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HIGHER MULTI-COURANT ALGEBROIDS

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ABSTRACT: The binary bracket of a Courant algebroid structure on $(E, \langle\cdot,\cdot\rangle)$ can be extended to a n-ary bracket on $\Gamma(E)$, yielding a multi-Courant algebroid. These n-ary brackets form a Poisson algebra and were defined, in an algebraic setting, by Keller and Waldmann. We construct a higher geometric version of Keller-Waldmann Poisson algebra and define higher multi-Courant algebroids. As Courant algebroid structures can be seen as degree 3 functions on a graded symplectic manifold of degree 2, higher multi-Courant structures can be seen as functions of degree $n \geq 3$ on that graded symplectic manifold.

KEYWORDS: Courant algebroid, graded symplectic manifold, graded Poisson algebra.
MATH. SUBJECT CLASSIFICATION (2020): 53D17, 17B70, 58A50.

1. Introduction

Aiming at interpreting the bracket on the Whitney sum $TM \oplus T^*M$ of the tangent and cotangent bundle of a smooth manifold M, proposed by Courant in [2], Liu, Weinstein and Xu [7] introduced the concept of Courant algebroid on a vector bundle $E \to M$. This vector bundle is equipped with a fiberwise symmetric bilinear form $\langle\cdot,\cdot\rangle$, a Leibniz bracket on the space $\Gamma(E)$ of sections and a morphism of vector bundles $\rho : E \to TM$, called the anchor, satisfying a couple of compatibility conditions. In [10], Roytenberg described a Courant algebroid as a degree 2 symplectic graded manifold $\mathscr{F}_E$ together with a degree 3 function Θ satisfying $\{\Theta, \Theta\} = 0$, where $\{\cdot,\cdot\}$ is the graded Poisson bracket corresponding to the graded symplectic structure. The morphism ρ and the Leibniz bracket on $\Gamma(E)$ are recovered as derived brackets (see 2.2).

The Courant bracket, or its no skew-symmetric version called Dorfman bracket, is a binary bracket. The first attempt to extend it to a n-ary bracket was given, in purely algebraic terms, by Keller and Waldmann in [5]. They built a graded Poisson algebra $\mathscr{C}$ of degree -2 whose degree 3 elements that are closed with respect to the graded Poisson bracket correspond to Courant structures. The graded Poisson algebra $\mathscr{C}$, that we call Keller-Waldmann

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Poisson algebra, is a complex that controls deformation. Keller-Waldmann algebra elements are n-ary brackets and each bracket comes with a symbol. In degree 3, the symbol is the anchor of the Courant structure.

We consider the geometric counterpart of the Keller-Waldmann Poisson algebra, and our starting point is a vector bundle $E \to M$ equipped with a fiberwise symmetric bilinear form $\langle \cdot, \cdot \rangle$. This is also the setting in [1], where the first author has started the study of the Keller-Waldmann algebra under a geometric point of view. In this case, the Keller-Waldmann Poisson algebra is denoted by $\mathscr{C}(E)$ and its elements are pre-multi-Courant brackets on $\Gamma(E)$. The prefix *pre* means that elements $C \in \mathscr{C}(E)$ do not need to close with respect to the Poisson bracket, denoted by $[\cdot, \cdot]_{\scriptscriptstyle KW}$. If $[C, C]_{\scriptscriptstyle KW} = 0$, the triple $(E, \langle \cdot, \cdot \rangle, C)$ is a multi-Courant algebroid. At this point it is important to notice that, for $n \neq 2$, what is called n-Courant bracket in Remark 3.2 of [5] is not the same as our n-ary Courant bracket, because we require the closedness with respect to the $[\cdot, \cdot]_{\scriptscriptstyle KW}$ bracket, while in [5] the authors ask the closedness with respect to a different bracket. For $n = 2$, the two brackets coincide (see Remark 3.12).

Very recently, Cueca and Mehta [3] showed that there is an isomorphism of graded commutative algebras between $(\mathscr{F}_E, \cdot)$ and $(\mathscr{C}(E), \wedge)$, where $\cdot$ and $\wedge$ denote the associative graded commutative products of the two Poisson algebras $\mathscr{F}_E$ and $\mathscr{C}(E)$, respectively. They also remarked that the isomorphism is indeed a Poisson isomorphism, but they don't prove this since they don't exhibit the Poisson bracket on $\mathscr{C}(E)$.

The main goal of this paper is to give a higher version of the Keller-Waldmann Poisson algebra and define higher multi-Courant algebroids. This means that we consider higher (pre-)multi-Courant brackets on $\Gamma(\wedge^{\geq 1} E)$, and not only on $\Gamma(E)$. Each higher (pre-)multi-Courant bracket has an associated symbol and it is the extension by derivation of a (pre-)multi-Courant bracket. This construction leads to a graded Poisson algebra $\mathscr{C}(\wedge^{\geq 1} E)$ with a Poisson bracket $[\![\cdot, \cdot]\!]$ that extends $[\cdot, \cdot]_{\scriptscriptstyle KW}$.

In literature we find several Courant bracket extensions, in different directions (see [13] and references therein). In [13] Zambon defines higher analogues of Courant algebroids, replacing the vector bundle $TM \oplus T^*M$, originally considered in [2], by $TM \oplus \wedge^p T^*M$, $p \geq 0$. In an algebraic setting, Roytenberg [11] extends the usual Courant bracket to a n-ary bracket on $\Gamma(E)$, and each n-ary bracket comes with a collection of symbols that control the defect of their skew-symmetry and also the skew-symmetry of the bracket. In [8], under the perspective of Loday-infinity algebras and using Voronov's derived bracket construction [12], Peddir defines n-ary Dorfman brackets on $\Gamma(E)$ and $C^\infty(M)$. Having started from the Keller-Waldmann algebra, whose elements are n-ary brackets on $\Gamma(E)$, we were led to an extension of Courant algebroid structures on $E \to M$ in two fold: the binary bracket is replaced by a n-ary bracket and the latter is a bracket on sections of $\wedge^{\geq 1}E$. Of course, the symbol goes along the bracket.

The paper is organized in the following way. In Section 2 we make a very brief summary of Roytenberg's graded Poisson bracket construction [10] and we recall the Courant algebroid definition. Section 3 is devoted to Keller-Waldmann Poisson algebra where we clarify and detail many aspects that are not covered in [5]. One of them is the explicit formula for the Poisson bracket $[\cdot,\cdot]_{_{KW}}$ on $\mathscr{C}(E)$, that is not given in [5] because the bracket is defined recursively there. To achieve this, we consider the binary case of a bracket introduced in [9], built using the interior product of two elements of $\mathscr{C}(E)$. We introduce the concept of multi-Courant algebroid on $(E, \langle\cdot,\cdot\rangle)$ as a n-ary element $C \in \mathscr{C}(E)$ that is closed under the bracket $[\cdot,\cdot]_{_{KW}}$. We point out an alternative definition for the Keller-Waldmann Poisson algebra, already presented in [5], that is needed in the remaining sections of the paper. In this setting, each $C \in \mathscr{C}(E)$ is in a one-to-one correspondence with $\widetilde{C}$, the latter being obtained from C and $\langle\cdot,\cdot\rangle$. In Section 4, we extend the symmetric bilinear form $\langle\cdot,\cdot\rangle$ to $\Gamma(\wedge^\bullet E)$ and prove that it coincides with the restriction of $[\cdot,\cdot]_{_{KW}}$ to $\Gamma(\wedge^{\geq 1}E)$. Then, we define higher (pre-)multi-Courant structures on $(E, \langle\cdot,\cdot\rangle)$. These are multilinear maps from $\Gamma(\wedge^{\geq 1}E) \times \overset{(n)}{\ldots} \times \Gamma(\wedge^{\geq 1}E)$ to $\Gamma(\wedge^\bullet E)$ which are derivations in each entry, together with a symbol that takes values on the space of derivations $Der(C^\infty(M), \Gamma(\wedge^\bullet E))$. All these data should satisfy some compatible conditions involving $\langle\cdot,\cdot\rangle$. The extension by derivation in each entry of every $\widetilde{C}$ is a higher (pre-)multi-Courant structure on E. Higher (pre-)multi-Courant brackets form a graded Poisson algebra $(\mathscr{C}(\wedge^{\geq 1}E), \wedge, [\![\cdot,\cdot]\!])$ of degree -2. In Section 5 we see how the higher Keller-Waldmann Poisson algebra $(\mathscr{C}(\wedge^{\geq 1}E), \wedge, [\![\cdot,\cdot]\!])$ is related to Roytenberg's Poisson algebra $(\mathscr{F}_E, \cdot, \{\cdot,\cdot\})$. We start by establishing a Poisson isomorphism between the Keller-Waldmann Poisson algebra $(\mathscr{C}(E), \wedge, [\cdot,\cdot]_{_{KW}})$ and Roytenberg's Poisson algebra and we show that this Poisson isomorphism gives rise to a Poisson isomorphim between the higher Keller-Waldmann algebra $(\mathscr{C}(\wedge^{\geq 1}E), \wedge, [\![\cdot,\cdot]\!])$ and Roytenberg's Poisson algebra.

Notation. Let τ be a permutation of n elements, $n \geq 1$; we denote by $\mathrm{sgn}(\tau)$ the sign of τ. We denote by $Sh(i, n-i)$ the set of $(i, n-i)$-unshuffles, i.e., permutations τ that satisfy the inequalities $\tau(1) < \ldots < \tau(i)$ and $\tau(i+1) < \ldots < \tau(n)$. For a vector bundle $E \to M$, we denote by $\Gamma(\wedge^n E)$ the space of homogeneous E-multivectors of degree n and we set $\Gamma(\wedge^\bullet E) := \oplus_{n\geq 0}\Gamma(\wedge^n E)$, with $\Gamma(\wedge^0 E) = C^\infty(M)$, and $\Gamma(\wedge^{\geq 1} E) := \oplus_{n\geq 1}\Gamma(\wedge^n E)$. For $n < 0$, $\Gamma(\wedge^n E) = \{0\}$.

