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DESCENT FOR INTERNAL MULTICATEGORY FUNCTORS

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Abstract: We study descent for (generalized) internal multicategories. Two approaches to the problem are given. The first one relies on establishing the category of internal multicategories as an equalizer of categories of diagrams. The second approach follows the techniques developed by Ivan Le Creurer in his study of descent for internal categories.

Keywords: effective descent morphisms, Grothendieck descent theory, internal T-multicategory, pseudoequalizer, coherence.
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Introduction

The study of effective descent morphisms is in the core of Grothendieck Descent Theory (see *e.g.* [JT94, JST04]) and its applications (see, for instance, [BJ01]). Except for the case of locally cartesian closed categories, the full characterization of effective descent morphisms is far from trivial in general. The topological descent case is the main example of such non-trivial problem (see the characterization in [RT94] and the reformulation in [CH02]).

By studying descent for $\mathcal{V}$-categories and $(T, \mathcal{V})$-categories*, Clementino and Hofmann gave further descent results and understanding in various contexts, including, for instance, the reinterpretation of the topological results mentioned above and many other interesting connections (see, for instance, [CH04, CJ11, CH12, CH17]).

On one hand, since they were mainly concerned with *spaces*, their study focused on the case where $\mathcal{V}$ is a preorder and there is no way to trivially generalize their approach to more general monoidal categories $\mathcal{V}$. On the other hand, their work, together with the characterization of effective descent morphisms for the category of categories and internal categories (see [JST04, Section 6] and [Cr99]), has raised interest in further studying descent for *generalized* categorical structures.

With this in mind, [Lu18, Lemma 9.10] showed that, under suitable conditions, we can embed the category of $\mathcal{V}$-enriched categories (with $\mathcal{V}$ extensive) in the category of internal ones. From this embedding, [Lu18, Theorem 1.6] and Ivan Le Creurer's [Cr99] characterization, [Lu18, Theorem 1.6] gives effective descent morphisms for $\mathcal{V}$-categories. However, the literature lacks results for $(T, \mathcal{V})$-categories for a non-trivial T and an extensive $\mathcal{V}$.

The present note is part of a project which aims the study of descent and Janelidze-Galois theory within the realm of generalized multicategories and other categorical structures. Following the approach of [Lu18, Theorem 1.6], in order to study effective descent morphisms between more general (enriched) multicategories, the first step is to

*The notion of $(T, \mathcal{V})$-categories was introduced in [CT03].

study effective descent morphisms between *internal* multicategories[†] which is the subject of this paper.

We give two ways of approaching the problem. After fixing some notation on Section 1, and describing the category of multicategories as a full embedding in the pseudo-equalizer of categories of diagrams in Section 2, we discuss each approach in the two subsequent sections. We end the paper with a short discussion of the known examples of cartesian monads.

Section 3 has more of an expository value. We exploit the techniques of [Lu18] and give an example of how one can proceed to use those techniques to study effective descent morphisms. The basic idea is to fully embed the category we are studying in a bilimit of simple enough categories, and, then, use the classical descent result for full embeddings (see Theorem 3.2). This exposition is especially relevant to our future work where the second approach isn't possible.

We can study descent for internal structures exploiting and extending the techniques established in [Cr99]. That is our second approach to the case of internal multicategories, presented in Section 4. This approach has the advantage of refining our previous results. Namely, under suitable conditions, we find that functors f such that

- f_1 is an effecive descent morphism,
- f_2 is descent morphism,
- f_3 is an almost descent morphism.

to be effective descent morphisms between internal T-multicategories, where f_i are the morphisms between the objects of i-tuples of composable morphisms.

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1. Preliminaries

Given a diagram $J\colon \mathcal{B} \to \mathcal{C}$ with limit $(\lim J, \lambda)$, for any cone $\gamma_b\colon x \to Jb$ there exists a unique morphism $f\colon x \to \lim J$ such that $\gamma_b = \lambda_b \circ f$ for all b. We denote f as $(\gamma_b)_{b \in \mathsf{ob}\, \mathcal{B}}$. As an example, let $\mathcal{B}$

[†]By internal multicategories, we mean the notion introduced in [He00, Definition 4.2].

be a category with pullbacks, and let $\mathscr{C}$ be an internal category. The object of pairs of composable morphisms is given by the pullback

$$\begin{tikzcd} \mathscr{C}_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & \mathscr{C}_1 \arrow[d, "d_1"] \\ \mathscr{C}_1 \arrow[r, "d_0"'] & \mathscr{C}_0 \end{tikzcd}$$

thus, if we have morphisms $g\colon X \to \mathscr{C}_1$ and $f\colon X \to \mathscr{C}_1$ with $d_1 \circ g = d_0 \circ f$, we write $(g,\, f)$ for the uniquely determined morphism $X \to \mathscr{C}_2$. Furthermore, we denote the internal composition by $g \bullet f = d_1 \circ (g,\, f)$, where $d_1\colon \mathscr{C}_2 \to \mathscr{C}_1$ is the composition morphism. Likewise, we can talk about tuples of composable "morphisms", an idea we apply to T–multicategories.

Another remark on notation: in a category $\mathcal{B}$ with a choice of pullbacks, we write

$$\begin{tikzcd} v \arrow[r, "\varepsilon_f"] \arrow[d, "p^* f"'] & w \arrow[d, "f"] \\ x \arrow[r, "p"'] & y \end{tikzcd}$$

for the given choice of pullback of f along p. It is clear that the change-of-base $p^*\colon \mathcal{B}/y \to \mathcal{B}/x$ defines a functor right adjoint to

$$p_!\colon \mathcal{B}/x \to \mathcal{B}/y$$

with counit ε. For a morphism $h\colon f \to g$ in $\mathcal{B}/y$ (that is, $f = g \circ h$), write p^*_h for the unique morphism $p^* f \to p^* g$ such that $\varepsilon_g \circ p^*_h = h \circ \varepsilon_f$.

