \ni A \mapsto \overline{A} \in \mathcal{V}\text{-}\mathsf{Cat}[\mathcal{I}, \mathcal{A}]$$

in which $\overline{A} : \mathcal{I} \to \mathcal{A}$ is the only $\mathcal{V}$-functor from the unit $\mathcal{V}$-category $\mathcal{I}$ to $\mathcal{A}$ such that $\overline{A}0 = A$. Moreover, for any $A, B \in \mathcal{A}$, the hom-object $\mathcal{A}(A, B)$ is the end $\int_{\mathcal{I}} \mathcal{A}(\overline{A}-, \overline{B}-)$ which gives the hom-object $\mathcal{V} - \mathsf{Cat}[\mathcal{I}, \mathcal{A}](\overline{A}, \overline{B})$. We get that, for any $\mathcal{V}$-functor $P : \mathcal{A} \to \mathcal{B}$, the morphism $P_{A,B}$ is essentially $\mathcal{V}\text{-}\mathsf{Cat}[\mathcal{I}, \mathcal{A}](\overline{A}, \overline{B})$.

Therefore $\mathcal{V}\text{-}\mathsf{Cat}[\mathcal{I}, P]$ is $\mathcal{V}$-fully faithful if and only if P is $\mathcal{V}$-fully faithful.

(a) $\Rightarrow$ (e). Given a $\mathcal{V}$-category $\mathcal{C}$ and $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{A}$, we have that

$$\mathcal{V}\text{-}\mathsf{Cat}[\mathcal{C}, P]_{F,G} : \mathcal{V}\text{-}\mathsf{Cat}[\mathcal{C}, \mathcal{A}](F, G) \to \mathcal{V}\text{-}\mathsf{Cat}[\mathcal{B}, \mathcal{A}](PF, PG)$$

is, by definition, the morphism

$$\int_{C \in \mathcal{C}} P_{(FC,GC)} : \int_{C \in \mathcal{C}} \mathcal{A}(FC, GC) \to \int_{C \in \mathcal{C}} \mathcal{B}(PFC, PGC) \qquad (4.0.1)$$

induced by the $\mathcal{V}$-natural transformation between the $\mathcal{V}$-functors $\mathcal{A}(F-,G-)$ and $\mathcal{B}(PF-,PG-)$ whose components are given by

$$P_{FA,GB} : \mathcal{A}(FA,GB) \to \mathcal{B}(PFA,PGB). \tag{4.0.2}$$

Since P is $\mathcal{V}$-fully faithful, we have that (4.0.2) is invertible and, hence, (4.0.1) is invertible. ∎

Recall that the counit of an adjunction $F_0 \dashv G_0$ between ordinary categories is invertible if and only if there is any natural isomorphism between F_0G_0 and the identity [12, Lemma 1.3].[§] From Lemma 4.2, we obtain:

Lemma 4.3. *Given a $\mathcal{V}$-adjunction $(\varepsilon,\eta) : F \dashv G : \mathcal{A} \to \mathcal{B}$, the $\mathcal{V}$-functor G is $\mathcal{V}$-fully faithful if and only if there is any (ordinary) natural isomorphism*

$$F_0G_0 \to \mathrm{id}_{\mathcal{A}_0}.$$

Proof: By Lemma 4.2, G is $\mathcal{V}$-fully faithful if and only if G_0 is fully faithful. But G_0 is fully faithful in Cat if and only if the counit ε_0 is invertible[¶], if and only if there is any (ordinary) natural isomorphism $F_0G_0 \to \mathrm{id}_{\mathcal{A}_0}$. ∎

On one hand, following Definition 1.1, a $\mathcal{V}$-functor $P : \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories is said a lax epimorphism in the 2-category $\mathcal{V}$-Cat if the (ordinary) functor

$$\mathcal{V}\text{-Cat}(P,\mathcal{C}) : \mathcal{V}\text{-Cat}(\mathcal{B},\mathcal{C}) \to \mathcal{V}\text{-Cat}(\mathcal{A},\mathcal{C})$$

is fully faithful, for all $\mathcal{V}$-categories $\mathcal{C}$. On the other hand, the notion of $\mathcal{V}$-fully faithful functor and Lemma 4.2 inspire the following definition.