2. Preliminaries

2.1. Graded Poisson bracket. We briefly recall the construction of a graded Poisson algebra introduced in [10]. Let $E \to M$ be a vector bundle equipped with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and denote by $E[m]$ the graded manifold obtained by shifting the fibre degree by m. Let $p^*(T^*[2]E[1])$ be the graded symplectic manifold which is the pull-back of $T^*[2]E[1]$ by the map $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, . \rangle)$. We denote by $\mathscr{F}_E := \oplus_{n\geq 0}\mathscr{F}_E^n$ the graded algebra of functions on $p^*(T^*[2]E[1])$, with $\mathscr{F}_E^0 = C^\infty(M)$ and $\mathscr{F}_E^1 = \Gamma(E)$ and, consequently, $\Gamma(\wedge^n E) \subset \mathscr{F}_E^n$. The graded algebra $\mathscr{F}_E$ is equipped with the canonical Poisson bracket $\{\cdot, \cdot\}$ of degree -2, determined by the graded symplectic structure, so that we have a graded Poisson algebra structure on $\mathscr{F}_E$. The Poisson bracket of functions of degrees 0 and 1 is given by

$$\{f, g\} = 0, \quad \{f, e\} = 0 \quad \text{and} \quad \{e, e'\} = \langle e, e' \rangle,$$

for all $e, e' \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

2.2. Courant structures. Recall that, given a vector bundle $E \to M$ equipped with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a *Courant structure* on $(E, \langle \cdot, \cdot \rangle)$ is a pair $(\rho, [\cdot, \cdot])$, where $\rho : E \to TM$ is a morphism of vector bundles called the *anchor*, and $[\cdot, \cdot]$ is a $\mathbb{R}$-bilinear bracket on $\Gamma(E)$, called the *Dorfman bracket*, such that

$$\rho(u)\cdot\langle v, w\rangle = \langle [u, v], w\rangle + \langle v, [u, w]\rangle, \qquad \rho(u)\cdot\langle v, w\rangle = \langle u, [v, w] + [w, v]\rangle, \tag{1}$$

and

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]], \tag{2}$$

for all $u, v, w \in \Gamma(E)$. The bracket $[\cdot,\cdot]$ equips the space $\Gamma(E)$ of sections of E with a *Leibniz algebra* structure. Skipping Equation (2) yields a *pre-Courant structure* on $(E, \langle\cdot,\cdot\rangle)$.

There is a one-to-one correspondence between pre-Courant structures $(\rho, [\cdot,\cdot])$ on $(E, \langle\cdot,\cdot\rangle)$ and functions $\Theta \in \mathscr{F}^3_E$, while for Courant structures the function Θ is such that $\{\Theta, \Theta\} = 0$ [10]. In this case, the hamiltonian vector field $X_\Theta = \{\Theta, \cdot\}$ on the graded manifold $p^*(T^*[2]E[1])$ is a homological vector field, and so $\big(p^*(T^*[2]E[1]), X_\Theta\big)$ is a Q-manifold.

The anchor and Dorfman bracket associated to a given $\Theta \in \mathscr{F}^3_E$ can be defined, for all $e, e' \in \Gamma(E)$ and $f \in C^\infty(M)$, by the derived bracket expressions:

$$\rho(e) \cdot f = \{f, \{e, \Theta\}\} \quad \text{and} \quad [e, e'] = \{e', \{e, \Theta\}\}.$$

3. Multi-Courant structures and Keller-Waldmann Poisson algebra

In this section we deepen the study of the Keller-Waldmann Poisson algebra.

3.1. Multi-Courant structures. Let $E \to M$ be a vector bundle equipped with a fibrewise non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. The next definition is taken from [5], but within a geometrical perspective. $\mathfrak{X}(M)$ denotes the space of vector fields on a manifold M.

Definition 3.1. A *n-ary pre-Courant structure* on $(E, \langle\cdot,\cdot\rangle)$ is a multilinear n-bracket on $\Gamma(E)$, $n \geq 0$,

$$C : \Gamma(E) \times \overset{(n)}{\cdots} \times \Gamma(E) \to \Gamma(E)$$

for which there exists a map σ_C, called the *symbol* of C,

$$\sigma_C : \Gamma(E) \times \overset{(n-1)}{\cdots} \times \Gamma(E) \to \mathfrak{X}(M),$$

such that for all $e, e', e_1, \dots, e_{n-1} \in \Gamma(E)$, we have

$$\sigma_C(e_1, \dots, e_{n-1}) \cdot \langle e, e' \rangle = \langle C(e_1, \dots, e_{n-1}, e), e' \rangle + \langle e, C(e_1, \dots, e_{n-1}, e') \rangle \tag{3}$$

and, for $n \geq 2$ and $1 \leq i \leq n-1$, the following $n-1$ conditions hold:

$$\begin{aligned} \langle C(e_1, \dots, e_i, e_{i+1}, \dots, e_n) + C(e_1, \dots, e_{i+1}, e_i, \dots, e_n), e \rangle \\ = \sigma_C(e_1, \dots, \widehat{e_i}, \widehat{e_{i+1}}, \dots, e_n, e) \cdot \langle e_i, e_{i+1} \rangle, \end{aligned} \tag{4}$$

with $\widehat{e_i}$ meaning the absence of e_i. A 0-ary pre-Courant structure is simply an element $e \in \Gamma(E)$ (with vanishing symbol). The triple $(E, \langle\cdot,\cdot\rangle, C)$ is called an *n-ary pre-Courant algebroid*. When we don't want to specify the arity of C, we call it a *pre-multi-Courant* structure and the triple $(E, \langle\cdot,\cdot\rangle, C)$ is a *pre-multi-Courant algebroid*.

For $n = 1$, C is derivative endomorphism [6] with symbol $\sigma_C \in \mathfrak{X}(M)$. When $n = 2$, conditions (3) and (4) coincide with (1) for $\rho = \sigma_C$. So, as it would be expected, Definition 3.1 generalizes the notion of pre-Courant structure on $(E, \langle\cdot,\cdot\rangle)$.

We denote by $\mathscr{C}^{n+1}(E)$ the space of all n-ary pre-Courant structures on E and set

$$\mathscr{C}(E) = \oplus_{n\geq 0}\mathscr{C}^n(E),$$

with $\mathscr{C}^0(E) = C^\infty(M)$ and $\mathscr{C}^1(E) = \Gamma(E)$.

Remark 3.2. The symbol σ_C of $C \in \mathscr{C}^{n+1}(E)$ is uniquely determined by C [5]. The uniqueness of σ_C allows to consider an extension of C, also denoted by C, on the graded space $\Gamma(\wedge^{\leq 1}E) = C^\infty(M) \oplus \Gamma(E)$, where $f \in C^\infty(M)$ has degree 0 and $e \in \Gamma(E)$ has degree 1. The extension of C is a degree $1 - n$ bracket,

$$C : \Gamma(\wedge^{\leq 1}E)\times \overset{(n)}{\cdots} \times\Gamma(\wedge^{\leq 1}E) \to \Gamma(\wedge^{\leq 1}E),$$

with symbol

$$\sigma_C : \Gamma(\wedge^{\leq 1}E)\times \overset{(n-1)}{\cdots} \times\Gamma(\wedge^{\leq 1}E) \to \mathfrak{X}(M),$$

such that, for all $e_i \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$C(e_1, \dots, e_{n-1}, f) = \sigma_C(e_1, \dots, e_{n-1}) \cdot f \tag{5}$$

and

$$C(e_1, \dots, \overset{\overset{i}{\downarrow}}{f}, \dots, e_{n-1}) = C(e_1, \dots, \overset{\overset{j}{\downarrow}}{f}, \dots, e_{n-1}),$$

for all $1 \leq i, j \leq n$.

By degree reasons, C vanishes when applied to at least two functions,

$$C(e_1, \dots, f, \dots, g, \dots, e_{n-2}) = 0.$$

Assuming that σ_C vanishes when applied to at least one function,

$$\sigma_C(e_1, \dots, f, \dots, e_{n-2}) = 0,$$

Equations (3) and (4), with the obvious adaptations, are satisfied.

3.2. Keller-Waldmann Poisson algebra. Given $C \in \mathscr{C}^{n+1}(E)$, $n \geq 1$, and $e \in \Gamma(E)$, we denote by $\imath_e C$ the element of $\mathscr{C}^n(E)$ defined by

$$\imath_e C(e_1, \ldots, e_{n-1}) = C(e, e_1, \ldots, e_{n-1}), \tag{6}$$

for all $e_1, \ldots, e_{n-1} \in \Gamma(E)$, with symbol given by

$$\sigma_{\imath_e C}(e_1, \ldots, e_{n-2}) = \sigma_C(e, e_1, \ldots, e_{n-2}).$$

If $C = e_1$, $\imath_e e_1 = \langle e, e_1 \rangle$ and we set $\imath_e f := 0$.

If we consider the extension of C as in Remark 3.2 we may define, for $f \in C^\infty(M)$,

$$\imath_f C(e_1, \ldots, e_{n-1}) = C(f, e_1, \ldots, e_{n-1}) = \sigma_C(e_1, \ldots, e_{n-1}) \cdot f,$$

for all $e_1, \ldots, e_{n-1} \in \Gamma(E)$.

The space $\mathscr{C}(E)$ is endowed with an associative graded commutative product $\wedge$ of degree zero [5] defined as follows[1]:

$$\begin{cases} f \wedge g = fg = g \wedge f \\ f \wedge e = fe = e \wedge f, \end{cases}$$

for all $f, g \in C^\infty(M)$ and $e \in \Gamma(E)$, and such that, for all $e \in \Gamma(E)$, $\imath_e$ is a derivation of $(\mathscr{C}(E), \wedge)$:

$$\imath_e(C_1 \wedge C_2) = \imath_e C_1 \wedge C_2 + (-1)^n C_1 \wedge \imath_e C_2,$$

for all $C_1 \in \mathscr{C}^n(E)$ and $C_2 \in \mathscr{C}(E)$. For $C_1 \in \mathscr{C}^n(E)$ and $C_2 \in \mathscr{C}^m(E)$, with $n, m \geq 1$, $C_1 \wedge C_2$ is equivalently given by [5]:

$$\begin{aligned} C_1 \wedge C_2\,(e_1, \ldots, e_{n+m-1}) = \\ &\sum_{\tau \in Sh(n,m-1)} \operatorname{sgn}(\tau) \left\langle C_1\left(e_{\tau(1)}, \ldots, e_{\tau(n-1)}\right), e_{\tau(n)} \right\rangle C_2\left(e_{\tau(n+1)}, \ldots, e_{\tau(n+m-1)}\right) \\ &+ (-1)^{nm} \sum_{\tau \in Sh(m,n-1)} \operatorname{sgn}(\tau) \\ &\left\langle C_2\left(e_{\tau(1)}, \ldots, e_{\tau(m-1)}\right), e_{\tau(m)} \right\rangle C_1\left(e_{\tau(m+1)}, \ldots, e_{\tau(n+m-1)}\right), \end{aligned} \tag{7}$$

for all $e_1, \ldots, e_{n+m-1} \in \Gamma(E)$.

[1]Our signs are different from those in [5] and coincide with [1].