The category $\mathsf{Desc}(p)$ of descent data for a morphism $p\colon x \to y$ in $\mathcal{B}$ is defined as the category of algebras for the monad $p^* p_!$. Explicitly, objects are pairs of morphisms (a, γ) satisfying

– $p^*(p \circ a) = a \circ \gamma$, that is, γ is a morphism $p^*(p \circ a) \to a$ in $\mathcal{B}/x$,
– $\gamma \circ p^*_{\varepsilon_{p\circ a}} = \gamma \circ p^*_\gamma$, which is the multiplication law (note that $p \circ a \circ \gamma = p \circ p^*(p \circ a)$, so that we may apply p^*),
– $\gamma \circ (a, \mathrm{id}) = \mathrm{id}$, where (a, id) is the unique morphism such that $a = p^*(p \circ a) \circ (a, \mathrm{id})$ and $\mathrm{id} = \varepsilon_{p\circ a} \circ (a, \mathrm{id})$, which is the unit law,

and a morphism $(a, \gamma) \to (b, \theta)$ of descent data is a morphism f with $a = b \circ f$ such that $f \circ \gamma = \theta \circ p_f^*$.

By the Bénabou-Roubaud theorem‡, this is equivalent to the classical formulation of the descent category w.r.t. the basic (bi)fibration.

The following lemma seems to be fairly commonly used in proofs about effective descent morphisms, but it sometimes is omitted in the literature. The proof of this result is given in Appendix A.

Lemma 1.1. $\mathcal{K}^p \colon \mathcal{B}/y \to \mathsf{Desc}(p)$ *is essentially surjective if and only if, for all descent data* (a, γ)*, there is* f *such that* $p^* f \cong a$ *and* $\varepsilon_f \circ \gamma = \varepsilon_f \circ \varepsilon_{p \circ a}$.

Throughout, we assume $T = (T, m, e)$ is a cartesian monad on a category $\mathcal{B}$ with pullbacks.

2. Multicategories and pseudoequalizers

In this section, we describe the category $\mathsf{Cat}(T, \mathcal{B})$ of internal T-multicategories in $\mathcal{B}$ as a strict equalizer of categories of diagrams in $\mathcal{B}$. As every strict equalizer is a full subcategory of the corresponding pseudo-equalizer, we get a full embedding of $\mathsf{Cat}(T, \mathcal{B})$ in a pseudo-equalizer involving the same categories. In Section 3, we use this embedding to study the effective descent morphisms of $\mathsf{Cat}(T, \mathcal{B})$. Essential to this study is the characterization of the internal T-multicategories among the objects of the pseudo-equalizer via coherence conditions to which this section is devoted to.

As defined in [He00], a T-multicategory internal to $\mathcal{B}$ is a monad in the bicategory $\mathsf{Span}_T(\mathcal{B})$, and a functor between two such T-multicategories is a monad morphism considering the usual proarrow equipment

$$\mathcal{B} \to \mathsf{Span}_T(\mathcal{B});$$

these define a category $\mathsf{Cat}(T, \mathcal{B})$. Explicitly, a T-multicategory is given by an object x_0 of $\mathcal{B}$, together with a span

$$Tx_0 \xleftarrow{d_1} x_1 \xrightarrow{d_0} x_0$$

‡It was originally proven in [BR70]. See, for instance, [JT94, pag. 258] or [Lu18, Theorem 7.4 and Theorem 8.5] for generalizations.

and two morphisms, given by dashed arrows below

$$
\begin{tikzcd}
 & x_2 \arrow[dl, "m\circ Td_1\circ d_2"'] \arrow[dr, "d_0\circ d_0"] \arrow[dd, dashed, "d_1"] & \\
Tx_0 & & x_0 \\
 & x_1 \arrow[ul, "d_1"] \arrow[ur, "d_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x_0 \arrow[dl, "e"'] \arrow[dr, "\mathrm{id}_{x_0}"] \arrow[dd, dashed, "s_0"] & \\
Tx_0 & & x_0 \\
 & x_1 \arrow[ul, "d_1"] \arrow[ur, "d_0"'] &
\end{tikzcd}
$$

which make the triangles commute, where

$$
\begin{tikzcd}
x_2 \arrow[r, "d_2"] \arrow[d, "d_0"'] & Tx_1 \arrow[d, "Td_0"] \\
x_1 \arrow[r, "d_1"'] & Tx_0
\end{tikzcd}
$$

is a pullback diagram. Thus, we say that a pair $g\colon a \to x_1$, $f\colon a \to Tx_1$ is *composable* if $d_1 g = (Td_0)f$, we write $(g, f)\colon a \to x_2$ for the uniquely defined morphism, and we let $g \bullet f = d_1(g, f)$. Likewise, define $k \bullet_T f = (Td_1)(k, h)$ for $k\colon a \to Tx_1$ and $h\colon a \to TTx_1$ such that $(Td_1)k = (TTd_0)h$ (T-composable).

The identity properties of the monad guarantee that $1_{d_0 f} \bullet ef = f = f \bullet 1_{d_1 f}$, and the associativity property guarantees that

$$h \bullet (g \bullet_T f) = (h \bullet g) \bullet mf,$$

where we are implicitly given the following pullback diagram

$$
\begin{tikzcd}
x_3 \arrow[r, "d_3"] \arrow[d, "d_0"'] & Tx_2 \arrow[d, "Td_0"] \\
x_2 \arrow[r, "d_2"'] & Tx_1
\end{tikzcd}
$$

for $h\colon a \to x_1$, $g\colon a \to Tx_1$ and $f\colon a \to TTx_1$ such that h, g are composable and g, f are T-composable. Moreover, a functor $p\colon x \to y$ between internal T-multicategories is given by a pair of morphisms $p_0\colon x_0 \to y_0$ and $p_1\colon x_1 \to y_1$ such that $d_i \circ p_1 = (T^i p_0) \circ d_i$ for $i = 0, 1$, $1_{p_0} = p_1 1$ and $p_1 g \bullet p_1 f = p_1(g \bullet f)$.