Definition 4.4. A $\mathcal{V}$-functor $J : \mathcal{A} \to \mathcal{B}$ (between small $\mathcal{V}$-categories) is a *$\mathcal{V}$-lax epimorphism* if, for any $\mathcal{C}$ in $\mathcal{V}$-Cat, the $\mathcal{V}$-functor

$$\mathcal{V}\text{-Cat}[J,\mathcal{C}] : \mathcal{V}\text{-Cat}[\mathcal{B},\mathcal{C}] \to \mathcal{V}\text{-Cat}[\mathcal{A},\mathcal{C}]$$

is $\mathcal{V}$-fully faithful.

Assumption 4.5. Until now, we are assuming that $\mathcal{V}_0$, and then also the $\mathcal{V}$-category $\mathcal{V}$, is complete (Assumption 4.1). From now on, we assume furthermore that $\mathcal{V}_0$ is also cocomplete.

Theorem 4.6. *Given a $\mathcal{V}$-functor $J : \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, the following conditions are equivalent.*

§See [12, Lemma 1.3] or [17] for further results on non-canonical isomorphisms.

¶Consider the diagram (1.0.1) for the case of adjunction between ordinary categories.

(a) J is a $\mathcal{V}$-lax epimorphism.
*(b) J is a lax epimorphism in the 2-category $\mathcal{V}$-*Cat.
(c) The functor $\mathcal{V}\text{-Cat}(J,\mathcal{V}) : \mathcal{V}\text{-Cat}(\mathcal{B},\mathcal{V}) \to \mathcal{V}\text{-Cat}(\mathcal{A},\mathcal{V})$ is fully faithful.
(d) The $\mathcal{V}$-functor $\mathcal{V}\text{-Cat}[J,\mathcal{V}] : \mathcal{V}\text{-Cat}[\mathcal{B},\mathcal{V}] \to \mathcal{V}\text{-Cat}[\mathcal{A},\mathcal{V}]$ is $\mathcal{V}$-fully faithful.

(e) There is a $\mathcal{V}$-natural isomorphism $Lan_J\mathcal{B}(B,J-) \cong \mathcal{B}(B,-)$ ($\mathcal{V}$-natural in $B \in \mathcal{B}^{op}$).
(f) The $\mathcal{V}$-functor $\mathcal{V}\text{-Cat}[J,\mathcal{C}] : \mathcal{V}\text{-Cat}[\mathcal{B},\mathcal{C}] \to \mathcal{V}\text{-Cat}[\mathcal{A},\mathcal{C}]$ is $\mathcal{V}$-fully faithful for every (possibly large) $\mathcal{V}$-category $\mathcal{C}$.

Proof: (a) $\Rightarrow$ (b). It follows from the implication (a) $\Rightarrow$ (b) of Lemma 4.2. Namely, given a (small) $\mathcal{V}$-category $\mathcal{C}$, since $\mathcal{V}\text{-Cat}[J,\mathcal{C}]$ is $\mathcal{V}$-fully faithful, we get that $\mathcal{V}\text{-Cat}[J,\mathcal{C}]_0 = \mathcal{V}\text{-Cat}(J,\mathcal{C})$ is fully faithful.

(b) $\Rightarrow$ (c). Given any $\mathcal{V}$-functors $F,G : \mathcal{B} \to \mathcal{V}$, we denote by $P : \mathcal{C} \to \mathcal{V}$ the full inclusion of the (small) sub-$\mathcal{V}$-category of $\mathcal{V}$ whose objects are in the image of F or in the image of G.

