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OVERDAMPED DYNAMICS OF A FALLING INEXTENSIBLE NETWORK: EXISTENCE OF SOLUTIONS

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Abstract: We study the equations of overdamped motion of an inextensible triod with three fixed ends and a free junction under the action of gravity. The problem can be expressed as a system of PDE that involves unknown Lagrange multipliers and non-standard boundary conditions related to the freely moving junction. It can also be formally interpreted as a gradient flow of the potential energy on a certain submanifold of the Otto-Wasserstein space of probability measures. We prove global existence of generalized solutions to this problem.

Keywords: gradient flow, triod, curvature, inextensible string, unknown Lagrange multiplier.
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1. Introduction

An *inextensible network* is a union of several inextensible strings that meet at some of their endpoints called *junctions*. The study of inextensible networks from the mathematical perspective was started a long time ago by Chebyshev Ghys (2011) and Rivlin Rivlin (1955), aiming at modelling textile fabrics. Aside from Novaga and Pozzi (2020), we are however not aware of any investigation of evolutionary behavior of inextensible networks. Our paper thus seems to be one of the first contributions to this particular field. On the other hand, there has been a major recent activity on well-posedness of geometric flows describing time-evolving extensible networks, see Mantegazza et al. (2004); Magni et al. (2016); Garcke et al. (2019, 2020); Dall'Acqua et al. (2019, 2021); Kröner et al. (2021) and the survey Mantegazza et al. (2021). Whereas the authors of Mantegazza et al. (2004); Magni et al. (2016); Kröner et al. (2021) deal with variants of the mean curvature flow for networks, Novaga and Pozzi (2020) and the other

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mentioned articles consider *elastic flows* (interpolations between the mean curvature flow and the Willmore flow).

The main technical difficulties that appear in the study of networks in contrast with the evolution of single strings are due to the rather non-standard boundary conditions at the junction points. Accordingly, to fix the ideas, we decided to restrict ourselves to the simplest possible network with only one junction, the so-called *triod*, cf. Dall'Acqua et al. (2019); Mantegazza et al. (2004). Our triod consists of three inextensible strings (*arms* of the triod) that meet at a common point (junction), and the remaining ends are fixed at three distinct points of $\mathbb{R}^d$. Note that the junction is moving in an unknown way and thus constitutes a kind of a *free boundary*.

The literature on flows of networks cited above is concerned with variational evolution driven by "intrinsic" energies (related to the length or curvature). In this paper we investigate the gradient flow of an "extrinsic" energy, namely, the potential energy determined by an external force (gravity), with respect to a suitable geometry, cf. Shi and Vorotnikov (2019a). As we explain below, this models the overdamped motion of a falling inextensible network (triod).

It is impossible to discuss evolution of inextensible networks without referring to the state of the art for single inextensible cords (we will not touch upon the extensible cords because the amount of the corresponding literature is enormous). Various elastic flows of inextensible strings were studied in Wen (1993); Koiso (1996); Öelz (2011); Oelz (2014); Okabe (2007, 2008); Lin et al. (2015); Okabe et al. (2020). The presence of elastic forces contributes towards non-degenerate parabolicity of the flows and helps to overcome the difficulties caused by the Lagrange multipliers related to the inextensibility constraint; in our situation such forces and hence such an advantage are missing. Our paper has particularly been influenced by Shi and Vorotnikov (2019a) (that studied the overdamped dynamics of a falling whip) and Shi and Vorotnikov (2019b) (that dealt with the "uniformly compressing" counterpart of the mean curvature flow).

The full dynamical equations (Hamiltonian systems) governing the motion of inextensible strings (with or without elastic forces) are very tricky. The literature about the solvability of the corresponding initial-boundary value problems is scarce and includes the studies near the equilibrium

Reeken (1979), locally in time Preston (2011), and globally in time in various severely relaxed senses Grothaus and Marheineke (2016); Sengül and Vorotnikov (2017).

The equations of motion of an inextensible triod in the ambient space $\mathbb{R}^d$ subject to gravity force can be derived from the least action principle by following the road map from (Sengül and Vorotnikov, 2017, Section 2.6). Assuming for simplicity that the length of each arm of the triod is equal to 1, the initial-boundary value problem reads

$$\begin{cases} \partial_{tt}\eta^i = \partial_s\left(\sigma^i\partial_s\eta^i\right) + g, \\ |\partial_s\eta^i| = 1, \end{cases} \tag{1}$$

subject to the boundary conditions

$$\begin{cases} \eta^1(t,0) = \eta^2(t,0) = \eta^3(t,0), \\ \eta^i(t,1) = \alpha^i(1), \\ \sigma^1\partial_s\eta^1 + \sigma^2\partial_s\eta^2 + \sigma^3\partial_s\eta^3 = 0 \text{ at } s = 0 \text{ for all } t, \end{cases} \tag{2}$$

and the initial condition

$$\eta^i(0,s) = \alpha^i(s). \tag{3}$$

Here $\eta^i = \eta^i(t,s) \in \mathbb{R}^d$, $i = 1,2,3$, is the position vector at time $t \geq 0$ of the particle that is labelled by the arc length parameter s and belongs the ith arm of the triod. For each i, the scalar function $\sigma^i = \sigma^i(t,s)$ is the Lagrange multiplier (that is often referred to as the *tension*) coming from the inextensibility of the ith arm. Finally, g is a constant gravity vector for which we assume w.l.o.g. that $|g| = 1$, and $\alpha^i(s)$ determines the initial configuration of the triod. Note that $s = 1$ corresponds to the fixed ends, and $s = 0$ corresponds to the (moving) junction.

From the geometrical point of view, a natural inifinite-dimensional configuration manifold for the evolving triods is

$$\mathcal{A} = \{\eta = (\eta^1,\eta^2,\eta^3) : \eta^i \in H^2\left(0,1;\mathbb{R}^d\right),$$
$$\eta^1(0) = \eta^2(0) = \eta^3(0),\ \eta^i(1) = \alpha^i(1),\ |\partial_s\eta^i(s)| = 1\ \forall s \in [0,1]\}$$

viewed as a submanifold of $L^2(0,1;\mathbb{R}^{3d})$ (and hence equipped with a weak Riemannian metric). Observe that the tangent space at a "point" η is

$$T_\eta\mathcal{A} = \{v = (v^1, v^2, v^3) : v^i \in H^2\left(0,1;\mathbb{R}^d\right),\ v^1(0) = v^2(0) = v^3(0),$$
$$v^i(1) = 0,\ \partial_s\eta^i(s)\cdot\partial_s v^i(s) = 0\}.$$

Note that we never employ Einstein's summation convention. Then (1), (2) is at least formally equivalent to Newton's equation

$$\nabla_{\dot\eta}\dot\eta = -\nabla_{\mathcal{A}}E\left(\eta\right). \tag{4}$$

Here

$$E(\eta) := \sum_{i=1}^{3}\int_0^1 -g\cdot\eta^i(s)\,ds \tag{5}$$

is the potential energy of a triod.

The Riemannian manifold $\mathcal{A}$ (as well as its counterparts for single strings, cf. Preston (2012); Shi and Vorotnikov (2019a,b)) has some interesting features. It can be viewed, cf. Shi and Vorotnikov (2019b), as a submanifold of the Otto-Wasserstein space of probability measures Otto (2001); Villani (2003, 2008) from the optimal transport theory (this in particular implies that the geodesic distance on $\mathcal{A}$ does not vanish, being bounded from below by the Wasserstein distance, which is in stark contrast with the underlying geometry of the mean curvature, Willmore and similar flows, cf. Michor and Mumford (2005, 2006, 2007); Bauer et al. (2012, 2014)). It can also be regarded as a particular case ($m = 1$) of the manifolds of m-dimensional *incompressible membranes* (in other words, of volume preserving immersions), cf. Bauer et al. (2016); Molitor (2017). The opposite borderline case $m = d$ tallies with Arnold's formalism Arnold (1966); Arnold and Khesin (1998) for ideal incompressible fluids or rather, even more specifically, with the motion of fluid patches in $\mathbb{R}^d$, which has recently been studied Liu et al. (2019) from a similar perspective. However, in Arnold's case ($m = d$) the manifold has a Lie group structure, which allows one to work in the corresponding Lie algebra (i.e., in the mechanical language, to use the Eulerian coordinates). In our case $m = 1$, there is no Lie algebra structure, and the Lagrangian description as in (1), (2), (3) seems to be unavoidable.

If the fall of the triod is overdamped by a heavily dense environment, the equations of motion (1) become

$$\begin{cases} \partial_t \eta^i = \partial_s \left(\sigma^i \partial_s \eta^i \right) + g, \\ |\partial_s \eta^i| = 1. \end{cases} \tag{6}$$

We refer to Shi and Vorotnikov (2019a) for the details of the derivation in the case of a single cord. It is also possible to directly obtain the overdamped flow (6) from the full dynamical equation (1) by employing the quadratic change of time, cf. Brenier and Duan (2018). Finally, our problem (6), (2) can be realized as the gradient flow of the potential energy E on the manifold $\mathcal{A}$, i.e.,

$$\dot{\eta} = -\nabla_{\mathcal{A}} E\left(\eta\right). \tag{7}$$

In light of the previous discussion (see also Preston (2011, 2012); Sengül and Vorotnikov (2017); Thess et al. (1999)) equation (1) has much in common with the Euler equation of ideal incompressible fluid. In the same spirit, the overdamped equation (6) is comparable to the Muskat problem (also known as the incompressible porous medium equation) that received a lot of attention during the last decade, see Córdoba et al. (2011); Székelyhidi Jr (2012); Constantin et al. (2016); Castro et al. (2021) and the references therein.

In this article, we are interested in constructing global in time solutions to (6), (2), (3). We deal with generalized solutions, which allows us to consider not necessarily smooth but merely rectifiable triods.

In what follows, we denote $\Omega := (0,1)$, $\mathbb{Q}_t := (0,t) \times \Omega$ for $t \in (0,\infty]$ and $\mathbf{g}(s) := (g,g,g) \in L^2(\Omega;\mathbb{R}^{3d})$.

Remark 1.1 (Initial data). We fix once and for all Lipschitz initial data $\alpha^i \in W^{1,\infty}(\Omega)^d$, $i = 1,2,3$, satisfying the compatibility conditions

$$\alpha^1(0) = \alpha^2(0) = \alpha^3(0) = 0 \tag{8}$$

and

$$|\partial_s \alpha^i(s)| = 1 \text{ a.e. in } \Omega. \tag{9}$$

Since (9) is only required to hold almost everywhere, the arms of the triod can have shape of any rectifiable curve at the initial moment. Note that we have also w.l.o.g. assumed that the junction is located at the origin at the initial moment. We will moreover assume that the arms of the triod are not fully straight at the initial moment which means that $|\alpha^i(1)| < 1$ (since the length of each arm is equal to 1), $i = 1,2,3$. Finally, we will assume that the three points $\alpha^1(1), \alpha^2(1)$ and $\alpha^3(1)$ are vertices of a triangle with circumradius $R < 1$.

Our goal is to prove the following main result.

Theorem 1.2 (Global existence of generalized solutions)**.** *For every initial configuration $\alpha^i(s) \in W^{1,\infty}(\Omega)^d$, $i = 1,2,3$, meeting the assumptions of Remark 1.1, there exists a generalized solution to (6), (2), (3) in $\mathcal{Q}_\infty$. Moreover, those solutions satisfy $\sigma^i(t,s) \geq 0$ for almost every $(t,s) \in \mathcal{Q}_\infty$.*

Note that the precise definition of a generalized solution is lengthy and will be introduced in Definition 5.1.

Observe that $\mathcal{A}$, being a formal submanifold of the Otto-Wasserstein space, is a metric space with a non-degenerate (Riemannian) distance. Nevertheless, $\mathcal{A}$ is neither a complete metric space nor a geodesic space. Accordingly, the theory of gradient flows in metric spaces, cf. Ambrosio et al. (2008); Villani (2008), does not sound to be applicable to well-posedness of our flow (7).