The symbol of $C_1 \wedge C_2$ is given by

$$\begin{aligned}
&\sigma_{C_1\wedge C_2}(e_1,\dots,e_{n+m-2})\cdot f = \\
&\quad \sum_{\tau\in Sh(n,m-2)} \operatorname{sgn}(\tau) \\
&\quad \left\langle C_1\left(e_{\tau(1)},\dots,e_{\tau(n-1)}\right), e_{\tau(n)}\right\rangle \sigma_{C_2}\left(e_{\tau(n+1)},\dots,e_{\tau(n+m-2)}\right)\cdot f \\
&\quad + \sum_{\tau\in Sh(n-2,m)} \operatorname{sgn}(\tau) \\
&\quad \left(\sigma_{C_1}\left(e_{\tau(1)},\dots,e_{\tau(n-2)}\right)\cdot f\right) \left\langle C_2\left(e_{\tau(n-1)},\dots,e_{\tau(n+m-3)}\right), e_{\tau(n+m-2)}\right\rangle,
\end{aligned} \tag{8}$$

for all $e_1,\dots,e_{n+m-2}\in\Gamma(E)$ and $f\in C^\infty(M)$.

Remark 3.3. A homogeneous $P\in\Gamma(\wedge^p E)$, $P = e_1\wedge\ldots\wedge e_p$, with $e_i\in\Gamma(E)=\mathscr{C}^1(E)$, can be seen as an element of $\mathscr{C}^p(E)$. From the definition and properties of the interior product, we may obtain an explicit expression for $P(e'_1,\dots,e'_{p-1})$, with $e'_1,\dots,e'_{p-1}\in\Gamma(E)$, by means of products of type $\langle e'_i, e_j\rangle$ (see also Equations (22) and (23)). Furthermore, Equation (8) yields $\sigma_P = 0$. Conversely, If C is an element of $\mathscr{C}^n(E)$ with $\sigma_C = 0$, then $C\in\Gamma(\wedge^n E)$ (see Lemma 5.7). For $P,Q\in\Gamma(\wedge^\bullet E)$, $P\wedge Q$ is the usual exterior product.

Definition 3.4. [5] The space $\mathscr{C}(E)$ is endowed with a graded Lie bracket of degree -2,

$$[\cdot,\cdot]_{KW} : \mathscr{C}^n(E)\times\mathscr{C}^m(E)\to\mathscr{C}^{n+m-2}(E),$$

uniquely defined, for all $f,g\in C^\infty(M)$, $e,e'\in\Gamma(E)$, $D\in\mathscr{C}^2(E)$, $C_1\in\mathscr{C}^n(E)$ and $C_2\in\mathscr{C}(E)$ by [2],

i) $[f,g]_{KW}=0$,
ii) $[f,e]_{KW}=0=[e,f]_{KW}$,
iii) $[e,e']_{KW}=\langle e,e'\rangle$,
iv) $[f,D]_{KW}=\sigma_D\cdot f=-[D,f]_{KW}$,
v) $[e,C_1]_{KW}=(-1)^{n+1}[C_1,e]_{KW}=\imath_e C_1$

[2] Our signs in (iv) and (v) are different from those in [5] and coincide with [1].

and, by recursion,

$$\imath_e[C_1, C_2]_{KW} = [e, [C_1, C_2]_{KW}]_{KW} = [[e, C_1]_{KW}, C_2]_{KW} + (-1)^n[C_1, [e, C_2]_{KW}]_{KW}. \tag{9}$$

In [5], it is proved that

$$[C_1, C_2 \wedge C_3]_{KW} = [C_1, C_2]_{KW} \wedge C_3 + (-1)^{nm} C_2 \wedge [C_1, C_3]_{KW}, \tag{10}$$

for all $C_1 \in \mathscr{C}^n(E)$, $C_2 \in \mathscr{C}^m(E)$ and $C_3 \in \mathscr{C}(E)$. Summing up, we have:

Proposition 3.5. [5] *The triple $(\mathscr{C}(E), \wedge, [\cdot,\cdot]_{KW})$ is a graded Poisson algebra of degree -2, that we call Keller-Waldmann Poisson algebra.*

From Remark 3.3, Definition 3.4 and Equation (10), we have:

Corollary 3.6. *For all $P \in \Gamma(\wedge^p E)$ and $Q \in \Gamma(\wedge^q E)$, $\sigma_{[P,Q]_{KW}} = 0$.*

Let V be a vector space and set $\mathfrak{g} = V^{\otimes(n-1)}$, for a fixed $n \in \mathbb{N}$. We denote by $\mathfrak{L}^p$ the space of linear maps from $\mathfrak{g}^{\otimes p} \otimes V$ to V and set $\mathfrak{L} = \oplus_{p\geq 0}\mathfrak{L}^p$, with $\mathfrak{L}^0 = \mathfrak{g}$. In [9] a bilinear bracket of degree zero on $\mathfrak{L}$,

$$[\cdot,\cdot]^{n\mathfrak{L}} : \mathfrak{L}^p \times \mathfrak{L}^q \to \mathfrak{L}^{p+q},$$

was introduced. We don't need its explicit definition which can be found in [9]. However, the important feature of $[\cdot,\cdot]^{n\mathfrak{L}}$ in the present work is that, since $[\cdot,\cdot]_{KW}$ is nothing but $-[\cdot,\cdot]^{2\mathfrak{L}}$, we may have an explicit expression for $[\cdot,\cdot]_{KW}$ that is not given in Definition 3.4, where the bracket is defined recursively.

Given $C_1 \in \mathscr{C}^n(E)$ and $C_2 \in \mathscr{C}^m(E)$, $n, m \geq 1$, the definition in [9] yields

$$[C_1, C_2]_{KW} = \imath_{C_1} C_2 - (-1)^{nm} \imath_{C_2} C_1, \tag{11}$$

with $\imath_{C_2} C_1 \in \mathscr{C}^{n+m-2}(E)$ defined, for all $e_1, \ldots, e_{n+m-3} \in \Gamma(E)$, as follows:

$$\imath_{C_2} C_1(e_1, \ldots, e_{n+m-3}) = \sum \mathrm{sgn}(J, I)\,(-1)^t \\ C_1(e_{i_1}, \ldots, e_{i_t}, C_2(e_{j_1}, \ldots, e_{j_{m-1}}), e_{i_{t+1}}, \ldots, e_{i_{n-2}}), \tag{12}$$

where the sum is over all shuffles $I = \{i_1 < \ldots < i_{n-2}\} \subset \{1, \ldots, n+m-3\} = N$. The $j's$ and t are defined by $\{j_1 < \ldots < j_{m-1}\} = N \backslash I$, $i_{t+1} = j_{m-1}+1$ or, in case $j_{m-1} = n+m-3$, $t := n-2$. The pair (J, I) denotes the permutation

$(j_1,\dots,j_{m-1},i_1,\dots,i_{n-2})$ of N. When $C_2 = e \in \mathscr{C}^1(E) = \Gamma(E)$, $\imath_e C_1$ is given by Equation (6).

Lemma 3.7. *The interior product $\imath_{C_2}C_1 \in \mathscr{C}^{n+m-2}(E)$ defined in Equation (12) is equivalently given by*

$$\imath_{C_2}C_1(e_1,\dots,e_{n+m-3}) = \sum_{k=m-1}^{n+m-3}\ \sum_{\tau\in Sh(k-(m-1),m-2)} \operatorname{sgn}(\tau)(-1)^{mk}$$
$$C_1(e_{\tau(1)},\dots,e_{\tau(k-(m-1))}, C_2(e_{\tau(k-(m-2))},\dots,e_{\tau(k-1)},e_k),e_{k+1},\dots,e_{n+m-3}), \tag{13}$$

with $C_1 \in \mathscr{C}^n(E)$ and $C_2 \in \mathscr{C}^m(E)$, $m \geq 1$.

Proof: We need to prove that (12) can be rewritten as (13). Let us consider a permutation $(i_1,\dots,i_t,j_1,\dots,j_{m-2},j_{m-1},i_{t+1},\dots,i_{n-2})$ of $N=\{1,\dots,n+m-3\}$, as in (12). It is easy to see that the last $n-t-2$ permuted indices, $(i_{t+1},\dots,i_{n-2})$, must coincide with the last $n-t-2$ elements of N: $(t+m,\dots,n+m-3)$. Then,

$$\begin{aligned}(i_1,\dots,i_t,j_1,\dots,j_{m-1},i_{t+1},\dots,i_{n-2}) &=\\ =(i_1,\dots,i_t,j_1,\dots,j_{m-1},t+m,\dots,n+m-3).&\end{aligned}$$

Moreover, the index t in (12) can be equivalently defined by setting

$$j_{m-1} = t+m-1.$$

Then, permutations $(i_1,\dots,i_t,j_1,\dots,j_{m-2},j_{m-1},i_{t+1},\dots,i_{n-2})$ considered in (12) can be rewritten as permutations

$$(i_1,\dots,i_t,j_1,\dots,j_{m-2},t+m-1,t+m,\dots,n+m-3), \tag{14}$$

where t takes values from 0 to $n-2$.[3] In addition, if we define the index k by setting $k := t+m-1$, then permutation (14) corresponds to

$$(\tau(1),\dots,\tau(k-(m-1),\tau(k-(m-2)),\dots,\tau(k-1),k,k+1,\dots,n+m-3),$$

where $\tau\in Sh(k-(m-1),m-2)$.

[3] When $t=0$, we have the trivial permutation $(\underbrace{1,\dots,m-1}_{j_1,\dots,j_{m-1}},\underbrace{m,\dots,n+m-3}_{i_1,\dots,i_{n-2}})$.

Finally, we need to rewrite the sign in (12), using the unshuffle permutation τ:

$$\begin{aligned}
&\operatorname{sgn}(J,I)\times(-1)^t = \operatorname{sgn}(j_1,\dots,j_{m+1},i_1,\dots,i_{n-2})\times(-1)^t\\
=\quad &\operatorname{sgn}(\tau(k-(m-2)),\dots,\tau(k-1),k,\tau(1),\dots,\tau(k-(m-1),\\
&\qquad k+1,\dots,n+m-3)\times(-1)^{k-(m-1)}\\
=\quad &\operatorname{sgn}\big((\tau(k-(m-2)),\dots,\tau(k-1),k,\tau(1),\dots,\tau(k-(m-1))\big)\\
&\qquad \times(-1)^{k-(m-1)}\\
=\quad &(-1)^{(k-(m-1))(m-1)}\operatorname{sgn}(\tau(1),\dots,\tau(k-(m-1),\tau(k-(m-2)),\\
&\qquad \dots,\tau(k-1),k)\times(-1)^{k-(m-1)}\\
=\quad &(-1)^{(k-(m-1))m}\operatorname{sgn}(\tau) = (-1)^{km}\operatorname{sgn}(\tau).
\end{aligned}$$

Therefore, we can rewrite (12) as

$$\imath_{C_2}C_1(e_1,\dots,e_{n+m-3}) = \sum_{k=m-1}^{n+m-3}\ \sum_{\tau\in Sh(k-(m-1),m-2)} \operatorname{sgn}(\tau)(-1)^{km}$$
$$C_1(e_{\tau(1)},\dots,e_{\tau(k-(m-1))},C_2(e_{\tau(k-(m-2))},\dots,e_{\tau(k-1)},e_k),e_{k+1},\dots,e_{n+m-3}). \quad \blacksquare$$

Lemma 3.7 together with Equation (11), provide an explicit definition of the bracket $[\cdot,\cdot]_{KW}$.

