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FULLY NONLINEAR FREE TRANSMISSION PROBLEMS

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ABSTRACT: We examine a free transmission problem driven by fully nonlinear elliptic operators. Since the transmission interface is determined endogeneously, our analysis regards this object as a free boundary. By relating our problem with a pair of viscosity inequalities, we resort to approximation methods and prove that strong solutions are of class $C^{1,\text{Log-Lip}}$, locally. Then additional, natural conditions allow us to prove quadratic growth of the solutions away from branch points.

KEYWORDS: Free transmission problems; fully nonlinear operators; regularity of the solutions; quadratic growth at branch points.
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1. Introduction

We consider a fully nonlinear transmission problem of the form

$$\begin{aligned} F_1(D^2u) = 1 \quad &\text{in} \quad \Omega^+(u) \cap B_1 \\ F_2(D^2u) = 1 \quad &\text{in} \quad \Omega^-(u) \cap B_1 \end{aligned} \tag{1}$$

where $F_1, F_2 : S(d) \to \mathbb{R}$ are (λ, Λ)-elliptic operators, $\Omega^-(u) := \{x \in B_1 \mid u < 0\}$, and $\Omega^+(u) := \{x \in B_1 \mid u > 0\}$. We examine local regularity of the strong solutions to (1) and study their growth regime at branch points along the free boundary. In particular, we prove that solutions are locally $C^{1,\text{Log-Lip}}$-regular, with estimates. Under further conditions, we prove quadratic growth of the solutions away from branch points.

We emphasize the operators F_1 and F_2 are comparable only locally in $S(d)$. As a consequence, (1) differs from the usual obstacle problem. We also stress that discontinuities arise in the *diffusion process*, as the solutions change sign.

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Transmission problems comprise a class of models aimed at examining a variety of phenomena in heterogeneous media. The problems under the scope of this formulation include thermal and electromagnetic conductivity, composite materials and, more generally, diffusion processes driven by discontinuous laws.

Given a domain $\Omega \subset \mathbb{R}^d$, it gets split into mutually disjoint subregions $\Omega_i \Subset \Omega$ for $i = 1, \ldots, k$, for some $k \in \mathbb{N}$. The mechanism governing the problem is smooth within Ω_i, though possibly discontinuous across $\partial\Omega_i$. A paramount, subtle, aspect of the theory concerns the nature of those subregions.

In fact, $(\Omega_i)_{i=1}^k$ and the geometry of $\partial\Omega_i$ can be prescribed a priori. The alternative is $(\Omega_i)_{i=1}^k$ to be determined endogenously. The latter setting frames the theory in the context of free boundary problems. Both cases differ substantially; as a consequence, their analysis also requires distinct techniques. The vast majority of former studies on transmission problems presupposes *a priori knowledge* of the subregions Ω_i and their geometric properties. A work-horse of the theory is the divergence-form equation

$$\operatorname{div}\left(a(x)Du\right) = 0 \quad \text{in} \quad \Omega, \tag{2}$$

where the matrix-valued function $a(\cdot)$ is defined as

$$a(x) := a_i \ \text{ for } \ x \in \Omega_i,$$

for constant matrices a_i and $i = 1, \ldots, k$. Though smooth within every Ω_i, the coefficients of (2) can be discontinuous across $\partial\Omega_i$. This feature introduces genuine difficulties in the analysis.

The first formulation of a transmission problem appeared in [31] and addressed a topic in the realm of material sciences. More precisely, in elasticity theory. In that paper, the author proves the uniqueness of solutions for a model consisting of two subregions, which are known a priori. The existence of solutions is discussed in [31], although not examined in detail. See also [30].

The formulation in [31] motivated a number of subsequent studies [5, 12, 13, 14, 27, 19, 29, 33, 35, 34]. Those papers present a wide range of developments, including the existence of solutions for the transmission problem in [31] and the analysis of several variants. We refer the reader to [6] for an account of those results and methods.

Estimates and regularity results for the solutions to transmission problems have also been treated in the literature. In [26] the authors consider a bounded subdomain $\Omega \subset \mathbb{R}^d$, which is split into a finite number of subregions $\Omega_1, \Omega_2, \ldots, \Omega_k$, known a priori. The motivation is in the study of composite materials with closely spaced inclusions. A two-dimensional example is the cross-section of a fiber-reinforced material. The mathematical analysis amounts to the study of

$$\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial}{\partial x_j}u\right) = f \quad \text{in} \quad \Omega, \tag{3}$$

where

$$a(x) := \begin{cases} a_i(x) & \text{for} \quad x \in \Omega_i, \quad i = 1, \ldots, k \\ a_{k+1}(x) & \text{for} \quad x \in \Omega \setminus \cup_{i=1}^k \Omega_i. \end{cases}$$

Under natural assumptions on the data, the authors establish local Hölder continuity for the gradient of the solutions. From the applied perspective, the gradient encodes information on the stress of the material. Their findings imply bounds on the gradient *independent of the location of the fibers.* C.f. [3].