To achieve our goal, we will follow the strategy suggested by Shi and the second author Shi and Vorotnikov (2019a) for the evolution of a single string. It basically consists in approximation of the original gradient flow on $\mathcal{A}$ by suitable gradient flows on the flat ambient space $L^2(\Omega; \mathbb{R}^{3d})$. The idea is to derive uniform estimates for the approximating problem that would allow us to pass to the limit and to show that the limiting functions are solutions to (6), (2), (3). However, **because of the complicated boundary conditions (2), many of the estimates that were used in Shi and Vorotnikov (2019a) fail to be generalizable to our setting. This in particular applies to the crucial L^∞ estimate in the spirit of Ladyzhenskaya, Solonnikov and Uraltseva, cf. Ladyženskaja et al. (1968). We will manage to overcome these difficulties and to prove novel and more refined estimates by leveraging the gradient flow structure of the approximating problem much more thoroughly than in Shi and Vorotnikov (2019a). This will be combined with careful observations involving geometric properties of triods, the behaviour of the curvature and some convexity argument.**

Apart from that, in Shi and Vorotnikov (2019a) the existence of C^∞-smooth solutions to the approximating problem was immediate from Amann's theory, cf. Amann (1993). It is not applicable here anymore (again due to the boundary conditions), so we will solve our approximate problem by the theory of abstract evolution equations with pseudomonotone maps, cf. Roubiček (2013).

The paper is organized as follows. In Section 2, we heuristically motivate and then introduce the approximating problem. In Section 3, we prove its solvability. The main technical work is done in Section 4 where we establish various uniform estimates for the approximating problem. A highlight of that section is the crucial and ingenious L^∞ bound for the tension (Lemma 4.7). In Section 5, based on the results of Section 4, we will be able to pass to the limit and to prove Theorem 1.2. Our results still hold for the overdamped dynamics of a falling single cord with two fixed ends, see Remark 5.7 and Proposition 5.8.

2. Approximating problem

Let us now describe the way of approximation of our gradient flow that we plan to employ in order to prove Theorem 1.2.

We begin with some heuristics. Consider the extra variables $\kappa^i = \sigma^i \partial_s \eta^i$, $i = 1, 2, 3$. Then our system (6) can at least formally be rewritten as

$$\begin{cases} \partial_t \eta^i = \partial_s \kappa^i + g, \\ \kappa^i = \sigma^i \partial_s \eta^i, \\ \sigma^i = \kappa^i \cdot \partial_s \eta^i. \end{cases} \tag{10}$$

More precisely, the constraints $|\partial_s \eta^i| = 1$ yield $|\kappa^i| = |\sigma^i|$ and $\kappa^i = \operatorname{sgn}(\sigma^i)|\kappa^i|\partial_s \eta^i$. We make the ansatz $\sigma^i \geq 0$ (that will be a posteriori justified by Theorem 1.2) and infer $\kappa^i \cdot \partial_s \eta^i = \sigma^i$ (see also Remark 5.2 below for a related discussion). Note that we formally have $\partial_s \eta^i = \frac{\kappa^i}{|\kappa^i|}$, thus the map $\kappa^i \mapsto \partial_s \eta^i$ is not a diffeomorphism. To overcome this issue, we fix $\epsilon \in (0, 1)$ and introduce the auxiliary functions

$$F_\epsilon : \mathbb{R}^d \to \mathbb{R}^d, \; F_\epsilon(\kappa) := \epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}} \tag{11}$$

and

$$G_\epsilon(\tau) := (F_\epsilon)^{-1}(\tau).$$

Approximating the relations $\kappa^i \mapsto \partial_s \eta^i$ and $\partial_s \eta^i \mapsto \kappa^i$ by F_ϵ and G_ϵ, respectively, leads from (6), (2), (3) to the problem

$$\partial_t \eta_\epsilon^i = \partial_s \left(G_\epsilon \left(\partial_s \eta_\epsilon^i \right) \right) + g, \quad i = 1,2,3, \tag{12}$$

with the following initial and boundary conditions:

$$\begin{aligned} \eta_\epsilon^i(0,s) &= \alpha^i(s), \\ \eta_\epsilon^i(t,1) &= \alpha^i(1), \\ \eta_\epsilon^1(t,0) = \eta_\epsilon^2(t,0) &= \eta_\epsilon^3(t,0), \\ \sum_{i=1}^{3} G_\epsilon \left(\partial_s \eta_\epsilon^i \right) &= 0 \text{ at } s = 0 \text{ for all } t. \end{aligned} \tag{13}$$

Remark 2.1. Let us make an elementary observation that is very important in the sequel. The Euclidean norm $|F_\epsilon(\kappa)|$ depends only on $|\kappa|$ and is an increasing function of $|\kappa|$. If $|\kappa| = 1$, then by simple calculation $|F_\epsilon(\kappa)| > 1$. Consequently, if $|\tau| \le 1$, then $|G_\epsilon(\tau)| < 1$.

By explicit computation, ∇G_ϵ is positive-definite and

$$\lambda_\epsilon(\tau)|\xi|^2 \le \nabla G_\epsilon(\tau)\xi \cdot \xi \le \Lambda_\epsilon(\tau)|\xi|^2, \ \ \forall \xi \in \mathbb{R}^d, \tau \in \mathbb{R}^d, \tag{14}$$

where Λ_ϵ and λ_ϵ satisfy

$$\begin{aligned} \frac{1}{\epsilon + \epsilon^{-1/2}} \le \lambda_\epsilon(\tau) &= \frac{1}{\epsilon + \left(\epsilon + |G_\epsilon(\tau)|^2\right)^{-1/2}} \\ \Lambda_\epsilon(\tau) &= \frac{\epsilon^{-1}}{1 + \left(\epsilon + |G_\epsilon(\tau)|^2\right)^{-3/2}} \le \epsilon^{-1}. \end{aligned} \tag{15}$$

Motivated by the original system (10), given a solution η_ϵ to the approximating problem (12), (13) we define

$$\kappa_\epsilon^i := G_\epsilon\left(\partial_s \eta_\epsilon^i\right), \ \ \sigma_\epsilon^i := G_\epsilon\left(\partial_s \eta_\epsilon^i\right) \cdot \partial_s \eta_\epsilon^i. \tag{16}$$

Observe from the definition of G_ϵ that there exists a bounded smooth positive scalar function γ_ϵ such that $G_\epsilon(\tau) = \gamma_\epsilon(|\tau^2|)\tau$, $\tau \in \mathbb{R}^d$. In particular, this implies that

$$\sigma_\epsilon^i \ge 0. \tag{17}$$

Moreover, γ_ϵ is bounded away from 0 and ∞ (not uniformly w.r.t. ϵ). Let Γ_ϵ be the primitive of γ_ϵ with $\Gamma_\epsilon(0) = 0$. Set

$$Q_\epsilon(\tau) := \frac{1}{2}\Gamma_\epsilon(|\tau|^2).$$

Observe that

$$\nabla Q_\epsilon(\tau) = G_\epsilon(\tau). \tag{18}$$

Moreover, Q_ϵ can be computed explicitly:

$$Q_\epsilon(\tau) = \epsilon\left(\frac{|G_\epsilon(\tau)|^2}{2} - \frac{1}{\sqrt{\epsilon + |G_\epsilon(\tau)|^2}}\right) + \sqrt{\epsilon}. \tag{19}$$

By Remark 2.1, $Q(\tau) << 1$ if $|\tau| \le 1$.

We define the associated "total energy" of the approximating problem (12), (13) by

$$\mathcal{E}_\epsilon\left(\eta\right) := \begin{cases} \sum_{i=1}^{3}\left(\int_0^1 Q_\epsilon\left(\partial_s \eta^i\right) ds + \int_0^1 \left(-g\right)\cdot \eta^i ds\right) \\ \text{for } \eta \in AC^2(\Omega;\mathbb{R}^{3d}) \text{ satisfying } \eta(1) = \alpha(1),\ \eta^1(0) = \eta^2(0) = \eta^3(0); \\ +\infty \text{ for any } \eta \in L^2(\Omega;\mathbb{R}^{3d}) \text{ except those above.} \end{cases} \tag{20}$$

Then (12), (13) can at least formally be interpreted as a gradient flow, with respect to the flat Hilbertian structure of $L^2(\Omega;\mathbb{R}^{3d})$, that is driven by this functional, i.e.

$$\dot\eta = -\nabla_{L^2(\Omega;\mathbb{R}^{3d})}\mathcal{E}_\epsilon\left(\eta\right), \quad \eta(0) = \alpha.$$

We will return to this issue in the next section.

3. Evolution by pseudomonotone maps and solvability of the approximating problem

For the existence of the solution to the approximating problem, we use the theory of abstract evolution equations involving pseudomonotone maps. We prefer this approach (instead of directly employing the theory of gradient flows in Hilbert spaces, cf. Brézis (1973); Attouch et al. (2014)) because it automatically gives us the regularity of solution that is required for the manipulations of Section 4.

Let us start with introducing some concepts and definitions, mainly following the book Roubiček (2013). Let V be a separable reflexive Banach space, and V^* be the dual space of V. With use the bra-ket notation for the duality.

Definition 3.1. A mapping $A : V \to V^*$ is called monotone if $\forall u, v \in V$ we have $\langle A(u) - A(v), u - v\rangle \geq 0$.

Definition 3.2. A mapping $A : V \to V^*$ is called radially continuous if $\forall u, v \in V : t \mapsto \langle A(u + vt), v\rangle$ is continuous.

Definition 3.3. A mapping $A : V \to V^*$ is called pseudomonotone provided
(i) A is bounded (i.e., the image of any bounded set is bounded),
(ii) for any sequence $u_k \rightharpoonup u$ weakly with

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0$$

and for every $v \in V$ it is true that

$$\langle A(u), u - v\rangle \leq \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle.$$

We will need the following useful criterion of pseudomonotonicity from Brézis (1968).

Lemma 3.4. *A bounded, radially continuous and monotone mapping is pseudomonotone.*

Assume that there is a continuous embedding operator $i : V \to H$, and $i(V)$ is dense in H, where H is a Hilbert space. This generates the *Gelfand triple* $V \subset H \subset V^*$ by the following well-known observation. The adjoint operator $i^* : H^* \to V^*$ is continuous and, since $i(V)$ is dense in H, one-to-one. Since i is one-to-one, $i^*(H^*)$ is dense in V^*, and one may identify H^* with a dense subspace of V^*. Due to the Riesz representation theorem, one may also identify H with H^*. Moreover, the H-scalar product of $f \in H, u \in V$ coincides with the value of the functional f from V^* on the element $u \in V$:

$$(f, u)_H = \langle f, u\rangle. \tag{21}$$

Assume that there is a seminorm $|\cdot|_V$ on V that satisfies the "abstract Poncaré inequality"

$$\|u\|_V \lesssim \|u\|_H + |u|_V, \quad \forall u \in V,$$

where $\|\cdot\|_H$ is the Eucledian norm in H.

Definition 3.5. A mapping $A : V \to V^*$ is called semicoercive if for $u \in V$ we have
$$\langle A(u), u\rangle \geq c_0 |u|_V^2 - c_1|u|_V - c_2\|u\|_H^2,$$
where c_0, c_1 and c_2 are nonnegative constants.

Consider the following abstract initial value problem on the time interval $(0, T)$:
$$\frac{d}{dt}u + A\Big(u(t)\Big) = f(t), \; u(0) = u_0. \tag{22}$$
The following result can be found in (Roubiček, 2013, Theorem 8.18).

Theorem 3.6. *Let $A : V \to V^*$ be a pseudomonotone and semicoercive mapping and*

$f \in AC^2\Big([0, T], V^*\Big)$,
$u_0 \in V$ *is such that* $A(u_0) - f(0) \in H$,
$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq c_0|u_1 - u_2|_V^2 - c_2\|u_1 - u_2\|_H^2$ *for* $u_1, u_2 \in V$ *with some constants* $c_0, c_2 > 0$.

Then there exists $u \in W^{1,\infty}(0, T; H) \cap AC^2\Big([0, T]; V\Big)$ that solves the Cauchy problem (22) (the first equality in (22) holds in the space V^* for a.a. in $(0, T)$, whereas the second one holds in the space V).