For the sake of completeness, in the next lemma we give the explicit formula for the symbol of $\imath_{C_2}C_1$.

Lemma 3.8. *Given $C_1\in\mathscr{C}^n(E)$ and $C_2\in\mathscr{C}^m(E)$, the symbol of $\imath_{C_2}C_1\in\mathscr{C}^{n+m-2}(E)$ is given by*

$$\begin{aligned}
\sigma_{\imath_{C_2}C_1}(e_1,\dots,e_{n+m-4})\cdot f = &\sum_{k=m-1}^{n+m-4}\ \sum_{\tau\in Sh(k-(m-1),m-2)} (-1)^{m(k-(m-1))}sgn(\tau)\\
&\sigma_{C_1}\Big(e_{\tau(1)},\dots,e_{\tau(k-(m-1))},\\
&C_2\left(e_{\tau(k-(m-2))},\dots,e_{\tau(k-1)},e_k\right),e_{k+1},\dots,e_{n+m-4}\Big)\cdot f\\
&+\sum_{\tau\in Sh(n-2,m-2)}(-1)^{m(n-2)}sgn(\tau)\\
&\sigma_{C_1}\left(e_{\tau(1)},\dots,e_{\tau(n-2)}\right)\cdot\left(\sigma_{C_2}\left(e_{\tau(n-1)},\dots,e_{\tau(n+m-4)}\right)\cdot f\right),
\end{aligned}$$

for all $e_1, \dots, e_{n+m-4} \in \Gamma(E)$ *and* $f \in C^\infty(M)$.

Definition 3.9. A pre-multi-Courant structure $C \in \mathscr{C}^n(E)$, $n \geq 2$, is a *multi-Courant structure* if $[C, C]_{_{KW}} = 0$. In this case, the triple $(E, \langle\cdot,\cdot\rangle, C)$ is called a *multi-Courant algebroid.*

For $n = 3$, a multi-Courant structure is simply the usual Courant structure on $(E, \langle\cdot,\cdot\rangle)$.

Remark 3.10. Since the bracket $[\cdot,\cdot]_{_{KW}}$ is graded skew-symmetric, given $C \in \mathscr{C}^{2k}(E)$, $k \geq 1$, we always have $[C, C]_{_{KW}} = 0$. So, all $(2k-1)$-ary pre-Courant structures are $(2k-1)$-ary Courant structures.

Lemma 3.7, Definition 3.9 and Remark 3.10 yield the next proposition.

Proposition 3.11. *A pre-multi-Courant structure* $C \in \mathscr{C}^n(E)$*, with* n *odd, is a multi-Courant structure if and only if*

$$\sum_{k=n-1}^{2n-3} \sum_{\tau \in Sh(k-(n-1), n-2)} \operatorname{sgn}(\tau)(-1)^{nk}$$
$$C(e_{\tau(1)}, \dots, e_{\tau(k-(n-1))}, C(e_{\tau(k-(n-2))}, \dots, e_{\tau(k-1)}, e_k), e_{k+1}, \dots, e_{2n-3}) = 0,$$

for all $e_i \in \Gamma(E)$, $1 \leq i \leq 2n-3$.

Remark 3.12. Let $C \in \mathscr{C}^{n+1}(E)$ be a n-ary pre-Courant structure on $(E, \langle\cdot,\cdot\rangle)$. If C satisfies the *Filippov identity* [4]:

$$\begin{aligned} &C(e_1, \dots, e_{n-1}, C(e'_1, \dots, e'_n)) \\ &\qquad = \sum_{i=1}^{n} C(e'_1, \dots, e'_{i-1}, C(e_1, \dots, e_{n-1}, e'_i), e'_{i+1}, \dots, e'_n), \end{aligned} \tag{15}$$

for all $e_1, \dots, e_{n-1}, e'_1, \dots, e'_n \in \Gamma(E)$, we say that C is a n*-Filippov Courant structure* on E [4]. Notice that n-Filippov Courant structures are called n-Courant structures in [5].

If C is a n-Filippov Courant structure on $(E, \langle\cdot,\cdot\rangle)$, $\Gamma(E)$ is equipped with a n-Leibniz algebra structure. Thus, a 2-Filippov Courant structure on $(E, \langle\cdot,\cdot\rangle)$ is the same as a Courant algebroid structure on $(E, \langle\cdot,\cdot\rangle)$. However, comparing Equation (15) with the identity in Proposition 3.11, we see that, for $n \geq 3$, n-Filippov algebroids and n-ary Courant algebroids are different structures.

[4]Equation (15) means that $C(e_1, \dots, e_{n-1}, \cdot)$ is a derivation of C.

An interesting aspect of the bracket $[\cdot,\cdot]^{n\mathfrak{L}}$ introduced in [9], is that it characterizes n-Leibniz brackets as those which are closed with respect to it. Indeed, as it is proved in [9], Equation (15) is equivalent to $[C,C]^{n\mathfrak{L}}=0$.

3.3. An alternative definition. There is an alternative definition of pre-multi-Courant structure on $(E,\langle\cdot,\cdot\rangle)$ that we shall use in the next sections.

Given $C\in\mathscr{C}^n(E)$, $n\geq 1$, we may define a map

$$\widetilde{C}:\Gamma(E)\times\overset{(n)}{\cdots}\times\Gamma(E)\to C^\infty(M)$$

by setting

$$\widetilde{C}(e_1,\dots,e_n):=\langle C(e_1,\dots,e_{n-1}),e_n\rangle \tag{16}$$

and, for $C\in\mathscr{C}^0(E)=C^\infty(M)$, $\widetilde{C}=C$. Notice that for $C\in\mathscr{C}^1(E)$, $\widetilde{C}(e)=\langle C,e\rangle$, for all $e\in\Gamma(E)$.

As it is remarked in [5], Definition 3.1 can be reformulated using the maps $\widetilde{C}$. In particular, $\widetilde{C}$ is $C^\infty(M)$-linear in the last entry and Equations (3) and (4) are equivalent to

$$\begin{aligned}\widetilde{C}(e_1,\dots,e_i,e_{i+1},\dots,e_n)+\widetilde{C}(e_1,\dots,e_{i+1},e_i,\dots,e_n)=\\ =\sigma_C(e_1,\dots,\widehat{e_i},\widehat{e_{i+1}},\dots,e_n)\cdot\langle e_i,e_{i+1}\rangle.\end{aligned} \tag{17}$$

Let $\widetilde{\mathscr{C}}^n(E)$ be the collection of maps $\widetilde{C}$ defined by (16), and set $\widetilde{\mathscr{C}}(E)=\oplus_{n\geq 0}\widetilde{\mathscr{C}}^n(E)$. There is a degree zero product on $\widetilde{\mathscr{C}}(E)$, that we also denote by $\wedge$:

$$\widetilde{C_1}\wedge\widetilde{C_2}=\widetilde{C_1\wedge C_2}, \tag{18}$$

for all $\widetilde{C_1}\in\widetilde{\mathscr{C}}^m(E)$ and $\widetilde{C_2}\in\widetilde{\mathscr{C}}^n(E)$, $m,n\geq 0$. Explicitly,

$$\begin{aligned}&\widetilde{C_1}\wedge\widetilde{C_2}(e_1,\dots,e_{m+n})=\\ &\quad=\sum_{\tau\in Sh(m,n)}\operatorname{sgn}(\tau)\,\widetilde{C_1}(e_{\tau(1)},\dots,e_{\tau(m)})\,\widetilde{C_2}(e_{\tau(m+1)},\dots,e_{\tau(m+n)}),\end{aligned} \tag{19}$$

for all $e_1,\dots,e_{m+n}\in\Gamma(E)$. The map

$$\widetilde{\cdot}:\mathscr{C}(E)\to\widetilde{\mathscr{C}}(E),\quad C\in\mathscr{C}^n(E)\mapsto\widetilde{C}\in\widetilde{\mathscr{C}}^n(E),$$

is an isomorphism of graded commutative algebras [5].

We may define a degree -2 bracket on $\widetilde{\mathscr{C}}(E)$, by setting

$$\left[\widetilde{C_1}, \widetilde{C_2}\right]_{\widetilde{KW}} := \widetilde{[C_1, C_2]_{KW}} \tag{20}$$

i.e., given $\widetilde{C_1} \in \widetilde{\mathscr{C}}^m(E)$ and $\widetilde{C_2} \in \widetilde{\mathscr{C}}^n(E)$,

$$\left[\widetilde{C_1}, \widetilde{C_2}\right]_{\widetilde{KW}} (e_1, \dots, e_{m+n-2}) = \langle [C_1, C_2]_{KW}(e_1, \dots, e_{m+n-3}), e_{m+n-2}\rangle$$

for all $e_1, \dots, e_{m+n-2} \in \Gamma(E)$. In Section 5 we shall see that $[\cdot, \cdot]_{\widetilde{KW}}$ is the bracket referred in Remark 2.6 of [3].

By construction, the map

$$\tilde{\cdot} : (\mathscr{C}(E), \wedge, [\cdot, \cdot]_{KW}) \to (\widetilde{\mathscr{C}}(E), \wedge, [\cdot, \cdot]_{\widetilde{KW}})$$

is an isomorphism of graded Poisson algebras.

Remark 3.13. Given $C \in \mathscr{C}^n(E)$, due to (20) and the non-degeneracy of $\langle \cdot, \cdot \rangle$, we have

$$[C, C]_{KW} = 0 \;\Leftrightarrow\; \left[\widetilde{C}, \widetilde{C}\right]_{\widetilde{KW}} = 0$$

and therefore, Definition 3.9 can be given using either $\widetilde{C} \in \widetilde{\mathscr{C}}^n(E)$ or $C \in \mathscr{C}^n(E)$.

4. Higher multi-Courant structures and higher Keller-Waldmann Poisson algebra

Inspired by the generalization of the Lie bracket by the Schouten bracket, in this section we extend a pre-multi-Courant structure $C \in \mathscr{C}^{n+1}(E)$ on $(E, \langle \cdot, \cdot \rangle)$ to the space $\Gamma(\wedge^{\geq 1} E)$, asking the extension to be a derivation in each entry.

