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ON PRESHEAF SUBMONADS OF QUANTALE-ENRICHED CATEGORIES

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Abstract: This paper focus on the presheaf monad and its submonads on the realm of V-categories, for a quantale V. First we present two characterisations of presheaf submonads, both using V-distributors: one based on admissible classes of V-distributors, and other using Beck-Chevalley conditions on V-distributors. Then we focus on the study of the corresponding Eilenberg-Moore categories of algebras, having as main examples the formal ball monad and the so-called Lawvere monad.

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Introduction

Having as guideline Lawvere's point of view that it is worth to regard metric spaces as categories enriched in the extended real half-line $[0,\infty]_+$ (see [17]), we regard both the formal ball monad and the monad that identifies Cauchy complete spaces as its algebras – which we call here the *Lawvere monad* – as submonads of the presheaf monad on the category **Met** of $[0,\infty]_+$-enriched categories. This leads us to the study of general presheaf submonads on the category of V-enriched categories, for a given quantale V.

Here we expand on known general characterisations of presheaf submonads and their algebras, and introduce a new ingredient – conditions of Beck-Chevalley type – which allows us to identify properties of functors and natural transformations, and, most importantly, contribute to a new facet of the behaviour of presheaf submonads.

In order to do that, after introducing the basic concepts needed to the study of V-categories in Section 1, Section 2 presents the presheaf monad and a characterisation of its submonads using admissible classes of V-distributors

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which is based on [2]. Next we introduce the already mentioned Beck-Chevalley conditions (BC*) which resemble those discussed in [5], with V-distributors playing the role of V-relations. In particular we show that lax idempotency of a monad $\mathbb{T}$ on V-**Cat** can be identified via a BC* condition, and that the presheaf monad satisfies fully BC*. This leads to the use of BC* to present a new characterisation of presheaf submonads in Section 4.

The remaining sections are devoted to the study of the Eilenberg-Moore category induced by presheaf submonads. In Section 5, based on [2], we detail the relationship between the algebras, (weighted) cocompleteness, and injectivity. Next we focus on the algebras and their morphisms, first for the formal ball monad, and later for a general presheaf submonad. We end by presenting the relevant example of the presheaf submonad whose algebras are the so-called Lawvere complete V-categories [3], which, when $V=[0,\infty]_+$, are exactly the Cauchy complete (generalised) metric spaces, while their morphisms are the V-functors which preserve the limits for Cauchy sequences.

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1. Preliminaries

Our work focus on V-categories (or V-enriched categories, cf. [7, 17, 15]) in the special case of V being a quantale.

Throughout V is a *commutative and unital quantale*; that is, V is a complete lattice endowed with a symmetric tensor product $\otimes$, with unit $k\neq\bot$, commuting with joins, so that it has a right adjoint hom; this means that, for $u,v,w\in V$,

$$u\otimes v\le w \;\Leftrightarrow\; v\le \hom(u,w).$$

As a category, V is a complete and cocomplete (thin) symmetric monoidal closed category.

Definition 1.1. A *V-category* is a pair (X,a) where X is a set and $a\colon X\times X\to V$ is a map such that:

(R) for each $x\in X$, $k\le a(x,x)$;

(T) for each $x,x',x''\in X$, $a(x,x')\otimes a(x',x'')\le a(x,x'')$.

If (X,a), (Y,b) are V-categories, a *V-functor* $f\colon (X,a)\to(Y,b)$ is a map $f\colon X\to Y$ such that

(C) for each $x,x'\in X$, $a(x,x')\le b(f(x),f(x'))$.

The category of V-categories and V-functors will be denoted by V-**Cat**. Sometimes we will use the notation $X(x,y) = a(x,y)$ for a V-category (X,a) and $x,y \in X$.

We point out that V has itself a V-categorical structure, given by the right adjoint to $\otimes$, hom; indeed, $u\otimes k \leq u \;\Rightarrow\; k \leq \hom(u,u)$, and $u\otimes \hom(u,u')\otimes \hom(u',u'') \leq u' \otimes \hom(u',u'') \leq u''$ gives that $\hom(u,u') \otimes \hom(u',u'') \leq \hom(u,u'')$. Moreover, for every V-category (X,a), one can define its *opposite V-category* $X^{\rm op} = (X,a^\circ)$, with $a^\circ(x,x') = a(x',x)$ for all $x,x' \in X$.

Examples 1.2. (1) For $V = \mathbf{2} = (\{0 < 1\}, \wedge, 1)$, a $\mathbf{2}$-category is an *ordered set* (not necessarily antisymmetric) and a $\mathbf{2}$-functor is a *monotone map*. We denote **2-Cat** by **Ord**.

(2) The lattice $V = [0,\infty]$ ordered by the "greater or equal" relation $\geq$ (so that $r \wedge s = \max\{r,s\}$, and the supremum of $S \subseteq [0,\infty]$ is given by $\inf S$) with tensor $\otimes = +$ will be denoted by $[0,\infty]_+$. A $[0,\infty]_+$-category is a *(generalised) metric space* and a $[0,\infty]_+$-functor is a *non-expansive map* (see [17]). We denote $[0,\infty]_+$-**Cat** by **Met**. We note that

$$\hom(u,v) = v \ominus u := \max\{v-u, 0\},$$

for all $u,v \in [0,\infty]$.

If instead of $+$ one considers the tensor product $\wedge$, then $[0,\infty]_\wedge$-**Cat** is the category **UMet** of *ultrametric spaces* and *non-expansive maps.*

(3) The complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$ is isomorphic to $[0,\infty]$ via the map $[0,1] \to [0,\infty]$, $u \mapsto -\ln(u)$ where $-\ln(0) = \infty$. Under this isomorphism, the operation $+$ on $[0,\infty]$ corresponds to the multiplication $*$ on $[0,1]$. Denoting this quantale by $[0,1]_*$, one has $[0,1]_*$-**Cat** isomorphic to the category **Met** $= [0,\infty]_+$-**Cat** of (generalised) metric spaces and non-expansive maps.

Since $[0,1]$ is a frame, so that finite meets commute with infinite joins, we can also consider it as a quantale with $\otimes = \wedge$. The category $[0,1]_\wedge$-**Cat** is isomorphic to the category **UMet**.

Another interesting tensor product in $[0,1]$ is given by the *Łukasiewicz tensor* $\odot$ where $u \odot v = \max(0, u+v-1)$; here $\hom(u,v) = \min(1, 1-u+v)$. Then $[0,1]_\odot$-**Cat** is the category of *bounded-by-1 (generalised) metric spaces* and *non-expansive maps.*

(4) We consider now the set

$$\Delta = \{\varphi\colon [0,\infty] \to [0,1] \mid \text{for all } \alpha \in [0,\infty]\colon \varphi(\alpha) = \bigvee_{\beta<\alpha} \varphi(\beta)\},$$

of *distribution functions*. With the pointwise order, it is a complete lattice. For $\varphi, \psi \in \Delta$ and $\alpha \in [0,\infty]$, define $\varphi \otimes \psi \in \Delta$ by

$$(\varphi \otimes \psi)(\alpha) = \bigvee_{\beta+\gamma\leq\alpha} \varphi(\beta) * \psi(\gamma).$$

Then $\otimes : \Delta \times \Delta \to \Delta$ is associative and commutative, and

$$\kappa : [0,\infty] \to [0,1],\ \alpha \mapsto \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{else} \end{cases}$$

is a unit for $\otimes$. Finally, $\psi \otimes - : \Delta \to \Delta$ preserves suprema since, for all $u \in [0,1]$, $u * -\colon [0,1] \to [0,1]$ preserves suprema. A Δ-category is a *(generalised) probabilistic metric space* and a Δ-functor is a *probabilistic non-expansive map* (see [13] and references there).

We will also make use of two additional categories we describe next, the category V-**Rel**, of sets and V-relations, and the category V-**Dist**, of V-categories and V-distributors.

Objects of V-**Rel** are sets, while morphisms are V-relations, i.e., if X and Y are sets, a *V-relation* $r\colon X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$ is a map $r\colon X \times Y \to V$. Composition of V-relations is given by *relational composition*, so that the composite of $r\colon X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$ and $s\colon Y {\longrightarrow\!\!\!\!\!\!\mapsto} Z$ is given by

$$(s \cdot r)(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z),$$

for every $x \in X$, $z \in Z$. Identities in V-**Cat** are simply identity relations, with $1_X(x,x') = k$ if $x = x'$ and $1_X(x,x') = \bot$ otherwise. The category V-**Rel** has an involution $(\ )^\circ$, assigning to each V-relation $r\colon X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$ the V-relation $r^\circ\colon Y {\longrightarrow\!\!\!\!\!\!\mapsto} X$ defined by $r^\circ(y,x) = r(x,y)$, for every $x \in X$, $y \in Y$.

Since every map $f\colon X \to Y$ can be thought as a V-relation through its graph $f_\circ\colon X \times Y \to V$, with $f_\circ(x,y) = k$ if $f(x) = y$ and $f_\circ(x,y) = \bot$ otherwise, there is an injective on objects and faithful functor $\mathbf{Set} \to V$-**Rel**. When no confusion may arise, we use also f to denote the V-relation $f_\circ$.

The category V-**Rel** is a 2-category, when equipped with the 2-cells given by the pointwise order; that is, for $r, r' \colon X \relbar\joinrel\mapsto Y$, one defines $r \le r'$ if, for all $x \in X$, $y \in Y$, $r(x,y) \le r'(x,y)$. This gives us the possibility of studying adjointness between V-relations. We note in particular that, if $f \colon X \to Y$ is a map, then $f_\circ \cdot f^\circ \le 1_Y$ and $1_X \le f^\circ \cdot f_\circ$, so that $f_\circ \dashv f^\circ$.