Going back to an internal description, we may denote

– $s_0 = (\mathrm{id},\, Ts_0 \circ d_1)\colon x_1 \to x_2$,
– $s_1 = (s_0 \circ d_0,\, e)\colon x_1 \to x_2$,
– $d_1 = (d_0 \circ d_0,\, Td_1 \circ d_3)$,
– $d_2 = (d_1 \circ d_0,\, m \circ Td_2 \circ d_3)$,

so the above data can be organized in the following diagram

$$
\begin{tikzcd}
x_0 \arrow[r, shift left, "s_0"] \arrow[dr, "e"'] & x_1 \arrow[l, shift left, "d_0"] \arrow[d, "d_1"'] \arrow[r, shift left=3, "s_1"] \arrow[r, shift left=1, "s_0"] \arrow[dr, "e"'] & x_2 \arrow[l, shift left=1, "d_0"] \arrow[l, shift left=3, "d_1"] \arrow[d, "d_2"'] & x_3 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=2, "d_2"] \arrow[d, "d_3"'] \\
& Tx_0 \arrow[r, shift left, "Ts_0"] & Tx_1 \arrow[l, shift left, "Td_0"] \arrow[d, "Td_1"'] & Tx_2 \arrow[l, shift right, "Td_0"'] \arrow[l, shift left, "Td_1"] \arrow[d, "Td_2"'] \\
& & TTx_0 \arrow[ul, "m"] & TTx_1 \arrow[ul, "m"]
\end{tikzcd}
$$

which is also given in [Bu71]. This leads us to the following diagrammatical description of an internal T-multicategory:

Theorem 2.1. *For a cartesian monad (T, m, e) on a category $\mathcal{B}$ with pullbacks, $\mathsf{Cat}(T, B)$ is the strict equalizer of the following diagram:*

$$
[\mathcal{S},\mathcal{B}] \underset{\Gamma}{\overset{S^*_-}{\rightrightarrows}} [\mathcal{S}_T,\mathcal{B}] \times [\mathcal{S}_{m_0},\mathcal{B}] \times [\mathcal{S}_{m_1},\mathcal{B}] \times [\mathcal{S}_{e_0},\mathcal{B}] \times [\mathcal{S}_{e_1},\mathcal{B}] \times [\mathcal{S}_{p_0},\mathcal{B}] \times [\mathcal{S}_{p_1},\mathcal{B}] \tag{1}
$$

Moreover, $\mathsf{Cat}(T, \mathcal{B})$ has pullbacks and the canonical functor

$$
\iota\colon \mathsf{Cat}(T, \mathcal{B}) \to [\mathcal{S}, \mathcal{B}]
$$

preserves them.

The category $\mathcal{S}$ is generated by the following diagram

$$
\begin{tikzcd}
x_0 \arrow[r, shift left, "s_0"] \arrow[dr, "e_0"'] & x_1 \arrow[l, shift left, "d_0"] \arrow[r, shift left=3, "s_1"] \arrow[r, shift left=1, "s_0"] \arrow[d, "d_1"'] \arrow[dr, "e_1"] & x_2 \arrow[l, shift left=1, "d_0"] \arrow[l, shift right=1, "d_1"] \arrow[d, "d_2"'] & x_3 \arrow[l, shift left=2, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift right=2, "d_2"] \arrow[d, "d_3"'] \\
 & x_0' \arrow[r, shift left, "s_0'"] & x_1' \arrow[l, shift left, "d_0'"] \arrow[d, "d_1'"'] & x_2' \arrow[l, shift right, "d_0'"'] \arrow[l, shift left, "d_1'"] \arrow[d, "d_2'"'] \\
 & & x_0'' \arrow[ul, "m_0"] & x_1'' \arrow[ul, "m_1"]
\end{tikzcd}
$$

with relations resembling the simplicial conditions

- $d_{1+i} \circ s_i = e_i \colon x_i \to x_i'$, $i = 0, 1$,
- $d_0 \circ s_0 = \mathrm{id} \colon x_0 \to x_0$,
- $d_1 \circ d_1 = m_0 \circ d_1' \circ d_2 \colon x_2 \to x_0'$,
- $d_0 \circ d_1 = d_0 \circ d_0 \colon x_2 \to x_0$,
- $d_2 \circ s_0 = s_0' \circ d_1 \colon x_1 \to x_1'$,
- $d_0 \circ s_0 = \mathrm{id} \colon x_1 \to x_1$,
- $d_0 \circ s_1 = s_0 \circ d_0 \colon x_1 \to x_1$,
- $d_1 \circ s_0 = d_1 \circ s_1 = \mathrm{id} \colon x_1 \to x_1$,
- $d_2 \circ d_2 = m_1 \circ d_2' \circ d_3 \colon x_3 \to x_1$,
- $d_0 \circ d_2 = d_1 \circ d_0 \colon x_3 \to x_1$,
- $d_2 \circ d_1 = d_1' \circ d_3 \colon x_3 \to x_1$,
- $d_0 \circ d_1 = d_0 \circ d_0 \colon x_3 \to x_1$
- $d_1 \circ d_2 = d_1 \circ d_1 \colon x_3 \to x_1$,

$\mathcal{S}_I$, $\mathcal{S}_T$, $\mathcal{S}_{m_i}$, $\mathcal{S}_{e_i}$, $\mathcal{S}_{s_i}$, and $\mathcal{S}_{p_i}$ for $i = 0, 1$ are subcategories of $\mathcal{S}$, respectively given by

$$
\begin{tikzcd}
x_0 \arrow[r, shift left, "s_0"] & x_1 \arrow[l, shift left, "d_0"] \arrow[d, "d_1"'] & x_2 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[d, "d_2"] \\
 & x_0' & x_1'
\end{tikzcd}
$$

$$\begin{tikzcd}
x'_0 \arrow[r, shift left, "s'_0"] & x'_1 \arrow[l, shift left, "d'_0"] \arrow[d, "d'_1"'] & x'_2 \arrow[l, shift right, "d'_0"'] \arrow[l, shift left, "d'_1"] \arrow[d, "d'_2"] \\
& x''_0 & x''_1
\end{tikzcd}$$

$$\begin{tikzcd}
x''_i \arrow[r, "m_i"] & x'_i
\end{tikzcd}$$

$$\begin{tikzcd}
x_i \arrow[r, "e_i"] & x'_i
\end{tikzcd}$$

$$\begin{tikzcd}
x'_{i+1} \arrow[r, "d'_0"] & x'_i & x_{i+1} \arrow[l, "d_{i+1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
x_{i+2} \arrow[r, "d_0"] \arrow[d, "d_{i+2}"'] & x_{i+1} \arrow[d, "d_{i+1}"] \\
x'_{i+1} \arrow[r, "d'_0"'] & x'_i
\end{tikzcd}$$

