Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 21–41

DIFFERENTIAL IDENTITIES AND POLYNOMIAL GROWTH OF THE CODIMENSIONS

CARLA RIZZO, RAFAEL BEZERRA DOS SANTOS AND ANA CRISTINA VIEIRA

Abstract: Let A be an associative algebra over a field F of characteristic zero and let L be a Lie algebra over F. If L acts on A by derivations, then such an action determines an action of its universal enveloping algebra $U(L)$ and in this case we refer to A as algebra with derivations or L-algebra.

Here we give a complete characterization of the ideal of differential identities of finite dimensional L-algebras A in case the corresponding sequence of differential codimensions $c_n^L(A)$, $n \geq 1$, is polynomially bounded. As a consequence, we also characterize L-algebras with multiplicities of the differential cocharacter bounded by a constant. Moreover, along the way we classify up to L-PI-equivalence the finite dimensional L-algebras of almost polynomial growth.

Keywords: polynomial identity, differential identity, variety of algebras, codimension growth.
Math. Subject Classification (2020): Primary 16R10, 16R50; Secondary 16W25, 16P90.

1. Introduction

This paper deals with differential identities of algebras over a field F of characteristic zero. More precisely, if A is an associative algebra over F and L is a Lie algebra acting on A by derivations, then this action can be naturally extended to an action of the universal enveloping algebra $U(L)$ of L and in this case we say that A is an algebra with derivations or an L-algebra. Then a differential identity of the L-algebra A is a polynomial in non-commuting variables $x^d = d(x)$, $d \in U(L)$, vanishing in A. Such identities have been studied in later years (see for example [6, 9, 12, 17, 19]) and they are a natural generalization of polynomial identities of algebras.

It is well-known that in the ordinary case the polynomial identities satisfied by a given associative algebra A can be measured through its sequence of codimensions $c_n(A)$, $n \geq 1$, i.e., where $c_n(A)$ is the dimension of the space

Received October 27, 2021.
C. Rizzo was supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.
A.C. Vieira was partially supported by CNPq.

P_n of multilinear polynomials in n variables modulo the polynomial identities $\mathrm{Id}(A)$ of A. Such a sequence was introduced by Regev in [16] and, in characteristic zero, gives an actual quantitative measure of the identities satisfied by a given algebra. The most important feature of the sequence of codimensions proved in [16] is that if A is an associative algebra satisfying a non trivial polynomial identity (PI-algebra), then $c_n(A)$ is exponentially bounded. Later Kemer in [11] showed that such codimensions are either polynomially bounded or grow exponentially (no intermediate growth is allowed).

In light of above, it is convenient to use the language of varieties of algebras. Recall that if $\mathcal{V} = \mathrm{var}(A)$ is the variety generated by an algebra A, then the growth of $\mathcal{V}$ is the growth of the sequence $c_n(\mathcal{V}) = c_n(A)$, $n \geq 1$. Also we say that $\mathcal{V}$ has polynomial growth if $c_n(\mathcal{V})$, $n \geq 1$, is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n(\mathcal{V})$, $n \geq 1$, is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Much effort has been put into the study of varieties of polynomial growth. In this setting a celebrated theorem of Kemer characterizes them as follows. If G is the infinite dimensional Grassmann algebra over F and UT_2 is the algebra of 2×2 upper triangular matrices over F, then a variety of algebras $\mathcal{V}$ has polynomial growth if and only if G, $UT_2 \notin \mathcal{V}$. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth. Similar results were also proved in the setting of varieties of graded algebras [5, 20] and algebras with involution [4].

Inspired by the above results it is natural to expect that a similar conclusion holds for varieties of L-algebras. In fact, in analogy with the ordinary case, one defines the sequence of differential codimensions $c_n^L(A)$, $n \geq 1$, of an L-algebra A. In case A is a finite dimensional L-algebra, Gordienko in [8] proved that $c_n^L(A)$ is exponentially bounded. As a consequence, it follows that the differential codimensions of a finite dimensional algebra are either polynomially bounded or grow exponentially.

Our purpose here is to characterize L-varieties $\mathcal{V}$, i.e., variety of algebras with derivations, having polynomial growth and we reach our goal in the setting of varieties generated by finite dimensional L-algebras A. In this setting, we prove that $\mathcal{V}$ has almost polynomial growth if and only if $UT_2, UT_2^\varepsilon \notin \mathcal{V}$, where UT_2 is the L-algebra of 2×2 upper triangular matrices over F where L acts trivially on it and UT_2^ε is the L-algebra UT_2 with $F\varepsilon$-action, where ε is the inner derivation induced by e_{11}, where e_{ij}'s denote the usual matrix units (see [6, 19]). As a consequence, there are only two varieties with derivations generated by a finite dimensional algebra with almost polynomial growth.

Similarly to the ordinary case, another two useful invariants can be attached to an algebra with derivations A: the sequence of differential cocharacter $\chi_n^L(A)$, $n \geq 1$, where $\chi_n^L(A)$ is the character of the S_n-module of multilinear differential polynomials in n variables modulo the differential identities $\mathrm{Id}^L(A)$ of A, and the differential colength sequence $l_n^L(A)$, $n \geq 1$, where $l_n^L(A)$ is the sum of the corresponding multiplicities of $\chi_n^L(A)$.

It is well-known that, in case A is a finite dimensional L-algebra, the multiplicities of the differential cocharacter are polynomially bounded (see [8]). Thus it seems interesting to characterize the differential cocharacter sequence when stronger conditions hold for the multiplicities. In this perspective, motivated by the results for ordinary algebras [14], for graded algebras [3, 15] and for algebras with involution [18, 21], we characterize the differential identities when the corresponding multiplicities are bounded by a constant. In particular we prove that the multiplicities in $\chi_n^L(A)$ are bounded by a constant if and only if differential codimensions of A grow polynomially, and, consequently, we get another characterization of L-varieties of polynomial growth. Also as a direct consequence of this results we have that $c_n^L(A)$ is polynomially bounded if and only if $l_n^L(A)$ is bounded by a constant.

We give also three others characterizations of L-varieties $\mathcal{V}$ of polynomial growth: the first one in terms of the L-exponent of $\mathcal{V}$, the second in terms of the structure of an algebra generating $\mathcal{V}$ and the last one in terms of the shape of the diagrams of the irreducible S_n-characters appearing with non-zero multiplicity in the nth differential cocharacter of $\mathcal{V}$.

2. Preliminaries

Throughout this paper F will denote a field of characteristic zero and L a Lie algebra over F. Let A be an associative algebra over F. Recall that a derivation of A is a linear map $\delta : A \to A$ such that

$$\delta(ab) = \delta(a)b + a\delta(b), \text{ for all } a, b \in A.$$

In particular, an inner derivation induced by $a \in A$ is the derivation $\mathrm{ad}_a : A \to A$ of A defined by $\mathrm{ad}_a(b) = [a, b] = ab - ba$, for all $b \in A$. The set of all derivations of A is a Lie algebra denoted by $\mathrm{Der}(A)$, and the set $\mathrm{ad}(A)$ of all inner derivations of A is a Lie subalgebra of $\mathrm{Der}(A)$.

If L acts on A by derivations, then by the Poincaré-Birkhoff-Witt Theorem, the L-action on A can be naturally extended to an $U(L)$-action, where $U(L)$ is the universal enveloping algebra of L. In this way A becomes a left $U(L)$-module and we call it algebra with derivations or L-algebra.