Objects of V-**Dist** are V-categories, while morphisms are V-distributors (also called V-bimodules, or V-profunctors); i.e., if (X,a) and (Y,b) are V-categories, a *V-distributor* – or, simply, a *distributor* – $\varphi \colon (X,a) \relbar\joinrel\circ\joinrel\rightarrow (Y,b)$ is a V-relation $\varphi \colon X \relbar\joinrel\mapsto Y$ such that $\varphi \cdot a \le \varphi$ and $b \cdot \varphi \le \varphi$ (in fact $\varphi \cdot a = \varphi$ and $b \cdot \varphi = \varphi$ since the other inequalities follow from (R)). Composition of distributors is again given by relational composition, while the identities are given by the V-categorical structures, i.e. $1_{(X,a)} = a$. Moreover, V-**Dist** inherits the 2-categorical structure from V-**Rel**.

Each V-functor $f \colon (X,a) \to (Y,b)$ induces two distributors, $f_* \colon (X,a) \relbar\joinrel\circ\joinrel\rightarrow (Y,b)$ and $f^* \colon (Y,b) \relbar\joinrel\circ\joinrel\rightarrow (X,a)$, defined by $f_*(x,y) = Y(f(x),y)$ and $f^*(y,x) = Y(y,f(x))$, that is, $f_* = b \cdot f_\circ$ and $f^* = f^\circ \cdot b$. These assignments are functorial, as we explain below.

First we define 2-cells in V-**Cat**: for $f, f' \colon (X,a) \to (Y,b)$ V-functors, $f \le f'$ when $f^* \le (f')^*$ as distributors, so that

$$f \le f' \quad \Leftrightarrow \quad \forall x \in X,\ y \in Y,\ Y(y,f(x)) \le Y(y,f'(x)).$$

V-**Cat** is then a 2-category, and we can define two 2-functors

$$\begin{array}{rcl} (\ )_* \colon V\text{-}\mathbf{Cat}^{\mathrm{co}} & \longrightarrow & V\text{-}\mathbf{Dist} \\ X & \longmapsto & X \\ f & \longmapsto & f_* \end{array} \quad \text{and} \quad \begin{array}{rcl} (\ )^* \colon V\text{-}\mathbf{Cat}^{\mathrm{op}} & \longrightarrow & V\text{-}\mathbf{Dist} \\ X & \longmapsto & X \\ f & \longmapsto & f^* \end{array}$$

Note that, for any V-functor $f \colon (X,a) \to (Y,b)$,

$$f_* \cdot f^* = b \cdot f_\circ \cdot f^\circ \cdot b \le b \cdot b = b \text{ and } f^* \cdot f_* = f^\circ \cdot b \cdot b \cdot f_\circ \ge f^\circ \cdot f_\circ \cdot a \ge a;$$

hence every V-functor induces a pair of adjoint distributors, $f_* \dashv f^*$. A V-functor $f \colon X \to Y$ is said to be *fully faithful* if $f^* \cdot f_* = a$, i.e. $X(x,x') = Y(f(x),f(x'))$ for all $x, x' \in X$, while it is *fully dense* if $f_* \cdot f^* = b$, i.e. $Y(y,y') = \bigvee_{x \in X} Y(y,f(x)) \otimes Y(f(x),y')$, for all $y, y' \in Y$. A fully faithful V-functor $f \colon X \to Y$ does not need to be an injective map; it is so in case

X and Y are separated V-categories (as defined below).

Remark 1.3. In V-**Cat** adjointness between V-functors

$$Y \underset{f}{\overset{g}{\rightleftarrows}} X \quad (\top)$$

can be equivalently expressed as:

$f \dashv g \Leftrightarrow f_* = g^* \ \Leftrightarrow g^* \dashv f^* \ \Leftrightarrow (\forall x \in X)\,(\forall y \in Y)\, X(x, g(y)) = Y(f(x), y).$

In fact the latter condition encodes also V-functoriality of f and g; that is, if $f\colon X \to Y$ and $g\colon Y \to X$ are maps satisfying the condition

$$(\forall x \in X)\ (\forall y \in Y)\ \ X(x, g(y)) = Y(f(x), y),$$

then f and g are V-functors, with $f \dashv g$.

Furthermore, it is easy to check that, given V-categories X and Y, a map $f\colon X \to Y$ is a V-functor whenever f_* is a distributor (or whenever f^* is a distributor).

The order defined on V-**Cat** is in general not antisymmetric. For V-functors $f, g\colon X \to Y$, one says that $f \simeq g$ if $f \le g$ and $g \le f$ (or, equivalently, $f^* = g^*$). For elements x, y of a V-category X, one says that $x \le y$ if, considering the V-functors $x, y\colon E = (\{*\}, k) \to X$ (where $k(*, *) = k$) defined by $x(*) = x$ and $y(*) = y$, one has $x \le y$; or, equivalently, $X(x, y) \ge k$. Then, for any V-functors $f, g\colon X \to Y$, $f \le g$ if, and only if, $f(x) \le g(x)$ for every $x \in X$.

Definition 1.4. A V-category Y is said to be *separated* if, for $f, g\colon X \to Y$, $f = g$ whenever $f \simeq g$; equivalently, if, for all $x, y \in Y$, $x \simeq y$ implies $x = y$.

The tensor product $\otimes$ on V induces a tensor product on V-**Cat**, with $(X, a) \otimes (Y, b) = (X \times Y, a \otimes b) = X \otimes Y$, where $(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$. The V-category E is a $\otimes$-neutral element. With this tensor product, V-**Cat** becomes a monoidal closed category. Indeed, for each V-category X, the functor $X \otimes (\ )\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ has a right adjoint $(\ )^X$ defined by $Y^X = (V\text{-}\mathbf{Cat}(X, Y), [\![\ ,\ ]\!])$, with $[\![f, g]\!] = \bigwedge_{x \in X} Y(f(x), g(x))$ (see [7, 17, 15] for details).

It is interesting to note the following well-known result (see, for instance, [3, Theorem 2.5]).

Theorem 1.5. *For V-categories X and Y, and a V-relation $\varphi\colon X \to\!\!\!\!\!\!+\; Y$, the following conditions are equivalent:*

(i) $\varphi\colon X \to\!\!\!\!\!\!\circ\; Y$ *is a distributor;*
(ii) $\varphi\colon X^{\mathrm{op}} \otimes Y \to (V, \hom)$ *is a V-functor.*

In particular, the V-categorical structure a of (X, a) is a V-distributor $a\colon (X, a) \to\!\!\!\!\!\!\circ\; (X, a)$, and therefore a V-functor $a\colon (X, a)^{\mathrm{op}} \otimes (X, a) \to (V, \hom)$, which induces, via the closed monoidal structure of V-**Cat**, the *Yoneda V-functor* $y_X\colon (X, a) \to (V, \hom)^{(X,a)^{\mathrm{op}}}$. Thanks to the theorem above, $V^{X^{\mathrm{op}}}$ can be equivalently described as

$$PX := \{\varphi\colon X \to\!\!\!\!\!\!\circ\; E \mid \varphi\ V\text{-distributor}\}.$$

Then the structure $\widetilde{a}$ on PX is given by

$$\widetilde{a}(\varphi, \psi) = [\![\varphi, \psi]\!] = \bigwedge_{x \in X} \hom(\varphi(x), \psi(x)),$$

for every $\varphi, \psi\colon X \to\!\!\!\!\!\!\circ\; E$, where by $\varphi(x)$ we mean $\varphi(x, *)$, or, equivalently, we consider the associated V-functor $\varphi\colon X \to V$. The Yoneda functor $y_X\colon X \to PX$ assigns to each $x \in X$ the distributor $x^*\colon X \to\!\!\!\!\!\!\circ\; E$, where we identify again $x \in X$ with the V-functor $x\colon E \to X$ assigning x to the (unique) element of E. Then, for every $\varphi \in PX$ and $x \in X$, we have that

$$[\![y_X(x), \varphi]\!] = \varphi(x),$$

as expected. In particular y_X is a fully faithful V-functor, being injective on objects (i.e. an injective map) when X is a separated V-category. We point out that $(V, \hom)$ is separated, and so is PX for every V-category X.

For more information on V-**Cat** we refer to [12, Appendix].

2. The presheaf monad and its submonads

The assignment $X \mapsto PX$ defines a functor $P\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$: for each V-functor $f\colon X \to Y$, $Pf\colon PX \to PY$ assigns to each distributor $X \xrightarrow{\varphi}\!\!\!\!\!\!\circ\;\; E$ the distributor $Y \xrightarrow{f^*}\!\!\!\!\!\!\circ\;\; X \xrightarrow{\varphi}\!\!\!\!\!\!\circ\;\; E$. It is easily checked that the Yoneda functors $(y_X\colon X \to PX)_X$ define a natural transformation $y\colon 1 \to P$. Moreover, since, for every V-functor f, the adjunction $f_* \dashv f^*$ yields an adjunction $Pf = (\ ) \cdot f^* \dashv (\ ) \cdot f_* =: Qf$, Py_X has a right adjoint, which we denote by $m_X\colon PPX \to PX$. It is straightforward to check that $\mathbb{P} = (P, m, y)$ is a

2-monad on V-**Cat** – the so-called *presheaf monad* –, which, by construction of m_X as the right adjoint to Py_X, is lax idempotent (see [11] for details).