The vectorial setting is the subject of [25]. In that paper the authors extend the developments reported in [26] to systems. Moreover, they produce bounds for higher derivatives of the solutions.

In [1] the authors consider a domain with two subregions, which are supposed to be ε-apart, for some $\varepsilon > 0$. Within each subregion, the divergence-form equation is governed by a constant coefficient k. Conversely, outside those subregions the diffusivity coefficient is equal to 1. By setting $k = +\infty$, the authors frame the problem in the context of perfect conductivity.

In this setting, it is known that bounds on the gradient deteriorate as the two subregions approach each other. The analysis in [1] yields blow up rates for the gradient bounds as $\varepsilon \to 0$. The case of multiple inclusions, covering perfect conductivity and insulation ($k = 0$), is discussed in [2]. See also [7].

Recently, new developments have been obtained under minimal regularity requirements for the transmission interfaces. In [11] the authors consider a smooth and bounded domain Ω and fix $\Omega_1 \Subset \Omega$, defining $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. They suppose the boundary of the transmission interface $\partial\Omega_1$ to be of class $C^{1,\alpha}$ and prove existence, uniqueness and $C^{1,\alpha}(\overline{\Omega}_i)$-regularity of the solutions to the problem, for $i = 1, 2$. Their argument imports regularity from flat problems, through a new stability result; see [11, Theorem 4.2].

Another class of transmission problems concerns models where the subregions of interest are determined endogenously. For example, given $\Omega \subset \mathbb{R}^d$, one would consider

$$\Omega_1 : \{x \in \Omega \mid u(x) < 0\} \qquad \text{and} \qquad \Omega_2 : \{x \in \Omega \mid u(x) > 0\},$$

where $u : \Omega \to \mathbb{R}$ solves a prescribed equation. Roughly speaking, knowledge of the solution is required to determine the subregions of the domain where distinct diffusion phenomena take place. In this context, a further structure arises, namely, the free interface, or free boundary. Here, in addition to the analysis of the solutions, properties of the free boundary are also of central interest.

In [15] the authors examine the (p, q)-functional

$$J_{p,q}(v) := \int_\Omega \left(|Dv^+|^p + |Dv^-|^q\right) \mathrm{d}x. \tag{4}$$

Heuristically, in the region where v is positive, the functional satisfies a p-growth regime, whereas in the region where v is negative, a q-growth regime is in force. Though the functional in (4) is discontinuous, and distinct regimes drive the process in distinct subregions of the domain, such discontinuities depend on the sign of the argument v.

Among the findings in [15], we mention the existence of minimizers for $J_{p,q}$ and their Hölder continuity. In addition, the authors prove the free boundary is of class $C^{1,\alpha}$ with respect to the p-harmonic measure $\Delta_p u^+$. Finally, they conclude that $\Delta_p u^+$ is supported on a set of σ-finite $(d-1)$-dimensional Hausdorff measure.

We remark that (1) relates to (and is very much inspired by) the literature on the fully nonlinear obstacle problem. To the best of our knowledge, the fully nonlinear obstacle problem was first examined in [23]; see also [24]. In [16] the authors introduced the so-called unconstrained free boundary problems, which are driven by fully nonlinear operators. This class of models accommodate a variety of distinct formulations, unifying the approach to regularity of the solutions and the analysis of the free boundary; see also [17, 22].

Still in the context of the obstacle problem governed by fully nonlinear operators, we mention the issue of non-transversality; see for instance [20, 21]. By examining the intersection of the fixed and the free boundaries, one can extract geometrical information on the latter. In addition, the techniques involved in this analysis have important spillovers on the classification of blow-up limits.

In the present paper we study $W^{2,d}$-strong solutions to (1). We start by noticing that a $W^{2,d}$-solution to (1) is a continuous viscosity solution to

$$\min\left(F_1(D^2u), F_2(D^2u)\right) \leq 1 \qquad \text{in } B_1 \tag{5}$$

and

$$\max\left(F_1(D^2u), F_2(D^2u)\right) \geq -1 \qquad \text{in } B_1. \tag{6}$$

We emphasize the importance of (5)-(6), even in the context of $W^{2,d}$-strong solutions. Although it is clear that a solution $u \in W^{2,d}_{\text{loc}}(B_1)$ is, for instance, α-Hölder continuous for every $\alpha \in (0,1)$, this inclusion does not ensure *universal estimates* for u. Because our analysis relies on the precompactness of strong solutions to (1), this type of estimates is critical. By noticing that strong solutions to (1) are viscosity solutions to (5)-(6), we access a maximum principle, stability results, and a Krylov-Safonov theory.

By requiring F_1 and F_2 to satisfy a near convexity condition we prove that solutions to (1) are locally of class $C^{1,\text{Log-Lip}}$, with the appropriate estimates. This is done through approximation methods; see [8, 9]. Our first main result reads as follows.