and write S^*_I, S^*_T, $S^*_{m_i}$, $S^*_{e_i}$, $S^*_{s_i}$ and $S^*_{p_i}$ for the restriction functors. Also write x^*_0 and $x^*_1\colon [\mathcal{S}, \mathcal{B}] \to \mathcal{B}$ for the projections. With these, S^*_-, and Γ are the uniquely determined functors given by the following:

$$\begin{tikzcd}
{[\mathcal{S}, \mathcal{B}]} \arrow[r, "S^*_T"] \arrow[d, "S^*_I"'] & {[\mathcal{S}_T, \mathcal{B}]} \\
{[\mathcal{S}_I, \mathcal{B}]} \arrow[r, "r^*"'] & {[\mathcal{S}_T, \mathcal{B}]} \arrow[u, "T_*"']
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathcal{S}, \mathcal{B}]} \arrow[rr, "S^*_{m_i}"] \arrow[dr, "x^*_i"'] & & {[\mathcal{S}_m, \mathcal{B}]} \\
& \mathcal{B} \arrow[ur, "\hat{m}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathcal{S}, \mathcal{B}]} \arrow[rr, "S^*_{e_i}"] \arrow[dr, "x^*_i"'] & & {[\mathcal{S}_{e_i}, \mathcal{B}]} \\
& \mathcal{B} \arrow[ur, "\hat{e}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathcal{S},\mathcal{B}]} \arrow[rr, "S^*_{p_i}"] \arrow[dr, "S^*_{s_i}"'] & & {[\mathcal{S}_{p_i},\mathcal{B}]} \\
& {[\mathcal{S}_{s_i},\mathcal{B}]} \arrow[ur, "\mathsf{Ran}_{\iota_i}"'] &
\end{tikzcd}$$

where $\hat{m}$, $\hat{e}$ are (pullback-preserving) functors induced by the respective (cartesian) natural transformations, $r\colon S_T \to S_I$ is the projection (note that S_T is free while S_I is not), and Ran_{ι_i} is the right Kan extension (take the pullback) of the inclusion $\iota_i\colon S_{s_i} \to S_{p_i}$.

Note that in general, the strict equalizer is a full subcategory of the pseudo-equalizer: for functors $F, G\colon \mathcal{C} \to \mathcal{D}$, the category $\mathsf{PsEq}(F,G)$ is the category whose objects are pairs (c,ϕ) where c is an object of $\mathcal{C}$ and $\iota\colon Fy \to Gy$ is an isomorphism, and morphisms $(c,\phi) \to (d,\psi)$ are morphisms $f\colon c \to d$ such that $Gf \circ \phi = \psi \circ Ff$. Thus, the full embedding may be given on objects by $x \mapsto (x,\mathrm{id})$. Moreover, since pullbacks in $\mathsf{Cat}(T,\mathcal{B})$ and $\mathsf{PsEq}(S^*_{-},\Gamma)$ are calculated pointwise in $[\mathcal{S},\mathcal{B}]$, the full embedding preserves pullbacks.

Lemma 2.2. *The inclusion of* $\mathsf{Cat}(T,\mathcal{B})$ *into* $\mathsf{PsEq}(S^*_{-},\Gamma)$ *is full and preserves pullbacks.*

Given an object (y,ι) of $\mathsf{PsEq}(S^*_{-},\Gamma)$, ι can be explicitly described as a family of isomorphisms making the appropriate squares commute:

$$\begin{tikzcd}
y'_0 \arrow[r, bend left, "s'_0"] \arrow[ddd, "\iota^T_0"'] & y'_1 \arrow[l, bend left, "d'_0"] \arrow[d, "d'_1"] \arrow[ddd, bend left=60, "\iota^T_1"] & y'_2 \arrow[l, shift right, "d'_0"'] \arrow[l, shift left, "d'_1"] \arrow[d, "d'_2"] \arrow[ddd, bend left=60, "\iota^T_2"] \\
& y''_0 \arrow[d, "\iota^T_3"] & y''_1 \arrow[d, "\iota^T_4"] \\
& Ty'_0 & Ty'_1 \\
Ty_0 \arrow[r, bend left, "Ts_0"] & Ty_1 \arrow[l, bend left, "Td_0"] \arrow[u, "Td_1"] & Ty_2 \arrow[l, shift right, "Td_0"'] \arrow[l, shift left, "Td_1"] \arrow[u, "Td_2"']
\end{tikzcd}$$

$$\begin{tikzcd} y''_i \arrow[r, "M"] \arrow[d, "\iota_0^{m_i}"'] & y'_i \arrow[d, "\iota_1^{m_i}"] \\ TTy_i \arrow[r, "m"] & Ty_i \end{tikzcd} \qquad \begin{tikzcd} y_i \arrow[r, "e_i"] \arrow[d, "\iota_0^{e_i}"'] & y'_i \arrow[d, "\iota_1^{e_i}"] \\ y_i \arrow[r, "e"'] & Ty_i \end{tikzcd}$$

$$\begin{tikzcd} y_{i+2} \arrow[rrr, "d_0"] \arrow[ddd, "d_{i+2}"'] & & & y_{i+1} \arrow[ddd, "d_{i+1}"] \\ & \cdot \arrow[ul, "\iota_0^{p_i}"] \arrow[r] \arrow[d] & y_{i+1} \arrow[ur, "\iota_1^{p_i}"] \arrow[d, "d_{i+1}"] & \\ & y'_{i+1} \arrow[r, "D_0"] \arrow[dl, "\iota_2^{p_i}"] & y'_i \arrow[dr, "\iota_3^{p_i}"] & \\ y'_{i+1} \arrow[rrr, "D_0"'] & & & y'_i \end{tikzcd}$$

And the following lemma gives a characterization of those objects (y, ι) which are isomorphic to a T-multicategory.