Next we present a characterisation of the submonads of $\mathbb{P}$ which is partially in [2]. We recall that, given two monads $\mathbb{T} = (T, \mu, \eta)$, $\mathbb{T}' = (T', \mu', \eta')$ on a category $\mathbf{C}$, a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{T}'$ is a natural transformation $\sigma\colon T \to T'$ such that

$$\begin{array}{ccc} 1 & \xrightarrow{\eta} & T \\ & \searrow^{\eta'} & \downarrow{\scriptstyle\sigma} \\ & & T' \end{array} \qquad\qquad \begin{array}{ccccc} TT & \xrightarrow{\sigma_T} & T'T & \xrightarrow{T'\sigma} & T'T' \\ {\scriptstyle\mu}\downarrow & & & & \downarrow{\scriptstyle\mu'} \\ T & & \xrightarrow[\sigma]{\qquad\qquad} & & T' \end{array} \tag{2.i}$$

By *submonad of* $\mathbb{P}$ we mean a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** with a monad morphism $\sigma : \mathbb{T} \to \mathbb{P}$ such that σ_X is an embedding (i.e. both fully faithful and injective on objects) for every V-category X.

Definition 2.1. Given a class Φ of V-distributors, for every V-category X let

$$\Phi X = \{\varphi\colon X \,{\circ\!\!\!\longrightarrow}\, E \mid \varphi \in \Phi\}$$

have the V-category structure inherited from the one of PX. We say that Φ is *admissible* if, for every V-functor $f\colon X \to Y$ and V-distributors $\varphi\colon Z \,{\circ\!\!\!\longrightarrow}\, Y$ and $\psi\colon X \,{\circ\!\!\!\longrightarrow}\, Z$ in Φ,

(1) $f^* \in \Phi$;
(2) $\psi \cdot f^* \in \Phi$ and $f^* \cdot \varphi \in \Phi$;
(3) $\varphi \in \Phi \;\Leftrightarrow\; (\forall y \in Y)\; y^* \cdot \varphi \in \Phi$;
(4) for every V-distributor $\gamma\colon PX \,{\circ\!\!\!\longrightarrow}\, E$, if the restriction of γ to ΦX belongs to Φ, then $\gamma \cdot (y_X)_* \in \Phi$.

Lemma 2.2. *Every admissible class* Φ *of* V*-distributors induces a submonad* $\Phi = (\Phi, m^\Phi, y^\Phi)$ *of* $\mathbb{P}$.

Proof: For each V-category X, equip ΦX with the initial structure induced by the inclusion $\sigma_X\colon \Phi X \to PX$, that is, for every $\varphi, \psi \in \Phi X$, $\Phi X(\varphi, \psi) = PX(\varphi, \psi)$. For each V-functor $f\colon X \to Y$ and $\varphi \in \Phi X$, by condition (2), $\varphi \cdot f^* \in \Phi$, and so Pf (co)restricts to $\Phi f\colon \Phi X \to \Phi Y$.

Condition (1) guarantees that $y_X\colon X \to PX$ corestricts to $y_X^\Phi\colon X \to \Phi X$.

Finally, condition (4) guarantees that $m_X\colon PPX \to PX$ also (co)restricts to $m_X^\Phi : \Phi\Phi X \to \Phi X$: with $\gamma\colon \Phi X \,{\circ\!\!\!\longrightarrow}\, E$, also $\widetilde{\gamma} := \gamma \cdot (\sigma_X)^*\colon PX \,{\circ\!\!\!\longrightarrow}\, E$

belongs to Φ by (2), and then, since γ is the restriction of $\widetilde{\gamma}$ to ΦX, by (4) $m_X(\widetilde{\gamma}) = \gamma\cdot(\sigma_X)^*\cdot(y_X)_* = \gamma\cdot(\sigma_X)^*\cdot(\sigma_X)_*\cdot(y_X^\Phi)_* = \gamma\cdot(y_X^\Phi)_*\in\Phi$.

By construction, $(\sigma_X)_X$ is a natural transformation, each σ_X is an embedding, and σ makes diagrams (2.i) commute. ■

Theorem 2.3. *For a 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-**Cat**, the following assertions are equivalent:*

(i) $\mathbb{T}$ *is isomorphic to Φ, for some admissible class of V-distributors Φ.*
(ii) $\mathbb{T}$ *is a submonad of* $\mathbb{P}$.

Proof: (i) $\Rightarrow$ (ii) follows from the lemma above.

(ii) $\Rightarrow$ (i): Let $\sigma\colon\mathbb{T}\to\mathbb{P}$ be a monad morphism, with σ_X an embedding for every V-category X, which, for simplicity, we assume to be an inclusion. First we show that

$$\Phi=\{\varphi\colon X{-\!\!\circ\!\!\to} Y \mid \forall y\in Y\ y^*\cdot\varphi\in TX\} \tag{2.ii}$$

is admissible. In the sequel $f\colon X\to Y$ is a V-functor.

(1) For each $x\in X$, $x^*\cdot f^* = f(x)^*\in TY$, and so $f^*\in\Phi$.

(2) If $\psi\colon X{-\!\!\circ\!\!\to} Z$ is a V-distributor in Φ, and $z\in Z$, since $z^*\cdot\psi\in TX$, $Tf(z^*\cdot\psi)=z^*\cdot\psi\cdot f^*\in TY$, and therefore $\psi\cdot f^*\in\Phi$ by definition of Φ. Now, if $\varphi\colon Z{-\!\!\circ\!\!\to} Y\in\Phi$, then, for each $x\in X$, $x^*\cdot f^*\cdot\varphi = f(x)^*\cdot\varphi\in TZ$ because $\varphi\in\Phi$, and so $f^*\cdot\varphi\in\Phi$.

(3) follows from the definition of Φ.

(4) If the restriction of $\gamma\colon PX{-\!\!\circ\!\!\to} E$ to TX, i.e., $\gamma\cdot(\sigma_X)_*$, belongs to Φ, then $\mu_X(\gamma\cdot(\sigma_X)_*)=\gamma\cdot(\sigma_X)_*\cdot(\eta_X)_*=\gamma\cdot(y_X)_*$ belongs to TX. ■

We point out that, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. This assertion is shown at the end of next section, making use of the Beck-Chevalley conditions we study next. (We note that the arguments of [6, Prop. 16.2], which states conditions under which a submonad of a lax idempotent monad is still lax idempotent, cannot be used directly here.)

3. The presheaf monad and Beck-Chevalley conditions

In this section our aim is to show that $\mathbb{P}$ verifies some interesting conditions of Beck-Chevalley type, that resemble the BC conditions studied in [5]. We recall from [5] that a commutative square in **Set**

$$\begin{array}{ccc} W & \xrightarrow{l} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is said to be a *BC-square* if the following diagram commutes in **Rel**

$$\begin{array}{ccc} W & \overset{l_\circ}{\relbar\joinrel\mapsto} & Z \\ {\scriptstyle g^\circ}\uparrow & & \uparrow{\scriptstyle h^\circ} \\ X & \underset{f_\circ}{\relbar\joinrel\mapsto} & Y, \end{array}$$

where, given a map $t\colon A \to B$, $t_\circ\colon A {\relbar\joinrel\mapsto} B$ denotes the relation defined by t and $t^\circ\colon B {\relbar\joinrel\mapsto} A$ its opposite. Since $t_\circ \dashv t^\circ$ in **Rel**, this is in fact a kind of Beck-Chevalley condition. A **Set**-endofunctor T is said to satisfy BC if it preserves BC-squares, while a natural transformation $\alpha\colon T \to T'$ between two **Set**-endofunctors satisfies BC if, for each map $f\colon X \to Y$, its naturality square

$$\begin{array}{ccc} TX & \xrightarrow{\alpha_X} & T'X \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle T'f} \\ TY & \xrightarrow[\alpha_Y]{} & T'Y \end{array}$$

is a BC-square.

In our situation, for endofunctors and natural transformations in V-**Cat**, the role of **Rel** is played by V-**Dist**.

Definition 3.1. A commutative square in V-**Cat**

$$\begin{array}{ccc} (W,d) & \xrightarrow{l} & (Z,c) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

is said to be a *BC*-square* if the following diagram commutes in V-**Dist**

$$\begin{array}{ccc} (W,d) & \overset{l_*}{\longrightarrow\!\!\!\!\!\circ\;} & (Z,c) \\ {\scriptstyle g^*}\uparrow\circ & & \uparrow\circ{\scriptstyle h^*} \\ (X,a) & \underset{f_*}{\longrightarrow\!\!\!\!\!\circ\;} & (Y,b) \end{array} \tag{3.i}$$

(or, equivalently, $h^* \cdot f_* \leq l_* \cdot g^*$).

Remarks 3.2. (1) For a V-functor $f\colon (X,a) \to (Y,b)$, to be fully faithful is equivalent to

$$\begin{array}{ccc} (X,a) & \xrightarrow{1} & (X,a) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

being a BC*-square (exactly in parallel with the characterisation of monomorphisms via BC-squares).

(2) We point out that, contrarily to the case of BC-squares, in BC*-squares the horizontal and the vertical arrows play different roles; that is, the fact that diagram (3.i) is a BC*-square is not equivalent to

$$\begin{array}{ccc} (W,d) & \xrightarrow{g} & (X,a) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,c) & \xrightarrow[h]{} & (Y,b) \end{array}$$

being a BC*-square; it is indeed equivalent to its *dual*

$$\begin{array}{ccc} (W,d^\circ) & \xrightarrow{g} & (X,a^\circ) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,c^\circ) & \xrightarrow[h]{} & (Y,b^\circ) \end{array}$$

being a BC*-square.

Definitions 3.3. (1) A *functor* $T\colon V$-**Cat** $\to V$-**Cat** *satisfies BC** if it preserves BC*-squares.

(2) Given two endofunctors T, T' on V-**Cat**, a *natural transformation* $\alpha\colon T \to T'$ *satisfies BC** if the naturality diagram

$$\begin{array}{ccc} TX & \xrightarrow{\alpha_X} & T'X \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle T'f} \\ TY & \xrightarrow[\alpha_Y]{} & T'Y \end{array}$$

is a BC*-square for every morphism f in V-**Cat**.