Theorem 1 (Local $C^{1,\text{Log-Lip}}$-regularity)**.** *Let $u \in W^{2,d}_{\text{loc}}(B_1)$ be a strong solution to* (1)*. Suppose Assumption A1-A2 are in force. Then, $u \in C^{1,\text{Log-Lip}}_{\text{loc}}(B_1)$ and there exists $C > 0$ such that*

$$\sup_{x \in B_r(x_0)} |u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C r^2 \ln \frac{1}{r},$$

for every $x_0 \in B_{1/2}$ and $r \in (0, 1/4)$. In addition, $C = C(d, \lambda, \Lambda, \|u\|_{L^\infty(B_1)})$.

Remark 1. We notice the optimal regularity of the solutions to (1) is unknown. Of particular interest is whether or not the end-point $W^{2,\infty}$-regularity is available for strong solutions of this problem.

After examining the local regularity of solutions, we turn our attention to the so-called branch points. In brief, such points lie at the interface of $\Omega^+(u)$, $\Omega^-(u)$ and $\{u = 0\}$. From a rigorous viewpoint, a point on the free boundary $\Gamma(u) := (\partial\Omega^+(u) \cup \partial\Omega^-(u)) \cap B_1$ can be of three different types.

First, $x_0 \in \Gamma(u)$ is a one-phase point if

$$x_0 \in \left(\partial\Omega^{\pm}(u) \setminus \partial\Omega^{\mp}(u)\right) \cap B_1.$$

If this is the case, the local regularity of the solutions and properties of the free boundary follow from the fully nonlinear obstacle problem [23]. Alternatively, $x_0 \in \Gamma(u)$ may behave as a two-phase point; that is,

$$x_0 \in \left(\partial\Omega^+(u) \cap \partial\Omega^-(u)\right) \cap B_1.$$

Among two-phase points, branch points are of particular interest, as they are at the interface of the positive and the negative phases with the region where the solutions vanishes. Formally, we say that $x^* \in \Gamma(u)$ is a branch point if

$$|B_r(x^*) \cap \{u = 0\}| > 0$$

for every $0 < r \ll 1$. We denote with $\Gamma_{\rm BR}(u) \subset \Gamma(u)$ the set of branch points. Under a small-density condition with respect to the negative phase, we prove a result on the quadratic growth of the solutions away from branch points.

This is done by requiring both F_1 and F_2 to be convex and supposing they are positively homogeneous of degree one. Here, a dyadic analysis builds upon the maximum principle and a scaling strategy, using the L^∞-norms of the solutions as a normalization factor. This machinery was introduced in [10] in the context of an obstacle problem driven by the Laplacian. In [24] the authors took this perspective to the fully nonlinear setting and developed a fairly complete analysis of the obstacle problem governed by fully nonlinear operators. We also refer the reader to [23].

We consider the quantity

$$V_r(x^*, u) := \frac{\mathrm{vol}\left(B_r(x^*) \cap \Omega^-(u)\right)}{r^d};$$

by supposing $V_r(x^*, u)$ is controlled for a branch point $x^* \in \Gamma_{\rm BR}(u)$, we are capable of proving quadratic growth for the solutions, away from x_0. We state our second main result in the sequel.

Theorem 2 (Quadratic growth away from branch points)**.** *Let* $u \in W^{2,d}_{\rm loc}(B_1)$ *be a strong solution to* (1)*. Suppose A1, A3, and A4, to be detailed further, are in force. Let* $x^* \in \Gamma_{\rm BR}(u)$ *be such that A5, yet to be presented, holds true at* x^**. Then the exists a universal constant* $C > 0$ *such that*

$$\sup_{x \in B_r(x^*)} |u(x)| \leq Cr^2$$

for every $0 < r \ll 1$.

We note Theorem 2 does not require F_1 and F_2 to be close, or even comparable, in any topology.

Remark 2. The small-density of the negative phase is critical in establishing Theorem 2. Were it reasonable to suppose it holds for every $x_0 \in \Gamma(u) \cap B_{1/2}$, the conclusion of Theorem 2 would hold for every such point. Then a clever scaling argument, as in [10, 24], would produce local $C^{1,1}$-regularity estimates for the solutions. However, to impose a small-density condition for every free boundary point $x_0 \in \Gamma(u) \cap B_{1/2}$ implies the negative phase has no effect on the problem. Ultimately, it turns (1) into a one-phase fully nonlinear obstacle problem, whose theory is currently well-understood and documented.

Remark 3. We notice the formulation in (1) includes the free transmission obstacle problem

$$F_1(D^2u)\chi_{\{u>0\}} + F_2(D^2u)\chi_{\{u<0\}} = \chi_{\{u\neq 0\}} \quad \text{in} \quad B_1, \tag{7}$$

in the sense that solutions to (7) also solve (1).

The remainder of this paper is organized as follows: Section 2 gathers elementary results and details the main assumptions under which we work. In Section 3 we study the regularity of the strong solutions to (1) and present the proof of Theorem 1. A fourth section examines the growth regime of the solutions away from branch points and puts forward the proof of Theorem 2.

2. Preliminaries

This section presents some preliminary material, as well as the main hypotheses we use in the paper. With $S(d)$ we denote the space of symmetric matrices of order d; when convenient, we identify $S(d) \sim \mathbb{R}^{\frac{d(d+1)}{2}}$. We start with the uniform ellipticity of the operators F_i.