Lemma 2.3. *An object (y, ι) of $\mathsf{PsEq}(S_-^*, \Gamma)$ is isomorphic to a T-multicategory if and only if the following coherence conditions hold*

(i) $\iota_1^{m_i} = \iota_1^{e_i} = \iota_i^T$ *for* $i = 0, 1$,
(ii) $T\iota_i^T \circ \iota_{3+i}^T = \iota_0^{m_i}$, *for* $i = 0, 1$,
(iii) $\iota_0^{e_i} = \mathrm{id}$ *for* $i = 0, 1$,
(iv) $\iota_j^{p_i} = \mathrm{id}$ *for* $i = 0, 1$ *and* $j = 0, 1, 2, 3$.

Proof: Given such an object, define a T-multicategory $\hat{y}$ such that $\hat{y}_0 = y_0$, $\hat{y}_1 = y_1$, and we consider the span

$$Ty_0 \xleftarrow{\iota_0^T \circ d_1} y_1 \xrightarrow{d_0} y_0,$$

so that we have $\hat{d}_1 = \iota_0^T \circ d_1$, $\hat{d}_0 = d_0$, and we let $\hat{d}_1 = d_1 \colon x_2 \to x_1$ and $\hat{s}_0 = s_0 \colon x_0 \to x_1$.

Observe that the rectangle below

$$\begin{tikzcd}
y_{2+i} \arrow[r, "d_{2+i}"] \arrow[d, "d_0"'] & y'_{1+i} \arrow[r, "\iota^T_{1+i}"] \arrow[d, "d'_0"] & Ty_{1+i} \arrow[d, "Td_0"] \\
y_{1+i} \arrow[r, "d_{1+i}"'] & y'_i \arrow[r, "\iota^T_i"'] & Ty_i
\end{tikzcd}$$

is a pullback square for $i = 0, 1$; the left square is a pullback by definition, while the right square commutes and since ι^T_i are isomorphisms for $i = 0, 1, 2$, it is also a pullback. Thus, we may let $\hat{d}_0 = d_0 \colon y_{2+i} \to y_{1+i}$ and $\hat{d}_{2+i} = \iota^T_{1+i} \circ d_{2+i} \colon y_{2+i} \to y_{1+i}$ for $i = 0, 1$.

First, we claim that each triangle below commutes:

$$\begin{tikzcd}
& y_2 \arrow[dl, "m \circ T\hat{d}_1 \circ \hat{d}_2"'] \arrow[dr, "d_0 \circ d_0"] \arrow[dd, "d_1"] & \\
Ty_0 & & y_0 \\
& y_1 \arrow[ul, "\hat{d}_1"] \arrow[ur, "d_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& y_0 \arrow[dl, "e"'] \arrow[dr, equal] \arrow[dd, "s_0"] & \\
Ty_0 & & y_0 \\
& y_1 \arrow[ul, "\hat{d}_1"] \arrow[ur, "d_0"'] &
\end{tikzcd}$$

Of course, both right triangles commute by definition, while

$$\begin{tikzcd}
y_2 \arrow[r, "d_2"] & y'_1 \arrow[r, "d'_1"] \arrow[d, "\iota^T_1"'] & y''_0 \arrow[r, "M"] \arrow[d, "\iota^T_3"] \arrow[dr, "\iota^m_0"] & y'_0 \arrow[dr, "\iota^m_1"] & \\
& Ty_1 \arrow[r, "Td_1"'] & Ty'_0 \arrow[r, "T\iota^T_0"'] & TTy_0 \arrow[r, "m"'] & Ty_0
\end{tikzcd}$$

where both squares commute by naturality and the triangle by coherence. Since $M \circ d'_1 \circ d_2 = d_1 \circ d_1$ by definition, the left left triangle commutes, and since $e = \iota^{e_0}_1 \circ e_0$, $\iota^{e_0}_1 = \iota^T_0$ and $e_0 = d_1 \circ s_0$, right left triangle commutes.

Now, we wish to define

- $\hat{s}_0 = (\mathrm{id}, Ts_0 \circ \hat{d}_1)$
- $\hat{s}_1 = (s_0 \circ d_0, e)$
- $\hat{d}_2 = (d_1 \circ d_0, m \circ T\hat{d}_2 \circ \hat{d}_3)$
- $\hat{d}_1 = (d_0 \circ d_0, Td_1 \circ \hat{d}_3)$

and then, we verify that $\hat{s}_i = s_i$ and $\hat{d}_i = d_i$, as they satisfy the same universal property, which concludes our proof. To make sure these expressions make sense, we must verify that

- $\hat{d}_1 = Td_0 \circ Ts_0 \circ \hat{d}_1$,
- $\hat{d}_1 \circ s_0 \circ d_0 = Td_0 \circ e$,
- $\hat{d}_1 \circ d_1 \circ d_0 = Td_0 \circ m \circ T\hat{d}_2 \circ \hat{d}_3$,
- $\hat{d}_1 \circ d_0 \circ d_0 = Td_0 \circ Td_1 \circ \hat{d}_3$,

Since $d_0 \circ s_0$ is the identity, the first equation is satisfied. We have

$$\iota_0^T \circ d_1 \circ s_0 \circ d_0 = \iota_1^{e_0} \circ e_0 \circ d_0 = e \circ d_0 = Td_0 \circ e,$$

which verifies the second. For the third and fourth, we get

$$\begin{aligned}
\iota_0^T \circ d_1 \circ d_0 \circ d_0 &= \iota_0^T \circ d_0' \circ d_2 \circ d_0 \\
&= Td_0 \circ \iota_1^T \circ d_0' \circ d_3 \\
&= Td_0 \circ Td_0 \circ \iota_2^T \circ d_3 \\
&= Td_0 \circ Td_1 \circ \hat{d}_3 \\
Td_0 \circ m \circ T\iota_1^T \circ Td_2 \circ \iota_2^T \circ d_3 &= m \circ TTd_0 \circ T\iota_1^T \circ Td_2 \circ \iota_2^T \circ d_3 \\
&= m \circ T\iota_0^T \circ Td_0' \circ Td_2 \circ \iota_2^T \circ d_3 \\
&= m \circ T\iota_0^T \circ Td_1 \circ Td_0 \circ \iota_2^T \circ d_3 \\
&= m \circ T\iota_0^T \circ Td_1 \circ \iota_1^T \circ d_0' \circ d_3 \\
&= m \circ T\iota_0^T \circ \iota_3^T \circ d_1' \circ d_2 \circ d_0 \\
&= m \circ \iota_0^{m_0} \circ d_1' \circ d_2 \circ d_0 \\
&= \iota_1^{m_0} \circ m_0 \circ d_1' \circ d_2 \circ d_0 \\
&= \iota_1^{m_0} \circ d_1 \circ d_1 \circ d_0
\end{aligned}$$

as desired.