Our next goal is to apply this theorem to show the existence and regularity of solutions to the approximating problem. It will be convenient to rewrite our approximating problem (12)-(13) with the help of the simple transformation
$$\xi^i(t, s) := \eta_\epsilon^i(t, s) - \alpha^i(s),$$
arriving at
$$\begin{cases} \partial_t \xi^i - \partial_s\Big(G_\epsilon\Big(\partial_s\Big(\xi^i + \alpha^i\Big)\Big)\Big) = g, \quad i = 1, 2, 3, \\ \xi^1(t, 0) = \xi^2(t, 0) = \xi^3(t, 0), \\ \xi^i(t, 1) = 0, \\ \xi^i(0, s) = 0, \\ \displaystyle\sum_{i=1}^{3} G_\epsilon\Big(\partial_s\Big(\xi^i + \alpha^i\Big)\Big)(t, 0) = 0. \end{cases} \tag{23}$$

Let us recast this system in the form of the Cauchy problem (22). Let $H = L^2 \times L^2 \times L^2\left(\Omega;\mathbb{R}^d\right)$ be the Hilbert space of triples with the natural scalar product. Consider the set

$$V := \{u =\{u^i\} \in AC^2 \times AC^2 \times AC^2\left(\overline{\Omega};\mathbb{R}^d\right) \text{ such that } u^i(1) = 0 \\ \text{and } u^1(0) = u^2(0) = u^3(0)\}.$$

It is a separable reflexive Banach space with the norm inherited from H^1. Define a seminorm on V by $|\{u^i\}|_V := \|\{\partial_s u^i\}\|_H$. The required Poincaré inequality obviously holds. Let $\boldsymbol{A} : V \to V^*$ be the mapping that is defined by duality as follows:

$$\langle \boldsymbol{A}(\xi), \zeta\rangle = \sum_{i=1}^{3} \int_0^1 G_\epsilon\left(\partial_s\left(\xi^i + \alpha^i\right)\right) \cdot \partial_s \zeta^i ds. \tag{24}$$

Then (23) rewrites as

$$\frac{d}{dt}\xi + \boldsymbol{A}\left(\xi(t)\right) = \mathbf{g},\ \xi(0) = 0. \tag{25}$$

Note that the last equality of (23) is hidden in the duality in (24).

In order to check that Theorem 3.6 is applicable to (25) we need to prove several auxiliary statements. For the sake of readability, we will omit the subscript ϵ coming from the approximating problem.

Lemma 3.7. *The mapping $\boldsymbol{A}$ satisfies the inequality*

$$\langle \boldsymbol{A}(\xi_1) - \boldsymbol{A}(\xi_2), \xi_1 - \xi_2\rangle \geq c_0 |\xi_1 - \xi_2|_V^2$$

for some constant $c_0 > 0$ (depending on ϵ) and any $\xi_1, \xi_2 \in V$.

Proof: Define $A^i : H^1\left(\Omega;\mathbb{R}^d\right) \to \left(H^1\left(\Omega;\mathbb{R}^d\right)\right)^*$ by

$$\langle A^i\left(\xi^i\right), \zeta\rangle = \int_0^1 G_\epsilon\left(\partial_s\left(\xi^i + \alpha^i\right)\right) \cdot \partial_s \zeta^i ds.$$

Throughout the rest of the proof, we omit the index i to avoid heavy notation in A^i, ξ_1^i, ξ_2^i and α^i. With this convention, it suffices to prove that

$$\langle A(\xi_1) - A(\xi_2), \xi_1 - \xi_2\rangle \geq c_0 \|\partial_s(\xi_1 - \xi_2)\|_{L^2}^2.$$

We compute

$$\langle A(\xi_1) - A(\xi_2), \xi_1 - \xi_2 \rangle =$$
$$\int_\Omega \Big[G\Big(\partial_s(\xi_1+\alpha)\Big) - G\Big(\partial_s(\xi_2+\alpha)\Big)\Big] \cdot \partial_s\Big((\xi_1+\alpha) - (\xi_2+\alpha)\Big) ds. \tag{26}$$

Let us denote $\mu := G\Big(\partial_s(\xi_1+\alpha)\Big)$ and $\gamma := G\Big(\partial_s(\xi_2+\alpha)\Big)$. Now, we use the relation between F and G and conclude that $F\big(\mu\big) = \partial_s(\xi_1+\alpha)$ and $F\big(\gamma\big) = \partial_s(\xi_2+\alpha)$. We can rewrite the right-hand side of (26) as

$$\begin{aligned}
&\int_\Omega (\mu-\gamma)\cdot \Big(F\big(\mu\big) - F\big(\gamma\big)\Big) ds \\
&= \int_\Omega \big(\mu-\gamma\big)\cdot \left(\epsilon\big(\mu-\gamma\big) + \frac{\mu}{\sqrt{\epsilon+|\mu|^2}} - \frac{\gamma}{\sqrt{\epsilon+|\gamma|^2}}\right) ds \\
&\geq \int_\Omega \epsilon|\mu-\gamma|^2 ds
\end{aligned}$$

because the map $r \mapsto \frac{r}{\sqrt{\epsilon+r^2}}$ is a gradient of a convex function. Observe that

$$|F\big(\mu\big) - F\big(\gamma\big)| \leq (\epsilon + \epsilon^{-1/2})|\mu-\gamma|$$

by the mean value theorem since the operator norm of the matrix $\nabla F(r)$ is bounded from above by $\epsilon+\epsilon^{-1/2}$, cf. (15).

Thus we conclude that

$$\begin{aligned}
\langle A(\xi_1) - A(\xi_2), \xi_1 - \xi_2 \rangle \geq \epsilon \int_\Omega |\mu-\gamma|^2 ds &\geq c_0 \int_\Omega |F(\mu) - F(\gamma)|^2 ds \\
&= c_0 \|\partial_s(\xi_1-\xi_2)\|_{L^2}^2.
\end{aligned}$$

■

Corollary 3.8. *The mapping A is monotone.*

Proof: It is clear from Lemma 3.7. ■

Corollary 3.9. *The mapping A is semicoercive.*

Proof: Employing Lemma 3.7 and Cauchy-Schwarz inequality, we see that

$$\begin{aligned}\langle \boldsymbol{A}(\xi), \xi\rangle &= \langle \boldsymbol{A}(\xi) - \boldsymbol{A}(0), \xi\rangle + \langle \boldsymbol{A}(0), \xi\rangle \\ &\geq |\xi|_V^2 + \langle \boldsymbol{A}(0), \xi\rangle \\ &= |\xi|_V^2 + \sum_{i=1}^{3}\int_0^1 G\left(\partial_s \alpha^i\right)\cdot \partial_s \xi^i ds \\ &\geq |\xi|_V^2 - \left\|\left\{G\left(\partial_s \alpha^i\right)\right\}\right\|_H |\xi|_V \\ &\geq |\xi|_V^2 - c_2|\xi|_V,\end{aligned}$$

where c_2 is a positive constant depending on α. ■

Lemma 3.10. *The mapping* $\boldsymbol{A}$ *is bounded.*

Proof: Indeed,

$$\begin{aligned}\langle \boldsymbol{A}(\xi), \zeta\rangle &= \sum_{i=1}^{3}\int_0^1 G\left(\partial_s\left(\xi^i + \alpha^i\right)\right)\cdot \partial_s \zeta^i ds \\ &\leq \left\|\left\{G\left(\partial_s\left(\xi^i + \alpha^i\right)\right)\right\}\right\|_H |\zeta|_V \\ &\lesssim \left\|\left\{\partial_s\left(\xi^i + \alpha^i\right)\right\}\right\|_H |\zeta|_V \\ &\leq |\xi + \alpha|_V \|\zeta\|_V.\end{aligned}$$

(We have used sublinearity of G). Since $|\alpha|_V$ is finite, this implies that $\|\boldsymbol{A}(\xi)\|_{V^*}$ is bounded provided $\|\xi\|_V$ is bounded. ■

Lemma 3.11. *The mapping* $\boldsymbol{A}$ *is radially continuous.*

Proof: Fix $\xi, \zeta \in V$ and let $\tau_n \to \tau$ be a sequence. Then it is easy to see that

$$\sum_{i=1}^{3} G\left(\partial_s\left(\xi^i + \tau_n \zeta^i + \alpha^i\right)\right)(x)\cdot \partial_s \zeta^i(x) \to \sum_{i=1}^{3} G\left(\partial_s\left(\xi^i + \tau \zeta^i + \alpha^i\right)\right)(x)\cdot \partial_s \zeta^i(x)$$

a.e. in Ω. The claim will follow from Lebesgue's dominated convergence theorem if there is a function in $L^1(\Omega)$ that dominates the left-hand side. But it is indeed the case since we can we leverage sublinearity of G to estimate

$$\left|\sum_{i=1}^{3} G\Big(\partial_s\big(\xi^i+\tau_n\zeta^i+\alpha^i\big)\Big)\cdot\partial_s\zeta^i\right| \le C|\partial_s(\xi+\tau_n\zeta+\alpha)|\cdot|\partial_s\zeta|$$
$$\le C(|\partial_s\xi|^2+|\partial_s\zeta|^2+|\partial_s\alpha|^2),$$

and the right-hand side is L^1 by the assumption. ∎

We can now legitimately use Theorem 3.6 in order to solve (25).

Corollary 3.12. *Given α as in Remark 1.1, the system* (25) *has a solution $\xi=\{\xi^i\}\in W^{1,\infty}(0,T;H)\cap AC^2\big([0,T];V\big)$ that is understood in the same sense as in Theorem 3.6.*

Returning back to the variable η and leveraging elementary properties of G_ϵ and ∇G_ϵ, we get the existence of approximate solutions.

Corollary 3.13. *Given α as in Remark 1.1, there exists a solution $\eta=\eta_\epsilon$ to* (12)-(13) *in $\mathfrak{Q}_T$ that belongs to the following regularity class:*

$$\eta^i\in W^{1,\infty}\big(0,T;L^2(\Omega)\big)^d\cap AC^2\Big([0,T];AC^2\big(\overline{\Omega}\big)\Big)^d,$$

$$\partial_s\eta^i\in AC^2\big([0,T];L^2(\Omega)\big)^d,$$

$$\kappa^i:=G_\epsilon(\partial_s\eta^i)\in L^\infty\big(0,T;L^2(\Omega)\big)^d,$$

$$\nabla G_\epsilon(\partial_s\eta^i)\in L^\infty\big(0,T;L^\infty(\Omega)\big)^d,$$

$$\partial_t\eta^i\in L^\infty\big(0,T;L^2(\Omega)\big)^d\cap L^2\big(0,T;H^1(\Omega)\big)^d,$$

$$\partial_s\kappa^i=\partial_s\Big(G_\epsilon\big(\partial_s\eta^i\big)\Big)\in L^\infty\big(0,T;L^2(\Omega)\big)^d\cap L^2\big(0,T;H^1(\Omega)\big)^d,$$

$$\partial_{ss}\eta^i\in L^\infty\big(0,T;L^2(\Omega)\big)^d.$$

Note that the norms of the solution $\eta=\eta_\epsilon$ in the corresponding spaces above may depend on ϵ. At this stage we cannot infer an L^∞ estimate on $\partial_s\eta$ (even ϵ-dependent) because we do not control $\partial_s\eta^i$ on $\partial\Omega$. Anyway, we will manage to establish a related bound in Corollary 4.8.

It is straightforward to see that $\eta = \eta_\epsilon$ from Corollary 3.13 coincides with the unique solution of the gradient flow

$$\dot{\eta} \in -\partial_{L^2(\Omega;\mathbb{R}^{3d})}\mathcal{E}_\epsilon\left(\eta\right) \tag{27}$$

in the sense of (Attouch et al., 2014, Theorem 17.2.3), where the driving functional $\mathcal{E}_\epsilon$ was defined in (20). This in particular implies that $t \mapsto \mathcal{E}_\epsilon\left(\eta(t)\right)$ is a continuous and non-increasing function.