A 1 (Uniform ellipticity)**.** *For $i = 1, 2$, we suppose the operator $F_i : S(d) \to \mathbb{R}$ to be (λ, Λ)-uniformly elliptic. That is, for $0 < \lambda \leq \Lambda$, it holds*

$$\lambda \|N\| \leq F_i(M + N) - F_i(M) \leq \Lambda \|N\|,$$

for every $M, N \in S(d)$, $N \geq 0$, and $i = 1, 2$. We also suppose $F_i(0) = 0$.

Uniform ellipticity relates closely with the extremal operators

$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i.$$

In fact, Assumption A1 can be rephrased as

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N) \le F_i(M) - F_i(N) \le \mathcal{M}^+_{\lambda,\Lambda}(M-N),$$

for every $M, N \in S(d)$, and $i = 1, 2$. For completeness, we recall the definition of viscosity solutions.

Definition 1 (C-viscosity solution)**.** *Let $G : S(d) \to \mathbb{R}$ be a (λ, Λ)-elliptic operator. We say that $u \in \mathrm{USC}(B_1)$ is a C-viscosity subsolution to*

$$G(D^2u) = 0 \qquad \text{in } B_1 \tag{8}$$

if, for every $\varphi \in C^2_{\mathrm{loc}}(B_1)$ and $x_0 \in B_1$, such that $u - \varphi$ attains a local maximum at x_0, we have

$$G(D^2\varphi(x_0)) \le 0.$$

Similarly, we say that $u \in \mathrm{LSC}(B_1)$ is a C-viscosity supersolution to (8) *if, for every $\varphi \in C^2_{\mathrm{loc}}(B_1)$ and $x_0 \in B_1$, such that $u - \varphi$ attains a local minimum at x_0, we have*

$$G(D^2\varphi(x_0)) \ge 0.$$

If $u \in C(B_1)$ is simultaneously a subsolution and a supersolution to (8)*, we say it is a viscosity solution to the equation.*

For $0 < \lambda \le \Lambda$ and $f \in C(B_1)$, we define $\overline{S}(\lambda, \Lambda, f)$ as the set of functions $u \in C(B_1)$ satisfying

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le f$$

in B_1, in the viscosity sense. Similarly, $\underline{S}(\lambda, \Lambda, f)$ is the set of functions $u \in C(B_1)$ satisfying

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f.$$

Finally, we set

$$S(\lambda, \Lambda, f) := \overline{S}(\lambda, \Lambda, f) \cap \underline{S}(\lambda, \Lambda, f)$$

and

$$S^*(\lambda, \Lambda, f) := \overline{S}(\lambda, \Lambda, -|f|) \cap \underline{S}(\lambda, \Lambda, |f|).$$

For a comprehensive account of the theory of C-viscosity solutions, we refer the reader to [9]. We proceed with the definition of $W^{2,d}$-strong solution.

Definition 2 ($W^{2,d}$-strong solution)**.** *We say that $u \in W^{2,d}_{\mathrm{loc}}(B_1)$ is a strong solution to*

$$G(D^2u(x)) = 0 \qquad \text{in } B_1$$

if u satisfies the equation at almost every $x \in B_1$.

We refer the reader to [18, Chapter 9] for further details on this class of solutions and their properties. In the sequel, we put forward two assumptions concerning the convexity of the opeators F_1 and F_2. We start with a near-convexity condition used in the context of local $C^{1,\text{Log-Lip}}$-regularity.

A 2 (Near-convexity condition). *For $i = 1, 2$, we suppose the operator $\overline{F}_i : S(d) \to \mathbb{R}$ satisfies a near-convexity condition. That is, there exists a convex, (λ, Λ)-elliptic operator $F : S(d) \to \mathbb{R}$ such that*

$$|F_i(M) - \overline{F}(M)| \leq \tau(1 + \|M\|),$$

for some small constant $\tau > 0$, yet to be determined.

When it comes to the analysis of branching points, we require F_1 and F_2 to be convex operators.

A 3 (Convexity). *For $i = 1, 2$, we suppose the operator $F_i : S(d) \to \mathbb{R}$ to be convex.*

The next assumption concerns homogeneity of degree 1. It plays a major role in the quadratic growth of the solutions. The argument towards quadratic growth in [10] uses the linearity of the Laplacian operator. In [24] the authors notice that in the fully nonlinear case the condition that parallels linearity is the homogeneity of degree 1.

A 4 (Homogeneity of degree one). *We suppose F_1 and F_2 to be homogeneous of degree one; that is, for every $\tau \in \mathbb{R}$ and $M \in S(d)$, we have*

$$F_i(\tau M) = \tau F_i(M),$$

for every $i = 1, 2$.