Finally, since the left square in (48) is a pullback for $i = 0, 1$, it follows that $s_0, s_1 \colon x_1 \to x_2$ and $d_1, d_2 \colon x_3 \to x_2$ are given by $(\mathrm{id}, s_0' \circ d_1)$, $(s_0 \circ d_0, e_1)$, $(d_1 \circ d_0, m_1 \circ d_2' \circ d_3)$ and $(d_0 \circ d_0, d_1' \circ d_3)$, respectively. But these are just $\hat{s}_0, \hat{s}_1, \hat{d}_1, \hat{d}_2$, respectively.

The converse is a corollary of our next, stronger result. ∎

Lemma 2.4. *Suppose that we have a pointwise epimorphism $f\colon (x,\mathrm{id}) \to (y,\iota)$ in $\mathsf{PsEq}(S^*_-,\Gamma)$. Then (y,ι) satisfies the coherence conditions.*

Proof: A morphism $f\colon (x,\mathrm{id}) \to (y,\iota)$ is a morphism $f\colon x \to y$ such that $Ff = Gf \circ \iota$, which translates to the following equations:

$$\begin{aligned}
&f_0 = f_0 \circ \iota_0^{e_0}\\
&f_1 = f_1 \circ \iota_0^{e_1} = f_1 \circ \iota_1^{p_0}\\
&f_2 = f_2 \circ \iota_1^{p_1}\\
&f_3 = f_3 \circ \iota_0^{p_1}\\
&f_0' = Tf_0 \circ \iota_0^{T} = Tf_0 \circ \iota_1^{m_0} = Tf_0 \circ \iota_1^{e_0} = f_0' \circ \iota_3^{p_0}\\
&f_1' = Tf_1 \circ \iota_1^{T} = Tf_1 \circ \iota_1^{m_1} = Tf_1 \circ \iota_1^{e_1} = f_1' \circ \iota_2^{p_0} = f_1' \circ \iota_3^{p_1}\\
&f_2' = Tf_2 \circ \iota_2^{T} = s_2' \circ \iota_2^{p_1}\\
&f_0'' = Tf_0' \circ \iota_3^{T} = TTf_0 \circ \iota_0^{m_0}\\
&f_1'' = Tf_1' \circ \iota_4^{T} = TTf_1 \circ \iota_0^{m_1}
\end{aligned}$$

and noting that f_i, f_i', f_i'' all are epimorphisms for all i we recover the coherences; just note that Tf_i and TTf_i are epimorphisms as well, and that $Tf_i' = TTf_i \circ T\iota_i^T$. ∎

3. First approach

We understand the effective descent morphisms of the pseudo-equalizer $\mathsf{PsEq}(S^*_-,\Gamma)$ in terms of the effective descent morphisms of the involved categories of diagrams by the following corollary of [Lu18, Theorem 9.2].

Theorem 3.1. *Suppose that we have a pseudo-equalizer of categories and pullback-preserving functors*

$$\mathsf{PsEq}(F,G) \xrightarrow{I} \mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$$

and let f be a morphism in the pseudo-equalizer. f is effective for descent whenever If is effective for descent and $FIf \cong GIf$ is a pullback-stable regular epimorphism.

Now, we study the effective descent morphisms of $\mathsf{Cat}(T,\mathcal{V})$ via the following classical descent result:

Theorem 3.2. *Let $U\colon \mathcal{C}\to\mathcal{D}$ be a fully faithful, pullback-preserving functor, and let π be a morphism in $\mathcal{C}$ such that $U\pi$ is effective for descent. Then π is effective for descent if and only if for all pullback diagrams of the form*

$$\begin{tikzcd} Up \arrow[r, "f^*(\pi)"] \arrow[d] & y \arrow[d, "f"] \\ Ue \arrow[r, "\pi"'] & Ub \end{tikzcd}$$

there exists an isomorphism $Ux\cong y$ for x an object of $\mathcal{C}$.

Applying these theorems to our setting, we get:

Corollary 3.3. *Let $p\colon x\to z$ be a functor of T-multicategories, and assume that*

- *p is effective for descent in $[\mathcal{S},\mathcal{B}]$, and*
- *$S^*_{-}p$ is a descent morphism.*

Then p is effective for descent if and only if for every pullback diagram of the form

$$\begin{tikzcd} (w,\mathrm{id}) \arrow[r] \arrow[d] & (y,\iota) \arrow[d] \\ (x,\mathrm{id}) \arrow[r, "(p{,}\mathrm{id})"'] & (z,\mathrm{id}) \end{tikzcd} \tag{2}$$

we have (y,ι) isomorphic to a T-multicategory.

Lemma 2.4 implies that in every pullback of the form (2) we have (y,ι) isomorphic to a T-multicategory; since (p,id) is effective for descent, it is in particular a pullback-stable epimorphism, hence (y,ι) is the codomain of an epimorphism with a T-multicategory domain, therefore the lemma applies. This is summed up in the following theorem:

Theorem 3.4. *If $p\colon x\to z$ is a functor of T-multicategories such that p is effective for descent in $[\mathcal{S},\mathcal{B}]$ and $S^*_{-}p$ is a descent morphism, then p is effective for descent in $\mathsf{Cat}(T,\mathcal{B})$.*

Since pointwise effective descent implies effective descent, we get the corollary:

Corollary 3.5. *Let $p \colon x \to z$ be a T-multicategory functor internal to $\mathcal{B}$. If Tp_1, Tp_2 and p_3 are effective for descent, then so is p.*

Proof: By the results in Appendix B (observe Tp_1 is a T-graph morphism), our hypothesis give pointwise effective descent in $[\mathcal{S}, \mathcal{B}]$, as well as in the projection. ■

4. Second approach

In this section, we extend the techniques of [Cr99, Chapter 3] to our setting, in order to refine our result on effective descent morphisms. We highlight that if $p \colon x \to y$ a functor of internal multicategories, note that if p_1 is a pullback-stable (regular) epimorphism, or of effective descent, then so is p_0 by Lemma B.3.