4.1. Extension of the bilinear form. We start by extending the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\Gamma(E)$ to $\Gamma(\wedge^\bullet E)$ as follows. Given two homogeneous elements $P \in \Gamma(\wedge^p E)$ and $Q \in \Gamma(\wedge^q E)$, with $p, q \geq 1$, $\langle P, Q\rangle \in \Gamma(\wedge^{p+q-2} E)$, i.e., $\langle \cdot, \cdot \rangle$ is a degree -2 operation. Moreover, $\langle \cdot, \cdot \rangle$ satisfies the following conditions:

i) $\langle P, Q\rangle = -(-1)^{pq}\langle Q, P\rangle$;
ii) $\langle f, R\rangle = \langle R, f\rangle = 0$;

iii)
$$\langle P, Q\wedge R\rangle = \langle P, Q\rangle \wedge R + (-1)^{pq} Q\wedge \langle P, R\rangle; \tag{21}$$
iv)
$$\langle P, \langle Q, R\rangle\rangle = \langle\langle P, Q\rangle, R\rangle + (-1)^{pq}\langle Q, \langle P, R\rangle\rangle,$$
for all $R \in \Gamma(\wedge^\bullet E)$ and $f \in C^\infty(M)$. Extending by bilinearity, $\langle\cdot,\cdot\rangle$ is defined in the whole $\Gamma(\wedge^\bullet E)$ and $(\Gamma(\wedge^\bullet E), \langle\cdot,\cdot\rangle)$ is a graded Lie algebra. Note that $\langle\cdot,\cdot\rangle$ is $C^\infty(M)$-linear in both entries.

Lemma 4.1. *Let $P = e_1\wedge\ldots\wedge e_p \in \Gamma(\wedge^p E)$ and $Q = e'_1\wedge\ldots\wedge e'_q \in \Gamma(\wedge^q E)$ be two homogeneous elements of $\Gamma(\wedge^{\geq 1} E)$. Then,*
$$\langle P, Q\rangle = \sum_{s=1}^{q}\sum_{k=1}^{p}(-1)^{k-s+p+1}\langle e_k, e'_s\rangle \widehat{P^k}\wedge\widehat{Q^s}, \tag{22}$$
where $\widehat{P^k} = e_1\wedge\ldots\wedge\widehat{e_k}\wedge\ldots\wedge e_p \in \Gamma(\wedge^{p-1}E)$ and $\widehat{Q^s} = e'_1\wedge\ldots\wedge\widehat{e'_s}\wedge\ldots\wedge e'_q \in \Gamma(\wedge^{q-1}E)$.

For $P \in \Gamma(\wedge^p E)$, $P \in \mathscr{C}^p(E)$ and is given by
$$P(e_1, \ldots, e_{p-1}) = \langle e_{p-1}, \ldots, \langle e_2, \langle e_1, P\rangle\rangle\ldots\rangle, \tag{23}$$
while $\widetilde{P} \in \widetilde{\mathscr{C}}^p(E)$ is given by
$$\widetilde{P}(e_1, \ldots, e_p) = \langle e_p, \ldots, \langle e_2, \langle e_1, P\rangle\rangle\ldots\rangle,$$
for all $e_1, \ldots e_p \in \Gamma(E)$.

Lemma 4.2. *For $P, Q \in \Gamma(\wedge^{\geq 1}E)$,*
$$\langle P, Q\rangle = [P, Q]_{_{KW}}.$$
Proof: Let us prove this result for homogeneous elements $P \in \Gamma(\wedge^p E)$ and $Q \in \Gamma(\wedge^q E)$. We shall use induction on $n = p + q$. For $n = 2$, we know from of Definition 3.4 (iii) that $\langle e, e'\rangle = [e, e']_{_{KW}}$, for all $e, e' \in \Gamma(E)$. Now, let us suppose that, for some $k \geq 2$ and for all homogeneous elements $P, Q \in \Gamma(\wedge^{\geq 1}E)$, such that $p + q \leq k$, we have $\langle P, Q\rangle = [P, Q]_{_{KW}}$. Let us consider $P, Q \in \Gamma(\wedge^{\geq 1}E)$, such that $p + q = k + 1$, we need to prove that $\langle P, Q\rangle = [P, Q]_{_{KW}}$. Since $p + q = k + 1 \geq 3$, we can suppose, without loss of generality,

that $q \geq 2$ and write $Q = \widehat{Q} \wedge e$, for some $e \in \Gamma(E)$. Then, using (21) and (10), we have

$$\begin{aligned}\langle P, Q\rangle &= \langle P, \widehat{Q} \wedge e\rangle \\ &= \langle P, \widehat{Q}\rangle \wedge e + (-1)^{p(q-1)}\widehat{Q} \wedge \langle P, e\rangle \\ &= [P, \widehat{Q}]_{_{KW}} \wedge e + (-1)^{p(q-1)}\widehat{Q} \wedge [P, e]_{_{KW}} \\ &= [P, \widehat{Q} \wedge e]_{_{KW}} \\ &= [P, Q]_{_{KW}}.\end{aligned}$$

■

4.2. Higher Multi-Courant algebroids. Now we introduce the main notion of this section. By $Der(C^\infty(M), \Gamma(\wedge^\bullet E))$ we denote the space of derivations of $C^\infty(M)$ with values in $\Gamma(\wedge^\bullet E)$.

Definition 4.3. A *higher pre-multi-Courant structure* on $(E, \langle\cdot,\cdot\rangle)$ is a multilinear map

$$\mathfrak{C} : \Gamma(\wedge^{\geq 1}E)\times \overset{(n)}{\dots} \times\Gamma(\wedge^{\geq 1}E) \to \Gamma(\wedge^\bullet E), \quad n \geq 0,$$

of degree $-n$, for which there exists a map $\sigma_{\mathfrak{C}}$, called the *symbol* of $\mathfrak{C}$,

$$\sigma_{\mathfrak{C}} : \Gamma(\wedge^{\geq 1}E)\times \overset{(n-2)}{\dots} \times\Gamma(\wedge^{\geq 1}E) \longrightarrow Der(C^\infty(M), \Gamma(\wedge^\bullet E)),$$

such that $\mathfrak{C}$ is $C^\infty(M)$-linear in the last entry and the following conditions hold:

$$\begin{aligned}\mathfrak{C}(P_1, \dots, P_i \wedge R, \dots, P_n) = (-1)^{p_i(p_{i+1}+\dots+p_n)}P_i \wedge \mathfrak{C}(P_1, \dots, R, \dots, P_n) \\ + (-1)^{r(p_i+\dots+p_n)}R \wedge \mathfrak{C}(P_1, \dots, P_i, \dots, P_n),\end{aligned} \tag{24}$$

$$\begin{aligned}\sigma_{\mathfrak{C}}(P_1, \dots, P_i \wedge R, \dots, P_{n-2}) = (-1)^{p_i(p_{i+1}+\dots+p_{n-2})}P_i \wedge \sigma_{\mathfrak{C}}(P_1, \dots, R, \dots, P_{n-2}) \\ + (-1)^{r(p_i+\dots+p_{n-2})}R \wedge \sigma_{\mathfrak{C}}(P_1, \dots, P_i, \dots, P_{n-2}),\end{aligned} \tag{25}$$

$$\begin{aligned}\mathfrak{C}(P_1, \dots, P_i, e, e', P_{i+1}, \dots, P_{n-2}) + \mathfrak{C}(P_1, \dots, P_i, e', e, P_{i+1}, \dots, P_{n-2}) = \\ = \sigma_{\mathfrak{C}}(P_1, \dots, P_i, P_{i+1}, \dots, P_{n-2})\left(\langle e, e'\rangle\right),\end{aligned} \tag{26}$$

for all $e, e' \in \Gamma(E)$ and for all homogeneous $P_i \in \Gamma(\wedge^{p_i}E)$, and $R \in \Gamma(\wedge^r E)$, where $p_i \geq 1$, $r \geq 1$ and $1 \leq i \leq n$. For $n = 0$, $\mathfrak{C} \in C^\infty(M)$.

The triple $(E, \langle\cdot,\cdot\rangle, \mathfrak{C})$ is called a *higher n-ary pre-Courant algebroid* or a *higher pre-multi-Courant algebroid*, if we don't want to specify the arity of $\mathfrak{C}$.

Notice that, for $P_i \in \Gamma(\wedge^{p_i}E)$, $1 \le i \le n-2$, and $f \in C^\infty(M)$,

$$\sigma_{\mathfrak{C}}(P_1, \dots, P_{n-2})(f) \in \Gamma(\wedge^{p_1+\dots+p_{n-2}-n+2}E).$$

Lemma 4.4. *If the bilinear form $\langle\cdot,\cdot\rangle$ is full* [5]*, $\sigma_{\mathfrak{C}}$ is $C^\infty(M)$-linear in the last entry.*

Proof: It is a direct consequence of (26) and the fact that $\langle\cdot,\cdot\rangle$ is full and $\mathfrak{C}$ is $C^\infty(M)$-linear in the last entry. ■

The space of higher n-ary pre-Courant structures on E is denoted by $\mathscr{C}^n(\wedge^{\ge 1}E)$ and we set

$$\mathscr{C}(\wedge^{\ge 1}E) = \oplus_{n\ge 0}\mathscr{C}^n(\wedge^{\ge 1}E),$$

with $\mathscr{C}^0(\wedge^{\ge 1}E) := C^\infty(M)$.

The alternative definition of pre-multi-Courant structure, introduced in 3.3, allows us to construct an example of higher pre-multi-Courant structure.