Given a basis $\mathcal{B} = \{h_i : i \in I\}$ of $U(L)$, we let $F\langle X|L\rangle$ be the free associative algebra over F with free formal generators $x_j^{h_i}$, $i \in I$, $j \in \mathbb{N}$. For all $h = \sum_{i\in I} \alpha_i h_i \in U(L)$, where only a finite number of $\alpha_i \in F$ are non-zero, we set $x^h := \sum_{i\in I} \alpha_i x^{h_i}$. We let $U(L)$ act on $F\langle X|L\rangle$ by setting

$$\gamma(x_{j_1}^{h_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{h_{i_n}}) = x_{j_1}^{\gamma h_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{h_{i_n}} + \dots + x_{j_1}^{h_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{\gamma h_{i_n}},$$

where $\gamma \in L$ and $x_{j_1}^{h_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{h_{i_n}} \in F\langle X|L\rangle$. In this way $F\langle X|L\rangle$ has a structure of L-algebra. We write $x_i := x_i^1$, $1 \in U(L)$, and we set $X = \{x_1, x_2, \dots\}$. Then the algebra $F\langle X|L\rangle$ is called the free associative algebra with derivations on the countable set X over F and its elements are called differential polynomials.

Let now A be an L-algebra. A polynomial $f(x_1, \dots, x_n) \in F\langle X|L\rangle$ is a differential identity of A, or an L-identity of A, if $f(a_1, \dots, a_n) = 0$ for all $a_i \in A$, and, in this case, we write $f \equiv 0$. We denote by

$$\mathrm{Id}^L(A) = \{f \in F\langle X|L\rangle : f \equiv 0 \text{ on } A\},$$

the T_L-ideal of differential identities of A, i.e., it is an ideal of $F\langle X|L\rangle$ invariant under the $U(L)$-action. In characteristic zero $\mathrm{Id}^L(A)$ is completely determined by its multilinear polynomials and for every $n \geq 1$ we denote by

$$P_n^L = \mathrm{span}\{x_{\sigma(1)}^{h_{i_1}} \dots x_{\sigma(n)}^{h_{i_n}} : \sigma \in S_n, h_i \in \mathcal{B}\}$$

the space of multilinear differential polynomials of degree n. Notice that in case $U(L)$ acts on A as a suitable finite dimensional subalgbera of the endomorphism algebra of A, then P_n^L is finite dimensional and similarly to the ordinary case we can define the following invariants.

The non-negative integer

$$c_n^L(A) = \dim_F \frac{P_n^L}{P_n^L \cap \mathrm{Id}^L(A)}, \quad n \geq 1,$$

is called the nth differential codimension of A.

Recall that the symmetric group S_n acts on the left on the space P_n^L as follows: for $\sigma \in S_n$, $\sigma(x_i^h) = x_{\sigma(i)}^h$. Since $P_n^L \cap \mathrm{Id}^L(A)$ is stable under this

S_n-action, the space

$$P_n^L(A) = \frac{P_n^L}{P_n^L \cap \mathrm{Id}^L(A)}$$

is a left S_n-module and its character, denoted by $\chi_n^L(A)$, is called the nth differential cocharacter of A. Since F is of characteristic zero, we can write

$$\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where λ is a partition of n, χ_λ is the irreducible S_n-character associated to λ and $m_\lambda \geq 0$ is the corresponding multiplicity.

Another numerical sequence that can be attached to a L-algebra A is the sequence of differential colengths. If $\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ is the nth differential cocharacter of A, then the nth differential colength of A is defined as

$$l_n^L(A) = \sum_{\lambda \vdash n} m_\lambda.$$

Let L be a Lie algebra and H be a Lie subalgebra of L. If A is an L-algebra, then by restricting the action, A can be regarded as a H-algebra. In this case we say that A is an L-algebra where L acts on it as the Lie algebra H and we restrict the T_L-ideal $\mathrm{Id}^L(A)$ to the T_H-ideal $\mathrm{Id}^H(A)$, i.e., in $\mathrm{Id}^L(A)$ we omit the differential identities $x^\gamma \equiv 0$, for all $\gamma \in L \backslash H$.

Notice that any algebra A can be regarded as L-algebra by letting L act on A trivially, i.e., L acts on A as the trivial Lie algebra and $U(L) \cong F$. Hence the theory of differential identities generalizes the ordinary theory of polynomial identities.

Recall that if A is an L-algebra, then the variety of algebras with derivations generated by A is denoted by $\mathrm{var}^L(A)$ and is called L-variety. The growth of $\mathcal{V} = \mathrm{var}^L(A)$ is the growth of the sequence $c_n^L(\mathcal{V}) = c_n^L(A)$, $n \geq 1$. We say that the L-variety $\mathcal{V}$ has polynomial growth if $c_n^L(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^L(\mathcal{V})$ is not polynomially bounded but every proper L-subvariety of $\mathcal{V}$ has polynomial growth.

We conclude this section by recalling some basic results concerning the sequence of cocharacters and colenghts which can be easily proved.

Remark 1. *Let A and B be two L-algebras such that*

$$\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \quad \text{and} \quad \chi_n^L(B) = \sum_{\lambda \vdash n} m'_\lambda \chi_\lambda.$$

1. *If $B \in var^L(A)$, then $m'_\lambda \leq m_\lambda$, for all $\lambda \vdash n$, and $l_n^L(B) \leq l_n^L(A)$, for all $n \leq 1$.*
2. *The direct sum $A \oplus B$ is also an L-algebra with L-action induced by the L-action by derivations defined on A and B. Moreover, if*
$$\chi_n^L(A \oplus B) = \sum_{\lambda \vdash n} \bar{m}_\lambda \chi_\lambda$$
is the decomposition of the nth differential cocharacter of $A \oplus B$, then $\bar{m}_\lambda \leq m_\lambda + m'_\lambda$, for all $\lambda \vdash n$.

3. Finite dimensional L-algebras and varieties of polynomial growth

In this section we shall characterize finite dimensional algebras with derivations generating varieties of polynomial growth.

We start by recalling some results on the structure of finite dimensional algebras with derivations.

Let L be a Lie algebra over F and A an L-algebra over F. An ideal (subalgebra) I of A is an L-ideal (subalgebra) if it is an ideal (subalgebra) such that $I^L \subseteq I$, where I^L denotes the set of all $h(a)$, for all $a \in I$ and $h \in U(L)$. The algebra A is L-simple if $A^2 \neq \{0\}$ and A has no non-trivial L-ideals.

Let A be a finite dimensional L-algebra over F. By Wedderburn-Malcev Theorem for associative algebras (see [7, Theorem 3.4.3]), we can write A as a direct sum of vector spaces
$$A = A_{ss} + J,$$
where A_{ss} is a maximal semisimple subalgebra of A and $J = J(A)$ is the Jacobson radical of A. Notice that J is always an L-ideal of A (see [10, Theorem 4.2]), but it may not exist an L-invariant Wedderburn-Malcev decomposition, i.e., it may happen that $A_{ss}^L \not\subseteq A_{ss}$, for every maximal semisimple subalgebra A_{ss} of A. However, we remark that the Wedderburn-Malcev decomposition always exists in case L is a semisimple Lie algebra (see [9, Theorem 4]). In what follows we give an example of an L-algebra that has no L-invariant Wedderburn-Malcev decomposition.