Theorem 4.1. *Let $p \colon x \to y$ be a functor of internal T–multicategories. If p_1 is an (pullback-stable) epimorphism, then so is p.*

Proof: Given functors q, r such that $qp = rp$, we have $q_i p_i = r_i p_i$, and therefore $q_i = r_i$ for $i = 0$, 1, hence $q = r$, thus p is an epimorphism. Since pullbacks are calculated pointwise, p must be pullback-stable whenever p_1 is. ■

Theorem 4.2. *Let p be a functor of internal T–multicategories. If*

- *p_1 is a (pullback-stable) regular epimorphism,*
- *p_2 is an (pullback-stable) epimorphism,*

then p is a (pullback-stable) regular epimorphism.

Proof: Consider the kernel pair r, s of p, and let $q \colon x \to z$ be a functor such that $q \circ r = q \circ s$. Then there exist unique morphisms k_0, k_1 such that $k_i p_i = q_i$ for $i = 0$, 1. We claim these morphisms define a functor

$y \to z$. We have

$$d_1 \circ k_1 \circ p_1 = d_1 \circ q_1 = Tq_0 \circ d_1 = Tk_0 \circ Tp_0 \circ d_1 = Tk_0 \circ d_1 \circ p_1 \tag{3}$$

$$d_0 \circ k_1 \circ p_1 = d_0 \circ q_1 = q_0 \circ d_0 = k_0 \circ p_0 \circ d_0 = k_0 \circ d_0 \circ p_1 \tag{4}$$

$$k_1 \circ d_1 \circ p_2 = k_1 \circ p_1 \circ d_1 = q_1 \circ d_1 = d_1 \circ q_2 = d_1 \circ k_2 \circ p_2, \tag{5}$$

and since p_1, p_2 are epimorphisms, cancellation allows us to conclude that k is a functor (note that k_2 is defined as $k_2(g,f) = (k_1 g, k_1 f)$, and hence $q_2 = k_2 p_2$).

Again, "relevant pointwise" pullback-stable is pullback-stable. ■

Theorem 4.3. *Let p be a functor of internal T–multicategories. If*

- *p_1 is an effective descent morphism,*
- *p_2 is a pullback-stable regular epimorphism,*
- *p_3 is a pullback-stable epimorphism,*

then p is effective for descent.

Proof: By the previous theorem, the comparison

$$\mathcal{K} \colon \mathsf{Cat}(T, \mathcal{B})/y \to \mathsf{Desc}(p)$$

is fully faithful. Hence, we aim to prove that $\mathcal{K}$ is also essentially surjective under our hypotheses, thereby concluding that p is effective for descent.

Suppose we are given a $p^* p_!$–algebra (a, γ), where $a \colon v \to x$ is a functor and $\gamma \colon u \to v$ is the algebra structure. We have equivalences $\mathcal{K}_i \colon \mathcal{B}/y_i \to \mathsf{Desc}(p_i)$, for $i = 0$, 1, and (a, γ) then determines algebras (a_i, γ_i) for $i = 0, 1$. Hence, there exist $f_i \colon w_i \to y_i$ and $h_i \colon v_i \to w_i$ such that the following diagram

$$\begin{tikzcd} v_i \arrow[r, "h_i"] \arrow[d, "a_i"'] & w_i \arrow[d, "f_i"] \\ x_i \arrow[r, "p_i"'] & y_i \end{tikzcd}$$

is a pullback square, and moreover, we have $h_i \circ \gamma_i = h_i \circ \varepsilon_{p_i \circ a_i}$. We claim that

- h_0, h_1 determine a functor $h \colon v \to w$,

– f_0, f_1 determine a functor $f\colon w \to y$,

so that the above lifts to a pullback diagram of T–multicategories.

The hypothesis that p_1, p_2 are pullback-stable regular epimorphisms implies that h_1, h_2 are regular epimorphisms. Taking kernel pairs and noting that T preserves pullbacks, we get

$$\begin{tikzcd}
u_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "d_i"'] & v_1 \arrow[r, "h_1"] \arrow[d, "d_i"] & w_1 \arrow[d, dashed, "d_i"] \\
T^i u_0 \arrow[r, shift left] \arrow[r, shift right] & T^i v_0 \arrow[r, "T^i h_0"'] & T^i w_0
\end{tikzcd}$$

$$\begin{tikzcd}
u_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "s_0"'] & v_0 \arrow[r, "h_0"] \arrow[d, "s_0"] & w_0 \arrow[d, dashed, "s_0"] \\
u_1 \arrow[r, shift left] \arrow[r, shift right] & v_1 \arrow[r, "h_1"'] & w_1
\end{tikzcd}$$

$$\begin{tikzcd}
u_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "d_1"'] & v_2 \arrow[r, "h_2"] \arrow[d, "d_1"] & w_2 \arrow[d, dashed, "d_1"] \\
u_1 \arrow[r, shift left] \arrow[r, shift right] & v_1 \arrow[r, "h_1"'] & w_1
\end{tikzcd}$$

therefore there exist unique morphisms making every right hand side square commute. Note that we define $h_2(g, f) = (h_1 g, (Th_1)f)$. Assuming w is in fact a T–multicategory, we may already conclude that h is a functor. The hypothesis that p_1, p_2, p_3 are pullback-stable epimorphisms implies that h_1, h_2, h_3 are epimorphisms. We have equations

$$\begin{aligned}
d_1 s_0 h_0 &= (Th_0) d_i s_0 = (Th_0)e = e h_0 \\
d_0 s_0 h_0 &= h_0 d_0 s_0 = h_0 \\
d_1 d_1 h_2 &= (Th_0) d_1 d_1 = (Th_0) m (Td_1) d_2 = m (Td_1) d_h k_2 \\
d_0 d_1 h_2 &= h_0 d_0 d_1 = h_0 d_0 d_0 = d_0 d_0 h_2 \\
d_1 s_i h_1 &= h_1 d_1 s_i = h_1 s_0 d_0 = s_0 d_0 h_1 \\
d_1 d_2 h_3 &= h_1 d_1 d_2 = h_1 d_1 d_1 = d_1 d_1 h_3
\end{aligned}$$

and by cancellation, we conclude w is a T–multicategory (proving our assumption) and, similarly, we can show that f is a functor, following the same strategy as in the previous lemma. This shows that p^* is essentially surjective.