Given $\widetilde{C} \in \widetilde{\mathscr{C}}^n(E)$, we denote by $\overline{\widetilde{C}}$ its extension by derivation in each entry, i.e., $\overline{\widetilde{C}}$ and $\widetilde{C}$ coincide on sections of E and, furthermore, $\overline{\widetilde{C}}$ satisfies

$$\begin{aligned}\overline{\widetilde{C}}(P_1, \dots, P_i \wedge e, \dots, P_n) &= (-1)^{p_i(p_{i+1}+\dots+p_n)} P_i \wedge \overline{\widetilde{C}}(P_1, \dots, e, \dots, P_n) \\ &\quad + (-1)^{p_i+\dots+p_n} e \wedge \overline{\widetilde{C}}(P_1, \dots, P_i, \dots, P_n), \qquad (27)\end{aligned}$$

for all homogeneous $P_i \in \Gamma(\wedge^{p_i}E)$, $p_i \ge 1$, $1 \le i \le n$, and $e \in \Gamma(E)$. For $f \in \widetilde{\mathscr{C}}^0(E) = C^\infty(M)$, we set $\bar{f} = f$. Moreover, we associate to $\overline{\widetilde{C}}$ the map

$$\sigma_{\overline{\widetilde{C}}} : \Gamma(\wedge^{\ge 1}E) \times \overset{(n-2)}{\dots} \times \Gamma(\wedge^{\ge 1}E) \to Der(C^\infty(M), \Gamma(\wedge^\bullet E)), \;\; n \ge 2,$$

[5] The bilinear form is said to be *full* if $\langle\cdot,\cdot\rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ is surjective.

that coincides with σ_C on sections of E and is the extension by derivation in each entry of σ_C, i.e., for all $f \in C^\infty(M)$,

$$\begin{aligned} \sigma_{\overset{\approx}{C}}(P_1, \dots, P_i \wedge e, \dots, P_{n-2})\,(f) = \\ &= (-1)^{p_i(p_{i+1}+\ldots+p_{n-2})} P_i \wedge \sigma_{\overset{\approx}{C}}(P_1, \dots, e, \dots, P_{n-2})\,(f) \\ &+ (-1)^{p_i+\ldots+p_{n-2}} e \wedge \sigma_{\overset{\approx}{C}}(P_1, \dots, P_i, \dots, P_{n-2})\,(f). \end{aligned} \tag{28}$$

Lemma 4.5. *For $\widetilde{C} \in \widetilde{\mathscr{C}}^n(E)$, $\overset{\approx}{C}$ defined by Equation (27) is an element of $\mathscr{C}^n(\wedge^{\geq 1}E)$, with symbol given by Equation (28).*

Proof: Applying repeatedly (27) (resp. (28)), we obtain (24) (resp. (25)). Also, it is immediate that $\overset{\approx}{C}$ is $C^\infty(M)$-linear in the last entry.

It remains to prove that, for all $e, e' \in \Gamma(E)$ and for all homogeneous $P_i \in \Gamma(\wedge^{p_i}E)$, $p_i \geq 1$, we have

$$\begin{aligned} \overset{\approx}{C}(P_1, \dots, P_i, e, e', P_{i+1}, \dots, P_{n-2}) + \overset{\approx}{C}(P_1, \dots, P_i, e', e, P_{i+1}, \dots, P_{n-2}) = \\ = \sigma_{\overset{\approx}{C}}(P_1, \dots, P_i, P_{i+1}, \dots, P_{n-2})\,(\langle e, e' \rangle)\,. \end{aligned} \tag{29}$$

Let us prove this by induction on $p_1 + \ldots + p_{n-2}$.
When

$$p_1 + \ldots + p_{n-2} = n - 2,$$

then $p_i = 1$, for $i = 1, \dots, n-2$ and (29) reduces to (17), which is satisfied by $\overset{\approx}{C}$ and $\sigma_{\overset{\approx}{C}}$. Now, suppose that (29) is satisfied for all $P_1, \dots, P_{n-2}$ such that

$$n - 2 \leq p_1 + \ldots + p_{n-2} \leq k,$$

for some $k \geq n-2$, and let us prove it for $P_1, \dots, P_{n-2}$ such that

$$p_1 + \ldots + p_{n-2} = k + 1.$$

Because $k + 1 \geq n - 1$, there is at least one $j \in \{1, \dots, n-2\}$ such that $p_j \geq 2$ and then we can write $P_j = \widehat{P}_j \wedge u$, with $u \in \Gamma(E)$. Then,

$$
\begin{aligned}
&\overline{\widetilde{C}}(P_1,\ldots,\widehat{P}_j\wedge u,\ldots,e,e',\ldots,P_{n-2})+\overline{\widetilde{C}}(P_1,\ldots,\widehat{P}_j\wedge u,\ldots,e',e,\ldots,P_{n-2}\\
&\quad=(-1)^{(p_j-1)(p_{j+1}+\ldots+p_{n-2})}\,\widehat{P}_j\wedge\overline{\widetilde{C}}(P_1,\ldots,u,\ldots,e,e',\ldots,P_{n-2})\\
&\qquad\qquad+(-1)^{(p_j-1+p_{j+1}\ldots+p_{n-2})}\,u\wedge\overline{\widetilde{C}}(P_1,\ldots,\widehat{P}_j,\ldots,e,e',\ldots,P_{n-2})\\
&\qquad+(-1)^{(p_j-1)(p_{j+1}+\ldots+p_{n-2})}\,\widehat{P}_j\wedge\overline{\widetilde{C}}(P_1,\ldots,u,\ldots,e',e,\ldots,P_{n-2})\\
&\qquad+(-1)^{(p_j-1+p_{j+1}\ldots+p_{n-2})}\,u\wedge\overline{\widetilde{C}}(P_1,\ldots,\widehat{P}_j,\ldots,e',e,\ldots,P_{n-2})\\
&\quad=(-1)^{(p_j-1)(p_{j+1}+\ldots+p_{n-2})}\,\widehat{P}_j\wedge\Big(\overline{\widetilde{C}}(P_1,\ldots,u,\ldots,e,e',\ldots,P_{n-2})+\\
&\qquad\qquad+\overline{\widetilde{C}}(P_1,\ldots,u,\ldots,e',e,\ldots,P_{n-2})\Big)\\
&\qquad+(-1)^{(p_j-1+p_{j+1}\ldots+p_{n-2})}\,u\wedge\Big(\overline{\widetilde{C}}(P_1,\ldots,\widehat{P}_j,\ldots,e,e',\ldots,P_{n-2})\\
&\qquad\qquad+\overline{\widetilde{C}}(P_1,\ldots,\widehat{P}_j,\ldots,e',e,\ldots,P_{n-2})\Big)\\
&\quad=(-1)^{(p_j-1)(p_{j+1}+\ldots+p_{n-2})}\,\widehat{P}_j\wedge\sigma_{\widetilde{C}}(P_1,\ldots,u,\ldots,\widehat{e},\widehat{e'},\ldots,P_{n-2})\left(\langle e,e'\rangle\right)\\
&\qquad+(-1)^{(p_j-1+p_{j+1}\ldots+p_{n-2})}\,u\wedge\sigma_{\widetilde{C}}(P_1,\ldots,\widehat{P}_j,\ldots,\widehat{e},\widehat{e'},\ldots,P_{n-2})\left(\langle e,e'\rangle\right)\\
&\quad=\sigma_{\widetilde{C}}(P_1,\ldots,\widehat{P}_j\wedge u,\ldots,\widehat{e},\widehat{e'},\ldots,P_{n-2})\left(\langle e,e'\rangle\right). \qquad\blacksquare
\end{aligned}
$$

Next proposition establishes a relation between $\widetilde{\mathscr{C}}(E)$ and $\mathscr{C}(\wedge^{\geq 1}E)$.

Proposition 4.6. *There is a one-to-one correspondence between $\widetilde{\mathscr{C}}(E)$ and $\mathscr{C}(\wedge^{\geq 1}E)$ such that, for all $n\geq 1$,*

$$
\begin{aligned}
\bar{\cdot}:\widetilde{\mathscr{C}}^n(E) &\to \mathscr{C}^n(\wedge^{\geq 1}E)\\
\widetilde{C} &\mapsto \overline{\widetilde{C}},
\end{aligned}
$$

with $\overline{\widetilde{C}}$ given by Equation (27). For $n=0$, $\bar{\cdot}$ is the identity map.

Proof: Given $\mathfrak{C}\in\mathscr{C}^n(\wedge^{\geq 1}E)$, its restriction to $\Gamma(E)$ satisfies (17) so that $\mathfrak{C}|_{\Gamma(E)}\in\widetilde{\mathscr{C}}^n(E)$. It is obvious that $\overline{\mathfrak{C}|_{\Gamma(E)}}=\mathfrak{C}$.

Now, if $\overline{\widetilde{C}}_1 = \overline{\widetilde{C}}_2 \in \mathscr{C}^n(\wedge^{\geq 1}E)$, obviously $\overline{\widetilde{C}}_1|_{\Gamma(E)} = \overline{\widetilde{C}}_2|_{\Gamma(E)}$, which means $\widetilde{C}_1 = \widetilde{C}_2$. ∎

Having the one-to-one correspondence given by Proposition 4.6, and if there is no ambiguity, in the sequel we shall write very often $\overline{\widetilde{C}}$ instead of $\mathfrak{C}$.

Remark 4.7. Let us explain why we consider $\overline{\widetilde{C}}$, the extension of $\widetilde{C} \in \widetilde{\mathscr{C}}(E)$ by derivation in each argument, instead of $\overline{C}$, the extension of $C \in \mathscr{C}(E)$ by derivation in each argument. The reason comes from what should be the extension by derivation of Equation (4) in Definition 3.1. The corresponding condition that $\overline{C}$ should satisfy is

$$\begin{aligned}\langle \overline{C}(P_1, \ldots, e, e', \ldots, P_{n-3}) + \overline{C}(P_1, \ldots, e', e, \ldots, P_{n-3}), P_{n-2}\rangle =\\ = \sigma_{\overline{C}}(P_1, \ldots, \widehat{e_i}, \widehat{e_{i+1}}, \ldots, P_{n-3}, P_{n-2})(\langle e, e'\rangle),\end{aligned}$$

for all $e, e' \in \Gamma(E)$ and for all homogeneous $P_i \in \Gamma(\wedge^{p_i}E)$. But in this expression, the right hand side is derivative with respect to each argument $P_i, i = 1, \ldots, n-3$ while the left hand side is not. On the contrary, Equation (26) is fully derivative on both sides.

4.3. Higher Keller-Waldmann Poisson algebra. The space $\mathscr{C}(\wedge^{\geq 1}E)$ is endowed with an associative graded commutative product of degree zero, that we denote by $\wedge$ [6], defined as follows. Given $\overline{\widetilde{C}}_1 \in \mathscr{C}^r(\wedge^{\geq 1}E)$ and $\overline{\widetilde{C}}_2 \in \mathscr{C}^s(\wedge^{\geq 1}E)$, set

$$\overline{\widetilde{C}}_1 \wedge \overline{\widetilde{C}}_2 := \overline{\widetilde{C}_1 \wedge \widetilde{C}_2}, \tag{30}$$

where the product $\wedge$ on the right-hand side is the one defined by Equation (19). Using Equation (18), we may write

$$\overline{\widetilde{C}}_1 \wedge \overline{\widetilde{C}}_2 = \overline{\widetilde{C_1 \wedge C_2}}.$$

The space $\mathscr{C}(\wedge^{\geq 1}E)$ is endowed with the following bracket of degree -2,

$$\begin{aligned}[\![\cdot,\cdot]\!] : \mathscr{C}^r(\wedge^{\geq 1}E) \times \mathscr{C}^s(\wedge^{\geq 1}E) &\to \mathscr{C}^{r+s-2}(\wedge^{\geq 1}E)\\ (\overline{\widetilde{C}}_1, \overline{\widetilde{C}}_2) &\mapsto \left[\!\!\left[\overline{\widetilde{C}}_1, \overline{\widetilde{C}}_2\right]\!\!\right] := \overline{\left[\widetilde{C}_1, \widetilde{C}_2\right]_{\widetilde{KW}}}.\end{aligned}$$

[6]Although we use the same notation, this product is not the one defined in $\mathscr{C}(E)$.