Example 2. *Let UT_2^δ be the L-algebra of 2×2 upper triangular matrices where L acts on it as the 1-dimensional Lie algebra spanned by the inner derivation $\delta = \mathrm{ad}_{e_{12}}$. Suppose that $UT_2^\delta = A_{ss} + J$ for some maximal* *semisimple subalgebra* A_{ss} *of* UT_2^δ *such that* $A_{ss}^L \subseteq A_{ss}$. *Since* $\delta = \mathrm{ad}_{e_{12}}$, $J = span_F\{e_{12}\}$ *and* $A_{ss}^L \subseteq A_{ss}$, *it follows that* $[A_{ss}, J] \subseteq A_{ss}$. *On the other hand , since* J *is an ideal of* UT_2^δ, $[A_{ss}, J] \subseteq J$. *Thus it follows that* $[A_{ss}, J] \subseteq A_{ss} \cap J = \{0\}$. *But since* $J = span_F\{e_{12}\}$, *we have that* $[J, J] = \{0\}$. *This says that the center of* UT_2^δ *contains* J, *that is no true. Therefore* $A_{ss}^L \nsubseteq A_{ss}$, *for all maximal semisimple subalgebra* A_{ss} *of* UT_2^δ. *Thus* UT_2^δ *has no* L*-invariant Wedderburn-Malcev decomposition.*

In [8], Gordienko proved that if A is a finite dimensional L-algebra, then the sequence of differential codimensions $c_n^L(A)$ is exponentially bounded. Moreover, the author proved that the limit $\lim_{n\to\infty} \sqrt[n]{c_n^L(A)}$ exists and is a non-negative integer. In this case, this limit is called the L-exponent of A and is denoted by $\exp^L(A)$. In particular, we have the following.

Theorem 3. [8, Theorems 1 and 3] *Let* A *be a finite dimensional algebra over a field of characteristic zero. If* L *is a Lie algebra acting on* A *by derivations, then there exist constants* $C_1, C_2, r_1, r_2, C_1 > 0$, *and a positive integer* d *such that*

$$C_1 n^{r_1} d^n \leq c_n^L(A) \leq C_2 n^{r_2} d^n, \text{ for all } n \in \mathbb{N}.$$

Hence, $\exp^L(A) = d$. *Moreover, If* $J = J(A)$ *is the Jacobson radical of* A *and* $A/J = \overline{A}_1 \oplus \cdots \oplus \overline{A}_m$, *then*

$$d = \max\{\dim(\overline{A}_{i_1} \oplus \overline{A}_{i_2} \oplus \cdots \oplus \overline{A}_{i_k}) : A_{i_1}^L A^+ A_{i_2}^L A^+ \cdots A^+ A_{i_k}^L \neq \{0\}\},$$

where $i_r \neq i_s, 1 \leq r, s \leq n$, $A^+ = A + F \cdot 1$ *and* A_i *is a subalgebra of* A *(not necessary* L*-invariant) such that* $\pi(A_i) = \overline{A}_i$, *for all* $1 \leq i \leq m$, *where* $\pi : A \to A/J$ *is the natural projection.*

As a consequence we have the following corollaries.

Corollary 4. *If* A *is a finite dimensional* L*-algebra, the sequence* $c_n^L(A)$, $n \geq 1$, *either is polynomially bounded or growth exponentially.*

Corollary 5. *Let* A *be a finite dimensional algebra over a field of characteristic zero. Then the sequence* $c_n^L(A)$, $n \geq 1$, *is polynomially bounded if and only if* $\exp^L(A) \leq 1$.

As in the ordinary case, we have the following remark (see [7, Lemma 7.2.1]).

Remark 6. *If A and B are L-algebras, then $A\oplus B$ has an induced structure of L-algebra and $c_n^L(A\oplus B)\le c_n^L(A)+c_n^L(B)$. As a consequence,* $\exp^L(A\oplus B)=\max\{\exp^L(A),\exp^L(B)\}$.

Recall that if A and B are two L-algebras, then we say that A is T_L-equivalent to B, and we write $A\sim_{T_L} B$, if $\mathrm{Id}^L(A)=\mathrm{Id}^L(B)$. Notice that given an L-algebra A, A is T_L-equivalent to B if and only if $\mathrm{var}^L(A)=\mathrm{var}^L(B)$.

Lemma 7. *Let F be a field of characteristic zero, $\bar{F}$ the algebraic closure of F and A a finite dimensional L-algebra over $\bar{F}$, where L is a Lie algebra over $\bar{F}$ acting on A by derivations. Suppose that* $\dim_{\bar{F}} A/J(A)\le 1$. *Then $A\sim_{T_L} B$ for some finite dimensional L-algebra B over F with* $\dim_F B/J(B)\le 1$.

Proof: Since $\dim_{\bar{F}} A/J(A)\le 1$, it follows that either $A\cong \bar{F}+J(A)$ or $A=J(A)$ is a nilpotent algebra. Now we take an arbitrary basis $\{v_1,\dots,v_p\}$ of $J(A)$ over $\bar{F}$ and we let B be the L-algebra over F generated by $\mathcal{B}=\{1_{\bar{F}},v_1,\dots,v_p\}$ or $\mathcal{B}=\{v_1,\dots,v_p\}$ according as $A\cong\bar{F}+J(A)$ or $A=J(A)$, respectively.

Since A is finite dimensional over $\bar{F}$ and $J(A)$ is a nilpotent L-ideal of A, B is finite dimensional over F. Therefore B is a finite dimensional L-algebra and $\dim_F B/J(B)=\dim_{\bar{F}} A/J(A)\le 1$. Now notice that, as F-algebras, $\mathrm{Id}^L(A)\subseteq \mathrm{Id}^L(B)$. On the other hand, if f is a multilinear differential identity of B then f vanishes on $\mathcal{B}$. But $\mathcal{B}$ is a basis of A over $\bar{F}$. Hence $\mathrm{Id}^L(B)\subseteq \mathrm{Id}^L(A)$ and $A\sim_{T_L} B$. ■

Next theorem gives a characterization of L-varieties of polynomial growth in terms of the structure of the generating algebra.

Theorem 8. *Let L be a Lie algebra over a field F of characteristic zero and A be a finite dimensional L-algebra over F. Then $c_n^L(A)$, $n\ge 1$, is polynomially bounded if and only if $A\sim_{T_L} B_1\oplus\cdots\oplus B_m$, where $B_1,\dots,B_m$ are finite dimensional L-algebras over F such that* $\dim B_i/J(B_i)\le 1$, *for all $1\le i\le m$.*

Proof: Suppose first that $A\sim_{T_L} B$ where $B=B_1\oplus\cdots\oplus B_m$, with $B_1,\dots,B_m$ finite dimensional L-algebras over F such that $\dim B_i/J(B_i)\le 1$, for all $1\le i\le m$. Then, by Theorem 3, $c_n^L(B_i)$ is polynomially bounded, for all $1\le i\le m$, and $c_n^L(A)=c_n^L(B)\le c_n^L(B_1)+\cdots+c_n^L(B_m)$. Thus $c_n^L(A)$ is polynomially bounded.

Conversely, suppose that $c_n^L(A)$ is polynomially bounded. Assume first that F is algebraically closed. Let $A = A_{ss} + J$ where A_{ss} is a semisimple subalgebra and $J = J(A)$ is the Jacobson radical of A. By Theorem 3, it follows that $A_{ss} = A_1 \oplus \cdots \oplus A_l$ with $A_1 \cong \cdots \cong A_l \cong F$ and $A_i^L A^+ A_k^L = \{0\}$, for all $1 \leq i, k \leq l$, $i \neq k$.