4. Uniform estimates of the approximate solutions

In this section we derive various uniform (in ϵ) estimates for the approximating solutions η^i_ϵ obtained in Corollary 3.13. These bounds are crucial for passing to the limit in Section 5. In the sequel, C will always stand for a constant independent of ϵ. For the sake of readability, we drop the dependence on ϵ in the subscripts and write $\eta^i = \eta^i_\epsilon$, $G = G_\epsilon$, $\alpha^i = \alpha^i_\epsilon$, etc., until the proof of Lemma 4.7.

Lemma 4.1 (Energy estimate). *Let $\eta = \{\eta^i\}$ be a solution of the approximating problem* (12)-(13) *in $\mathbb{Q}_T$ as constructed in Corollary 3.13. Then*

$$\mathcal{E}(\alpha) + \sum_{i=1}^{3}\left(\int_{\mathbb{Q}_T} |\partial_t\eta^i|^2 + |\nabla G\left(\partial_s\eta^i\right)\partial_{ss}\eta^i|^2 \, dsdt\right) + \|\eta\|^2_{L^\infty\left(0,T;L^1(\Omega)\right)} \le C. \tag{28}$$

Here the constant may only depend on α and T, but not on ϵ.

Proof: We first establish a uniform bound (w.r.t. ϵ) on the initial energies. Indeed, since $|\partial_s\alpha^i(s)| = 1$, Remark 2.1 implies that $|G(\partial_s\alpha^i(s))| < 1$, and using the explicit definition of Q given in (19), we get that the first terms (for each i) in the expansion

$$\mathcal{E}(\alpha) = \sum_{i=1}^{3}\left(\int_0^1 Q\left(\partial_s\alpha^i(s)\right)ds + \int_0^1 \left(-g\right)\cdot\alpha^i(s)\,ds\right)$$

are uniformly bounded. The second terms are obviously uniformly bounded.

We now prove (28). Take the $L^2(\Omega)$-inner product of (12) and $\partial_t\eta^i$ and integrate over $\mathbb{Q}_t$, $t \in (0,T]$. We obtain

$$\sum_{i=1}^{3}\int_{\mathbb{Q}_t} |\partial_t\eta^i|^2 \, dsdt = \sum_{i=1}^{3}\int_{\mathbb{Q}_t} \partial_s G\left(\partial_s\eta^i\right)\cdot\partial_t\eta^i \, dsdt + \sum_{i=1}^{3}\int_{\mathbb{Q}_t} g\cdot\partial_t\eta^i \, dsdt$$

Then, we perform an integration by parts and also integrate the last term over time, ending up with

$$
\begin{aligned}
\sum_{i=1}^{3}\int_{\mathfrak{Q}_{\mathrm{t}}}|\partial_t\eta^i|^2\,dsdt =& -\sum_{i=1}^{3}\int_{\mathfrak{Q}_{\mathrm{t}}}G\big(\partial_s\eta^i\big)\cdot\partial_{st}\eta^i\,dsdt+\sum_{i=1}^{3}\int_{\Omega}g\cdot\eta^i(\mathrm{t})\,ds\\
&-\sum_{i=1}^{3}\int_{\Omega}g\cdot\alpha^i\,ds+\sum_{i=1}^{3}\int_0^{\mathrm{t}}\underbrace{G\big(\partial_s\eta^i\big)\cdot\partial_t\eta^i}_{\text{at } s=1}dt\\
&-\sum_{i=1}^{3}\int_0^{\mathrm{t}}\underbrace{G\big(\partial_s\eta^i\big)\cdot\partial_t\eta^i}_{\text{at } s=0}dt\\
=& -\sum_{i=1}^{3}\int_{\mathfrak{Q}_{\mathrm{t}}}G\big(\partial_s\eta^i\big)\cdot\partial_{st}\eta^i\,dsdt+\sum_{i=1}^{3}\int_{\Omega}g\cdot\eta^i(\mathrm{t})\,ds\\
&-\sum_{i=1}^{3}\int_{\Omega}g\cdot\alpha^i\,ds+\sum_{i=1}^{3}\int_0^{\mathrm{t}}\underbrace{\underbrace{G\big(\partial_s\eta^i(1)\big)\cdot\partial_t\alpha^i(1)}_{\partial_t\alpha^i=0}}_{=0}dt\\
&-\sum_{i=1}^{3}\int_0^{\mathrm{t}}\underbrace{\underbrace{G\big(\partial_s\eta^i(0)\big)\cdot\partial_t\bar{\eta}}_{\sum_{i=1}^{3}G(\partial_s\eta^i(0))=0}}_{=0}dt.
\end{aligned}
$$

Here $\bar{\eta}(t)$ denotes the spatial position of the junction. Consequently,

$$
\begin{aligned}
\sum_{i=1}^{3}\int_{\mathfrak{Q}_{\mathrm{t}}}|\partial_t\eta^i|^2\,dsdt =& -\sum_{i=1}^{3}\int_{\mathfrak{Q}_{\mathrm{t}}}G\big(\partial_s\eta^i\big)\cdot\partial_{st}\eta^i\,dsdt\\
&+\sum_{i=1}^{3}\int_{\Omega}g\cdot\eta^i(\mathrm{t})\,ds-\sum_{i=1}^{3}\int_{\Omega}g\cdot\alpha^i\,ds. \qquad (29)
\end{aligned}
$$

For the first term on the right-hand side, we observe that

$$G(\partial_s \eta^i)\cdot \partial_{st}\eta^i = \partial_t Q\left(\partial_s \eta^i\right), \tag{30}$$

cf. (18), where Q is defined as in (19). In view of (30), (29) becomes

$$\sum_{i=1}^{3}\int_{\mathbb{Q}_t} |\partial_t \eta^i|^2 \, dsdt + \sum_{i=1}^{3}\int_\Omega Q\left(\partial_s \eta^i\right)(\mathrm{t},\cdot) + \int_\Omega \left(-\mathbf{g}\right)\cdot \eta\,(\mathrm{t},\cdot)\, ds$$
$$= \sum_{i=1}^{3}\int_\Omega Q\left(\partial_s \alpha^i(s)\right) ds + \int_\Omega \left(-\mathbf{g}\right)\cdot \alpha(s)\, ds,$$

whence*

$$\sum_{i=1}^{3}\int_{\mathbb{Q}_t} |\partial_t \eta^i|^2 \, dsdt + \boldsymbol{\mathcal{E}}\left(\eta(\mathrm{t})\right) = \boldsymbol{\mathcal{E}}(\alpha). \tag{31}$$

Using $Q \geq 0$ and the definition of $\boldsymbol{\mathcal{E}}$, we derive that

$$\boldsymbol{\mathcal{E}}\left(\eta(\mathrm{t})\right) \geq -\|\eta(\mathrm{t})\|_{L^1(\Omega)}\|\mathbf{g}\|_{L^\infty(\Omega)}. \tag{32}$$

Hence,

$$\frac{1}{3}\|\eta(\mathrm{t})\|^2_{L^1(\Omega)} \leq \sum_{i=1}^{3}\|\eta^i(\mathrm{t})\|^2_{L^1(\Omega)} = \sum_{i=1}^{3}\left(\int_\Omega |\eta^i(\mathrm{t})|\, ds\right)^2$$
$$= \sum_{i=1}^{3}\left(\int_\Omega |\alpha^i(s)|\, ds + \int_{\mathbb{Q}_t} \partial_t|\eta^i|\, dsdt\right)^2 \leq 2\|\alpha\|^2_{L^2(\Omega)} + 2\sum_{i=1}^{3}\left(\int_{\mathbb{Q}_t} |\partial_t\eta^i|\, dsdt\right)^2$$
$$\leq 2\|\alpha\|^2_{L^2(\Omega)} + 2\mathrm{t}\sum_{i=1}^{3}\int_{\mathbb{Q}_t} |\partial_t\eta^i|^2\, dsdt \leq 2\|\alpha\|^2_{L^2(\Omega)} + 2T\boldsymbol{\mathcal{E}}(\alpha) + 2T\|\eta(\mathrm{t})\|_{L^1(\Omega)}\|\mathbf{g}\|_{L^\infty(\Omega)}.$$

Simple algebra implies that $\|\eta(\mathrm{t})\|_{L^1(\Omega)}$ is uniformly bounded. Consequently, $\sum_{i=1}^{3}\int_{\mathbb{Q}_T} |\partial_t\eta^i|^2\, dsdt$ is uniformly bounded. On the other hand, from the

*Of course, equality (31) is a generic property of gradient flows and at least the fact that its right-hand side is greater than or equal to the left-hand one follows from the general theory, cf. Attouch et al. (2014). We decided to present a direct and explicit proof here in order to help the reader to perceive the non-standard boundary conditions of the problem "by touching".

equality $\partial_s\Big(G\big(\partial_s\eta^i\big)\Big) = \partial_t\eta^i - g$ we deduce

$$\begin{aligned}\sum_{i=1}^{3}\int_{\mathbb{Q}_T}|\nabla G(\partial_s\eta^i)\partial_{ss}\eta^i|^2\,dsdt &= \sum_{i=1}^{3}\int_{\mathbb{Q}_T}|\partial_s\big(G(\partial_s\eta^i)\big)|^2\,dsdt\\ &\le 2\sum_{i=1}^{3}\int_{\mathbb{Q}_T}|\partial_t\eta^i|^2\,dsdt + 6\int_{\mathbb{Q}_T}|g|^2\,dsdt \le C.\end{aligned}$$

■

In view of (32) we simultaneously proved the following.

Corollary 4.2. *The energy of the approximating problem $\mathcal{E}\big(\eta(t)\big)$ is bounded from below for all $t\in[0,T]$ uniformly in ϵ.*

Since $\eta^i(0) = \alpha^i$ does not depend on ϵ, the uniform regularity can immediately be improved by the Poincaré inequality.

Corollary 4.3. *The norm $\|\eta^i\|_{L^\infty\left(0,T;L^2(\Omega)\right)}$ is uniformly bounded with respect to ϵ.*

For the the subsequent family of estimates will need to bound the time away from zero by some constant $\delta > 0$.

Lemma 4.4. *Given $\delta > 0$, the norm $\|\partial_t\eta\|_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$ is bounded uniformly in ϵ.*

Proof: By (Attouch et al., 2014, Theorem 17.2.3), the right derivative $\partial_t^+\eta$ exists for all times, and the expression $\|\partial_t^+\eta\,(t)\|^2_{L^2(\Omega)}$ is non-increasing in time. Using (Attouch et al., 2014, formula (17.79)), we obtain

$$\begin{aligned}\mathcal{E}(\alpha) - \mathcal{E}\big(\eta\,(\delta)\big) &\ge \limsup_{h\searrow 0}\mathcal{E}\Big(\eta\big(h\big)\Big) - \mathcal{E}\big(\eta\,(\delta)\big)\\ &= \int_0^\delta \|\partial_t\eta\,(t)\|^2_{L^2(\Omega)}dt\\ &\ge \int_0^\delta \|\partial_t^+\eta\,(\delta)\|^2_{L^2(\Omega)}\\ &= \delta\|\partial_t^+\eta\,(\delta)\|^2_{L^2(\Omega)}.\end{aligned}$$

By (28) and Corollary 4.2, the left-hand side is bounded from above uniformly in ϵ. Hence, $\|\partial_t^+ \eta^i(\delta)\|_{L^2(\Omega)} \leq C/\delta$.

Since $\|\partial_t^+ \eta(t)\|_{L^2(\Omega)}$ is non-increasing in time, we infer that

$$\|\partial_t \eta^i\|_{L^\infty\left(\delta,T;L^2(\Omega)\right)} = \|\partial_t^+ \eta^i\|_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$$

is bounded uniformly in ϵ. ■

We now derive uniform bounds for κ^i that were defined in (16). We start with the following lemma.