As a consequence of Equation (20) and Lemma 4.2, we have:

Lemma 4.8. *For $P, Q \in \Gamma(\wedge^{\geq 1} E)$,*

$$\left[\!\left[\overline{\widetilde{P}}, \overline{\widetilde{Q}} \right]\!\right] = \overline{\widetilde{[P,Q]_{_{KW}}}} = \overline{\widetilde{\langle P, Q \rangle}}.$$

Theorem 4.9. *The triple $(\mathscr{C}(\wedge^{\geq 1}E), \wedge, [\![\cdot,\cdot]\!])$ is a graded Poisson algebra of degree -2, that we call the higher Keller-Waldmann Poisson algebra.*

Proof: Bilinearity and graded skew-symmetry of $[\![\cdot,\cdot]\!]$ are obvious. Let us take $\overline{\widetilde{C}}_i \in \mathscr{C}(\wedge^{\geq 1}E)$, $i = 1, 2, 3$. Since

$$\left[\!\left[\left[\!\left[\overline{\widetilde{C}}_1, \overline{\widetilde{C}}_2 \right]\!\right], \overline{\widetilde{C}}_3 \right]\!\right] = \left[\!\left[\overline{[\widetilde{C}_1, \widetilde{C}_2]_{_{KW}}}, \overline{\widetilde{C}}_3 \right]\!\right] = \overline{\left[\left[\widetilde{C}_1, \widetilde{C}_2 \right]_{\widetilde{KW}}, \widetilde{C}_3 \right]_{\widetilde{KW}}},$$

the graded Jacobi identity of $[\![\cdot,\cdot]\!]$ follows from the graded Jacobi identity of $[\cdot,\cdot]_{\widetilde{KW}}$. Analogously for the Leibniz rule, since

$$\left[\!\left[\overline{\widetilde{C}}_1, \overline{\widetilde{C}}_2 \wedge \overline{\widetilde{C}}_3 \right]\!\right] = \left[\!\left[\overline{\widetilde{C}}_1, \overline{\widetilde{C}_2 \wedge \widetilde{C}_3} \right]\!\right] = \overline{\left[\widetilde{C}_1, \widetilde{C}_2 \wedge \widetilde{C}_3 \right]_{\widetilde{KW}}}. \quad \blacksquare$$

It is now obvious that

$$\bar{\cdot} : \left(\widetilde{\mathscr{C}}(E), \wedge, [\cdot,\cdot]_{\widetilde{KW}} \right) \to \left(\mathscr{C}(\wedge^{\geq 1}E), \wedge, [\![\cdot,\cdot]\!] \right)$$

is an isomorphism of graded Poisson algebras.

Definition 4.10. A higher pre-multi-Courant structure $\mathfrak{C} \equiv \overline{\widetilde{C}} \in \mathscr{C}^n(\wedge^{\geq 1}E)$, $n \geq 2$, is a higher multi-Courant structure if $[\![\mathfrak{C}, \mathfrak{C}]\!] = 0$. In this case, the triple $(E, \langle\cdot,\cdot\rangle, \mathfrak{C})$ is called a *higher multi-Courant algebroid.*

Note that, because the bracket $[\![\cdot,\cdot]\!]$ is skew-symmetric, all $\mathfrak{C} \in \mathscr{C}^{2k}(\wedge^{\geq 1}E)$, $k \geq 1$, are higher multi-Courant structures.

5. On Cueca-Mehta isomorphism

In this section we consider the graded algebras $(\mathscr{F}_E, \cdot)$, with $\mathscr{F}_E = C^\infty\big(p^*\big(T^*[2]E[1]\big)\big)$ (see 2.1), and $\left(\widetilde{\mathscr{C}}(E), \wedge\right) \simeq (\mathscr{C}(E), \wedge)$. The isomorphism

$$\widetilde{\Upsilon} : (\mathscr{F}_E, \cdot) \to \left(\widetilde{\mathscr{C}}(E), \wedge \right),$$

introduced in [3], maps $\Theta \in \mathscr{F}_E^n$, $n \geq 1$, into $\widetilde{\Upsilon}(\Theta) \in \widetilde{\mathscr{C}}^n(E)$ given by

$$\widetilde{\Upsilon}(\Theta)(e_1, e_2, \ldots, e_n) = \{e_n, \ldots, \{e_2, \{e_1, \Theta\}\} \ldots\} \in C^\infty(M), \tag{31}$$

with symbol

$$\sigma_{\widetilde{\Upsilon}(\Theta)}(e_1, \ldots, e_{n-1}) \cdot f = \{f, \{e_{n-1}, \ldots, \{e_1, \Theta\}\} \ldots\},$$

for all $e_1, \ldots, e_n \in \Gamma(E)$ and $f \in C^\infty(M)$. For $n = 0$, and for all $f \in \mathscr{F}_E^0 = C^\infty(M)$, $\widetilde{\Upsilon}(f) = f$. Moreover, $\widetilde{\Upsilon}$ is an isomorphism of graded commutative algebras [3]:

$$\widetilde{\Upsilon}(\Theta \cdot \Theta') = \widetilde{\Upsilon}(\Theta) \wedge \widetilde{\Upsilon}(\Theta'), \quad \Theta, \Theta' \in \mathscr{F}_E. \tag{32}$$

The isomorphism $\widetilde{\Upsilon}$ induces an isomorphism

$$\Upsilon : (\mathscr{F}_E, \cdot) \to (\mathscr{C}(E), \wedge)$$

that maps $\Theta \in \mathscr{F}_E^n$, $n \geq 1$, into $\Upsilon(\Theta) \in \mathscr{C}^n(E)$ defined by

$$\langle \Upsilon(\Theta)(e_1, \ldots, e_{n-1}), e_n \rangle = \widetilde{\Upsilon}(\Theta)(e_1, \ldots, e_n),$$

for all $e_1, \ldots, e_n \in \Gamma(E)$, and $\Upsilon(f) = f$, for all $f \in C^\infty(M)$. Due to the non-degeneracy of $\langle \cdot, \cdot \rangle$, Υ is well-defined and, since $\{\cdot, \cdot\}$ is also non-degenerate, we have

$$\Upsilon(\Theta)(e_1, e_2, \ldots, e_{n-1}) = \{e_{n-1}, \ldots, \{e_2, \{e_1, \Theta\}\} \ldots\} \in \Gamma(E). \tag{33}$$

In particular, $\Upsilon(e) = e$, for all $e \in \Gamma(E)$. The symbol of $\Upsilon(\Theta)$ is given by

$$\sigma_{\Upsilon(\Theta)}(e_1, \ldots, e_{n-2}) \cdot f = \{f, \{e_{n-2}, \ldots, \{e_1, \Theta\}\} \ldots\}, \tag{34}$$

for all $f \in C^\infty(M)$.

Remark 5.1. Equations (33) and (34) show that, in $\mathscr{F}_E$, the extension of $C \in \mathscr{C}^n(E)$ considered in Remark 3.2 and, in particular Equation (5), appears in a natural way.

Moreover, $\widetilde{\Upsilon}$ being an isomorphism of graded commutative algebras, Υ inherits the same property, as it is shown in the next lemma.

Lemma 5.2. *For every $\Theta \in \mathscr{F}_E^n$ and $\Theta' \in \mathscr{F}_E^m$,*

$$\Upsilon(\Theta \cdot \Theta') = \Upsilon(\Theta) \wedge \Upsilon(\Theta'). \tag{35}$$

Proof: Using (7), (32) and the $C^\infty(M)$-linearity of $\langle\cdot,\cdot\rangle$ we have, for all $e_1,\dots,e_{m+n}$,

$$\left\langle \Upsilon(\Theta)\wedge\Upsilon(\Theta')(e_1,\dots,e_{m+n-1}), e_{m+n}\right\rangle =$$

$$= \left\langle \sum_{\tau\in Sh(m,n-1)} \mathrm{sgn}\,(\tau)\widetilde{\Upsilon}(\Theta)(e_{\tau(1)},\dots,e_{\tau(m)})\,\Upsilon(\Theta')(e_{\tau(m+1)},\dots,e_{\tau(m+n-1)}), e_{m+n}\right\rangle$$

$$+(-1)^{mn}\left\langle \sum_{\tau\in Sh(n,m-1)} \mathrm{sgn}\,(\tau)\right.$$

$$\left.\widetilde{\Upsilon}(\Theta')(e_{\tau(1)},\dots,e_{\tau(n)})\,\Upsilon(\Theta)(e_{\tau(n+1)},\dots,e_{\tau(n+m-1)}), e_{m+n}\right\rangle$$

$$= \sum_{\tau\in Sh(m,n-1)} \mathrm{sgn}\,(\tau)\widetilde{\Upsilon}(\Theta)(e_{\tau(1)},\dots,e_{\tau(m)})\widetilde{\Upsilon}(\Theta')(e_{\tau(m+1)},\dots,e_{\tau(m+n-1)}, e_{m+n})$$

$$+(-1)^{mn}\sum_{\tau\in Sh(n,m-1)} \mathrm{sgn}\,(\tau)$$

$$\widetilde{\Upsilon}(\Theta')(e_{\tau(1)},\dots,e_{\tau(n)})\widetilde{\Upsilon}(\Theta)(e_{\tau(n+1)},\dots,e_{\tau(n+m-1)}, e_{m+n})$$

$$= \sum_{\tau\in Sh(m,n)} \mathrm{sgn}\,(\tau)\widetilde{\Upsilon}(\Theta)(e_{\tau(1)},\dots,e_{\tau(m)})\widetilde{\Upsilon}(\Theta')(e_{\tau(m+1)},\dots,e_{\tau(m+n-1)}, e_{\tau(m+n)})$$

$$= \widetilde{\Upsilon}(\Theta)\wedge\widetilde{\Upsilon}(\Theta')(e_1,\dots,e_{m+n}) = \widetilde{\Upsilon}(\Theta\cdot\Theta')(e_1,\dots,e_{m+n})$$
$$= \langle \Upsilon(\Theta\cdot\Theta')(e_1,\dots,e_{m+n-1}), e_{m+n}\rangle\,.$$