Before proceeding with further assumptions, we gather some notation used throughout the paper. We denote by $\Omega^+(u)$ the subset of the unit ball where $u > 0$, whereas $\Omega^-(u)$ stands for the set where $u < 0$. That is,

$$\Omega^+(u) := \{x \in B_1 \mid u(x) > 0\} \quad \text{and} \quad \Omega^-(u) := \{x \in B_1 \mid u(x) < 0\}.$$

When referring to the set where $u \neq 0$ it is convenient to use the notation $\Omega(u) := \Omega^+(u) \cup \Omega^-(u)$. With $\Gamma(u)$ we denote the union of the topological boundaries of Ω^+ and Ω^-. I.e.,

$$\Gamma(u) := \left(\partial\Omega^+(u) \cup \partial\Omega^-(u)\right) \cap B_1.$$

We say that $x^* \in \Gamma(u)$ is a branch point if

$$|B_r(x^*) \cap \Sigma(u)| > 0$$

for every $0 < r < 1$. The set of branch points is denoted with $\Gamma_{\mathrm{BR}}(u)$. We also denote with $\Sigma(u)$ the set where u vanishes:

$$\Sigma(u) = \{x \in B_1 \mid u(x) = 0\}.$$

A further condition on the problem regards the subregion $\Omega^-(u)$; it is critical in proving quadratic growth of the solutions through the set of methods used in the paper. For $x^* \in \partial\Omega$ and $0 < r \ll 1$, we consider the quantity

$$V_r(x^*, u) := \frac{\mathrm{vol}\,(B_r(x^*) \cap \Omega^-(u))}{r^d}. \tag{9}$$

For ease of notation, we set $V_r(0, u) =: V_r(u)$.

A 5 (Normalized volume of $\Omega^-(u)$)**.** *Let $x^* \in \Gamma_{\mathrm{BR}}(u)$ be fixed. We suppose there exists $C_0 > 0$, to be determined later, such that*

$$V_r(x^*, u) \leq C_0$$

for every $r \in (0, 1/2)$.

The former assumption imposes a control on the size of the subregion where u is negative, ina. vicinity of $x^* \in \Gamma_{\mathrm{BR}}(u)$. It resonates on the geometry of the free boundary. In the next section we examine the regularity of strong solutions to (1). In particular, we present the proof of Theorem 1.

3. Local regularity of solutions

In this section we detail the proof of Theorem 1. We start by relating (1) with viscosity inequalities of the form

$$\min\left(F_1(D^2u), F_2(D^2u)\right) \leq 1 \qquad \text{in } B_1 \tag{10}$$

and

$$\max\left(F_1(D^2u), F_2(D^2u)\right) \geq -1 \qquad \text{in } B_1 \tag{11}$$

Lemma 1. *Let $u \in W^{2,d}_{\mathrm{loc}}(B_1)$ be a strong solution to* (1)*. Suppose A1 holds true. Then u is a C-viscosity solution to the inequalities* (10)-(11)*.*

The proof of Lemma 1 follows from standard computations and the maximum principle for $W^{2,d}$-functions, see [28, Corollary 3] and [4]. In addition, if u is a continuous viscosity solution to (10)-(11) we also have $u \in \overline{S}(\lambda,\Lambda,1)\cap S^*(\lambda,\Lambda,1)$. In fact, because

$$\mathcal{M}^-_{\lambda,\Lambda}(M)\leq F_i(M)\leq \mathcal{M}^+_{\lambda,\Lambda}(M)$$

holds for $i=1,2$, we have

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u)\leq \min\big(F_1(D^2u),F_2(D^2u)\big)\leq 1$$

and

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u)\geq \max\big(F_1(D^2u),F_2(D^2u)\big)\geq -1.$$

As a consequence to the former inclusion we derive the Hölder continuity for the strong solutions to (1), with universal estimates.

Lemma 2 (Hölder continuity)**.** *Let $u\in W^{2,d}_{\rm loc}(B_1)$ be a strong solution to* (1) *and suppose A1 holds. Then $u\in C^\alpha_{\rm loc}(B_1)$, for some $\alpha\in(0,1)$, and there exists $C>0$ such that*

$$\|u\|_{C^\alpha(B_{1/2})}\leq C\left(\|u\|_{L^\infty(B_1)}+\|f\|_{L^d(B_1)}\right).$$

In addition, $\alpha=\alpha(\lambda,\Lambda,d)$ and $C=C(\lambda,\Lambda,d)$.

For a proof of Lemma 2, see [9, Lemma 4.10]. In the sequel we prove that solutions to (1) satisfy a quadratic growth *away from the free boundary.*

3.1. Proof of Theorem 1. We continue with an approximation lemma.

Proposition 1. *Let $u\in W^{2,d}_{\rm loc}(B_1)$ be a $W^{2,d}$-strong solution to* (1)*. Suppose A1 and A2 hold true. Given $\delta>0$, there exists $0<\tau_0\ll 1$ such that, if the parameter $\tau>0$ in A2 satisfies $\tau<\tau_0$, there exists $h\in C^{2,\alpha}_{\rm loc}(B_{9/10})$ with*

$$\|u-h\|_{L^\infty(B_{8/9})}\leq\delta$$

and

$$\|h\|_{C^{2,\alpha}(B_{8/9})}\leq C,$$

for some universal constant $C>0$ and some universal exponente $\alpha\in(0,1)$.