Lemma 4.5. *For fixed $\delta > 0$, $\partial_s \kappa^i$ and the product $|\kappa^i||\partial_{ss}\eta^i - \epsilon \partial_s \kappa^i|$ are bounded in $L^\infty\left(\delta, T; L^2(\Omega)\right)$ uniformly with respect to ϵ, $i = 1, 2, 3$.*

Proof: By Lemma 4.4, we know that $\|\partial_t \eta^i\|_{L^\infty\left(\delta,T;L^2(\Omega)\right)} \leq C$. Since $\partial_t \eta^i = \partial_s \kappa^i + g$, we infer that $\partial_s \kappa^i$ is bounded in $L^\infty\left(\delta, T; L^2(\Omega)\right)$ uniformly with respect to ϵ. We differentiate both sides of the equality

$$\partial_s \eta^i = F_\epsilon(\kappa^i) = \epsilon \kappa^i + \frac{\kappa^i}{\sqrt{\epsilon + |\kappa^i|^2}}$$

with respect to s to get

$$\partial_{ss}\eta^i = \epsilon \partial_s \kappa^i + \frac{\partial_s \kappa^i}{\sqrt{\epsilon + |\kappa^i|^2}} - \frac{\kappa^i \left(\partial_s \kappa^i \cdot \kappa^i\right)}{(\epsilon + |\kappa^i|^2)^{3/2}}.$$

We multiply this equality by $\sqrt{\epsilon + |\kappa^i|^2}$ and deduce

$$\partial_{ss}\eta^i \sqrt{\epsilon + |\kappa^i|^2} = \epsilon \partial_s \kappa^i \sqrt{\epsilon + |\kappa^i|^2} + \partial_s \kappa^i - \frac{\kappa^i \left(\partial_s \kappa^i \cdot \kappa^i\right)}{\sqrt{\epsilon + |\kappa^i|^2}}.$$

We reorganize the equality above to obtain

$$\left(\partial_{ss}\eta^i - \epsilon \partial_s \kappa^i\right) \sqrt{\epsilon + |\kappa^i|^2} = \partial_s \kappa^i - \frac{\kappa^i \left(\partial_s \kappa^i \cdot \kappa^i\right)}{\sqrt{\epsilon + |\kappa^i|^2}}.$$

The right-hand side is bounded in $L^\infty\left(\delta, T; L^2(\Omega)\right)$ uniformly with respect to ϵ, hence so is the left-hand side. Consequently, $|\kappa^i||\partial_{ss}\eta^i - \epsilon \partial_s \kappa^i|$ is bounded in $L^\infty\left(\delta, T; L^2(\Omega)\right)$ uniformly with respect to ϵ. ■

Lemma 4.6. *Let p^1, p^1, p^3 be three points in $\mathbb{R}^d$ whose convex hull is a triangle containing the origin. Assume that $|p^i| \geq 1$, $i = 1, 2, 3$. Then the circumradius of that triangle is greater than or equal to* 1.

Proof: It suffices to prove that there is no $p \in \mathbb{R}^d$ with $|p^i - p| < 1$, $i = 1, 2, 3$. Indeed, if such p exists, then $p^i \cdot p \geq \frac{1}{2}|p^i|^2 - \frac{1}{2}|p^i - p|^2 > 0$. Since the origin belongs to the convex hull, we infer $0 > 0$, a contradiction. ∎

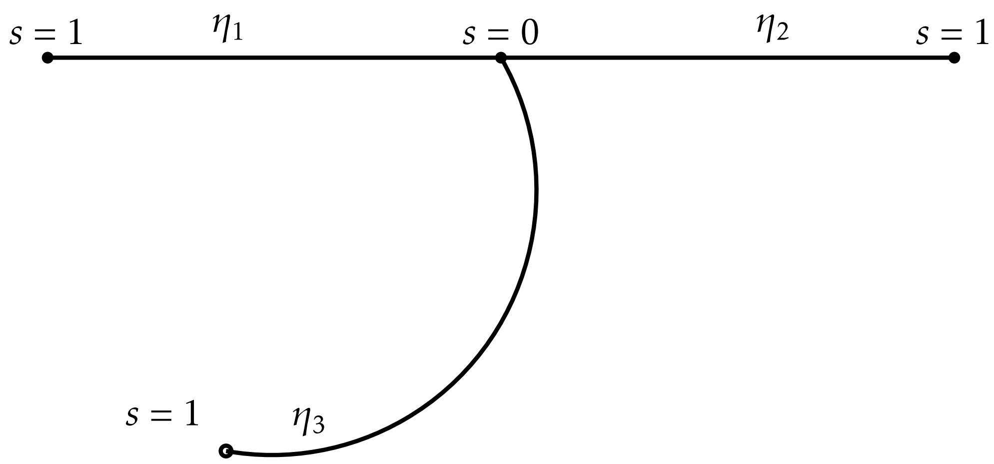

FIGURE 1. Symbolic depiction of the 1st scenario in Lemma 4.7: two arms of the triod tend to the straight position

Now we assemble all the ingredients to get the crucial L^∞ bounds for κ and $\partial_s \eta$.

Lemma 4.7. *Given $\delta > 0$, the norm $\|\kappa^i\|_{L^\infty\left(\delta,T;L^\infty(\Omega)\right)}$ is uniformly bounded with respect to ϵ.*

Proof: From now on, we do not omit the subscript ϵ. However, in this proof we decided to swap the sub- and superindices for the sake of convenience and readability.

Step 1. We argue by contradiction. Assume that there is a sequence $\epsilon^n \to 0$ such that

$$\|\kappa_1^{\epsilon^n}\|_{L^\infty\left(\delta,T;L^\infty(\Omega)\right)} \to +\infty.$$

Here, without loss of generality, we have chosen the generic i to be equal to 1. By the regularity of $\partial_s \kappa^{\epsilon^n}$ and $\partial_{ss} \eta^{\epsilon^n}$ there exists a set Υ_n of full measure in $[\delta, T]$ such that $\kappa_i^{\epsilon^n}(t)$ and $\eta_i^{\epsilon^n}(t)$ are C^1-smooth in $\overline{\Omega}$ whereas $\partial_{ss} \eta_i^{\epsilon^n}(t) \in$

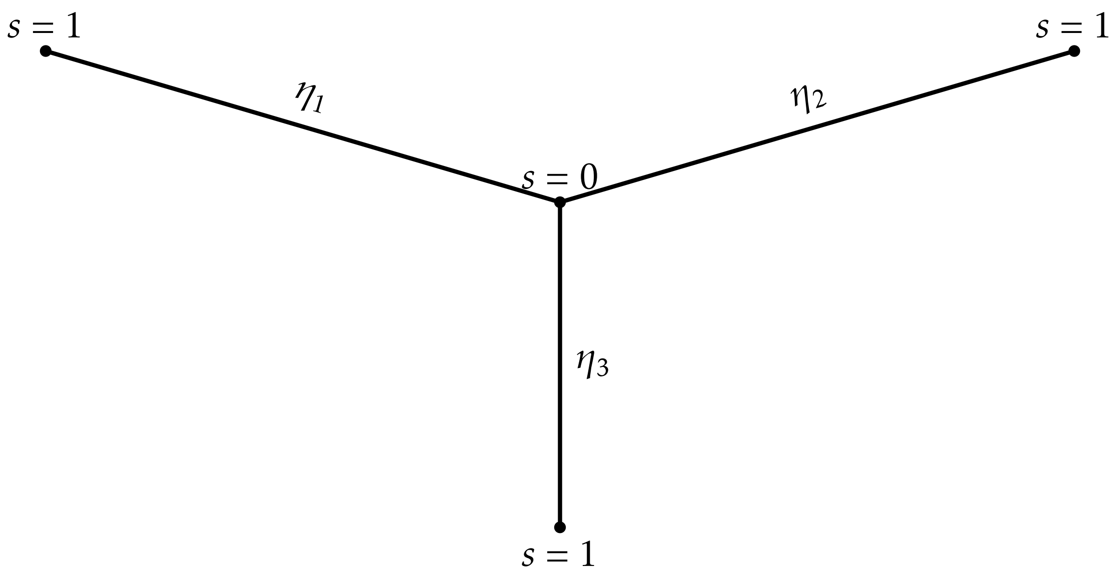

FIGURE 2. Symbolic depiction of the 2nd scenario in Lemma 4.7: all the arms of the triod tend to the straight position

$L^2(\Omega)$ for every i and every $t \in \mathfrak{T}_n$. Furthermore, by Lemma 4.5 without loss of generality we can assume that $\partial_s \kappa_i^{\epsilon^n}(t)$ and $|\kappa_i^{\epsilon^n}(t,\cdot)||\partial_{ss}\eta_i^{\epsilon^n}(t,\cdot) - \epsilon^n \partial_s \kappa_i^{\epsilon^n}(t,\cdot)|$ are bounded in $L^2(\Omega)$ uniformly w.r.t. n and $t \in \mathfrak{T}_n$. Let $\mathfrak{T} := \cap_{n\in\mathbb{N}} \mathfrak{T}_n$. Then there is a sequence $(t^n, s^n) \in \mathfrak{T} \times \Omega$ such that $|\kappa_1^{\epsilon^n}(t^n, s^n)| \to +\infty$ as $n \to \infty$. Thus,

$$\kappa_1^{\epsilon^n}\left(t^n, s\right) = \underbrace{\kappa_1^{\epsilon^n}\left(t^n, s^n\right)}_{\to+\infty} + \underbrace{\int_{s^n}^{s} \partial_\xi \kappa_1^{\epsilon^n}\left(t^n, \xi\right) \partial\xi}_{\leq C}$$

when $n \to \infty$. Accordingly, $|\kappa_1^{\epsilon^n}(t^n)| \to +\infty$ uniformly in s.

Step 2. By the boundary conditions,

$$\sum_{i=1}^{3} \kappa_i^{\epsilon^n}(t^n, 0) = 0. \tag{33}$$

By the previous step, $|\kappa_1^{\epsilon^n}(t^n, 0)| \to +\infty$. Hence we have two possible scenarios symbolically pictured in Figures 1 and 2, respectively. The first option is $|\kappa_2^{\epsilon^n}(t^n, 0)| \to +\infty$ and $|\kappa_3^{\epsilon^n}(t^n, 0)| \leq C$ as $n \to +\infty$ (up to swapping the second and the third arms). The second one is $|\kappa_2^{\epsilon^n}(t^n, 0)| \to +\infty$ and $|\kappa_3^{\epsilon^n}(t^n, 0)| \to +\infty$ as $n \to +\infty$.

Step 3. We start by examining the second scenario. An argument similar to the one of Step 1 shows that $|\kappa_i^{\epsilon^n}(t^n)| \to +\infty$ uniformly in s, $i = 1,2,3$. Since $t^n \in \Upsilon$, we know that

$$|\kappa_i^{\epsilon^n}\left(t^n,\cdot\right)||\partial_{ss}\eta_i^{\epsilon^n}\left(t^n,\cdot\right) - \epsilon\partial_s\kappa_i^{\epsilon^n}\left(t^n,\cdot\right)|$$

is uniformly bounded in $L^2(\Omega)$. Hence,

$$|\partial_{ss}\eta_i^{\epsilon^n}\left(t^n,\cdot\right) - \epsilon^n\partial_s\kappa_i^{\epsilon^n}\left(t^n,\cdot\right)| \to 0$$

in $L^2(\Omega)$ as $n \to +\infty$. On the other hand, $\partial_s\kappa_i^{\epsilon^n}(t^n)$ is uniformly bounded in $L^2(\Omega)$, whence

$$|\epsilon^n\partial_s\kappa_i^{\epsilon^n}\left(t^n\right)| \to 0$$

in $L^2(\Omega)$. We conclude that $|\partial_{ss}\eta_i^{\epsilon^n}(t^n,\cdot)| \to 0$ in $L^2(\Omega)$ as $n \to +\infty$. By Remark 2.1, $|\kappa_i^{\epsilon^n}(t^n,\cdot)| \geq 1$ implies $|\partial_s\eta_i^{\epsilon^n}(t^n,\cdot)| \geq 1$ (assuming n to be large enough).