Set $B_1 = A_1 + J, \ldots, B_l = A_l + J$. Since $A_i^L \subseteq A_i + J$ for all $1 \leq i \leq l$, and J is an L-ideal of A, B_i is an L-subalgebra of A, for all $1 \leq i \leq l$. We claim that

$$\mathrm{Id}^L(A) = \mathrm{Id}^L(B_1) \cap \cdots \cap \mathrm{Id}^L(B_l) \cap \mathrm{Id}^L(J).$$

Clearly $\mathrm{Id}^L(A) \subseteq \mathrm{Id}^L(B_1) \cap \cdots \cap \mathrm{Id}^L(B_l) \cap \mathrm{Id}^L(J)$. Now let $f \in \mathrm{Id}^L(B_1) \cap \cdots \cap \mathrm{Id}^L(B_l) \cap \mathrm{Id}^L(J)$ and suppose that f is not a differential identity of A. We may clearly assume that f is multilinear. Moreover, by choosing a basis of A as the union of a basis of A_{ss} and a basis of J, it is enough to evaluate f on this basis. Let $u_1, \ldots, u_t$ be elements of this basis such that $f(u_1, \ldots, u_t) \neq 0$. Since $f \in \mathrm{Id}^L(J)$, at least one element, say u_s, does not belong to J. Then $u_s \in B_r$, for some r. Recalling that $A_i^L A_k^L \subseteq A_i^L A^+ A_k^L = \{0\}$, for all $i \neq k$, we must have that $u_1, \ldots, u_t \in A_r \cup J$. Thus $u_1, \ldots, u_t \in A_r + J = B_r$ and this contradicts the fact that f is a differential identity of B_r. This prove the claim. The proof is completed by noticing that $\mathrm{Id}^L(B_1 \oplus \cdots \oplus B_l \oplus J) = \mathrm{Id}^L(B_1) \cap \cdots \cap \mathrm{Id}^L(B_l) \cap \mathrm{Id}^L(J)$ and $\dim B_i/J(B_i) = 1$, for all $1 \leq i \leq l$.

In case F is arbitrary, we consider the algebra $\bar{A} = A \otimes_F \bar{F}$, where $\bar{F}$ is the algebraic closure of F and $\bar{A} = A \otimes_F \bar{F}$ is endowed with the induced L-action $(a \otimes \alpha)^\gamma = a^\gamma \otimes \alpha$, for $\gamma \in L$, $a \in A$ and $\alpha \in \bar{F}$. Clearly, over F, $\mathrm{var}^L(A) = \mathrm{var}^L(\bar{A})$. Moreover, the differential codimensions of A over F coincide with the differential codimensions of $\bar{A}$ over $\bar{F}$. Thus, by hypothesis, it follows that the differential codimensions of $\bar{A}$ are polynomially bounded. But then, by the first part of the proof, $\bar{A} \sim_{T_L} B_1 \oplus \cdots \oplus B_m$ where $B_1, \ldots, B_m$ are finite dimensional L-algebras over $\bar{F}$ such that $\dim_{\bar{F}} B_i/J(B_i) \leq 1$, for all $1 \leq i \leq m$. By Lemma 7 there exist finite dimensional L-algebras $C_1, \ldots, C_m$ over F such that, for all i, $C_i \sim_{T_L} B_i$ and $\dim_F C_i/J(C_i) \leq 1$. It follows that $\mathrm{Id}^L(A) = \mathrm{Id}^L(\bar{A}) = \mathrm{Id}^L(B_1 \oplus \cdots \oplus B_m) = \mathrm{Id}^L(C_1 \oplus \cdots \oplus C_m)$ and we are done. ■

4. L-varieties of almost polynomial growth

In this section we shall introduce two finite dimensional L-algebras generating L-varieties of almost polynomial growth and we shall prove that are the only two finite dimensional algebras with derivations generating varieties of almost polynomial growth.

Let L be any Lie algebra over F and let consider the algebra UT_2 of 2×2 upper triangular matrices over F where L acts trivially on it. Since $x^\gamma \equiv 0$, for all $\gamma \in L$, is a differential identity of UT_2, we are dealing with ordinary identities. Thus by [11] we have the following.

Theorem 9. *The algebra UT_2 generates a variety of algebras with derivations of almost polynomial growth.*

Recall also that in the ordinary case by [13], [11] and by the proof of Lemma 3.5 in [1], we have the following results.

Theorem 10.

1. $\mathrm{Id}^L(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle_{T_L}$.
2. $c_n^L(UT_2) = 2^{n-1}(n-2) + 2$.
3. *If $\chi_n^L(UT_2) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ is the nth differential cocharacter of UT_2, then*

$$m_\lambda = \begin{cases} 1, & \text{if } \lambda = (n) \\ q+1, & \text{if } \lambda = (p+q, p) \text{ or } \lambda = (p+q, p, 1) \\ 0 & \text{in all other cases} \end{cases} .$$

As a consequence it follows that

$$l_n^L(UT_2) = \frac{n^2 - n + 2}{2}. \tag{1}$$

In [6], Giambruno and Rizzo introduced another algebra with derivations generating a variety of almost polynomial growth. They considered UT_2^ε to be the L-algebra UT_2 where L acts on it as the 1-dimensional Lie algebra spanned by the inner derivation $\varepsilon = \mathrm{ad}_{e_{11}}$, where e_{ij}'s are the usual matrix units. The authors proved the following.

Theorem 11. [6, Theorems 5 and 12]

1. $\mathrm{Id}^L(UT_2^\varepsilon) = \langle x_1^{\varepsilon^2} - x_1^\varepsilon,\ x_1^\varepsilon x_2^\varepsilon,\ [x_1, x_2]^\varepsilon - [x_1, x_2] \rangle_{T_L}$;
2. $c_n^L(UT_2^\varepsilon) = 2^{n-1}n - 1$.

3. *If* $\chi_n^L(UT_2^\varepsilon) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda$ *is the nth differential cocharacter of* UT_2^ε, *then*

$$m_\lambda = \begin{cases} n+1, & \text{if } \lambda = (n) \\ 2(q+1), & \text{if } \lambda = (p+q,p) \\ q+1, & \text{if } \lambda = (p+q,p,1) \\ 0 & \text{in all other cases} \end{cases}.$$

Theorem 12. [6, Theorem 15] *The algebra* UT_2^ε *generates a variety of algebras with derivations of almost polynomial growth.*

Notice that as a consequence of Theorem 11 we get that

$$l_n^L(UT_2^\varepsilon) = \begin{cases} \frac{3n^2-2n+4}{4}, & \text{if } n \text{ is even} \\ \frac{3n^2-2n+3}{4}, & \text{if } n \text{ is odd} \end{cases}. \tag{2}$$

Denote by UT_2^η the L-algebra UT_2 where L acts on it as the 1-dimensional Lie algebra spanned by a derivation η of UT_2. Notice that since any derivation of UT_2 is inner (see [2]), $\mathrm{Der}(UT_2)$ is the 2-dimensional metabelian Lie algebra with basis $\{\varepsilon, \delta\}$, where $\varepsilon = \mathrm{ad}_{e_{11}}$ and $\delta = \mathrm{ad}_{e_{12}}$. Then $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$. In [19] the author proved the following.

Theorem 13. [19, Theorem 12] *Let* $\eta = \alpha\varepsilon + \beta\delta \in \mathrm{Der}(UT_2)$ *such that* $\alpha, \beta \in F$ *are not both zero.*

1. *If* $\alpha \neq 0$, *then* $\mathrm{Id}^L(UT_2^\eta) = \langle x_1^{\eta^2} - \alpha x_1^\eta,\ x_1^\eta x_2^\eta,\ [x_1,x_2]^\eta - \alpha[x_1,x_2]\rangle_{T_L}$. *Otherwise,* $\mathrm{Id}^L(UT_2^\eta) = \langle x_1^{\eta^2},\ x_1^\eta x_2^\eta,\ [x_1,x_2]^\eta\rangle_{T_L}$.
2. $c_n^L(UT_2^\eta) = 2^{n-1}n + 1$.

As a consequence we get the following corollary.

Corollary 14. *If* $\alpha \neq 0$, *then* $\mathrm{Id}^L(UT_2^\eta) = \mathrm{Id}^L(UT_2^\varepsilon)$. *Otherwise,* $\mathrm{Id}^L(UT_2^\eta) \subseteq \mathrm{Id}^L(UT_2)$.

A basic result we shall need in what follows is the following.