(3) *A 2-monad* $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** is said to satisfy *fully BC** if T, μ, and η satisfy BC*.

Remark 3.4. In the case of **Set** and **Rel**, since the condition of being a BC-square is equivalent, under the Axiom of Choice (AC), to being a weak pullback, a **Set**-monad $\mathbb{T}$ *satisfies fully BC* if, and only if, it is *weakly cartesian* (again, under (AC)). This, together with the fact that there are relevant **Set**-monads – like for instance the ultrafilter monad – whose functor and multiplication satisfy BC but the unit does not, led the authors of [5] to name such monads as *BC-monads*. This is the reason why we use *fully BC** instead of BC* to identify these V-**Cat**-monads.

As a side remark we recall that, still in the **Set**-context, a partial BC-condition was studied by Manes in [18]: for a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$ to be *taut* requires that T, μ, η satisfy BC for commutative squares where f is monic.

Our first use of BC* is the following characterisation of lax idempotency for a 2-monad $\mathbb{T}$ on V-**Cat**.

Proposition 3.5. *Let* $\mathbb{T} = (T, \mu, \eta)$ *be a 2-monad on* V-**Cat**.

(1) The following assertions are equivalent:

(i) $\mathbb{T}$ *is lax idempotent.*

(ii) *For each V-category X, the diagram*

$$\begin{array}{ccc} TX & \xrightarrow{T\eta_X} & TTX \\ {\scriptstyle \eta_{TX}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow[\mu_X]{} & TX \end{array} \tag{3.ii}$$

is a BC-square.*

(2) If $\mathbb{T}$ is lax idempotent, then μ satisfies BC.*

Proof: (1) (i) $\Rightarrow$ (ii): The monad $\mathbb{T}$ is lax idempotent if, and only if, for every V-category X, $T\eta_X \dashv \mu_X$, or, equivalently, $\mu_X \dashv \eta_{TX}$. These two conditions are equivalent to $(T\eta_X)_* = (\mu_X)^*$ and $(\mu_X)_* = (\eta_{TX})^*$. Hence $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ as claimed.

(ii) $\Rightarrow$ (i): From $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ it follows that

$$(\mu_X)_* = (\mu_X)_*(\mu_X)^*(\mu_X)_* = (\mu_X \cdot T\eta_X)_*(\eta_{TX})^* = (\eta_{TX})^*,$$

that is, $\mu_X \dashv \eta_{TX}$.

(2) BC* for μ follows directly from lax idempotency of $\mathbb{T}$, since

$$\begin{array}{ccc} TTX & \overset{(\mu_X)_*}{-\!\!\circ\!\!\rightarrow} & TX \\ {\scriptstyle (TTf)^*}\uparrow\circ & & \uparrow\circ{\scriptstyle (Tf)^*} \\ TTY & \underset{(\mu_Y)_*}{-\!\!\circ\!\!\rightarrow} & TY \end{array} \quad = \quad \begin{array}{ccc} TTX & \overset{(\eta_{TX})^*}{-\!\!\circ\!\!\rightarrow} & TX \\ {\scriptstyle (TTf)^*}\uparrow\circ & & \uparrow\circ{\scriptstyle (Tf)^*} \\ TTY & \underset{(\eta_{TY})^*}{-\!\!\circ\!\!\rightarrow} & TY \end{array}$$

and the latter diagram commutes trivially. ■

Remark 3.6. Thanks to Remarks 3.2 we know that, if we invert the role of η_{TX} and $T\eta_X$ in (3.ii), we get a characterisation of oplax idempotent 2-monad: $\mathbb{T}$ is oplax idempotent if, and only if, the diagram

$$\begin{array}{ccc} TX & \xrightarrow{\eta_{TX}} & TTX \\ {\scriptstyle T\eta_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow[\mu_X]{} & TX \end{array}$$

is a BC*-square.

Theorem 3.7. *The presheaf monad $\mathbb{P} = (P, \mathfrak{m}, \mathfrak{y})$ satisfies fully BC*.*

Proof: (1) *P satisfies BC**: Given a BC*-square

$$\begin{array}{ccc} (W,d) & \xrightarrow{l} & (Z,c) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

in V-**Cat**, we want to show that

$$\begin{array}{ccc} PW & \xrightarrow[\circ]{(Pl)_*} & PZ \\ {\scriptstyle (Pg)^*}\uparrow\circ & \geq & \circ\uparrow{\scriptstyle (Ph)^*} \\ PX & \xrightarrow[(Pf)_*]{\circ} & PY. \end{array} \tag{3.iii}$$

For each $\varphi \in PX$ and $\psi \in PZ$, we have

$$\begin{aligned}
(Ph)^*(Pf)_*(\varphi,\psi) &= (Ph)^\circ \cdot \widetilde{b} \cdot Pf(\varphi,\psi) \\
&= \widetilde{b}(Pf(\varphi), Ph(\psi)) \\
&= \bigwedge_{y\in Y} \hom(\varphi \cdot f^*(y), \psi \cdot h^*(y)) \\
&\leq \bigwedge_{x\in X} \hom(\varphi \cdot f^* \cdot f_*(x), \psi \cdot h^* \cdot f_*(x)) \\
&\leq \bigwedge_{x\in X} \hom(\varphi(x), \psi \cdot l_* \cdot g^*(x)) && (\varphi \leq \varphi \cdot f^* \cdot f_*,\ \text{(3.iii) is BC*}) \\
&= \widetilde{a}(\varphi, \psi \cdot l_* \cdot g^*) \\
&\leq \widetilde{a}(\varphi, \psi \cdot l_* \cdot g^*) \otimes \widetilde{c}(\psi \cdot l_* \cdot l^*, \psi) && (\text{because } \psi \cdot l_* \cdot l^* \leq \psi) \\
&= \widetilde{a}(\varphi, Pg(\psi \cdot l_*)) \otimes \widetilde{c}(Pl(\psi \cdot l_*), \psi) \\
&\leq \bigvee_{\gamma \in PW} \widetilde{a}(\varphi, Pg(\gamma)) \otimes \widetilde{c}(Pl(\gamma), \psi) \\
&= (Pl)_*(Pg)^*(\varphi,\psi).
\end{aligned}$$

(2) μ *satisfies BC**: For each V-functor $f\colon X \to Y$, from the naturality of $\mathfrak{y}$ it follows that the following diagram

$$\begin{array}{ccc} PPX & \xrightarrow[\circ]{(\mathfrak{y}_{PX})^*} & PX \\ {\scriptstyle (PPf)^*}\uparrow\circ & & \circ\uparrow{\scriptstyle (Pf)^*} \\ PPY & \xrightarrow[(\mathfrak{y}_{PY})^*]{\circ} & PY \end{array}$$

commutes. Lax idempotency of $\mathbb{P}$ means in particular that $\mathfrak{m}_X \dashv \mathfrak{y}_{PX}$, or, equivalently, $(\mathfrak{m}_X)_* = (\mathfrak{y}_{PX})^*$, and therefore the commutativity of this diagram shows BC* for $\mathfrak{m}$.

(3) *y satisfies BC**: Once again, for each V-functor $f\colon (X,a)\to(Y,b)$, we want to show that the diagram

$$\begin{array}{ccc} X & \xrightarrow{(y_X)_*} & PX \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle (Pf)^*} \\ Y & \xrightarrow[(y_Y)_*]{} & PY \end{array}$$

commutes. Let $y\in Y$ and $\varphi\colon X{\longrightarrow\!\!\!\!\!\circ}\, E$ belong to PX. Then

$$\begin{aligned} ((Pf)^*(y_Y)_*)(y,\varphi) &= ((Pf)^\circ\cdot\widetilde{b}\cdot y_Y)(y,\varphi)=\widetilde{b}(y_Y(y),Pf(\varphi))=Pf(\varphi)(y)\\ &=\bigvee_{x\in X} b(y,f(x))\otimes\varphi(x)=\bigvee_{x\in X} b(y,f(x))\otimes\widetilde{a}(y_X(x),\varphi)\\ &=(\widetilde{a}\cdot y_X\cdot f^\circ\cdot b)(y,\varphi)=(y_X)_*\cdot f^*(y,\varphi), \end{aligned}$$

as claimed. ■

Corollary 3.8. *Let $\mathbb{T}=(T,\mu,\eta)$ on V-**Cat** be a 2-monad on V-**Cat**, and $\sigma\colon\mathbb{T}\to\mathbb{P}$ be a monad morphism, pointwise fully faithful. Then $\mathbb{T}$ is lax idempotent.*

Proof: We know that $\mathbb{P}$ is lax idempotent, and so, for every V-category X, $(m_X)_*=(y_{PX})^*$. Consider diagram (2.i). The commutativity of the diagram on the right gives that $(\mu_X)_*=(\sigma_X)^*(\sigma_X)_*(\mu_X)_*=(\sigma_X)^*(m_X)_*(P\sigma_X)_*(\sigma_{TX})_*$; using the equality above, and preservation of fully faithful V-functors by $\mathbb{P}$ – which follows from BC* – we obtain:

$$\begin{aligned} (\mu_X)_* &= (\sigma_X)^*(y_{PX})^*(P\sigma_X)_*(\sigma_{TX})_*=(\sigma_X)^*(\eta_{PX})^*(\sigma_{PX})^*(P\sigma_X)_*(\sigma_{TX})_*=\\ &=(\eta_{TX})^*\cdot(\sigma_{TX})^*(P\sigma_X)^*(P\sigma_X)_*(\sigma_{TX})_*=(\eta_{TX})^*. \end{aligned}$$ ■

4. Presheaf submonads and Beck-Chevalley conditions

In this section, for a general 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-**Cat**, we relate its BC* properties with the existence of a (sub)monad morphism $\mathbb{T}\to\mathbb{P}$. We remark that a necessary condition for $\mathbb{T}$ to be a submonad of $\mathbb{P}$ is that TX is separated for every V-category X, since PX is separated and separated V-categories are stable under monomorphisms.