Step 4. The idea now is to compare the triangle[†] formed by the points $p_i^n := \eta_i^{\epsilon^n}(t^n,0) + \partial_s\eta_i^{\epsilon^n}(t^n,0)$ with the fixed triangle formed by $\eta_i^{\epsilon^n}(t^n,1) = \alpha_i(1)$, $i = 1,2,3$. Observe that

$$|\partial_s\eta_i^{\epsilon^n}\left(t^n,0\right) - \partial_s\eta_i^{\epsilon^n}\left(t^n,\xi\right)| = \left|\int_0^\xi \partial_{ss}\eta_i^{\epsilon^n}\left(t^n\right)\,ds\right|$$

$$\leq \int_0^\xi |\partial_{ss}\eta_i^{\epsilon^n}\left(t^n\right)|\,ds \leq \sqrt{\int_0^1 |\partial_{ss}\eta_i^{\epsilon^n}(t^n)|^2\,ds} \to 0 \text{ uniformly in } \xi \text{ as } n \to \infty.$$

Hence,

$$\begin{aligned}|p_i^n - \alpha_i(1)| &= |\eta_i^{\epsilon^n}\left(t^n,0\right) - \eta_i^{\epsilon^n}\left(t^n,1\right) + \partial_s\eta_i^{\epsilon^n}\left(t^n,0\right)| \\ &= \left|\int_0^1 \partial_s\eta_i^{\epsilon^n}\left(t^n,0\right) - \partial_s\eta_i^{\epsilon^n}\left(t^n,s\right)ds\right| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Since the circumradius is a continuous function of the vertices of a triangle, we must have that the circumradius of the three points p_i^n is less than 1 for n sufficiently large. Since the junction point $\eta_i^{\epsilon^n}(t^n,0)$ does not depend on i, the circumradius of the three points $\tilde{p}_i^n := \partial_s\eta_i^{\epsilon^n}(t^n,0)$ is the same as the previous one. By Step 3, $|\tilde{p}_i^n| \geq 1$. Moreover, since $\sum_{i=1}^3 \kappa_i^{\epsilon^n}(t^n,0) = 0$ and

[†]It is clear from the proof below that these points do not lie on the same straight line, at least for n large enough.

$\tilde{p}_i^n = F_{\epsilon^n}(\kappa_i^{\epsilon^n}(t^n,0))$, we conclude that the convex hull of $\{\tilde{p}_i^n\}$ contains the origin. We arrive at a contradiction because by Lemma 4.6 the circumradius of $\{\tilde{p}_i^n\}$ must be greater than or equal to 1.

Step 5. We now study the first scenario. Define p_i^n and $\tilde{p}_i^n$ as in Step 4. The plan is to look at the angle θ_n between the position vectors of $\tilde{p}_1^n$ and $\tilde{p}_2^n$ and to obtain a contradiction from that.

We first show that θ_n cannot tend to π. Indeed, mimicking the arguments of Steps 3 and 4, we can prove that for $i = 1,2$ one has $|\partial_s \eta_i^{\epsilon^n}(t^n,\cdot)| \geq 1$ with n large enough, $|\partial_{ss}\eta_i^{\epsilon^n}(t^n,\cdot)| \to 0$ in $L^2(\Omega)$ and

$$|p_i^n - \alpha_i(1)| \to 0 \text{ as } n \to \infty.$$

Hence,

$$|\tilde{p}_1^n - \tilde{p}_2^n| = |p_1^n - p_2^n| \to |\alpha_1(1) - \alpha_2(1)| < 2.$$

Since we have $|\tilde{p}_1^n| \geq 1, |\tilde{p}_2^n| \geq 1$, the angle θ_n cannot converge to π.

Now take the wedge product of relation (33) with the vector

$$\frac{1}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)||\kappa_2^{\epsilon^n}(t^n,0)|}\partial_s \eta_1^{\epsilon^n}\left(t^n,0\right)$$

to obtain

$$\frac{\kappa_2^{\epsilon^n}(t^n,0)}{|\kappa_2^{\epsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|} + \frac{\kappa_3^{\epsilon^n}(t^n,0)}{|\kappa_2^{\epsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|} = 0.$$

Since $|\kappa_2^{\epsilon^n}(t^n,0)| \to +\infty$ and $|\kappa_3^{\epsilon^n}(t^n,0)| \leq C$, the second term converges to 0. Consequently,

$$|\sin\theta_n| = \left|\frac{\partial_s \eta_2^{\epsilon^n}(t^n,0)}{|\partial_s \eta_2^{\epsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|}\right| = \left|\frac{\kappa_2^{\epsilon^n}(t^n,0)}{|\kappa_2^{\epsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|}\right| \to 0$$

as $n \to \infty$.

To obtain a contradiction, it remains to observe that θ_n cannot tend to 0. Indeed, taking the scalar product of relation (33) with

$$\frac{1}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)||\kappa_2^{\epsilon^n}(t^n,0)|}\partial_s \eta_1^{\epsilon^n}\left(t^n,0\right)$$

we get

$$\frac{\kappa_1^{\epsilon^n}(t^n,0)}{|\kappa_2^{\epsilon^n}(t^n,0)|} \cdot \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|} + \frac{\kappa_2^{\epsilon^n}(t^n,0)}{|\kappa_2^{\epsilon^n}(t^n,0)|} \cdot \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|} + \frac{\kappa_3^{\epsilon^n}(t^n,0)}{|\kappa_2^{\epsilon^n}(t^n,0)|} \cdot \frac{\partial_s \eta_1^{\epsilon^n}(t^n,0)}{|\partial_s \eta_1^{\epsilon^n}(t^n,0)|} = 0. \tag{34}$$

The first term is equal to $\frac{\sigma_1^{\epsilon^n}}{|\kappa_2^{\epsilon^n}(t^n,0)||\partial_s\eta_1^{\epsilon^n}(t^n,0)|} \geq 0$ by (16) and (17). The third term converges to 0. Accordingly, the second term, which is equal to $\cos\theta_n$, cannot tend to 1. ∎

Corollary 4.8. *Given $\delta > 0$, the norm $\|\eta^i_{\epsilon^n}\|_{L^\infty\left(\delta,T;W^{1,\infty}(\Omega)\right)}$ is uniformly bounded with respect to ϵ.*

Proof: Since $\partial_s\eta^i_{\epsilon^n} = F_{\epsilon^n}(\kappa^i_{\epsilon^n})$ and the sequence $\{\epsilon_n\}$ is bounded, Lemma 4.7 yields a uniform L^∞ bound for $\partial_s\eta^i$. By Lemma 4.1, $\|\eta^i\|_{L^\infty\left(\delta,T;L^1(\Omega)\right)}$ is also uniformly bounded with respect to ϵ, and the claim follows by the mean value theorem. ∎

Lemma 4.9. *Given $\delta > 0$, the norm $\|\sigma^i_{\epsilon^n}\|_{L^\infty\left(\delta,T;H^1(\Omega)\right)}$ is bounded uniformly in ϵ.*

Proof: In view of Lemma 4.7 and Corollary 4.8, the $L^\infty\left(\delta,T;L^\infty(\Omega)\right)$-bound for σ immediately follows from the equality $\sigma^i = \partial_s\eta^i\cdot\kappa^i$. Differentiating this equality w.r.t. s we obtain

$$\partial_s\sigma^i = \partial_s\kappa^i\cdot\partial_s\eta^i + \kappa^i\cdot\partial_{ss}\eta^i.$$

We estimate the two terms on the right-hand side separately. Firstly, a uniform $L^\infty\left(\delta,T;L^2(\Omega)\right)$ bound for $\partial_s\kappa^i$ has been already established, cf. Lemma 4.5. This together with Corollary 4.8 implies the uniform boundedness of $\partial_s\kappa^i\cdot\partial_s\eta^i$ in $L^\infty\left(\delta,T;L^2(\Omega)\right)$.

Now, we estimate $\kappa^i\cdot\partial_{ss}\eta^i$. From the explicit expression of λ_{ϵ^n} in (15), for $\tau\in\mathbb{R}^d$ we have

$$\lambda_{\epsilon^n}(\tau) = \frac{\sqrt{\epsilon^n + |G_{\epsilon^n}(\tau)|^2}}{\epsilon^n\sqrt{\epsilon^n+|G_{\epsilon^n}(\tau)|^2}+1} \geq \frac{|G_{\epsilon^n}(\tau)|}{\epsilon^n|G_{\epsilon^n}(\tau)|+1}. \tag{35}$$

Thus,

$$\begin{aligned}|G_{\epsilon^n}\left(\partial_s\eta^i\right)||\partial_{ss}\eta^i| &\leq \left(\epsilon^n|G_{\epsilon^n}\left(\partial_s\eta^i\right)|+1\right)|\lambda_{\epsilon^n}\partial_{ss}\eta^i|\\ &\leq \left(\epsilon^n|\kappa^i|+1\right)|\nabla G_{\epsilon^n}\left(\partial_s\eta^i\right)\partial_{ss}\eta^i|.\end{aligned}$$

By Lemma 4.7, $|\kappa^i|$ is uniformly bounded in $L^\infty\left(\delta,T;L^\infty(\Omega)\right)$, whence

$$|\kappa^i\cdot\partial_{ss}\eta^i| \leq |G_{\epsilon^n}\left(\partial_s\eta^i\right)||\partial_{ss}\eta^i| \leq C|\nabla G_{\epsilon^n}\left(\partial_s\eta^i\right)\partial_{ss}\eta^i| = C|\partial_s\kappa^i|.$$

Since the right-hand side is uniformly bounded in $L^\infty\left(\delta, T; L^2(\Omega)\right)$, so is the left-hand side and, consequently, the spatial derivative $\partial_s\sigma^i$ itself. ■

5. Existence of generalized solutions

We are now at the position to define generalized solutions to the original problem (6), (2), (3) and to prove their existence.

Definition 5.1. Given initial data $\alpha^i(s) \in W^{1,\infty}(\Omega)^d$ as in Remark 1.1, we call a pair $\left(\eta^i, \sigma^i\right)$ a generalized solution to (6), (2), (3) in $\mathfrak{Q}_\infty$ if

(i)
- $\eta^i \in L^\infty_{loc}\left((0,\infty; W^{1,\infty}(\Omega)\right)^d \cap C_{loc}\left((0,\infty); C\left(\overline{\Omega}\right)\right)^d \cap AC^2_{loc}\left([0,\infty); L^2(\Omega)\right)^d$,
- $\partial_t\eta^i \in L^\infty_{loc}\left((0,\infty); L^2(\Omega)\right)^d \cap L^2_{loc}\left([0,\infty); L^2(\Omega)\right)^d$,
- $\sigma^i \in L^\infty_{loc}\left((0,\infty); AC^2(\Omega)\right)$,
- $\sigma^i\partial_s\eta^i \in L^\infty_{loc}\left((0,\infty); AC^2(\Omega)\right)^d$.

(ii) Each pair $\left(\eta^i, \sigma^i\right)$ satisfies for a.e. $(t,s) \in \mathfrak{Q}_\infty$

$$\partial_t\eta^i(t,s) = \partial_s\left(\sigma^i(t,s)\,\partial_s\eta^i(t,s)\right) + g, \tag{36}$$

$$\sigma^i(t,s)\left(|\partial_s\eta^i(t,s)|^2 - 1\right) = 0, \tag{37}$$

$$|\partial_s\eta^i(t,s)| \leq 1, \tag{38}$$

as well as the initial conditions

$$\eta^i(0,s) = \alpha^i(s)$$

and the boundary conditions

$$\eta^1(t,0) = \eta^2(t,0) = \eta^3(t,0),$$
$$\eta^i(t,1) = \alpha^i(1),$$
$$\sum_{i=1}^{3} \sigma^i(t,0)\,\partial_s\eta^i(t,0) = 0.$$

(iii) The solutions η^i satisfy the energy dissipation inequality

$$\sum_{i=1}^{3}\int_\Omega |\partial_t\eta^i(t,s)|^2\,ds \leq \sum_{i=1}^{3}\int_\Omega g\cdot\partial_t\eta^i(t,s)\,ds \tag{39}$$

for a.e. $t \in (0,\infty)$.

Remark 5.2. Note that (37), (38) is a minor relaxation of the non-convex constraint

$$|\partial_s \eta^i(t,s)| = 1. \tag{40}$$

However, this is not a banal convexification of the constraint since (37) is still not convex. The new constraints (37), (38) naturally appear from the (η, σ, κ)-formulation (10), cf. (42) in the proof below. Moreover, if a generalized (in the sense of Definition 5.1) solution (η, σ) is C^2-smooth, then it automatically satisfies the strong constraint (40). This claim can be shown by following the lines of (Shi and Vorotnikov, 2019a, Remark 4.2), (Shi and Vorotnikov, 2019b, Remark 3.20). As in Shi and Vorotnikov (2019a), our generalized solutions are, generally speaking, not unique. Yet this has nothing to do with the fact that we slightly relaxed the constraint (40). As a matter of fact, non-uniqueness can persist even if the strong constraint (40) is imposed, cf. (Shi and Vorotnikov, 2019a, Remark 6.5).