It should be noted that $\mathcal{V}\text{-Cat}(J,\mathcal{C})_{F,G}$ is a bijection by hypothesis, and $\mathcal{V}\text{-Cat}(\mathcal{A},P)_{F,G}, \mathcal{V}\text{-Cat}(\mathcal{B},P)_{F,G}$ are bijections since P is $\mathcal{V}$-fully faithful. Therefore, since the diagram

$$\begin{tikzcd}
\mathcal{V}\text{-Cat}(\mathcal{B},\mathcal{C})(F,G) \arrow[r, "\mathcal{V}\text{-Cat}(J,\mathcal{C})_{F,G}"] \arrow[d, "\mathcal{V}\text{-Cat}(\mathcal{B},P)_{F,G}"'] & \mathcal{V}\text{-Cat}(\mathcal{A},\mathcal{C})(F\cdot J, G\cdot J) \arrow[d, "\mathcal{V}\text{-Cat}(\mathcal{A},P)_{F,G}"] \\
\mathcal{V}\text{-Cat}(\mathcal{B},\mathcal{V})(F,G) \arrow[r, "\mathcal{V}\text{-Cat}(J,\mathcal{V})_{F,G}"'] & \mathcal{V}\text{-Cat}(\mathcal{A},\mathcal{V})(F\cdot J, G\cdot J)
\end{tikzcd} \tag{4.0.3}$$

commutes, we conclude that $\mathcal{V}\text{-Cat}(J,\mathcal{V})_{F,G}$ is also a bijection. This proves that $\mathcal{V}\text{-Cat}(J,\mathcal{V})$ is fully faithful.

(c) $\Rightarrow$ (d). Since $\mathcal{V}$ is complete, we have that $\mathcal{V}\text{-Cat}[J,\mathcal{V}]$ has a right $\mathcal{V}$-adjoint given by the (pointwise) Kan extensions Ran_J. Therefore, assuming that $\mathcal{V}\text{-Cat}(J,\mathcal{V})$ is fully faithful, we conclude that $\mathcal{V}\text{-Cat}[J,\mathcal{V}]$ is $\mathcal{V}$-fully faithful by Lemma 4.2.

(d) $\Rightarrow$ (e). Since $\mathcal{V}$ is cocomplete, we have that $\mathrm{Lan}_J \dashv \mathcal{V}\text{-Cat}[J,\mathcal{V}]$. Therefore, assuming that $\mathcal{V}\text{-Cat}[J,\mathcal{V}]$ is $\mathcal{V}$-fully faithful, we have the $\mathcal{V}$-natural isomorphism $\epsilon : \mathrm{Lan}_J(-\cdot J) \cong \mathrm{id}_{\mathcal{V}\text{-Cat}[\mathcal{B},\mathcal{V}]}$ given by the counit.

Denoting by $\mathcal{Y}_{\mathcal{B}^{\mathrm{op}}}$ the Enriched Yoneda Embedding (see, for instance, [13, 2.4]), we have that $\epsilon^{-1} * \mathrm{id}_{\mathcal{Y}_{\mathcal{B}^{\mathrm{op}}}}$ gives an isomorphism $\mathrm{Lan}_J \mathcal{B}(B, J-) \cong \mathcal{B}(B, -)$ ($\mathcal{V}$-natural in $B \in \mathcal{B}^{\mathrm{op}}$).

(e) $\Rightarrow$ (f). Let $\mathcal{C}$ be any (possibly large) $\mathcal{V}$-category. We consider the $\mathcal{V}$-functor $\mathcal{V}\text{-}\mathsf{Cat}[J, \mathcal{C}]$ and its factorization

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Cat}[\mathcal{B},\mathcal{C}] \arrow[r, "\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]_{\mathrm{Im}}"] \arrow[rr, bend right, "\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]"'] & \mathrm{Im}(\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]) \arrow[r] & \mathcal{V}\text{-}\mathsf{Cat}[\mathcal{A},\mathcal{C}] \end{tikzcd} \tag{4.0.4}$$

into a bijective on objects $\mathcal{V}$-functor $\mathcal{V}\text{-}\mathsf{Cat}[J, \mathcal{C}]_{\mathrm{Im}}$ and the $\mathcal{V}$-full inclusion

$$\mathrm{Im}(\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]) \to \mathcal{V}\text{-}\mathsf{Cat}[\mathcal{A},\mathcal{C}]$$

of the sub-$\mathcal{V}$-category $\mathrm{Im}(\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}])$ whose objects are in the image of $\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]$. We prove below that $\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]$ is $\mathcal{V}$-fully faithful by proving that $\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}]_{\mathrm{Im}}$ is $\mathcal{V}$-fully faithful.