Proof: For ease of presentation we split the proof into three main steps.

Step 1 - We argue by contradiction; suppose the statement of the proposition is false. Then there exist sequences $(u_n)_{n\in\mathbb{N}}$, $(F_1^n)_{n\in\mathbb{N}}$ and $(F_2^n)_{n\in\mathbb{N}}$ such that:

(1) F_i^n satisfies A1 for $i = 1, 2$, and every $n \in \mathbb{N}$. Moreover,

$$|F_i^n(M) - F(M)| \leq \frac{1}{n}(1 + \|M\|) \tag{12}$$

for every $i = 1, 2$, $n \in \mathbb{N}$, and $M \in S(d)$;

(2) u_n is a viscosity solution to

$$\min\left(F_1^n(D^2u_n), F_2^n(D^2u_n)\right) \leq \frac{1}{n} \tag{13}$$

and

$$\max\left(F_1^n(D^2u_n), F_2^n(D^2u_n)\right) \geq -\frac{1}{n} \tag{14}$$

in $B_{9/10}$, with $u_n = u$ on $\partial B_{9/10}$, for every $n \in \mathbb{N}$;

(3) there exists $\delta_0 > 0$ for which

$$\|u_n - h\|_{L^\infty(B_{8/9})} > \delta_0$$

for every $h \in C^2(B_{9/10})$ with $\|h\|_{C^2(B_{8/9})} \leq C$, and every $n \in \mathbb{N}$.

Step 2 - Because of (13)-(14) and Lemma 2, we learn that $\|u_n\|_{C^\beta(B_{9/10})} \leq C$ for every $n \in \mathbb{N}$, for some universal constant $C > 0$. As a consequence, it converges locally uniformly, through a subsequence if necessary, to a function $u_\infty \in C^{\beta/2}_{\text{loc}}(B_{9/10})$. Also, (12) ensures that F_1^n and F_2^n converge locally uniformly on $S(d)$ to the convex operator $\overline{F}$ from A2. The stability of viscosity sub and supersolutions implies

$$\overline{F}(D^2u_\infty) = 0 \quad \text{in} \quad B_{9/10}.$$

Step 3 - Because $\overline{F}$ is convex, we infer $u_\infty \in C^{2,\alpha}_{\text{loc}}(B_{9/10})$, with $\|u_\infty\|_{C^{2,\alpha}(B_{8/9})} \leq C$, for some universal constant $C > 0$ and some universal exponent $\alpha \in (0, 1)$. Set $h := u_\infty$ to get a contradiction and complete the proof. ■

Proposition 2. *Let $u \in W^{2,d}_{\text{loc}}(B_1)$ be a $W^{2,d}$-strong solution to* (1)*. Suppose A1 and A2 hold true. There exists $0 < \tau_0 \ll 1$ such that, if the parameter $\tau > 0$ in A2 satisfies $\tau < \tau_0$, one can find $0 < \rho \ll 1$ and a sequence of quadratic polynomials $(P_n)_{n\in\mathbb{N}}$, with*

$$P_n(x) := a_n + \mathrm{b}_n \cdot x + \frac{x \cdot C_n x}{2},$$

with

$$\|u - P_n\|_{L^\infty(B_{\rho^n})} \leq \rho^{2n}, \tag{15}$$

$$\overline{F}(C_n) = 0, \tag{16}$$

and

$$|a_n - a_{n-1}| + \rho^{n-1}|\mathrm{b}_n - \mathrm{b}_{n-1}| + \rho^{2(n-1)}|C_n - C_{n-1}| \leq C\rho^{2(n-1)}, \tag{17}$$

for every $n \in \mathbb{N}$.

Proof: We resort to an induction argument; for ease of clarity, we split the proof into four steps.

Step 1 - We consider the base case. Set $P_0 := 0$; let $h \in C^{2,\alpha}_{\rm loc}(B_{9/10})$ be the δ-approximating function whose existence follows from Proposition 1 and define

$$P_1(x) := h(0) + Dh(0) \cdot x + \frac{x \cdot D^2h(0)x}{2}.$$

We verify (15)-(17) in the case $n = 1$. Notice that

$$\sup_{x \in B_\rho} |u(x) - P_1(x)| \leq \sup_{x \in B_{\rho^n}} |u(x) - h(x)| + \sup_{x \in B_{\rho^n}} |h(x) - P_1(x)| \leq \delta + C\rho^{2+\alpha}.$$

By choosing

$$\delta := \frac{\rho}{2} \qquad \text{and} \qquad \rho := \left(\frac{1}{2C}\right)^{\frac{1}{\alpha}}$$

one ensure (15) holds. Because h is the approximating function from Proposition 1, we have (16). Finally, (17) follows from the $C^{2,\alpha}$-estimates available for h.