For convenience, we first pass to the limit on finite time intervals. In what follows, we use the shortcut $\mathfrak{Q}_T^* := (\delta, T) \times \Omega$.

Proposition 5.3. *Fix $T > 0$ and a small $\delta > 0$. Let η_ϵ be a solution to* (12) *in $\mathfrak{Q}_T$ with the initial/boundary conditions* (13) *as constructed in Section 3. Let $\left(\kappa^i, \sigma^i\right)$ be defined as in* (16)*. Then (up to selecting a subsequence ϵ^n) there exists a limit $(\eta^i, \sigma^i, \kappa^i)$ such that as $\epsilon \to 0$ we have*

$\eta_\epsilon^i \to \eta^i$ weakly in $L^\infty\left(\delta, T; W^{1,\infty}(\Omega)\right)^d$, strongly in $C\left(\overline{\mathfrak{Q}_T^*}\right)^d$ and weakly in $L^2(\mathfrak{Q}_T)^d$,*
$\partial_t \eta_\epsilon^i \to \partial_t \eta^i$ weakly- in $L^\infty\left(\delta, T; L^2(\Omega)\right)^d$ and weakly in $L^2(\mathfrak{Q}_T)^d$,*
$\sigma_\epsilon^i \to \sigma^i$ weakly- in $L^\infty\left(\delta, T; H^1(\Omega)\right)$,*
$\kappa_\epsilon^i \to \kappa^i$ weakly- in $L^\infty\left(\delta, T; H^1(\Omega)\right)$.*

The limit satisfies the relation

$$\kappa^i = \sigma^i \partial_s \eta^i \in L^\infty\left(\delta, T; H^1(\Omega)\right)$$

and solves (6)-(2) *in* $\mathfrak{Q}_T^*$ *in the sense that*

$$\begin{aligned}
&\partial_t \eta^i = \partial_s\left(\sigma^i \partial_s \eta^i\right) + g \text{ a.e. in } \mathfrak{Q}_T^*,\\
&\sigma^i\left(|\partial_s \eta^i|^2 - 1\right) = 0 \text{ a.e. in } \mathfrak{Q}_T^*,\\
&\eta^i(t,1) = \alpha^i(1),\\
&\eta^1(t,0) = \eta^2(t,0) = \eta^3(t,0),\\
&\sum_{i=1}^{3} \kappa^i = 0 \text{ at } s=0 \text{ for a.e. } t \in (\delta, T).
\end{aligned}$$

Remark 5.4. At this stage we don't discuss the validity of the initial condition $\eta^i(0,s) = \alpha^i(s)$ that is postponed until Remark 5.5.

Proof: The weak compactness results for η_ϵ^i, σ_ϵ^i and κ_ϵ^i follow immediately from the estimates above. By the Aubin-Lions-Simon theorem,

$$L^\infty\left(\delta, T; W^{1,\infty}(\Omega)\right) \cap W^{1,\infty}\left(\delta, T; L^2(\Omega)\right) \subset C\left([\delta, T]; C\left(\overline{\Omega}\right)\right) \tag{41}$$

and the embedding is compact, which implies strong compactness of η_ϵ^i in $C\left([\delta, T]; C\left(\overline{\Omega}\right)\right)$.

Let us show that

$$\kappa^i = \sigma^i \partial_s \eta^i, \quad \sigma^i = \kappa^i \cdot \partial_s \eta^i \tag{42}$$

a.e. in $\mathfrak{Q}_T^*$. Since both sides of the equalities (42) are integrable on $\mathfrak{Q}_T^*$, it suffices to prove (42) in the sense of the distributions, i.e., that for any $\phi^i \in L^2\left(\delta, T; H_0^1(\Omega)\right)$

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \kappa^i \phi^i ds dt = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \sigma^i \eta^i \partial_s \phi^i ds dt - \sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \partial_s \sigma^i \eta^i \phi^i ds dt \tag{43}$$

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \sigma^i \phi^i ds dt = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \kappa^i \cdot \eta^i \partial_s \phi^i ds dt - \sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \partial_s \kappa^i \cdot \eta^i \phi^i ds dt. \tag{44}$$

Firstly, applying integration by parts to the equality $\sigma_\epsilon^i = \kappa_\epsilon^i \cdot \partial_s \eta_\epsilon^i$ we obtain

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \sigma_\epsilon^i \phi^i = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \kappa_\epsilon^i \cdot \eta_\epsilon^i \partial_s \phi^i ds dt - \sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \partial_s \kappa_\epsilon^i \cdot \eta_\epsilon^i \phi^i ds dt,$$

and due to the strong compactness property of $\{\eta_\epsilon\}$ given above we can pass to the limit to get (44). We now claim

$$\lim_{\epsilon\to 0}\left|\kappa_\epsilon^i\right|\left|\left|\partial_s\eta_\epsilon^i\right|^2-1\right|=0 \tag{45}$$

uniformly in $\mathfrak{Q}_T^*$. Before proving the claim we show how (43) follows from (45). Indeed, with (45) in our hand and noting that $\kappa_\epsilon^i|\partial_s\eta_\epsilon^i|^2=\left(\kappa_\epsilon^i\cdot\partial_s\eta_\epsilon^i\right)\partial_s\eta_\epsilon^i=\sigma_\epsilon^i\partial_s\eta_\epsilon^i$ we have for each $i=1,2,3$

$$\lim_{\epsilon\to 0}\|\sigma_\epsilon^i\partial_s\eta_\epsilon^i-\kappa_\epsilon^i\|_{L^\infty\left(\mathfrak{Q}_T^*\right)}=0. \tag{46}$$

In particular, for any $\phi^i\in L^2\left(\delta,T;H_0^1(\Omega)\right)$

$$\lim_{\epsilon\to 0}\sum_{i=1}^3\int_{\mathfrak{Q}_T^*}\kappa_\epsilon^i\phi^i\,dsdt=\lim_{\epsilon\to 0}\sum_{i=1}^3\int_{\mathfrak{Q}_T^*}\sigma_\epsilon^i\partial_s\eta_\epsilon^i\phi^i\,dsdt.$$

An integration by parts applied to the integral on the right-hand side gives

$$\lim_{\epsilon\to 0}\sum_{i=1}^3\int_{\mathfrak{Q}_T^*}\kappa_\epsilon^i\phi^i\,dsdt=\lim_{\epsilon\to 0}\sum_{i=1}^3\int_{\mathfrak{Q}_T^*}\left(-\sigma_\epsilon^i\eta_\epsilon^i\partial_s\phi_\epsilon^i-\partial_s\sigma_\epsilon^i\eta_\epsilon^i\phi^i\right)dsdt.$$

This together with the compactness properties established above yields (43).

We now provide a proof of (45). By the definition of F_ϵ in (11),

$$\begin{aligned}
|\partial_s\eta_\epsilon^i|-1&=\left|F_\epsilon\left(\kappa_\epsilon^i\right)\right|-1\\
&=\epsilon|\kappa_\epsilon^i|+\frac{|\kappa_\epsilon^i|}{\sqrt{\epsilon+|\kappa_\epsilon^i|^2}}-1\\
&=\epsilon|\kappa_\epsilon^i|-\frac{\epsilon}{\sqrt{\epsilon+|\kappa_\epsilon^i|^2}\left(\sqrt{\epsilon+|\kappa_\epsilon^i|^2}+|\kappa_\epsilon^i|\right)}.
\end{aligned}$$

Thus,

$$|\kappa_\epsilon^i|\left||\partial_s\eta_\epsilon^i|^2-1\right| = \left||\partial_s\eta_\epsilon^i|+1\right|\left|\epsilon|\kappa_\epsilon^i|^2 - \frac{\epsilon|\kappa_\epsilon^i|}{\sqrt{\epsilon+|\kappa_\epsilon^i|^2}\left(\sqrt{\epsilon+|\kappa_\epsilon^i|^2}+|\kappa_\epsilon^i|\right)}\right|$$
$$\leq \left||\partial_s\eta_\epsilon^i|+1\right|\left(\epsilon|\kappa_\epsilon^i|^2+\sqrt{\epsilon}\right).$$

This together with uniform L^∞ bounds on $\partial_s\eta_\epsilon^i$ and κ_ϵ^i yields (45).

Passing to the limit in $L^2\left(\mathbb{Q}_T^*\right)$ in $\partial_t\eta_\epsilon^i = \partial_s\kappa_\epsilon^i + g$ and using (42), we obtain $\partial_t\eta^i = \partial_s\left(\sigma^i\partial_s\eta^i\right)+g$. To get $\sigma^i\left(|\partial_s\eta^i|^2-1\right)=0$, it suffices to express κ^i from the first of the equalities (42) and substitute the result into second one.

Due to the strong uniform convergence of η_ϵ^i, we have $\alpha^i(1) = \eta_\epsilon^i(t,1) \to \eta^i(t,1)$, whence $\eta^i(t,1) = \alpha^i(1)$ for all $t\in[\delta,T]$. The condition $\eta_\epsilon^1(t,0) = \eta_\epsilon^2(t,0) = \eta_\epsilon^3(t,0)$ similarly passes to the limit. To check the validity of the boundary condition at $s=0$ for κ, we swap the variables t and s, noting that κ_ϵ^i are uniformly bounded and weakly-* converging in $H^1(0,1;L^\infty(\delta,T))$. Employing, for instance, (Zvyagin and Vorotnikov, 2008, Corollary 2.2.1), we get

$$H^1(0,1;L^\infty(\delta,T)) = AC^2([0,1];L^\infty(\delta,T)). \tag{47}$$

Hence, by the Aubin-Lions-Simon theorem, the embedding

$$H^1(0,1;L^\infty(\delta,T)) \subset C([0,1];H^{-1}(\delta,T))$$

is compact, whence we may assume that $\kappa_\epsilon^i \to \kappa^i$ strongly in $C([0,1];H^{-1}(\delta,T))$. Thus,

$$0 = \sum_{i=1}^3 \kappa_\epsilon^i(\cdot,0) \to \sum_{i=1}^3 \kappa^i(\cdot,0)$$

in $H^{-1}(\delta,T)$. Due to (47), $\sum_{i=1}^3 \kappa^i(\cdot,0) = 0$ in $L^\infty(\delta,T)$. ∎

Remark 5.5 (Initial conditions). By the Aubin-Lions-Simon theorem, the embedding

$$H^1(0,T;L^2(\Omega)) \subset C([0,T];H^{-1}(\Omega))$$

is compact. Since η_ϵ^i (w.l.o.g.) converge weakly in $H^1(0,T;L^2(\Omega))$ we can pass to the limit in the initial conditions to obtain $\eta^i(0,\cdot) = \alpha^i$ in $H^{-1}(\Omega)$. However, since $H^1(0,T;L^2(\Omega)) = AC^2(0,T;L^2(\Omega))$, the initial conditions actually hold in $L^2(\Omega)$.

Proposition 5.6. *Let $\left(\eta^i, \sigma^i\right)$ be the limiting solution obtained in Proposition 5.3. Then*

(i) $|\partial_s \eta^i (t,s)| \le 1$ *for a.e.* $(t,s) \in \mathfrak{Q}_T^*$*;*
(ii) $\sigma^i \ge 0$ *for a.e.* $(t,s) \in \mathfrak{Q}_T^*$*;*
(iii) (39) *holds for a.a.* $t \in (\delta, T)$*.*

We omit the proof since it follows the same lines as the proofs of (Shi and Vorotnikov, 2019a, Proposition 3.6 and Theorem 3).

Employing a diagonal argument and taking into account Proposition 5.6 and Remark 5.5, it is easy to deduce Theorem 1.2 from Proposition 5.3.