Given any $\mathcal{V}$-functor $G : \mathcal{A} \to \mathcal{C}$ in $\mathrm{Im}(\mathcal{V}\text{-}\mathsf{Cat}[J,\mathcal{C}])$, we have that $G = FJ$ for some $F : \mathcal{B} \to \mathcal{C}$. Since $\mathrm{Lan}_J \mathcal{B}(B, J-) \cong \mathcal{B}(B, -)$, we conclude that $\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), F\right)$ exists and, moreover, we have the isomorphisms

$$\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), F\right) \cong \lim(\mathcal{B}(B,-), F) \cong F(B) \tag{4.0.5}$$

by the (strong) Enriched Yoneda Lemma (see [13, Sections 2.4 and 4.1]).

Since $\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), F\right)$ exists, it follows as a consequence of the universal property of left Kan extensions that $\lim(\mathcal{B}(B, J-), F \cdot J)$ exists and is isomorphic to $\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), F\right)$ (see [13, Proposition 4.57]). Therefore, by (4.0.5) and by the formula for pointwise right Kan extensions (see [8, Theorem I.4.2] or, for instance, [13, Theorem 4.6]), we conclude that $\mathrm{Ran}_J(F \cdot J)$ exists and we have the isomorphism

$$\begin{aligned} \mathrm{Ran}_J(F \cdot J)B &\cong \lim(\mathcal{B}(B, J-), F \cdot J) \\ &\cong \lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), F\right) \\ &\cong \lim(\mathcal{B}(B,-), F) \\ &\cong F(B) \end{aligned}$$

$\mathcal{V}$-natural in $B \in \mathcal{B}$ and $F \in \mathcal{V}\text{-Cat}[\mathcal{B},\mathcal{C}]$.

Since we proved that $\mathrm{Ran}_J(F \cdot J)$ exists for any $G = F \circ J$ in $\mathrm{Im}(\mathcal{V}\text{-Cat}[J,\mathcal{C}])$, we conclude that $\mathcal{V}\text{-Cat}[J,\mathcal{C}]_{\mathrm{Im}}$ has a right $\mathcal{V}$-adjoint, which we may denote by Ran_J by abuse of language. Finally, by the natural isomorphism $\mathrm{Ran}_J(F \cdot J)B \cong F(B)$ above and Lemma 4.3, we conclude that $\mathcal{V}\text{-Cat}[J,\mathcal{C}]_{\mathrm{Im}}$ is $\mathcal{V}$-fully faithful.

(f) $\Rightarrow$ (a). Trivial. ∎

Remark 4.7. For $\mathcal{V}$ = Set, the equivalence (b) $\Leftrightarrow$ (c) of Theorem 4.6 was given in [1].

Remark 4.8 (Duality)**.** A morphism $J : \mathcal{A} \to \mathcal{B}$ is a lax epimorphism in $\mathcal{V}$-Cat if and only if $J^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a lax epimorphism in $\mathcal{V}$-Cat as well.

Indeed, since the 2-functor op : $\mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}^{\mathrm{co}}$ is invertible, it takes lax epimorphisms to lax epimorphisms. Thus, J is a lax epimorphism in $\mathcal{V}$-Cat if, and only if, $\mathrm{op}(J)$ is a lax epimorphism in $\mathcal{V}\text{-Cat}^{\mathrm{co}}$ which, by Remark 1.2, holds if and only if J^{op} is a lax epimorphism in $\mathcal{V}$-Cat.

Therefore, assuming that $\mathcal{V}_0$ is complete and cocomplete,

$$J \text{ is a } \mathcal{V}\text{-lax epimorphism} \quad \Leftrightarrow \quad J^{\mathrm{op}} \text{ is a } \mathcal{V}\text{-lax epimorphism}$$

by Theorem 4.6.

Recall that a $\mathcal{V}$-functor $J : \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories is *$\mathcal{V}$-dense* if and only if its density comonad $\mathrm{Lan}_J J$ is isomorphic to the identity on $\mathcal{A}$ (see [13, Theorem 5.1]). Dually, J is *$\mathcal{V}$-codense* if and only if the right Kan extension $\mathrm{Ran}_J J$ is the identity. (Several concrete examples of ($\mathcal{V}$-)codensity monads are given in [3].)