It remains to show that $h \circ \gamma = h \circ \varepsilon_{p \circ a}$, which is immediate since we have $h_i \circ \gamma_i = h_i \circ \varepsilon_{p_i \circ a_i}$ for $i = 0, 1$, and pullbacks are calculated pointwise. The result now follows by Lemma 1.1. ■

5. Note on examples

There are sparse examples of cartesian monads, and therefore sparse examples of categories of internal multicategories over a monad. For $\mathcal{B}$ finitely extensive with finite limits and pullback-stable nested countable unions, as in [Le04, Appendix D], the free category monad on graphs internal to $\mathcal{B}$ is cartesian, and therefore so is the free monoid monad W on $\mathcal{B}$. In fact, Leinster's construction is iterable, and most known examples fit into the above conditions. A family of examples outside of this setting is given by free monoid monads on extensive categories with finite limits. These are also cartesian; the idea is that the coproduct functor $\mathsf{Fam}(\mathcal{B}) \to \mathcal{B}$ preserves finite limits, so we may construct the required limit diagrams in $\mathsf{Fam}(\mathcal{B})$, allowing us to conclude that such monads preserve pullbacks and that the required naturality squares are pullbacks.

Another result of Leinster, Corollary 6.2.5 *ibid*: for $\mathscr{C}$ an internal T-multicategory, we can construct a cartesian monad $T_{\mathscr{C}}$ on $\mathcal{B}/\mathscr{C}_0$. We have an equivalence of categories

$$\mathsf{Cat}(T_{\mathscr{C}}, \mathcal{B}/\mathscr{C}_0) \cong \mathsf{Cat}(T, \mathcal{B})/\mathscr{C},$$

and since universal (regular) epimorphisms and effective descent remain unchanged on slice categories, this allows us to say something about effective descent of complicated internal multicategories in terms of simpler ones.

Appendix A. The Eilenberg-Moore comparsion

Lemma 1.1 is a fairly elementary observation. It has been, sometimes, implicitly assumed in the literature about effective descent morphisms. The instance of Le Creurer's argument in Proposition 3.2.4, where he implicitly uses this result, is of particular interest for our work.

Since we have defined descent data as algebra structures, we restrict our attention to this context. We state Lemma 1.1 as a result (Lemma A.1) about the Eilenberg-Moore comparison. It should be noted, however, that the result holds for much more general contexts, and hence its applicability in descent arguments does not depend on the Bénabou–Roubaud theorem.[§]

Lemma A.1. *Let* $(l \dashv u, \varepsilon, \eta) : \mathcal{A} \to \mathcal{B}$ *be an adjunction that induces a monad* $\mathtt{t} = (ul, u\varepsilon l, \eta)$*. An algebra* $(a \in \mathcal{B}, \gamma : ul\,(a) \to a)$ *is in the image of the Eilenberg-Moore comparison* $\mathcal{K} : \mathcal{A} \to \mathtt{t}\text{-}\mathsf{alg}$ *if, and only if, there is* $w \in \mathcal{A}$ *such that* $u(w) = a$ *and*

$$\varepsilon_w \circ l\,(\gamma) = \varepsilon_w \circ \varepsilon_{lu(w)}. \tag{6}$$

Proof: If the algebra (a, γ) is in the image of $\mathcal{K}$, we have that

$$(u(w), u\,(\varepsilon_w)) = (a, \gamma)$$

for some $w \in \mathcal{A}$. Hence, of course, (6) holds by naturality. Reciprocally, if

$$(a = u(w) \in \mathcal{B}, \gamma \colon ulu\,(w) \to u(w))$$

is an algebra structure such that (6) holds, then

$$\begin{aligned} \gamma &= u\,(\varepsilon_w) \cdot \eta_{u(w)} \cdot \gamma \\ &= u\,(\varepsilon_w) \cdot ul\,(\gamma) \cdot \eta_{ulu(w)} \\ &= u\,(\varepsilon_w) \cdot u\left(\varepsilon_{lu(w)}\right) \cdot \eta_{ulu(w)} \\ &= u\,(\varepsilon_w)\,. \end{aligned}$$

This proves that $(a, \gamma) = (u(w), u\,(\varepsilon_w)) = \mathcal{K}\,(w)$. ∎

[§]Although it is out of the scope of our work and we don't give further details, the observation has, for instance, a natural generalization in the context of [Lu18] which does not depend on the Bénabou–Roubaud theorem at all.

Appendix B. T-stability of pullback-stable classes

We establish in this appendix some auxiliary lemmas about preservation of pullback-stable classes.

Lemma B.1. *T creates any pullback-stable property in its essential image.*

Proof: If Tf satisfies a property P, stable under pullback, then the unit and multiplication naturality squares guarantee that f and TTf also satisfy P. ■

Corollary B.2. *If Tf is a pullback-stable (regular) epimorphism, effective for descent, then f and TTf also have the respective property.*

Lemma B.3. *Let $f\colon x \to y$ be a T-graph morphism, and let $\mathcal{E}$ be a class of epimorphisms, containing all split epimorphisms, closed under composition and cancellation. If f_1 is in $\mathcal{E}$, then so is f_0.*

Proof: Note that $d_0\colon x_1 \to x_0$ is a split epimorphism, hence $d_0 f_1 = f_0 d_0$ is in $\mathcal{E}$, and thus so is f_0. ■

We are interested in the cases $\mathcal{E}$ is the class of pullback-stable epimorphisms, descent morphisms and effective descent morphisms.

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