The non-degeneracy of $\langle\cdot,\cdot\rangle$ completes the proof. ■

Since $\Upsilon(f)=f$, for all $f\in C^\infty(M)$, and $\Upsilon(e)=e$, for all $e\in\Gamma(E)$, Lemma 5.2 yields

$$\Upsilon_{|\Gamma(\wedge^n E)} = \mathrm{id},\ n\geq 0. \tag{36}$$

Remark 5.3. In the case where $E=A\oplus A^*$, the above mentioned isomorphism was defined in [1] recursively by the following procedure. Taking into account that the algebra $(\mathscr{C}(E),\wedge)$ is generated by its terms of degrees 0, 1 and 2 [5], the map Υ is defined in $\mathscr{F}^0_{A\oplus A^*}$, $\mathscr{F}^1_{A\oplus A^*}$ and $\mathscr{F}^2_{A\oplus A^*}$ as follows:

$$\begin{cases} \Upsilon(f) = f, & f \in C^\infty(M); \\ \Upsilon(e) = e, & e \in \Gamma(E); \\ \Upsilon(\Theta) = \{\cdot, \Theta\}, & \Theta \in \mathscr{F}^2. \end{cases}$$

Then, Υ is extended to $\mathscr{F}_{A\oplus A^*}$ by linearity, asking that Equation (35) holds for all $\Theta \in \mathscr{F}^n_{A\oplus A^*}$ and $\Theta' \in \mathscr{F}^m_{A\oplus A^*}$. Using the Leibniz rule and the Jacobi identity for $\{\cdot,\cdot\}$, we conclude that the isomorphisms introduced in [3] an in [1] are the same.

Lemma 5.4. *Let Θ be a function in $\mathscr{F}^n_E$ and $e \in \Gamma(E)$. Then,*

$$\Upsilon(\{e,\Theta\}) = \imath_e(\Upsilon(\Theta)) = [e, \Upsilon(\Theta)]_{KW}.$$

Proof: For all $e_1, \dots, e_{n-2} \in \Gamma(E)$, we have

$$\begin{aligned} \Upsilon(\{e,\Theta\})(e_1,\dots,e_{n-2}) &= \{e_{n-2},\dots,\{e_1,\{e,\Theta\}\}\dots\} \\ &= \Upsilon(\Theta)(e,e_1,\dots,e_{n-2}) \\ &= \imath_e(\Upsilon(\Theta))(e_1,\dots,e_{n-2}). \end{aligned}$$

Thus,

$$\Upsilon(\{e,\Theta\}) = \imath_e(\Upsilon(\Theta)). \quad \blacksquare$$

Proposition 5.5. *Let $\Theta \in \mathscr{F}^n_E$ and $\Theta' \in \mathscr{F}^m_E$, $n, m \geq 0$. Then,*

$$\Upsilon(\{\Theta,\Theta'\}) = [\Upsilon(\Theta), \Upsilon(\Theta')]_{KW}. \tag{37}$$

Proof: The proof is done by induction on the sum $n+m$ of degrees of Θ and Θ'. First, let us prove directly that (37) holds for all possible cases such that $n+m \leq 2$. For $f, g \in C^\infty(M)$, $e \in \Gamma(E)$ and $\delta \in \mathscr{F}^2_E$, using Definition 3.4 and (36), we have:

i) $\Upsilon(\{f,g\}) = 0 = [\Upsilon(f), \Upsilon(g))]_{KW}$;
ii) $\Upsilon(\{f,e\}) = 0 = [\Upsilon(f), \Upsilon(e)]_{KW}$;
iii) $\Upsilon(\{e,e'\}) = \Upsilon(\langle e,e'\rangle) = \langle e,e'\rangle = [e,e']_{KW} = [\Upsilon(e), \Upsilon(e')]_{KW}$;
iv) $\Upsilon(\{f,\delta\}) = \{f,\delta\} = \sigma_{\Upsilon(\delta)} \cdot f = [\Upsilon(f), \Upsilon(\delta)]_{KW}$.

Now, let us assume that (37) holds for $n+m \leq k$, $k \geq 2$. Take $\Theta \in \mathscr{F}^n_E$ and $\Theta' \in \mathscr{F}^m_E$, with $n+m = k+1$. For every $e \in \Gamma(E)$, using (9), Lemma 5.4

and the Jacobi identity of $\{\cdot,\cdot\}$, we have

$$\begin{aligned}\imath_e(\Upsilon(\{\Theta,\Theta'\})) &= \Upsilon(\{e,\{\Theta,\Theta'\}\}) = \Upsilon\big(\{\{e,\Theta\},\Theta'\}\} + (-1)^n\{\Theta,\{e,\Theta'\}\}\big)\\ &= [\Upsilon(\{e,\Theta\}),\Upsilon(\Theta')]_{\scriptscriptstyle KW} + (-1)^n[\Upsilon(\Theta),\Upsilon(\{e,\Theta\})]_{\scriptscriptstyle KW}\\ &= [\imath_e(\Upsilon(\Theta)),\Upsilon(\Theta')]_{\scriptscriptstyle KW} + (-1)^n[\Upsilon(\Theta),\imath_e(\Upsilon(\Theta'))]_{\scriptscriptstyle KW}\\ &\overset{(9)}{=} \imath_e[\Upsilon(\Theta),\Upsilon(\Theta')]_{\scriptscriptstyle KW},\end{aligned}$$

where in the third equality we use the induction hypothesis. Since $e \in \Gamma(E)$ is arbitrary, (37) is proved. ■

We have proved the following.

Theorem 5.6. *The map $\Upsilon : (\mathscr{F}_E, \cdot, \{\cdot,\cdot\}) \to (\mathscr{C}(E), \wedge, [\cdot,\cdot]_{\scriptscriptstyle KW})$ is a degree zero isomorphism of graded Poisson algebras.*

Now, we prove a result announced in Remark 3.3.

Lemma 5.7. *If C is an element of $\mathscr{C}^n(E)$ with $\sigma_C = 0$, then $C \in \Gamma(\wedge^n E)$.*

Proof: Let us consider $C \in \mathscr{C}^n(E)$ such that $\sigma_C = 0$. Because Υ is an isomorphism, $C = \Upsilon(\Theta)$, for some $\Theta \in \mathscr{F}_E^n$. Using (34) and the Jacobi identity for $\{\cdot,\cdot\}$, we have

$$\begin{aligned}0 &= \sigma_C(e_1,\dots,e_{n-2})\cdot f = \{f,\{e_{n-2},\dots,\{e_1,\Theta\}\}\dots\}\\ &= \{e_{n-2},\dots,\{e_1,\{f,\Theta\}\}\dots\},\end{aligned}$$

for all $e_1,\dots,e_{n-2} \in \Gamma(E)$ and $f \in C^\infty(M)$. The non-degeneracy of $\{\cdot,\cdot\}$ implies that equation above is equivalent to

$$\{f,\Theta\} = 0, \text{ for all } f \in C^\infty(M),$$

which means (see [10]) that $\Theta \in \Gamma(\wedge^n E) \subset \mathscr{F}_E^n$. Therefore, since $\Upsilon_{|\Gamma(\wedge^n E)}$ is the identity map, we have $C \in \Gamma(\wedge^n E)$. ■

The isomorphism $\widetilde{\Upsilon} : \mathscr{F}_E \to \widetilde{\mathscr{C}}(E)$, defined by Equation (31), naturally gives rise to a map

$$\overline{\widetilde{\Upsilon}} : \mathscr{F}_E \to \mathscr{C}(\wedge^{\geq 1} E)$$

such that

$$\Theta \in \mathscr{F}_E^n \mapsto \overline{\widetilde{\Upsilon}}(\Theta) := \overline{\widetilde{\Upsilon}(\Theta)} \in \mathscr{C}^n(\wedge^{\geq 1} E),$$

where $\overline{\widetilde{\Upsilon}(\Theta)}$ is the extension by derivation in each entry of $\widetilde{\Upsilon}(\Theta) \in \widetilde{\mathscr{C}}^n(E)$.

Theorem 5.8. *The map* $\overline{\widetilde{\Upsilon}} : (\mathscr{F}_E, \cdot, \{\cdot,\cdot\}) \to \left(\mathscr{C}(\wedge^{\geq 1}E), \wedge, [\![\cdot,\cdot]\!]\right)$ *is a degree zero isomorphism of graded Poisson algebras.*

Proof: For every $\Theta \in \mathscr{F}_E^n$ and $\Theta' \in \mathscr{F}_E^m$, we have

$$\begin{aligned}\overline{\widetilde{\Upsilon}}(\{\Theta,\Theta'\}) = \overline{\widetilde{\Upsilon}(\{\Theta,\Theta'\})} = \overline{\widetilde{\Upsilon(\{\Theta,\Theta'\})}} = \overline{\widetilde{[\Upsilon(\Theta),\Upsilon(\Theta')]_{KW}}} \\ = \overline{\left[\widetilde{\Upsilon}(\Theta),\widetilde{\Upsilon}(\Theta')\right]_{\widetilde{KW}}} = \left[\!\!\left[\overline{\widetilde{\Upsilon}}(\Theta),\overline{\widetilde{\Upsilon}}(\Theta')\right]\!\!\right].\end{aligned}$$

Moreover, (30), (18) and (35) yield

$$\overline{\widetilde{\Upsilon}}(\Theta\cdot\Theta') = \overline{\widetilde{\Upsilon}}(\Theta)\wedge\overline{\widetilde{\Upsilon}}(\Theta'). \quad \blacksquare$$

Remark 5.9. We should stress that, although

$$\Upsilon(\Theta)(e_1,\dots,e_{n-1}) = \{e_{n-1},\dots,\{e_2,\{e_1,\Theta\}\}\dots\},$$

for all $e_1,\dots,e_{n-1} \in \Gamma(E)$, in general

$$\overline{\widetilde{\Upsilon}}(\Theta)(P_1,\dots,P_n) \neq \{P_n,\dots,\{P_2,\{P_1,\Theta\}\}\dots\},$$

for all $P_1,\dots,P_n \in \Gamma(\wedge^{\geq 1}E)$.

Notice that from the proof of Theorem 5.8, we get the following result.

Corollary 5.10. *The map* $\widetilde{\Upsilon} : (\mathscr{F}_E, \cdot, \{\cdot,\cdot\}) \to \left(\widetilde{\mathscr{C}}(E), \wedge, [\cdot,\cdot]_{\widetilde{KW}}\right)$ *is a degree zero isomorphism of graded Poisson algebras.*

Remark 5.11. The $[\cdot,\cdot]_{\widetilde{KW}}$ bracket, given by Equation (20), is the bracket announced, but not explicitly defined, in Remark 2.6 of [3].

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