We say that J is *absolutely $\mathcal{V}$-dense* if it is $\mathcal{V}$-dense and $\mathrm{Lan}_J J$ is preserved by any $\mathcal{V}$-functor $F : \mathcal{B} \to \mathcal{V}$. Dually, we define absolutely $\mathcal{V}$-codense $\mathcal{V}$-functor.

The following characterization of lax epimorphisms as absolutely dense functors was given in [1] for $\mathcal{V}$ = Set:

Theorem 4.9. *Given a $\mathcal{V}$-functor $J : \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, the following conditions are equivalent.*

(a) J is a $\mathcal{V}$-lax epimorphism.
(b) J is absolutely $\mathcal{V}$-dense.
(c) J is absolutely $\mathcal{V}$-codense.

Proof: (a) $\Rightarrow$ (b). Assume that J is a $\mathcal{V}$-lax epimorphism. By (e) of Theorem 4.6, we have that $\mathcal{B}(B,-) \cong \mathrm{Lan}_J \mathcal{B}(B,J-)$. Hence, since $\lim(\mathcal{B}(B,-), \mathrm{id}_{\mathcal{B}}) \cong B$

exists by the (strong) Enriched Yoneda Lemma, we have that

$$\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), \mathrm{id}_{\mathcal{B}}\right)$$

exists and is isomorphic to $\lim(\mathcal{B}(B,-), \mathrm{id}_{\mathcal{B}}) \cong B$ (in which isomorphisms are always $\mathcal{V}$-natural in B).

Moreover, from the existence of $\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), \mathrm{id}_{\mathcal{B}}\right)$, we get that

$$\lim(\mathcal{B}(B, J-), J)$$

exists and is isomorphic to $\lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), \mathrm{id}_{\mathcal{B}}\right) \cong B$ (see [13, Proposition 4.57]).

Finally, then, from the formula for pointwise right Kan extensions and the above, we get the $\mathcal{V}$-natural isomorphisms (in $B \in \mathcal{B}$)

$$\begin{aligned} B &\cong \lim(\mathcal{B}(B,-), \mathrm{id}_{\mathcal{B}}) \\ &\cong \lim\left(\mathrm{Lan}_J \mathcal{B}(B, J-), \mathrm{id}_{\mathcal{B}}\right) \\ &\cong \lim(\mathcal{B}(B, J-), J) \\ &\cong \mathrm{Ran}_J J(B). \end{aligned}$$

This proves that $\mathrm{Ran}_J J$ is the identity on $\mathcal{B}$. That is to say, J is $\mathcal{V}$-codense.

Moreover, assuming that J is a $\mathcal{V}$-lax epimorphism, by Remark 4.8, J^{op} is a $\mathcal{V}$-lax epimorphism and, hence, by the proved above, J^{op} is $\mathcal{V}$-codense. Therefore J is $\mathcal{V}$-dense.

By (d) of Theorem 4.6, we have that $\mathcal{V}\text{-}\mathrm{Cat}[J, \mathcal{V}]$ is $\mathcal{V}$-fully faithful. Since $\mathcal{V}$ is cocomplete, we get that Lan_J exists and there is an isomorphism

$$\mathrm{Lan}_J(F \cdot J) \cong F,$$

$\mathcal{V}$-natural in $F \in \mathcal{V}\text{-}\mathrm{Cat}[\mathcal{B}, \mathcal{V}]$, given by the counit of $\mathrm{Lan}_J \dashv \mathcal{V}\text{-}\mathrm{Cat}[J, \mathcal{V}]$. This shows that $\mathrm{Lan}_J J$ is preserved by any $\mathcal{V}$-functor $F : \mathcal{B} \to \mathcal{V}$.

(b) $\Rightarrow$ (a). Assume that J is absolutely $\mathcal{V}$-dense. We conclude that there is a natural isomorphism $\mathrm{Lan}_J(F \cdot J) \cong F$. Therefore, by Lemma 4.2, we conclude that $\mathcal{V}\text{-}\mathrm{Cat}[J, \mathcal{V}]$ is $\mathcal{V}$-fully faithful. By Theorem 4.6, this proves that J is a $\mathcal{V}$-lax epimorphism.