Theorem 15. [10, Theorem 4.3] *Let* $A = A_{ss} + J$ *be an algebra over* F, *where* A_{ss} *is a semisimple subalgebra and* $J = J(A)$ *is its Jacobson radical. Suppose* γ *is a derivation of* A. *Then* $\gamma = \mathrm{ad}_a + \gamma'$ *where* $a \in A$ *and* γ' *is a derivation of* A *such that* $\gamma'(A_{ss}) = 0$.

Next lemmas will be useful to establish a structural result about L-varieties of polynomial growth.

Lemma 16. *Let $A = A_1 \oplus A_2 + J$ be a finite dimensional L-algebra over an algebraically closed field F of characteristic zero, where $A_1 \cong A_2 \cong F$. If $A_1^L A_2^L \neq \{0\}$, then $A_1^L A_2 \neq \{0\}$ and $A_1 A_2^L \neq \{0\}$.*

Proof: Notice first that $A_1^L A_2 = \{0\}$ if and only if $A_1 A_2^L = \{0\}$. In fact, if $A_1^L A_2 = \{0\}$, then for all $\gamma \in L$ we have that $\gamma(e_1)e_2 = 0$, where $e_i \in A_i$ with $e_i^2 = e_i$, $i = 1, 2$. By definition of derivation it follows that $e_1\gamma(e_2) = -\gamma(e_1)e_2 = 0$. Thus by Poincaré-Birkhoff-Witt Theorem we have that $A_1 A_2^L = \{0\}$. Similarly it can be proved the converse.

Let now assume by contradiction that $A_1^L A_2 = A_1 A_2^L = \{0\}$. Then, since by hypothesis $A_1^L A_2^L \neq \{0\}$, there exist $h_1, h_2 \in U(L)$ such that $h_1, h_2 \notin \mathrm{span}_F\{1_{U(L)}\}$ and $h_1(e_1)h_2(e_2) \neq 0$. Without loss generality we may assume that $h_1 = \gamma_1 \ldots \gamma_r$, $\gamma_i \in L$, $i = 1, \ldots, r$, $r \geq 1$. We proceed by induction on r.

If $r = 1$, then by definition of derivation $\gamma_1(e_1 h_2(e_2)) = \gamma_1(e_1)h_2(e_2) + e_1\gamma_1(h_2(e_2))$. Since $A_1 A_2^L = \{0\}$, it follows that $h_1(e_1)h_2(e_2) = 0$, a contradiction. So let suppose that $r > 1$. We set $I = \{i_1, \ldots, i_p\}$ and $K = \{k_1, \ldots, k_t\}$ to be two disjoint subsets of $\{1, \ldots, r\}$ such that $i_1 < \cdots < i_p$, $p < r$, and $k_1 < \cdots < k_t$, $t < r$, respectively. If we denote $h_I = \gamma_{i_1} \cdots \gamma_{i_p}$ and $h_K = \gamma_{k_1} \cdots \gamma_{k_t}$, then by definition of derivation, we have that

$$h_1(e_1 h_2(e_2)) = h_1(e_1)h_2(e_2) + e_1 h_1(h_2(e_2)) + \sum_{I,K} h_I(e_1)h_K(h_2(e_2)).$$

Thus since $A_1 A_2^L = \{0\}$, it turns out that

$$h_1(e_1)h_2(e_2) = -\sum_{I,K} c_I(e_1)c_K(h_2(e_2)).$$

Hence by the induction hypothesis we have that $h_1(e_1)h_2(e_2) = \{0\}$, a contradiction and the claim is proved. ∎

Lemma 17. *Let $A = A_1 \oplus \cdots \oplus A_m + J$ be a finite dimensional L-algebra over an algebraically closed field F of characteristic zero, where $A_1 \cong \cdots \cong A_m \cong F$. If there exist $1 \leq i, k \leq m$, $i \neq k$, such that $A_i^L A^+ A_k^L \neq \{0\}$, then $UT_2^\eta \in \mathrm{var}^L(A)$, where $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$.*

Proof: Suppose that there exist $1 \leq i, k \leq m$, $i \neq k$, such that $A_i^L A^+ A_k^L \neq \{0\}$. Then we assume, as we may, that $i = 1$ and $k = 2$. Moreover, since $A' = A_1 \oplus A_2 + J$ is an L-subalgebra of A, we shall prove that $UT_2^\eta \in$

$\mathrm{var}^L(A') \subseteq \mathrm{var}^L(A)$, where $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$. Hence without loss of generality we may suppose that $A = A_1 \oplus A_2 + J$ and $A_1^L A^+ A_2^L \neq \{0\}$.

Let us assume first that $A_1^L A_2^L = \{0\}$. We claim that there exist elements $j \in J$, $e_i \in A_i$ with $e_i^2 = e_i$, $i = 1, 2$, such that $e_1 j e_2 \neq 0$. In fact, assume by contradiction that $A_1 J A_2 = \{0\}$. Since by hypothesis $A_1^L A^+ A_2^L \neq \{0\}$ and $A_1^L A_2^L = \{0\}$, it follows that $A_1^L J A_2^L \neq \{0\}$. Thus there exist $j \in J$, $e_i \in A_i$ with $e_i^2 = e_i$, $i = 1, 2$, such that $h_1(e_1) j h_2(e_2) \neq 0$, for some $h_1, h_2 \in U(L)$. Since $A_1 J A_2 = \{0\}$, then $h_1 \notin \mathrm{span}_F\{1_{U(L)}\}$ or $h_2 \notin \mathrm{span}_F\{1_{U(L)}\}$.

Suppose first that $h_1, h_2 \notin \mathrm{span}_F\{1_{U(L)}\}$, the other cases will follow analogously. Notice that we may assume $h_1, h_2 \in L$. In fact, if for example $h_1 \notin L$, then without loss generality we may suppose that $h_1 = \gamma_1 \dots \gamma_r$, $\gamma_i \in L$, $i = 1, \dots, r$, $r \geq 1$. Hence by definition of derivation and the idempotence of e_1 we have that

$$h_1(e_1) = h_1(e_1)e_1 + e_1 h_1(e_1) + \sum_{I,K} h_I(e_1) h_K(e_1), \tag{3}$$

where $I = \{i_1, \dots, i_p\}$ and $K = \{k_1, \dots, k_t\}$ are two disjoint subsets of $\{1, \dots, r\}$ such that $i_1 < \cdots < i_p$, $p < r$, and $k_1 < \cdots < k_t$, $t < r$, respectively, $h_I = \gamma_{i_1} \cdots \gamma_{i_p}$ and $h_K = \gamma_{k_1} \cdots \gamma_{k_t}$. Since $A_1 J A_2 = \{0\}$ and $h_1(e_1) j h_2(e_2) \neq 0$, it turn out that there exist $I = \{i_1, \dots, i_p\}$ and $K = \{k_1, \dots, k_t\}$ such that $h_I(e_1) h_K(e_2) j h_2(e_2) \neq 0$. Thus if $p = 1$ or $t = 1$, we have done. If $p, t > 1$, then we iterate the previous argument. Therefore it follows that $h_1 \in L$. Analogously it can be proved that $h_2 \in L$.

Since $h_1(e_1) j h_2(e_2) \neq 0$ with $h_1, h_2 \in L$, then by definition of derivation and the idempotence of e_1 and e_2, we have that

$$e_1 h_1(e_1) j e_2 h_2(e_2) + e_1 h_1(e_1) j h_2(e_2) e_2 + h_1(e_1) e_1 j e_2 h_2(e_2) + h_1(e_1) e_1 j h_2(e_2) e_2 \neq 0,$$

a contradiction since $h_i(e_i) \in J$, $i = 1, 2$. Hence $A_1 J A_2 \neq \{0\}$.