Step 2 - Now we formulate the induction hypothesis: suppose (15)-(17) have been verified for $n = k$. We examine the case $n = k + 1$. Let $v_k : B_1 \to \mathbb{R}$ be defined as

$$v_k(x) := \frac{u(\rho^k x) - P_k(\rho^k x)}{\rho^{2k}}.$$

It is clear from the induction hypothesis that v_k is a normalized viscosity solution to

$$\min\left(F_1(D^2v_k + C_k), F_2(D^2v_k + C_k)\right) \leq 1 \quad \text{in} \quad B_1$$

and

$$\max\left(F_1(D^2v_k + C_k), F_2(D^2v_k + C_k)\right) \geq -1 \quad \text{in} \quad B_1.$$

Also, A2 implies

$$\sup_{M \in S(d)} |F_i(M + C_k) - \overline{F}_k(M)| \leq \tau(1 + \|M\|),$$

where $\overline{F}_k$ is the convex operator defined as $\overline{F}_k(M) := \overline{F}(M + C_k)$. Because of the induction hypothesis, $\overline{F}(C_k) = 0$; hence, $\overline{F}(D^2 w) = 0$ and $\overline{F}_k(D^2 w) = 0$ have the same estimates.

As a consequence, if $0 < \tau < \tau_0$, Proposition 1 ensures the existence of $\tilde{h} \in C^{2,\alpha}_{\text{loc}}(B_{9/10})$, with $\|\tilde{h}\|_{C^{2,\alpha}(B_{8/9})} \leq C$ satisfying

$$\|v_k - \tilde{h}\|_{L^\infty(B_{8/9})} \leq \delta.$$

Arguing as in the former step, one concludes the existence of

$$\tilde{P}(x) := \tilde{a} + \tilde{\mathrm{b}} \cdot x + \frac{x \cdot \tilde{C} x}{2}$$

such that

$$\sup_{x \in B_\rho} \left| v_k(x) - \tilde{P}(x) \right| \leq \rho^2.$$

The induction assumption and the definition of v_k yield

$$\sup_{x \in B_{\rho^{k+1}}} |u(x) - P_{k+1}(x)| \leq \rho^{2(k+1)},$$

where P_{k+1} is given by

$$P_{k+1}(x) := a_k + \rho^{2k} \tilde{a} + (\mathrm{b}_k + \rho^k \tilde{\mathrm{b}}) \cdot x + \frac{x \cdot (C_k + \tilde{C}) x}{2}. \tag{18}$$

Because $\tilde{C} = D^2 \tilde{h}(0)$, it follows that $\overline{F}(C_{k+1}) = 0$. Defining a_{k+1}, b_{k+1} and C_{k+1} as in (18), one ensures that (17) is also satisfied at the $(k+1)$-level, and the proof is complete. ∎

Now we are in a position to detail the proof of Theorem 1.

Proof of Theorem 1: Once Proposition 2 is available, the proof of Theorem 1 follows from (by now) standard computations. See, for instance, [32, Proof of Theorem 2.6, p. 1398] ∎

4. Quadratic growth away from branch points

Let $x^* \in \Gamma_{\text{BR}}(u) \cap B_1$ be fixed. Consider the maximal subset of $\mathbb{N}$ whose elements j are such that

$$\sup_{x \in B_{2^{-j-1}}(x^*)} |u(x)| \geq \frac{1}{16} \sup_{x \in B_{2^{-j}}(x^*)} |u(x)|; \tag{19}$$

we denote such set by $\mathcal{M}(x^*, u)$.

Proposition 3. *Let $u \in W^{2,d}_{\text{loc}}(B_1)$ be a strong solution to* (1). *Suppose A1, A3, and A4 hold true. Let $x^* \in \Gamma_{\text{BR}}(u)$ and suppose A5 holds at x^*. There exists a choice of $C_0 > 0$ in A5 such that, if*

$$V_{2^{-j}}(x^*, u) < C_0 \tag{20}$$

for every $j \in \mathcal{M}(x^*, u)$, then

$$\sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{1}{C_0} 2^{-2j}, \qquad \forall j \in \mathcal{M}(x^*, u).$$

Proof: For ease of presentation, we split the proof into three steps.

Step 1 - Set $x^* = 0$ and $\mathcal{M}(u) := \mathcal{M}(0, u)$. We resort to a contradiction argument; suppose the statement of the proposition is false. Then, there exist sequences $(u_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$ such that u_n is a normalized strong solution to (1),

$$V_{\frac{1}{2^n}}(u_n) < \frac{1}{n}, \tag{21}$$

with

$$\sup_{x \in B_{2^{-j_n}}} |u_n(x)| > \frac{n}{2^{2j_n}}, \tag{22}$$

for every $j_n \in \mathcal{M}(u_n)$, and $n \in \mathbb{N}$. Because $\|u_n\|_{L^\infty(B_1)}$ is uniformly bounded, it follows from (22) that $j_n \longrightarrow \infty$. In particular, we may re-write (21) as

$$V_{\frac{1}{2^{j_n}}}(u_n) < \frac{1}{j_n}. \tag{23}$$

Step 2 - Now, we introduce an auxiliary function $v_n : B_1 \to \mathbb{R}$, given by

$$v_n(x) := \frac{u_n(2^{-j_n}x)}{\|u_n\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}}.$$