Theorem 4.1. *For a 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-**Cat** with TX separated for every V-category X, the following assertions are equivalent:*

(i) $\mathbb{T}$ *is a submonad of* $\mathbb{P}$.
(ii) $\mathbb{T}$ *is lax idempotent and satisfies BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.
(iii) $\mathbb{T}$ *is lax idempotent,* μ *and* η *satisfy BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.
(iv) $\mathbb{T}$ *is lax idempotent,* η *satisfies BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.

Proof: (i) $\Rightarrow$ (ii): By (i) there exists a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{P}$ with σ_X an embedding for every V-category X. By Corollary 3.8, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. Moreover, from $\sigma_X \cdot \eta_X = y_X$, with y_X, also η_X is fully faithful. (In fact this is valid for any monad with a monad morphism into $\mathbb{P}$.)

To show that $\mathbb{T}$ satisfies BC* we use the characterisation of Theorem 2.3; that is, we know that there is an admissible class Φ of distributors so that $\mathbb{T} = \Phi$. Then BC* for T follows directly from the fact that Φf is a (co)restriction of Pf, for every V-functor f.

BC* for η follows from BC* for y and full faithfulness of σ since, for any commutative diagram in V-**Cat**

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \xrightarrow{f} & \cdot \\ \downarrow & \boxed{1} & \downarrow & \boxed{2} & \downarrow \\ \cdot & \longrightarrow & \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

with $\boxed{1}\boxed{2}$ satisfying BC*, and f and g fully faithful, also $\boxed{1}$ satisfies BC*.

Thanks to Proposition 3.5, BC* for μ follows directly from lax idempotency of $\mathbb{T}$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i): For each V-category (X,a), we denote by $\widehat{a}$ the V-category structure on TX, and define the V-functor

$$(\ TX \xrightarrow{\sigma_X} PX\ ) = (\ TX \xrightarrow{y_{TX}} PTX \xrightarrow{Q\eta_X} PX\ );$$

that is, $\sigma_X(\mathfrak{x}) = (\ X \xrightarrow{\eta_X} TX \xrightarrow{\widehat{a}} TX \xrightarrow{\mathfrak{x}^\circ} E\ ) = \widehat{a}(\eta_X(\ ),\mathfrak{x})$. As a composite of fully faithful V-functors, σ_X is fully faithful; moreover, it is an embedding because, by hypothesis, TX and PX are separated V-categories.

To show that $\sigma = (\sigma_X)_X \colon T \to P$ *is a natural transformation*, that is, for each V-functor $f\colon X \to Y$, the outer diagram

$$\begin{array}{ccccc}
TX & \xrightarrow{y_{TX}} & PTX & \xrightarrow{Q\eta_X} & PX \\
{\scriptstyle Tf}\downarrow & \boxed{1} & \downarrow{\scriptstyle PTf} & \boxed{2} & \downarrow{\scriptstyle Pf} \\
TY & \xrightarrow[y_{TY}]{} & PTY & \xrightarrow[Q\eta_Y]{} & PY
\end{array}$$

commutes, we only need to observe that $\boxed{1}$ is commutative and BC* for η implies that $\boxed{2}$ is commutative.

It remains to show σ *is a monad morphism*: for each V-category (X,a) and $x \in X$,

$$(\sigma_X \cdot \eta_X)(x) = \widehat{a}(\eta_X(\;), \eta_X(x)) = a(-,x) = x^* = y_X(x),$$

and so $\sigma \cdot \eta = y$. To check that, for every V-category (X,a), the following diagram commutes

$$\begin{array}{ccccc}
TTX & \xrightarrow{\sigma_{TX}} & PTX & \xrightarrow{P\sigma_X} & PPX \\
{\scriptstyle \mu}\downarrow & & & & \downarrow{\scriptstyle m_X} \\
TX & & \xrightarrow[\sigma_X]{\qquad\qquad\qquad} & & PX,
\end{array}$$

let $\mathfrak{X} \in TTX$. We have

$$m_X \cdot P\sigma_X \cdot \sigma_{TX}(\mathfrak{X}) =$$

$$= (\; X \xrightarrow{y_X} PX \overset{\widetilde{a}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} PX \overset{\sigma_X^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TX \xrightarrow{\eta_{TX}} TTX \overset{\widehat{\widehat{a}}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TTX \overset{\mathfrak{X}^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} E \;)$$

$$= (\; X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TX \xrightarrow{\eta_{TX}} TTX \overset{\widehat{\widehat{a}}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TTX \overset{\mathfrak{X}^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} E \;),$$

since $\sigma_X^\circ \cdot \widetilde{a} \cdot y_X(x,\mathfrak{x}) = \widetilde{a}(y_X(x), \sigma_X(\mathfrak{x})) = \sigma_X(\mathfrak{x})(x) = \widehat{a} \cdot \eta_X(x,\mathfrak{x})$, and

$$\sigma_X \cdot \mu_X(\mathfrak{x}) = (\; X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TX \overset{\mu_X^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TTX \overset{\mathfrak{X}^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\;} E \;).$$

Hence the commutativity of the diagram follows from the equality $\widehat{\widehat{a}} \cdot \eta_{TX} \cdot \widehat{a} \cdot \eta_X = \mu_X^\circ \cdot \widehat{a} \cdot \eta_X$ we show next. Indeed,

$$\begin{aligned}
\widehat{\widehat{a}} \cdot \eta_{TX} \cdot \widehat{a} \cdot \eta_X &= (\eta_{TX})_*(\eta_X)_* = (\eta_{TX} \cdot \eta_X)_* = (T\eta_X \cdot \eta_X)_* = (T\eta_X)_*(\eta_X)_* \\
&= \mu_X^*(\eta_X)_* = \mu_X^\circ \cdot \widehat{a} \cdot \eta_X.
\end{aligned}$$

■

The proof of the theorem allows us to conclude immediately the following result.

Corollary 4.2. *Given a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** such that η satisfies BC*, there is a monad morphism $\mathbb{T} \to \mathbb{P}$ if, and only if, η is pointwise fully faithful.*

5. On algebras for submonads of $\mathbb{P}$: a survey

In the remainder of this paper we will study, given a submonad $\mathbb{T}$ of $\mathbb{P}$, the category $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ of (Eilenberg-Moore) $\mathbb{T}$-algebras. Here we collect some known results which will be useful in the following sections. We will denote by $\Phi(\mathbb{T})$ the admissible class of distributors that induces the monad $\mathbb{T}$ (defined in (2.ii)).

The following result, which is valid for any lax idempotent monad $\mathbb{T}$, asserts that, for any V-category, to be a $\mathbb{T}$-algebra is a property (see, for instance, [9] and [6]).

Theorem 5.1. *Let $\mathbb{T}$ be lax idempotent monad on V-**Cat**.*

(1) For a V-category X, the following assertions are equivalent:
   - (i) *$\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure on X;*
   - (ii) *there is a V-functor $\alpha\colon TX \to X$ with $\alpha \dashv \eta_X$ and $\alpha \cdot \eta_X = 1_X$;*
   - (iii) *there is a V-functor $\alpha\colon TX \to X$ such that $\alpha \cdot \eta_X = 1_X$;*
   - (iv) *$\alpha\colon TX \to X$ is a split epimorphism in V-**Cat**.*

(2) If (X, α) and (Y, β) are $\mathbb{T}$-algebra structures, then every V-functor $f\colon X \to Y$ satisfies $\beta \cdot Tf \leq f \cdot \alpha$.

Next we formulate characterisations of $\mathbb{T}$-algebras that can be found in [11, 2], using *injectivity* with respect to certain *embeddings*, and using the existence of certain *weighted colimits*, notions that we recall very briefly in the sequel.

Definition 5.2. [8] A V-functor $f\colon X \to Y$ is a *T-embedding* if Tf is a left adjoint right inverse; that is, there exists a V-functor $Tf_\sharp$ such that $Tf \dashv Tf_\sharp$ and $Tf_\sharp \cdot Tf = 1_{TX}$.

For each submonad $\mathbb{T}$ of $\mathbb{P}$, the class $\Phi(\mathbb{T})$ allows us to identify easily the T-embeddings.

Proposition 5.3. *For a V-functor $h\colon X \to Y$, the following assertions are equivalent:*

(i) *h is a T-embedding;*
(ii) *h is fully faithful and h_* belongs to $\Phi(\mathbb{T})$.*

In particular, P-embeddings are exactly the fully faithful V-functors.

Proof: (ii) $\Rightarrow$ (i): Let h be fully faithful with $h_* \in \Phi(\mathbb{T})$. As in the case of the presheaf monad, $\Phi h : \Phi X \to \Phi Y$ has always a right adjoint whenever $h_* \in \Phi(\mathbb{T})$, $\Phi^{\dashv} h := (-)\cdot h_* \colon \Phi Y \to \Phi X$; that is, for each distributor $\psi : Y {\longrightarrow\!\!\!\!\!\!\circ\;\;} E$ in ΦY, $\Phi^{\dashv} h(\psi) = \psi \cdot h_*$, which is well defined because by hypothesis $h_* \in \Phi(\mathbb{T})$. If h is fully faithful, that is, if $h^* \cdot h_* = (1_X)^*$, then $(\Phi^{\dashv} h \cdot \Phi h)(\varphi) = \varphi \cdot h^* \cdot h_* = \varphi$.