Let $j \in J$ be such that $e_1 j e_2 \neq 0$ and let B the algebra generated by $h(e_1)$, $h(e_2)$, $h(e_1 j e_2)$, for all $h \in U(L)$. Then B is an L-subalgebra of A and if I is the ideal generated by $h_1(e_1)$, $h_2(e_2)$, $h_3(e_1 j e_2) - e_1 h_3(e_1 j e_2) e_2$, $e_1 j e_2 - h_3(e_1 j e_2)$, for all $h_i \in U(L)$, $h_i \notin \mathrm{span}_F\{1_{U(L)}\}$, $1 \leq i \leq 3$, such that $h_3(e_1 j e_2) \neq 0$ and $h_i(e_i) \neq h_3(e_1 j e_2)$, $i = 1, 2$, then I is an L-ideal of B. Thus the algebra $\bar{B} = B/I$ is an L-algebra.

Let $\phi : \bar{B} \to UT_2^\eta$, where $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$, the linear map defined by $\phi(e_1 + I) = e_{11}$, $\phi(e_2 + I) = e_{22}$, $\phi(e_1 j e_2 + I) = e_{12}$. Then for appropriate choose of $\alpha, \beta \in F$, ϕ is an isomorphism of L-algebras and since $\bar{B} \in \mathrm{var}^L(A)$, the claim is proved.

Assume now that $A_1^L A_2^L \neq \{0\}$. By Lemma 16, it follows that $A_1^L A_2$, $A_1 A_2^L \neq \{0\}$. So, let $h \in U(L)$, $h \notin \mathrm{span}_F\{1_{U(L)}\}$, such that $h(e_1)e_2 \neq 0$ and let consider the algebra C generated by $\bar{h}(e_1)$, $\bar{h}(e_2)$, for all $\bar{h} \in U(L)$. Then C is an L-subalgebra of A. If we consider the ideal I generated by $h_1(e_1)$, $h_2(e_2)$, $h_3(h(e_1)e_2) - e_1 h_3(h(e_1)e_2)e_2$, $h(e_1)e_2 - h_3(h(e_1)e_2)$, for all $h_i \in U(L)$, $h_i \notin \mathrm{span}_F\{1_{U(L)}\}$, $1 \leq i \leq 3$, such that $h_3(h(e_1)e_2) \neq 0$ and for all $\bar{h} \in U(L)$, $h_i(e_i) \neq e_1\bar{h}(h(e_1)e_2)e_2$, $i = 1, 2$, then I is an L-ideal of C. Thus the algebra $\bar{C} = C/I$ is an L-algebra such that $\bar{C} \in \mathrm{var}^L(A)$. Thus in order to complete the proof is enough to show that $\bar{C}$ is isomorphic as L-algebra to UT_2^η, where $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$.

Notice that we may assume that $e_1 h(e_1)e_2 = h(e_1)e_2$. In fact, without loss generality we may suppose that $h_1 = \gamma_1 \ldots \gamma_r$, $\gamma_i \in L$, $i = 1, \ldots, r$, $r \geq 1$. We proceed by induction on r.

If $r = 1$, the claim ready follows from the definition of derivation. So let $r > 1$. Thus by (3), if $\sum_{I,K} h_I(e_1)h_K(e_2) = 0$, it follows that $h(e_1)e_2 = e_1h(e_1)e_2$. Otherwise there exist $\gamma_{l_1}, \ldots, \gamma_{l_s} \in L$, $s < r$, such that $\gamma_{l_1} \ldots \gamma_{l_s}(e_1)e_2 \neq 0$. By the inductive hypothesis $e_1\gamma_{l_1} \ldots \gamma_{l_s}(e_1)e_2 = \gamma_{l_1} \ldots \gamma_{l_s}(e_1)e_2$ and we have done.

Let $\psi : \bar{C} \to UT_2^\eta$ be the linear map defined by $\psi(e_1+I) = e_{11}$, $\psi(e_2+I) = e_{22}$, $\psi(h(e_1)e_2 + I) = e_{12}$, where $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$. Then for an opportune choose of $\alpha, \beta \in F$, ψ is an isomorphism of L-algebras and the proof is complete. ■

Next theorem gives us a characterization of the varieties of algebras with derivations of polynomial growth in terms of the L-algebras UT_2 and UT_2^ε.

Theorem 18. *Let L be a Lie algebra over a field F of characteristic zero and let A be a finite dimensional L-algebra over F. Then the sequence $c_n^L(A)$, $n \geq 1$, is polynomially bounded if and only if $UT_2, UT_2^\varepsilon \notin \mathrm{var}^L(A)$.*

Proof: First suppose that $c_n^L(A)$ is polynomially bounded. Since, by Theorem 11, UT_2 and UT_2^ε generate L-varieties of exponential growth, we have $UT_2, UT_2^\varepsilon \notin \mathrm{var}^L(A)$.

Now assume $UT_2, UT_2^\varepsilon \notin \mathrm{var}^L(A)$. Using an argument analogous to that used in the ordinary case (see [7, Theorem 4.1.9]), we can prove that the differential codimensions do not change upon extension of the base field and so we may assume F is algebraically closed.

By Wedderburn-Malcev theorem for ordinary algebras,

$$A = A_1 \oplus \cdots \oplus A_m + J,$$

where $J = J(A)$ is the Jacobson radical of A and A_i is a simple algebra, for all $1 \leq i \leq m$. Notice that since $UT_2 \notin \mathrm{var}^L(A)$, it follows that $A_i \cong F$, for all $1 \leq i \leq m$. Then, in order to finish the proof, by Theorem 3, it is enough to guarantee that $A_i^L A^+ A_k^L = \{0\}$, for all $1 \leq i, k \leq m$, $i \neq k$. Suppose to the contrary that there exist $1 \leq i, k \leq m$, $i \neq k$, such that $A_i^L A^+ A_k^L \neq \{0\}$. By Lemma 17, $UT_2^\eta \in \mathrm{var}^L(A)$, where $\eta = \alpha\varepsilon + \beta\delta$, for some $\alpha, \beta \in F$. Thus by Corollary 14 we reach a contradiction and the theorem is proved. ∎

As a consequence we have the following corollary.

Corollary 19. *UT_2 and UT_2^ε are the only finite dimensional algebras with derivations generating L-varieties of almost polynomial growth.*

5. Differential cocharacter of varieties of polynomial growth

In this section we give other characterizations of L-varieties $\mathcal{V}$ of polynomial growth through the behaviour of their sequences of cocharacters.

Theorem 20. *Let L be a Lie algebra over a field F of characteristic zero and let A be a finite dimensional L-algebra over F. Then $c_n^L(A)$, $n \geq 1$, is polynomially bounded if and only if there exists a constant q such that*

$$\chi_n^L(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda \chi_\lambda$$

and $J(A)^q = \{0\}$.

Proof: Notice that the decomposition of $\chi_n^L(A)$ into irreducible characters does not change under extensions of the base field. This fact can be proved following word by word the proof for the ordinary case (see for example [7, Theorem 4.1.9]). Also if $\bar{F}$ is the algebraic closure of F and $J(A)^q = \{0\}$, then $J(A \otimes_F \bar{F})^q = \{0\}$. Therefore we may assume, without loss of generality, that F is an algebraically closed field.

Suppose $c_n^L(A)$, $n \geq 1$, is polynomially bounded and let λ be a partition of n such that $|\lambda| - \lambda_1 \geq q$ and $m_\lambda \neq 0$. Then there exist $f \in P_n^L$ and a tableau T_λ such that $e_{T_\lambda} f \notin \mathrm{Id}^L(A)$. Let $\lambda' = (\lambda_1', \ldots, \lambda_t')$ be the conjugate partition of λ. Then $e_{T_\lambda} f$ is a linear combination of polynomials each alternating on t disjoint sets of $\lambda_1', \ldots, \lambda_t'$ variables, respectively. We shall reach a contradiction by proving that these polynomials g vanish in A.