Remark 5.7 (Single cord with two fixed ends). The results of the paper, mutatis mutandis, are valid for the overdamped fall of a single inextensible string with the ends fixed at two distinct spatial points (it suffices to observe that such a string can be viewed as a degenerate "triod" with one arm having zero length); remember that Shi and Vorotnikov (2019a) studied the case of one free and one fixed end (i.e., a "whip"). More precisely, we have the following result.

Proposition 5.8. *Given* $\alpha(s) \in W^{1,\infty}(\Omega)^d$ *satisfying* $|\alpha(0) - \alpha(1)| < 1$, $|\partial_s \alpha(s)| = 1$ *a.e. in* Ω*, there exists a generalized solution to*

$$\begin{cases} \partial_t \eta = \partial_s \left(\sigma \partial_s \eta\right) + g, \\ |\partial_s \eta| = 1, \\ \eta(t,0) = \alpha(0), \quad \eta(t,1) = \alpha(1), \\ \eta(0,s) = \alpha(s). \end{cases} \tag{48}$$

in $\mathfrak{Q}_\infty$*. Moreover,* $\sigma(t,s) \ge 0$ *for almost every* $(t,s) \in \mathfrak{Q}_\infty$*.*

References

Amann, H., 1993. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992). Teubner, Stuttgart. volume 133 of *Teubner-Texte Math.*, pp. 9–126.

Ambrosio, L., Gigli, N., Savaré, G., 2008. Gradient flows in metric spaces and in the space of probability measures. Lectures in Mathematics ETH Zürich. second ed., Birkhäuser Verlag, Basel.

Arnold, V., 1966. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier (Grenoble) 16, 319–361.

Arnold, V.I., Khesin, B.A., 1998. Topological methods in hydrodynamics. volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

Attouch, H., Buttazzo, G., Michaille, G., 2014. Variational analysis in Sobolev and BV spaces. volume 17 of *MOS-SIAM Series on Optimization*. Second ed., SIAM, Philadelphia, PA. URL: https://doi.org/10.1137/1.9781611973488, doi:10.1137/1.9781611973488.

Bauer, M., Bruveris, M., Michor, P.W., 2014. Overview of the geometries of shape spaces and diffeomorphism groups. J. Math. Imaging Vision 50, 60–97. URL: http://dx.doi.org/10.1007/s10851-013-0490-z, doi:10.1007/s10851-013-0490-z.

Bauer, M., Harms, P., Michor, P.W., 2012. Almost local metrics on shape space of hypersurfaces in n-space. SIAM J. Imaging Sci. 5, 244–310. URL: https://doi.org/10.1137/100807983, doi:10.1137/100807983.

Bauer, M., Michor, P.W., Müller, O., 2016. Riemannian geometry of the space of volume preserving immersions. Differential Geom. Appl. 49, 23–42. URL: http://dx.doi.org/10.1016/j.difgeo.2016.07.002, doi:10.1016/j.difgeo.2016.07.002.

Brenier, Y., Duan, X., 2018. From conservative to dissipative systems through quadratic change of time, with application to the curve-shortening flow. Arch. Ration. Mech. Anal. 227, 545–565. URL: https://doi.org/10.1007/s00205-017-1166-4, doi:10.1007/s00205-017-1166-4.

Brézis, H., 1968. Équations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier (Grenoble) 18, 115–175. URL: http://www.numdam.org/item?id=AIF_1968__18_1_115_0.

Brézis, H., 1973. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, No. 5, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York.

Castro, Á., Córdoba, D., Faraco, D., 2021. Mixing solutions for the Muskat problem. Inventiones mathematicae 226, 251–348.

Constantin, P., Córdoba, D., Gancedo, F., Rodriguez-Piazza, L., Strain, R.M., 2016. On the Muskat problem: global in time results in 2d and 3d. American Journal of Mathematics 138, 1455–1494.

Córdoba, A., Córdoba, D., Gancedo, F., 2011. Interface evolution: the Hele-Shaw and Muskat problems. Ann. of Math. (2) 173, 477–542. URL: https://doi.org/10.4007/annals.2011.173.1.10, doi:10.4007/annals.2011.173.1.10.

Dall'Acqua, A., Lin, C.C., Pozzi, P., 2019. Elastic flow of networks: long-time existence result. Geom. Flows 4, 83–136. URL: https://doi.org/10.1515/geofl-2019-0005, doi:10.1515/geofl-2019-0005.

Dall'Acqua, A., Lin, C.C., Pozzi, P., 2021. Elastic flow of networks: short-time existence result. J. Evol. Equ. 21, 1299–1344. URL: https://doi.org/10.1007/s00028-020-00626-6, doi:10.1007/s00028-020-00626-6.

Garcke, H., Menzel, J., Pluda, A., 2019. Willmore flow of planar networks. J. Differential Equations 266, 2019–2051. URL: https://doi.org/10.1016/j.jde.2018.08.019, doi:10.1016/j.jde.2018.08.019.

Garcke, H., Menzel, J., Pluda, A., 2020. Long time existence of solutions to an elastic flow of networks. Comm. Partial Differential Equations 45, 1253–1305. URL: https://

doi.org/10.1080/03605302.2020.1771364, doi:10.1080/03605302.2020.1771364.

Ghys, E., 2011. Sur la coupe des vêtements: variation autour d'un thème de Tchebychev. Enseign. Math. (2) 57, 165–208. URL: https://doi.org/10.4171/LEM/57-1-8, doi:10.4171/LEM/57-1-8.

Grothaus, M., Marheineke, N., 2016. On a nonlinear partial differential algebraic system arising in the technical textile industry: analysis and numerics. IMA Journal of Numerical Analysis 36, 1783–1803.

Koiso, N., 1996. On the motion of a curve towards elastica, in: Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992). Soc. Math. France, Paris. volume 1 of *Sémin. Congr.*, pp. 403–436.

Kröner, H., Novaga, M., Pozzi, P., 2021. Anisotropic curvature flow of immersed networks. Milan J. Math. 89, 147–186. URL: https://doi.org/10.1007/s00032-021-00329-8, doi:10.1007/s00032-021-00329-8.

Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N., 1968. Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I. Translated from the Russian by S. Smith.

Lin, C.C., Lue, Y.K., Schwetlick, H.R., 2015. The second-order L^2-flow of inextensible elastic curves with hinged ends in the plane. J. Elasticity 119, 263–291. URL: https://doi.org/10.1007/s10659-015-9518-5, doi:10.1007/s10659-015-9518-5.

Liu, J.G., Pego, R.L., Slepčev, D., 2019. Least action principles for incompressible flows and geodesics between shapes. Calculus of Variations and Partial Differential Equations 58, Paper No. 179.

Magni, A., Mantegazza, C., Novaga, M., 2016. Motion by curvature of planar networks, II. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 15, 117–144.

Mantegazza, C., Novaga, M., Tortorelli, V.M., 2004. Motion by curvature of planar networks. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3, 235–324.

Mantegazza, C., Pluda, A., Pozzetta, M., 2021. A survey of the elastic flow of curves and networks. Milan J. Math. 89, 59–121. URL: https://doi.org/10.1007/s00032-021-00327-w, doi:10.1007/s00032-021-00327-w.

Michor, P.W., Mumford, D., 2005. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. Doc. Math. 10, 217–245.

Michor, P.W., Mumford, D., 2006. Riemannian geometries on spaces of plane curves. J. Eur. Math. Soc. (JEMS) 8, 1–48. URL: http://dx.doi.org/10.4171/JEMS/37, doi:10.4171/JEMS/37.

Michor, P.W., Mumford, D., 2007. An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach. Appl. Comput. Harmon. Anal. 23, 74–113. URL: http://dx.doi.org/10.1016/j.acha.2006.07.004, doi:10.1016/j.acha.2006.07.004.

Molitor, M., 2017. Remarks on the space of volume preserving embeddings. Differential Geom. Appl. 52, 127–141. URL: http://dx.doi.org/10.1016/j.difgeo.2017.03.004, doi:10.1016/j.difgeo.2017.03.004.

Novaga, M., Pozzi, P., 2020. A second order gradient flow of p-elastic planar networks. SIAM J. Math. Anal. 52, 682–708. URL: https://doi.org/10.1137/19M1262292, doi:10.1137/19M1262292.

Oelz, D., 2014. Convergence of the penalty method applied to a constrained curve straightening flow. Commun. Math. Sci. 12, 601–621. URL: https://doi.org/10.4310/CMS.2014.v12.n4.a1, doi:10.4310/CMS.2014.v12.n4.a1.

Öelz, D.B., 2011. On the curve straightening flow of inextensible, open, planar curves. SeMA J. , 5–24.

Okabe, S., 2007. The motion of elastic planar closed curves under the area-preserving condition. Indiana Univ. Math. J. 56, 1871–1912. URL: https://doi.org/10.1512/iumj.2007.56.3015, doi:10.1512/iumj.2007.56.3015.

Okabe, S., 2008. The dynamics of elastic closed curves under uniform high pressure. Calc. Var. Partial Differential Equations 33, 493–521. URL: https://doi.org/10.1007/s00526-008-0179-0, doi:10.1007/s00526-008-0179-0.

Okabe, S., Pozzi, P., Wheeler, G., 2020. A gradient flow for the p-elastic energy defined on closed planar curves. Math. Ann. 378, 777–828. URL: https://doi.org/10.1007/s00208-019-01885-6, doi:10.1007/s00208-019-01885-6.

Otto, F., 2001. The geometry of dissipative evolution equations: the porous medium equation. Comm. Partial Differential Equations 26, 101–174. URL: http://dx.doi.org/10.1081/PDE-100002243, doi:10.1081/PDE-100002243.

Preston, S.C., 2011. The motion of whips and chains. Journal of Differential Equations 251, 504–550.

Preston, S.C., 2012. The geometry of whips. Annals of Global Analysis and Geometry 41, 281–305.

Reeken, M., 1979. Classical solutions of the chain equation I. Mathematische Zeitschrift 165, 143–169.

Rivlin, R.S., 1955. Plane strain of a net formed by inextensible cords. J. Rational Mech. Anal. 4, 951–974. URL: https://doi.org/10.1512/iumj.1955.4.54037, doi:10.1512/iumj.1955.4.54037.

Roubiček, T., 2013. Nonlinear partial differential equations with applications. volume 153 of *International Series of Numerical Mathematics*. Second ed., Birkhäuser/Springer Basel AG, Basel. URL: https://doi.org/10.1007/978-3-0348-0513-1, doi:10.1007/978-3-0348-0513-1.

Sengül, Y., Vorotnikov, D., 2017. Generalized solutions for inextensible string equations. J. Differential Equations 262, 3610–3641. URL: http://dx.doi.org/10.1016/j.jde.2016.11.040, doi:10.1016/j.jde.2016.11.040.

Shi, W., Vorotnikov, D., 2019a. The gradient flow of the potential energy on the space of arcs. Calc. Var. Partial Differential Equations 58, Paper No. 59. doi:10.1007/s00526-019-1524-1.

Shi, W., Vorotnikov, D., 2019b. Uniformly compressing mean curvature flow. J. Geom. Anal. 29, 3055–3097. doi:10.1007/s12220-018-00104-z.

Székelyhidi Jr, L., 2012. Relaxation of the incompressible porous media equation. Annales scientifiques de l'Ecole normale supérieure 45, 491–509.

Thess, A., Zikanov, O., Nepomnyashchy, A., 1999. Finite-time singularity in the vortex dynamics of a string. Phys. Rev. E (3) 59, 3637–3640. URL: https://doi.org/10.1103/PhysRevE.59.3637, doi:10.1103/PhysRevE.59.3637.

Villani, C., 2003. Topics in optimal transportation. American Mathematical Soc.

Villani, C., 2008. Optimal transport: old and new. Springer Science & Business Media.

Wen, Y., 1993. L^2 flow of curve straightening in the plane. Duke Math. J. 70, 683–698. URL: https://doi.org/10.1215/S0012-7094-93-07016-0, doi:10.1215/S0012-7094-93-07016-0.

Zvyagin, V.G., Vorotnikov, D.A., 2008. Topological approximation methods for evolutionary problems of nonlinear hydrodynamics. volume 12 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin. URL: http://dx.doi.org/10.1515/9783110208283, doi:10.1515/9783110208283.

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