(i) $\Rightarrow$ (ii): If $\Phi^{\dashv} h$ is well-defined, then $y^* \cdot h_*$ belongs to $\Phi(\mathbb{T})$ for every $y \in Y$, hence $h_* \in \Phi(\mathbb{T})$, by 2.1(3), and so $h_* \in \Phi(\mathbb{T})$. Moreover, if $\Phi^{\dashv} h \cdot \Phi h = 1_{\Phi X}$, then in particular $x^* \cdot h^* \cdot h_* = x^*$, for every $x \in X$, which is easily seen to be equivalent to $h^* \cdot h_* = (1_X)^*$. ∎

In V-**Dist**, given a V-distributor $\varphi \colon (X, a) {\longrightarrow\!\!\!\!\!\!\circ\;\;} (Y, b)$, the functor $(\ )\cdot\varphi$ preserves suprema, and therefore it has a right adjoint $[\varphi, -]$ (since the hom-sets in V-**Dist** are complete ordered sets):

$$\mathbf{Dist}(X, Z) \underset{(\ )\cdot\varphi}{\overset{[\varphi,-]}{\underset{\top}{\rightleftarrows}}} \mathbf{Dist}(Y, Z).$$

For each distributor $\psi \colon X {\longrightarrow\!\!\!\!\!\!\circ\;\;} Z$,

$$\begin{array}{ccc} X & \overset{\psi}{\longrightarrow\!\!\!\!\!\!\circ\;\;} & Z \\ {\scriptstyle\varphi}\big\downarrow\!\circ & \leq & \nearrow {\scriptstyle [\varphi,\psi]} \\ Y & & \end{array}$$

$[\varphi, \psi] \colon Y {\longrightarrow\!\!\!\!\!\!\circ\;\;} Z$ is defined by

$$[\varphi, \psi](y, z) = \bigwedge_{x \in X} \hom(\varphi(x, y), \psi(x, z)).$$

Definitions 5.4. (1) Given a V-functor $f \colon X \to Z$ and a distributor (here called *weight*) $\varphi \colon X {\longrightarrow\!\!\!\!\!\!\circ\;\;} Y$, a *$\varphi$-weighted colimit* of f (or simply a *φ-colimit* of f), whenever it exists, is a V-functor $g \colon Y \to Z$ such that $g_* = [\varphi, f_*]$. One says then that g *represents* $[\varphi, f_*]$.

(2) A V-category Z is called *φ-cocomplete* if it has a colimit for each weighted diagram with weight $\varphi \colon (X, a) {\longrightarrow\!\!\!\!\!\!\circ\;\;} (Y, b)$; i.e. for each V-functor $f \colon X \to Z$, the φ-colimit of f exists.

(3) Given a class Φ of V-distributors, a V-category Z is called Φ-*cocomplete* if it is φ-cocomplete for every $\varphi \in \Phi$. When $\Phi = V$-**Dist**, then Z is said to be *cocomplete.*

The proof of the following result can be found in [11, 2].

Theorem 5.5. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, for a V-category X the following assertions are equivalent:*

(i) *X is a $\mathbb{T}$-algebra.*
(ii) *X is injective with respect to T-embeddings.*
(iii) *X is $\Phi(\mathbb{T})$-cocomplete.*

$\Phi(\mathbb{T})$-cocompleteness of a V-category X is guaranteed by the existence of some special weighted colimits, as we explain next. (Here we present very briefly the properties needed. For more information on this topic see [19].)

Lemma 5.6. *For a distributor $\varphi\colon X \to Y$ and a V-functor $f\colon X \to Z$, the following assertions are equivalent:*

(i) *there exists the φ-colimit of f;*
(ii) *there exists the $(\varphi \cdot f^*)$-colimit of 1_Z;*
(iii) *for each $y \in Y$, there exists the $(y^* \cdot \varphi)$-colimit of f.*

Proof: (i) $\Leftrightarrow$ (ii): It is straightforward to check that

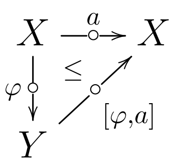
$$[\varphi, f_*] = [\varphi \cdot f^*, (1_Z)_*].$$

(i) $\Leftrightarrow$ (iii): Since $[\varphi, f_*]$ is defined pointwise, it is easily checked that, if g represents $[\varphi, f_*]$, then, for each $y \in Y$, the V-functor $E \xrightarrow{y} Y \xrightarrow{g} Z$ represents $[y^* \cdot \varphi, f_*]$.

Conversely, if, for each $y\colon E \to Y$, $g_y\colon E \to Z$ represents $[y^* \cdot \varphi, f_*]$, then the map $g\colon Y \to Z$ defined by $g(y) = g_y(*)$ is such that $g_* = [\varphi, f_*]$; hence, as stated in Remark 1.3, g is automatically a V-functor. ∎

Corollary 5.7. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, a V-category X is a $\mathbb{T}$-algebra if, and only if, $[\varphi, (1_X)_*]$ has a colimit for every $\varphi \in TX$.*

Remark 5.8. Given $\varphi\colon X \to\!\!\!\!\circ\; E$ in TX, in the diagram



$$[\varphi, a](*, x) = \bigwedge_{x' \in X} \hom(\varphi(x', *), a(x', x)) = TX(\varphi, x^*).$$

Therefore, if $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure, then

$$[\varphi, a](*, x) = TX(\varphi, x^*) = X(\alpha(\varphi), x),$$

that is, $[\varphi, a] = \alpha(\varphi)_*$; this means that α assigns to each distributor $\varphi\colon X \,{\circ\!\!\!\!\rightarrow}\, E$ the representative of $[\varphi, (1_X)_*]$.

Hence, we may describe the category of $\mathbb{T}$-algebras as follows.

Theorem 5.9. *(1) A map $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure if, and only if, for each distributor $\varphi\colon X \,{\circ\!\!\!\!\rightarrow}\, E$ in TX, $\alpha(\varphi)_* = [\varphi, (1_X)_*]$.*

(2) If X and Y are $\mathbb{T}$-algebras, then a V-functor $f\colon X \to Y$ is a $\mathbb{T}$-homomorphism if, and only if, f preserves φ-weighted colimits for any $\varphi \in TX$, i.e., if $x \in X$ represents $[\varphi, (1_X)_*]$, then $f(x)$ represents $[\varphi \cdot f^*, (1_Y)_*]$.

6. On algebras for submonads of $\mathbb{P}$: the special case of the formal ball monad

From now on we will study more in detail $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ for special submonads $\mathbb{T}$ of $\mathbb{P}$. In our first example, the formal ball monad $\mathbb{B}$, we will need to consider the (co)restriction of $\mathbb{B}$ and $\mathbb{P}$ to $V\text{-}\mathbf{Cat}_{\mathrm{sep}}$. We point out that the characterisations of $\mathbb{T}$-algebras of Theorem 5.5 remain valid for these (co)restrictions.

The space of formal balls is an important tool in the study of (quasi-)metric spaces. Given a metric space (X, d) its *space of formal balls* is simply the collection of all pairs (x, r), where $x \in X$ and $r \in [0, \infty[$. This space can itself be equipped with a (quasi-)metric. Moreover this construction can naturally be made into a monad on the category of (quasi-)metric spaces (cf. [10, 16] and references there).

This monad can readily be generalised to V-categories, using a V-categorical structure in place of the (quasi-)metric. We will start by considering an extended version of the formal ball monad, the *extended formal ball monad* $\mathbb{B}_\bullet$, which we define below.

Definitions 6.1. The *extended formal ball monad* $\mathbb{B}_\bullet = (B_\bullet, \eta, \mu)$ is given by the following:

– a functor $B_\bullet\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ which maps each V-category X to $B_\bullet X$ with underlying set $X \times V$ and

$$B_\bullet X((x,r),(y,s)) = \hom(r, X(x,y) \otimes s)$$

and every V-functor $f\colon X \to Y$ to the V-functor $B_\bullet f\colon B_\bullet X \to B_\bullet Y$ with $B_\bullet f(x,r) = (f(x),r)$;

– natural transformations $\eta\colon 1 \to B_\bullet$ and $\mu\colon B_\bullet B_\bullet \to B_\bullet$ with $\eta_X(x) = (x,k)$ and $\mu_X((x,r),s) = (x, r\otimes s)$, for every V-category X, $x \in X$, $r,s \in V$.

The *formal ball monad* $\mathbb{B}$ is the submonad of $\mathbb{B}_\bullet$ obtained when we only consider balls with radius different from $\bot$.

Remark 6.2. Note that $\mathbb{B}_\bullet X$ is not separated if X has more than one element (for any $x,y \in X$, $(x,\bot) \simeq (y,\bot)$), while, as shown in 6.13, for X separated, separation of $\mathbb{B}X$ depends on an extra property of the quantale V.

Using Corollaries 4.2 and 3.8, it is easy to check that

Proposition 6.3. *There is a pointwise fully faithful monad morphism $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$. In particular, $\mathbb{B}_\bullet$ is lax-idempotent.*

Proof: First of all let us check that η satisfies BC*, i.e., for any V-functor $f\colon X \to Y$,

$$\begin{array}{ccc} X & \xrightarrow{(\eta_X)_*} & B_\bullet X \\ {\scriptstyle f^*}\uparrow & \geq & \uparrow{\scriptstyle (B_\bullet f)^*} \\ Y & \xrightarrow[(\eta_Y)_*]{} & B_\bullet Y \end{array}$$

For $y \in Y$, $(x,r) \in B_\bullet X$,

$$\begin{aligned} ((B_\bullet f)^*(\eta_Y)_*)(y,(x,r)) &= B_\bullet Y((y,k),(f(x),r)) = Y(y,f(x)) \otimes r \\ &\le \bigvee_{z\in X} Y(y,f(z)) \otimes X(z,x) \otimes r \\ &= \bigvee_{z\in X} Y(y,f(z)) \otimes B_\bullet X((z,k),(x,r)) \\ &= ((\eta_X)_* f^*)(y,(x,r)). \end{aligned}$$

Then, by Corollary 4.2, for each V-category X, σ_X is defined as in the proof of Theorem 4.1, i.e. for each $(x,r) \in B_\bullet X$,

$$\sigma_X(x,r) = B_\bullet X((-,k),(x,r))\colon X \to V;$$

more precisely, for each $y \in X$, $\sigma_X(x,r)(y) = X(y,x) \otimes r$.