Let $A = A_1 \oplus \cdots \oplus A_m + J$, where $A_1, \ldots, A_m$ are simple algebras and $J = J(A)$ is the Jacobson radical, then by Theorem 3, $\dim A_i = 1$ and $A_i^L A^+ A_k^L = \{0\}$ for all $1 \leq i, k \leq m$, $i \neq k$. In order to get a non-zero value of g we must replace at most one variable with elements of a single component, say, A_i, and the others variables with elements of J. Since $\dim A_i = 1$, we can substitute at most one element of A_i in each alternating set. Thus we can substitute at most λ_1 elements from A_i. It follows that to get a non-zero value, we must substitute at least $|\lambda| - \lambda_1$ elements from J, but $|\lambda| - \lambda_1 \geq q$, and we reach a contradiction since $J^q = \{0\}$.

Suppose now that $\chi_n^L(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda \chi_\lambda$. Since $|\lambda| - \lambda_1 < q$, then $\lambda_1 > n - q$ and by the hook formula $d_\lambda = \deg \chi_\lambda = \dfrac{n!}{(n-q)!} \leq n^q$. Thus by [8, Theorem 5], it follows that

$$c_n^L(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda d_\lambda \leq n^q \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < q}} m_\lambda \leq C n^{q'}$$

for some constant C, q', and the claim is proved. ■

Next theorem give us a characterization of finite dimensional L-algebras with multiplicities of the nth differential cocharacter bounded by a constant.

We start by proving the following result.

Lemma 21. *Let A be a finite dimensional L-algebra over an algebraically closed field such that $\dim_F A/J(A) \leq 1$. Then there exists a constant C such that in $\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$*

$$m_\lambda \leq C,$$

for all $n \geq 1$.

Proof: Let $A = A_{ss} + J$ where A_{ss} is a semisimple subalgebra and $J = J(A)$ is the Jacobson radical of A. Since $\dim_F A/J(A) \leq 1$, it follows that either $A_{ss} \cong F$ or $A = J(A)$ is a nilpotent algebra. Clearly if A is a nilpotent algebra, we have nothing to prove. So let assume that $A_{ss} \cong F$.

Let now $d = \dim_F A$ and $\{a_1, \ldots, a_d\}$ be a basis of A where $a_1 \in A_{ss}$ and $a_2, \ldots, a_d \in J$. If q is the smallest positive integer such that $J^q = \{0\}$, we shall prove that $m_\lambda \leq dq^{dq}$, for all $\lambda \vdash n$.

Notice that since $\dim_F A/J(A) \leq 1$, by Theorem 3, $c_n^L(A)$ is polynomially bounded. Then, by Theorem 20, we get that $m_\lambda \neq 0$ if and only if $h(\lambda) \leq q$, where $h(\lambda)$ is the height of the partition $\lambda \vdash n$, i.e., the number of the rows of λ.

So let $\lambda \vdash n$ be a partition such that $h(\lambda) \leq q$. Consider the Young tableau T_λ of shape λ and the corresponding minimal essential idempotent e_{T_λ}. Then it is well-known that

$$e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (\operatorname{sgn} \tau) \sigma \tau$$

where R_{T_λ} and C_{T_λ} are the subgroups of row and column permutations of T_λ, respectively.

For all $1 \leq j \leq q$, let X_j be the set of variables whose indices lies in the ith row of T_λ. Thus, for any $f \in P_n^L$, the polynomial $e_{T_\lambda} f$ is symmetric in each set $X_1, \ldots, X_q$ and its variables are partitioned into the disjoint union of q subsets $X_1 \cup \cdots \cup X_q$. Notice that X_j may be empty if $h(\lambda) < j < q$.

Notice that for any $\rho \in S_n$, $\rho e_{T_\lambda} \neq 0$. Then it follows that, if $e_{T_\lambda} f \neq 0$, where f is a multilinear differential polynomial, then $e_{T_\lambda} f$ and $\rho e_{T_\lambda} f$ generate the same irreducible S_n-module.

Let $f_1, \ldots, f_m$ be a multilinear differential polynomial generating in $P_n^L(A)$ different isomorphic irreducible S_n-modules corresponding to the same partition. By the above, one can choose $\rho_1, \ldots, \rho_m \in S_n$ and a decomposition $X = X_1 \cup \cdots \cup X_q$ such that $\rho_1 f_1, \ldots, \rho_m f_m$ are simultaneously symmetric on X_j, $1 \leq j \leq q$. Thus without loss of generality, we may assume that $f_1, \ldots, f_m$ satisfy this condition.

Now assume by contradiction that $m = m_\lambda > C = dq^{dq}$ and prove that A satisfies a differential identity of the type

$$f = \beta_1 f_1 + \cdots + \beta_m f_m, \tag{4}$$

where $\beta_1, \ldots, \beta_m \in F$ are not all zero. Then we shall reach a contradiction since this will say that $f_1, \ldots, f_m$ are linearly dependent modulo $\mathrm{Id}^L(A)$.

Since f is multilinear, in order to verify that $f \equiv 0$, it is sufficient to verify that f has only zero value on elements of a basis of A. First let us define substitutions of special kind. Consider the non-negative integers $\alpha_1^j, \ldots, \alpha_d^j$ such that, for all $1 \leq j \leq q$,

$$\sum_{i=1}^{d} \alpha_i^j = |X_j|.$$

We say that an evaluation φ has type

$$(\alpha_1^j, \ldots, \alpha_d^j)$$

for $1 \leq j \leq q$, if we replace the variables from X in the following way: for fixed j, $1 \leq j \leq q$, we evaluate the first α_1^j variables from X_j by elements a_1, the next α_2^j in a_2, and so on up to the last α_d^j variables from X_j in a_d.

In order to get a non-zero value of f in (4), any substitution should satisfy the following condition

$$\sum_{i=2}^{d} \alpha_i^j \leq q-1,$$

for all $1 \leq j \leq q$, since $J^q = \{0\}$. Moreover, by definition we have also the following restriction

$$\alpha_1^j = |X_j| - \sum_{i=2}^{d} \alpha_i^j,$$

for all $1 \leq j \leq q$. Then for any $1 \leq j \leq q$, the number of distinct d-tuples $(\alpha_1^j, \ldots, \alpha_d^j)$ is less than q^d. Thus it follows that the total number N of distinct type of special substitutions is less than q^{dq}.

Let us consider all these N distinct special substitutions $\varphi_1, \ldots, \varphi_N$ and construct the matrix (b_{ij}), where, for all $1 \leq i \leq m$ and $1 \leq j \leq N$,

$$\varphi_j(f_i) = b_{ij}.$$

This matrix has m rows and N columns of elements of A. Since $m > dq^{dq} > dN$, the rows of (b_{ij}) are linearly dependent. Thus there exist $\beta_1, \ldots \beta_m \in F$ not all zero such that

$$\sum_{i=1}^{m} \beta_i b_{ij} = 0,$$

for all $1 \leq j \leq N$, i.e., the polynomial $f = \sum_{i=1}^{m} \beta_i f_i$ is zero under all special substitution $\varphi_1, \ldots, \varphi_N$. Therefore it is enough to show that this implies that $f \in \mathrm{Id}^L(A)$.

To this end, let ψ be any substitution by elements of the basis $\{a_1, \ldots, a_d\}$. Let l_1^j be the number of variables in X_j mapped by ψ in a_1; let l_2^j the number of variables in X_j mapped by ψ in a_2, and so on. Since f is simultaneously symmetric on $X_1, \ldots, X_q$, we get that, for all $\rho \in S_n$ such that $\rho(X_1) = X_1, \ldots \rho(X_q) = X_q$,

$$\psi(f) = \psi(\rho f) = (\psi\rho) f.$$

In particular, we can choose $\rho \in S_n$ such that $\psi\rho$ is the special substitution of the type $(l_1^j, \ldots, l_d^j)$. By the above, $\psi(f) = (\psi\rho)f = 0$ and $f \in \mathrm{Id}^L(A)$, a contradiction. This complete the proof. ∎

Theorem 22. *Let L be a Lie algebra over a field F of characteristic zero, A be a finite dimensional L-algebra over F and $\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ be its nth differential cocharacter. Then $c_n^L(A)$ is polynomially bounded if and only if there exists a constant C such that, for all $\lambda \vdash n$, the inequality*

$$m_\lambda \leq C$$

holds.