(a) $\Leftrightarrow$ (c). By Remark 4.8 and by the proved above, we conclude that

J is a $\mathcal{V}$-lax epimorphism $\Leftrightarrow$ J^{op} is absolutely $\mathcal{V}$-dense $\Leftrightarrow$ J is absolutely $\mathcal{V}$-codense. ∎

Remark 4.10. Of course, density and codensity are not enough for a functor to be a lax epimorphism: for 1 the terminal object in Cat, the functor $J : 1 \sqcup 1 \to 1$ is dense and codense, but not a lax epimorphism. Moreover, $\mathrm{Ran}_J J$ (respectively, $\mathrm{Lan}_J J$) is preserved by $F : 1 \to \mathrm{Set}$ if and only if the image of F is a preterminal object, *i.e.* the terminal set 1 (respectively, a preinitial object, *i.e.* the empty set $\emptyset$); see [18, Remark 4.14] and [17, Remark 4.5].

References

[1] J. Adámek, R. el Bashir, M. Sobral, J. Velebil. On functors which are lax epimorphisms. *Theory App. Categ.* 8 (29) (2001), 509-521.

[2] J. Adámek, H. Herrlich, G. Strecker. Abstract and concrete categories - The Joy of Cats. Originally published by: John Wiley and Sons, New York, 1990. Republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1-507.

[3] J. Adámek, L. Sousa, D-ultrafilters and their monads. *Adv. Math.* 377 (2021), 107486, 41 pp.

[4] J. Adámek, L. Sousa, J. Velebil. Kan injectivity in order-enriched categories. *Math. Structures Comput. Sci.* 25 (2015), no. 1, 6-45.

[5] G.J. Bird, G.M. Kelly, A.J. Power, R.H. Street. Flexible limits for 2-categories. *J. Pure Appl. Algebra* 61 (1989) 1-27.

[6] F. Borceux. *Handbook of Categorical Algebra*. Vol. 1, Cambridge Univ. Press, 1994.

[7] A. K. Bousfield, Constructions of factorization systems in categories. *J. Pure Appl. Algebra* 9 (1976/77), no. 2, 207–220.

[8] E.J. Dubuc. *Kan extensions in Enriched Category Theory*. Lecture Notes in Mathematics, Vol. 145, Springer-Verlag 1970.

[9] M. Dupont and E.M. Vitale. Proper factorization systems in 2-categories. *J. Pure Appl. Algebra* 179 (2003), 65-86.

[10] P.J. Freyd and G.M. Kelly. Categories of continuous functors I. *J. Pure Appl. Algebra* 2 (1972), 169-191.

[11] G.B. Im, G. M. Kelly. On classes of morphisms closed under limits. *J. Korean Math. Soc.* 23 (1986), 1-18.

[12] P.T. Johnstone and I. Moerdijk. Local maps of toposes. *Proc. London Math. Soc.*, (3) 58 (1989), 281-305.

[13] G.M. Kelly. *Basic concepts of enriched category theory*. London Math. Soc. Lecture Notes Series 64, Cambridge Univ. Press, 1982, *Repr. Theory Appl. Categ.* 10 (2005), 1-136.

[14] G.M. Kelly, Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.* 39 (1989), 301-317.

[15] G.M. Kelly, R. Street, Review of the elements of 2-categories. In *Lecture Notes in Math* 420, Springer-Verlag, 1974, 75-103.

[16] S. Lack, A 2-categories companion. In *Towards Higher Categories*, 105-191, IMA Vol. Math. Appl., 152, Springer, New York, 2010.

[17] F. Lucatelli Nunes. Pseudoalgebras and non-canonical isomorphisms. *Appl. Categ. Structures* **27** (2019), no. 1, 55–63.

[18] F. Lucatelli Nunes. Semantic Factorization and Descent. arXiv: 1902.01225. 2019.

[19] S. Mac Lane. *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics, 5, Springer-Verlag, New York, 1998.

[20] R. Street, R.F.C. Walters. A comprehensive factorization of a functor. *Bull. Amer. Math. Soc.* 79 (1973), 936-941.
[21] W. Tholen. Factorizations, localizations, and the orthogonal subcategory problem. *Math. Nachr.* 114 (1983), 63-85.

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