Moreover, σ_X is fully faithful: for each $(x,r),(y,s) \in B_\bullet X$,

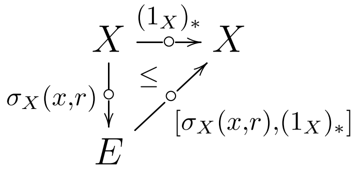
$$\begin{aligned} B_\bullet X((x,r),(y,s)) &= \hom(r, X(x,y) \otimes s) \geq \hom(X(x,x) \otimes r, X(x,y) \otimes s) \\ &\geq \bigwedge_{z \in X} \hom(X(z,x) \otimes r, X(z,y) \otimes s) = PX(\sigma(x,r), \sigma(y,s)). \end{aligned}$$

∎

It is clear that $\sigma \colon \mathbb{B}_\bullet \to \mathbb{P}$ is not pointwise monic; indeed, if $r = \bot$, then $\sigma_X(x,\bot) \colon X {\circ\!\!\to} E$ is the distributor that is constantly $\bot$, for any $x \in X$. Still it is interesting to identify the $\mathbb{B}_\bullet$-algebras via the existence of special weighted colimits.

Proposition 6.4. *For a V-category X, the following conditions are equivalent:*

(i) *X has a $\mathbb{B}_\bullet$-algebra structure $\alpha \colon B_\bullet X \to X$;*
(ii) $(\forall x \in X,\, r \in V)\,(\exists x \oplus r \in X)\,(\forall y \in X)\; X(x \oplus r, y) = \hom(r, X(x,y))$;
(iii) *for all $(x,r) \in B_\bullet X$, every diagram of the sort*



has a (weighted) colimit.

Proof: (i) ⇒ (ii): The adjunction $\alpha \dashv \eta_X$ gives, via Remark 1.3,

$$X(\alpha(x,r), y) = B_\bullet X((x,r),(y,k)) = \hom(r, X(x,y)).$$

For $x \oplus r := \alpha(x,r)$, condition (ii) follows.

(ii) ⇒ (iii): The calculus of the distributor $[\sigma_X(x,r),(1_X)_*]$ shows that it is represented by $x \oplus r$:

$$[\sigma_X(x,r),(1_X)_*](*,y) = \hom(r, X(x,y)).$$

(iii) ⇒ (i) For each $(x,r) \in B_\bullet X$, let $x \oplus r$ represent $[\sigma_X(x,r),(1_X)_*]$. In case $r = k$, we choose $x \oplus k = x$ to represent the corresponding distributor (any $x' \simeq x$ would fit here but x is the right choice for our purpose). Then $\alpha \colon B_\bullet X \to X$ defined by $\alpha(x,r) = x \oplus r$ is, by construction, left adjoint to η_X, and $\alpha \cdot \eta_X = 1_X$. ∎

The V-categories X satisfying (iii), and therefore satisfying the above (equivalent) conditions, are called *tensored*. This notion was originally introduced in the article [1] by Borceux and Kelly for general V-categories (for our special V-categories we suggest to consult [19]).

Note that, thanks to condition (ii), we get the following characterisation of tensored categories.

Corollary 6.5. *A V-category X is tensored if, and only if, for every $x \in X$,*

$$X \underset{x\oplus -}{\overset{X(x,-)}{\rightleftarrows}} V \quad (\top)$$

*is an adjunction in V-**Cat**.*

We now shift our attention to the formal ball monad $\mathbb{B}$. The characterisation of $\mathbb{B}_\bullet$-algebras given by the Proposition 6.4 may be adapted to obtain a characterisation of $\mathbb{B}$-algebras. Indeed, the only difference is that a $\mathbb{B}$-algebra structure $BX \to X$ does not include the existence of $x \oplus \bot$ for $x \in X$, which, when it exists, is the top element with respect to the order in X. Moreover, the characterisation of $\mathbb{B}$-algebras given in [10, Proposition 3.4] can readily be generalised to V-**Cat** as follows.

Proposition 6.6. *For a V-functor $\alpha\colon BX \to X$ the following conditions are equivalent.*

(i) *α is a $\mathbb{B}$-algebra structure.*
(ii) *For every $x \in X$, $r, s \in V \setminus \{\bot\}$, $\alpha(x,k) = x$ and $\alpha(x, r\otimes s) = \alpha(\alpha(x,r),s)$.*
(iii) *For every $x \in X$, $r \in V \setminus \{\bot\}$, $\alpha(x,k) = x$ and $X(x,\alpha(x,r)) \geq r$.*
(iv) *For every $x \in X$, $\alpha(x,k) = x$.*

Proof: By definition of $\mathbb{B}$-algebra, (i) $\Leftrightarrow$ (ii), while (i) $\Leftrightarrow$ (iv) follows from Theorem 5.1, since $\mathbb{B}$ is lax-idempotent. (iii) $\Rightarrow$ (iv) is obvious, and so it remains to prove that, if α is a $\mathbb{B}$-algebra structure, then $X(x,\alpha(x,r)) \geq r$, for $r \neq \bot$. But

$$X(x,\alpha(x,r)) \geq r \;\Leftrightarrow\; k \leq \hom(r, X(x,\alpha(x,r)) = X(\alpha(x,r),\alpha(x,r)),$$

because $\alpha(x,-) \dashv X(x,-)$ by Corollary 6.5. ∎

Since we know that, if X has a $\mathbb{B}$-algebra structure α, then $\alpha(x,r) = x \oplus r$, we may state the conditions above as follows.

Corollary 6.7. *If* $BX \xrightarrow{-\oplus-} X$ *is a* $\mathbb{B}$*-algebra structure, then, for* $x \in X$*,* $r, s \in V \setminus \{\bot\}$*:*

(1) $x \oplus k = x$*;*
(2) $x \oplus (r \otimes s) = (x \oplus r) \oplus s$*;*
(3) $X(x, x \oplus r) \geq r$*.*

Lemma 6.8. *Let* X *and* Y *be* V*-categories equipped with* $\mathbb{B}$*-algebra structures* $BX \xrightarrow{-\oplus-} X$ *and* $BY \xrightarrow{-\oplus-} Y$ *. Then a map* $f : X \to Y$ *is a* V*-functor if and only if*

$$f \text{ is monotone and } f(x) \oplus r \leq f(x \oplus r),$$

for all $(x, r) \in BX$*.*

Proof: Assume that f is a V-functor. Then it is, in particular, monotone, and, from Theorem 5.1 we know that $f(x) \oplus r \leq f(x \oplus r)$.

Conversely, assume that f is monotone and that $f(x) \oplus r \leq f(x \oplus r)$, for all $(x, r) \in BX$. Let $x, x' \in X$. Then $x \oplus X(x, x') \leq x'$ since $(x \oplus -) \dashv X(x, -)$ by Corollary 6.5, and then

$$\begin{aligned} f(x) \oplus X(x, x') &\leq f(x \oplus X(x, x')) && \text{(by hypothesis)} \\ &\leq f(x') && \text{(by monotonicity of } f\text{).} \end{aligned}$$

Now, using the adjunction $f(x) \oplus - \dashv Y(f(x), -))$, we conclude that

$$X(x, x') \leq Y(f(x), f(x')).$$

■

The following results are now immediate:

Corollary 6.9. *(1) Let* $(X, \oplus), (Y, \oplus)$ *be* $\mathbb{B}$*-algebras. Then a map* $f \colon X \to Y$ *is a* $\mathbb{B}$*-algebra morphism if and only if, for all* $(x, r) \in BX$*,*

$$f \text{ is monotone and } f(x \oplus r) = f(x) \oplus r.$$

(2) Let $(X, \oplus), (Y, \oplus)$ *be* $\mathbb{B}$*-algebras. Then a* V*-functor* $f \colon X \to Y$ *is a* $\mathbb{B}$*-algebra morphism if and only if, for all* $(x, r) \in BX$*,*

$$f(x \oplus r) \leq f(x) \oplus r.$$

Example 6.10. If $X \subseteq [0, \infty]$, with the V-category structure inherited from hom, then

(1) X is a $\mathbb{B}_\bullet$-algebra if, and only if, $X = [a, b]$ for some $a, b \in [0, \infty]$.

(2) X is a $\mathbb{B}$-algebra if, and only if, $X =]a,b]$ or $X = [a,b]$ for some $a, b \in [0,\infty]$.

Let X be a $\mathbb{B}_\bullet$-algebra. From Proposition 6.4 one has

$$(\forall x \in X, r \in [0,\infty])(\exists x \oplus r \in X)(\forall y \in X)\, y \ominus (x \oplus r) = (y \ominus x) \ominus r = y \ominus (x + r).$$

This implies that, if $y \in X$, then $y > x \otimes r \;\Leftrightarrow\; y > x + r$. Therefore, if $x + r \in X$, then $x \oplus r = x + r$, and, moreover, X is an interval: given $x, y, z \in [0,\infty]$ with $x < y < z$ and $x, z \in X$, then, with $r = y - x \in [0,\infty]$, $x + r = y$ must belong to X:

$$\begin{aligned} z \ominus (x \oplus r) = z - (x + r) = z - y > 0 \;\Rightarrow\; & z \ominus (x \oplus r) = z - (x \oplus r) = z - y \\ \Leftrightarrow\; & y = x \oplus r \in X. \end{aligned}$$

In addition, X must have bottom element (that is a maximum with respect to the classical order of the real half-line): for any $x \in X$ and $b = \sup X$, $x \oplus (b - x) = \sup\{z \in X \,;\, z \leq b\} = b \in X$. For $r = \infty$ and any $x \in X$, $x \oplus \infty$ must be the top element of X, so $X = [a,b]$ for $a, b \in [0,\infty]$.