Proof: Since the decomposition of $\chi_n^L(A)$ into irreducible characters do not change by extending the base field, we may assume that F is algebraically closed. Suppose now that $c_n^L(A)$, $n \geq 1$, is polynomially bounded, then the proof follows by Theorem 8, Remark 1 and Lemma 21.

Conversely, assume by contradiction that $c_n^L(A)$ is not polynomially bounded. Then by Theorem 18 $UT_2 \in \mathrm{var}^L(A)$ or $UT_2^\varepsilon \in \mathrm{var}^L(A)$. But by Theorems 10 and 11 the multiplicities in $\chi_n^L(UT_2)$ and in $\chi_n^L(UT_2^\varepsilon)$ are not bounded by a constant. Thus by Remark 1 we get a contradiction and the theorem is proved. ∎

As an important consequence, we shall prove the following corollary that relates the growth of the differential codimension sequence of a finite dimensional L-algebra A with its differential colength.

Corollary 23. *Let L be a Lie algebra over a field F of characteristic zero and let A be a finite dimensional L-algebra over F. Then $c_n^L(A)$, $n \geq 1$, is polynomially bounded if and only if $l_n^L(A) \leq k$, for some constant k and for all $n \geq 1$.*

Proof: Assume first that $c_n^L(A)$, $n \geq 1$, is polynomially bounded. By the previous theorem all non-zero multiplicities m_λ in

$$\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

are bounded by a constant C. On the other hand, by Theorem 20, $n-\lambda_1 \leq q$ as soon as $m_\lambda \neq 0$, where q is such that $J(A)^q = \{0\}$. Since the number of partition $n-\lambda_1 \leq q$ is less than q^2, we get

$$l_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \leq C \cdot q^2 = \text{const.}$$

Conversely, suppose that $l_n^L(A)$ is bounded by a constant. By using Remark 1 and equations (1) and (2) we get that $UT_2, UT_2^\varepsilon \notin \mathrm{var}^L(A)$. Thus by Theorem 18, $c_n^L(A)$ must be polynomially bounded. ∎

We now collect the results obtained in the following theorem which gives a complete characterization of the L-variety generated by a finite dimensional algebras with derivations of polynomial growth.

Theorem 24. *Let L be a Lie algebra over a field F of characteristic zero and let A be a finite dimensional L-algebra over F. Then the following conditions are equivalent:*

1. $c_n^L(A) \leq \alpha n^t$, *for some constant α, t, for all $n \geq 1$;*
2. $\exp^L(A) \leq 1$*;*
3. $UT_2, UT_2^\varepsilon \notin \mathrm{var}^L(A)$*;*
4. $A \sim_{T_L} B_1 \oplus \cdots \oplus B_m$*, with $B_1, \ldots, B_m$ finite dimensional L-algebras over F such that* $\dim B_i/J(B_i) \leq 1$*, for all $1 \leq i \leq m$;*
5. *There exists a constant q such that*
$$\chi_n^L(A) = \sum_{\substack{\lambda \vdash n \\ |\lambda|-\lambda_1 < q}} m_\lambda \chi_\lambda$$
and $J(A)^q = 0$;
6. *There exists a constant C such that in $\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$*
$$m_\lambda \leq C,$$
for all $n \geq 1$;
7. *there exists a constant k such that $l_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \leq k$, for all $n \geq 1$.*

References

[1] F. Benanti, A. Giambruno, I. Sviridova, *Asymptotics for the multiplicities in the cocharacters of some PI-algebras*, Proc. Amer. Math. Soc. **132**(2004), no. 3, 669–679.

[2] S.P. Coelho, C. Polcino Milies, *Derivations of upper triangular matrix rings*, Linear Algebra Appl. **187** (1993), 263–267.

[3] A. Cirrito, A. Giambruno, *Group graded algebras and multiplicities bounded by a constant*, J. Pure Appl. Algebra **217** (2013), no. 2, 259–268.

[4] A. Giambruno, S. Mishchenko, *On star-varieties with almost polynomial growth*, Algebra Colloq. 8 (2001) 33–42.

[5] A. Giambruno, S. Mishchenko, M. Zaicev, *Polynomial identities on superalgebras and almost polynomial growth*, Comm. Algebra 29 **(9)** (2001) 3787–3800 (Special issue dedicated to Alexei Ivanovich Kostrikin).

[6] A. Giambruno, C. Rizzo, *Differential identities, 2×2 upper triangular matrices and varieties of almost polynomial growth*, J. Pure Appl. Algebra **223** (2019), n. 4, 1710–1727.

[7] A. Giambruno, M. Zaicev, Polynomial identities and asymptotic methods, Math. Surv. Monogr., AMS, Providence, RI, **122** (2005).

[8] A.S. Gordienko, *Asymptotics of H-identities for associative algebras with an H-invariant radical*, J. Algebra **393** (2013), 92–101.

[9] A.S. Gordienko, M.V. Kochetov, *Derivations, gradings, actions of algebraic groups, and codimension growth of polynomial identities*, Algebr. Represent. Th. **17** (2014), no. 2, 539–563.

[10] G. Hochschild. *Semi-simple algebras and generalized derivations*, Amer. J. Math. **64** (1942), no. 1, 677–694.

[11] A.R. Kemer, *Varieties of finite rank*, Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk. **2** (1979), p. 73 (in Russian).

[12] V.K. Kharchenko, *Differential identities of semiprime rings*, Algebra Logic **18** (1979), 86–119.

[13] J.N. Malcev, *A basis for the identities of the algebra of upper triangular matrices*, Algebra i Logika **10** (1971), 393–400.

[14] S. P. Mischenko, A. Regev, M. V. Zaicev, *A characterization of P.I. algebras with bounded multiplicities of the cocharacters*, J. Algebra **219** (1999), no. 1, 356–368.

[15] F. C. Otera, *Finitely generated PI-superalgebras with bounded multiplicities of the cocharacters*, Comm. Algebra **33** (2005), no. 6, 1693–1707.

[16] A. Regev, *Existence of identities in $A \otimes B$*, Israel J. Math. **11** (1972), 131–152.

[17] C. Rizzo, *The Grassmann algebra and its differential identities*, Algebr. Represent. Theory **23** (2020), n. 1, 125–134.

[18] C. Rizzo, *Algebras with involution and multiplicities bounded by a constant*, J. Algebra **547** (2020), 556–578.

[19] C. Rizzo, *Growth of differential identities*, Polynomial Identities in Algebras, Springer INdAM Series **44** (2021), 383–400.

[20] A. Valenti, *Group graded algebras and almost polynomial growth*, J. Algebra **334** (2011) 247–254.

[21] A.C. Vieira, *Finitely generated algebras with involution and multiplicities bounded by a constant*, J. Algebra **422** (2015), 487–503.

Carla Rizzo

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

E-mail address: carlarizzo@mat.uc.pt

Rafael Bezerra dos Santos
Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Avenida Antônio Carlos 6627, 31123-970 Belo Horizonte, Brazil
E-mail address: rafaelsantos23@ufmg.br

Ana Cristina Vieira
Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Avenida Antônio Carlos 6627, 31123-970 Belo Horizonte, Brazil
E-mail address: anacris@ufmg.br