Clearly, $v_n(0) = 0$. In addition, $V_1(v_n) \longrightarrow 0$. Moreover, it follows from the definition of v_n that

$$\sup_{B_{1/2}} |v_n(x)| = 1 \tag{24}$$

and

$$\sup_{B_1} |v_n(x)| \leq 16.$$

We notice that A4 yields

$$\min\big(F_1(D^2v_n),F_2(D^2v_n)\big)\le\frac{\min\big(F_1(D^2u_n(2^{-j_n}x)),F_2(D^2u_n(2^{-j_n}x))\big)}{2^{2j_n}\left\|u_n\right\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}}.$$

Therefore,

$$\min\big(F_1(D^2v_n),F_2(D^2v_n)\big)\le\frac{1}{n}\frac{C\left\|u_n\right\|_{L^\infty\left(B_{2^{-j_n}}\right)}}{\left\|u_n\right\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}}\le\frac{C}{n}\le C_0, \tag{25}$$

for some $C_0>0$ and $n\gg 1$. On the other hand,

$$\begin{aligned}\max\big(F_1(D^2v_n),F_2(D^2v_n)\big)&\ge\frac{\max\big(F_1(D^2u_n(\frac{x}{2^{j_n}})),F_2(D^2u_n(\frac{x}{2^{j_n}}))\big)}{2^{2j_n}\left\|u_n\right\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}}\\&\ge -C_0.\end{aligned} \tag{26}$$

It follows from (25)-(26) that $(v_n)_{n\in\mathbb{N}}\subset S^*(\lambda,\Lambda,C_0)$. As a consequence, $v_n\in C^{\alpha}_{loc}(B_1)$ for every $n\in\mathbb{N}$, for some unknown $\alpha\in(0,1)$, with uniform estimates; see [9, Proposition 4.10]. Therefore, there exists v_∞ such that $v_n\longrightarrow v_\infty$ in $C^{\beta}_{\rm loc}(B_1)$, for every $0<\beta<\alpha$. Since $v_n(0)=0$ for every $n\in\mathbb{N}$ we infer that $v_\infty(0)=0$, whereas (24) leads to $\|v_\infty\|_{L^\infty(B_{1/2})}=1$. Because $V_1(v_n)\longrightarrow 0$, we conclude that $v_\infty\ge 0$ in B_1.

Step 3 - Standard stability results for viscosity solutions build upon (25) to ensure

$$\min\big(F_1(D^2v_\infty),F_2(D^2v_\infty)\big)\le 0 \qquad \text{in} \qquad B_1.$$

We conclude that $v_\infty\in\overline{S}(\lambda,\Lambda,0)$ attains an interior local minimum at the origin, which leads to a contradiction (see, for instance, [9, Proposition 4.9]). ∎

In Proposition 3 the constant $C_0>0$ informing A5 is determined. This quantity remains unchanged throughout the paper. The next result extrapolates the former analysis from $\mathcal{M}(x^*,u)$ to the entire set of natural numbers.

Proposition 4. *Let $u\in W^{2,d}_{\rm loc}(B_1)$ be a strong solution to* (1)*. Suppose A1, A3, and A4 hold true. Let $x^*\in\Gamma_{\rm BR}(u)$ and suppose A5 holds at x^*. Finally, suppose that for every $j\in\mathcal{M}(x^*,u)$ we have*

$$V_{2^{-j}}(x^*,u)<C_0,$$

for $C_0 > 0$ *fixed in* (20). *Then*

$$\sup_{x\in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{4}{C_0} 2^{-2j}, \qquad \forall j \in \mathbb{N}.$$

Proof: As before we set $x^* = 0$ and argue through a contradiction argument. Suppose the proposition is false. Let $m \in \mathbb{N}$ be the smallest natural number such that

$$\sup_{B_{2^{-m}}} |u(x)| > \frac{4}{C_0} 2^{-2m}. \tag{27}$$

We claim that $m-1 \in \mathcal{M}(u)$. Indeed,

$$\sup_{B_{2^{1-m}}} |u(x)| \leq \frac{4}{C_0} 2^{-2(m-1)} = \frac{16}{C_0} 2^{-2m} < 4 \sup_{B_{2^{-m}}} |u(x)|.$$

We conclude

$$\sup_{B_{2^{-m}}} |u(x)| \leq \sup_{B_{2^{1-m}}} |u(x)| \leq \frac{1}{C_0} 2^{-2(m-1)} = \frac{4}{C_0} 2^{-2m},$$

which contradicts (27) and completes the proof. ■

Consequential to Proposition 4 is the quadratic growth of u away from the branch point x^*. We detail this argument in the proof of Theorem 2.

Proof of Theorem 2: Find $j \in \mathbb{N}$ satisfying $2^{-(j+1)} \leq r < 2^{-j}$. It is straightforward to notice that

$$\sup_{B_r} |u(x)| \leq \sup_{B_{2^{-j}}} |u(x)| \leq C \left[\left(\frac{1}{2} \right)^{j+1-1} \right]^2 \leq Cr^2,$$

which ends the proof. ■

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