Conversely, if $X =]a,b]$, for $x \in X$ and $r \in [0,\infty[$, define $x \oplus r = x + r$ if $x + r \in X$ and $x \oplus r = b$ elsewhere. It is easy to check that condition (ii) of Proposition 6.4 is satisfied for $r \neq \infty$.

Analogously, if $X = [a,b]$, for $x \in X$ and $r \in [0,\infty]$, we define $x \oplus r$ as before in case $r \neq \infty$ and $x \oplus \infty = a$.

As we will see, (co)restricting $\mathbb{B}$ to V-$\mathbf{Cat}_{\text{sep}}$ will allows us to obtain some interesting results. Unfortunately X being separated does not entail BX being so. Because of this we will need to restrict our attention to the *cancellative* quantales which we define and characterize next.

Definition 6.11. A quantale V is said to be *cancellative* if

$$\forall r, s \in V,\, r \neq \bot :\; r = s \otimes r \;\Rightarrow\; s = k. \tag{6.i}$$

Remark 6.12. We point out that this notion of cancellative quantale does not coincide with the notion of cancellable ccd quantale introduced in [4]. On the one hand cancellative quantales are quite special, since, for instance, when V is a locale, and so with $\otimes = \wedge$ is a quantale, V is not cancellative since condition (6.i) would mean, for $r \neq \bot$, $r = s \wedge r \;\Rightarrow\; s = \top$. On the other hand, $[0,1]_\odot$, that is $[0,1]$ with the usual order and having as tensor product the Łukasiewicz sum, is cancellative but not cancellable. In addition we remark that every *value quantale* [16] is cancellative.

Proposition 6.13. *Let V be an integral quantale. The following assertions are equivalent:*

(i) *BV is separated;*
(ii) *V is cancellative;*
(iii) *If X is separated then BX is separated.*

Proof: (i) $\Rightarrow$ (ii): Let $r,s\in V$, $r\neq\bot$ and $r=s\otimes r$. Note that

$$BV((k,r),(s,r))=\hom(r,\hom(k,s)\otimes r)=\hom(r,s\otimes r)=\hom(r,r)=k$$

and

$$\begin{aligned}BV((s,r),(k,r))&=\hom(r,\hom(s,k)\otimes r)=\hom(r,\hom(s,k)\otimes s\otimes r)\\&=\hom(s\otimes r,s\otimes r)=k.\end{aligned}$$

Therefore, since BV is separated, $(s,r)=(k,r)$ and it follows that $s=k$.

(ii) $\Rightarrow$ (iii): If $(x,r)\simeq(y,s)$ in BX, then

$$BX((x,r),(y,s))=k\Leftrightarrow r\leq X(x,y)\otimes s,\ \text{and}$$
$$BX((y,s),(x,r))=k\Leftrightarrow s\leq X(y,x)\otimes r.$$

Therefore $r\leq s$ and $s\leq r$, that is $r=s$. Moreover, since $r\leq X(x,y)\otimes r\leq r$ we have that $X(x,y)=k$. Analogously, $X(y,x)=k$ and we conclude that $x=y$.

(iii) $\Rightarrow$ (i): Since V is separated it follows immediately from (iii) that BV is separated. ∎

We can now show that $\mathbb{B}$ is a submonad of $\mathbb{P}$ in the adequate setting. *From now on we will be working with a cancellative and integral quantale V, and $\mathbb{B}$ will be the (co)restriction of the formal ball monad to V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proposition 6.14. *Let V be a cancellative and integral quantale. Then $\mathbb{B}$ is a submonad of $\mathbb{P}$ in V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proof: Thanks to Proposition 6.3, all that remains is to show that σ_X is injective on objects, for any V-category X. Let $\sigma(x,r)=\sigma(y,s)$, or, equivalently, $X(-,x)\otimes r=X(-,y)\otimes s$. Then, in particular,

$$r=X(x,x)\otimes r=X(x,y)\otimes s\leq s=X(y,y)\otimes s=X(y,x)\otimes r\leq r.$$

Therefore $r=s$ and $X(y,x)=X(x,y)=k$. We conclude that $(x,r)=(y,s)$. ∎

Thanks to Theorem 5.5 $\mathbb{B}$-algebras are characterized via an injectivity property with respect to special embeddings. We end this section studying in more detail these embeddings. Since we are working in V-$\mathbf{Cat}_{\rm sep}$, a B-embedding $h\colon X\to Y$, being fully faithful, is injective on objects. Therefore, for simplicity, we may think of it as an inclusion. With $Bh_\sharp\colon BY\to BX$ the right adjoint and left inverse of $Bh\colon BX\to BY$, we denote $Bh_\sharp(y,r)$ by (y_r,r_y).

Lemma 6.15. *Let $h\colon X\to Y$ be a B-embedding. Then:*

(1) $(\forall y\in Y)\ (\forall x\in X)\ (\forall r\in V)\ BY((x,r),(y,r))=BY((x,r),(y_r,r_y))$*;*
(2) $(\forall\, y\in Y)\colon k_y=Y(y_k,y)$*;*
(3) $(\forall\, y\in Y)\ (\forall x\in X)\colon\ \ Y(x,y)=Y(x,y_k)\otimes Y(y_k,y)$*.*

Proof: (1) From $Bh_\sharp\cdot Bh=1_{BX}$ and $Bh\cdot Bh_\sharp\leq 1_{BY}$ one gets, for any $(y,r)\in BY$, $(y,r)\leq(y_r,r_y)$, i.e. $BY((y,r),(y_r,r_y))=\hom(r_y,Y(y_r,y)\otimes r)=k$. Therefore, for all $x\in X$, $y\in Y$, $r\in V$,

$$\begin{aligned}BY((x,r),(y,r))&\leq BX((x,r),(y_r,r_y))=BY((x,r),(y_r,r_y))\\&=BY((x,r),(y_r,r_y))\otimes BY((y_r,r_y),(y,r))\\&\leq BY((x,r),(y,r)),\end{aligned}$$

that is

$$BY((x,r),(y,r))=BY((x,r),(y_r,r_y)).$$

(2) Let $y\in Y$. Then

$$Y(y_k,y)=BY((y_k,k),(y,k))=BY((y_k,k),(y_k,k_y))=k_y.$$

(3) Let $y\in Y$ and $x\in X$. Then

$$\begin{aligned}Y(x,y)&=BY((x,k),(y,k))=BY((x,k),(y_k,k_y))\\&=Y(x,y_k)\otimes k_y=Y(x,y_k)\otimes Y(y_k,y).\end{aligned}$$

■

Proposition 6.16. *Let X and Y be V-categories. A V-functor $h\colon X\to Y$ is a B-embedding if and only if h is fully faithful and*

$$(\forall y\in Y)\ (\exists! z\in X)\ (\forall x\in X)\ \ Y(x,y)=Y(x,z)\otimes Y(z,y).\tag{6.ii}$$

Proof: If h is a B-embedding, then it is fully faithful by Proposition 5.3 and, for each $y\in Y$, $z=y_k\in X$ fulfils the required condition. To show that such z is unique, assume that $z,z'\in X$ verify the equality of condition (6.ii). Then

$$Y(z,y)=Y(z,z')\otimes Y(z',y)\leq Y(z',y)=Y(z',z)\otimes Y(z,y)\leq Y(z,y),$$

and therefore, because V is cancellative, $Y(z',z)=k$; analogously one proves that $Y(z,z')=k$, and so $z=z'$ because Y is separated.

To prove the converse, for each $y\in Y$ we denote by $\overline{y}$ the only $z\in X$ satisfying (6.ii), and define

$$Bh_\sharp(y,r)=(\overline{y},Y(\overline{y},y)\otimes r).$$

When $x\in X$, it is immediate that $\overline{x}=x$, and so $Bh_\sharp\cdot Bh=1_{BX}$. Using Remark 1.3, to prove that $Bh_\sharp$ is a V-functor and $Bh\dashv Bh_\sharp$ it is enough to show that

$$BX((x,r),Bh_\sharp(y,s))=BY(Bh(x,r),(y,s)),$$

for every $x\in X$, $y\in Y$, $r,s\in V$. By definition of $Bh_\sharp$ this means

$$BX((x,r),(\overline{y},Y(\overline{y},y)\otimes s))=BY((x,r),(y,s)),$$

that is,

$$\hom(r,Y(x,\overline{y})\otimes Y(\overline{y},y)\otimes s)=\hom(r,Y(x,y)\otimes s),$$

which follows directly from (6.ii). ■

Corollary 6.17. *In* **Met***, if $X\subseteq[0,\infty]$, then its inclusion $h\colon X\to[0,\infty]$ is a B-embedding if, and only if, X is a closed interval.*

Proof: If $X=[x_0,x_1]$, with $x_0,x_1\in[0,\infty]$, $x_0\le x_1$, then it is easy to check that, defining $\overline{y}=x_0$ if $y\le x_0$, $\overline{y}=y$ if $y\in X$, and $\overline{y}=x_1$ if $y\ge x_1$, for every $y\in[0,\infty]$, condition (6.ii) is fulfilled.

We divide the proof of the converse in two cases:

(1) If X is not an interval, i.e. if there exists $x,x'\in X$, $y\in[0,\infty]\setminus X$ with $x<y<x'$, then either $\overline{y}<y$, and then

$$0=y\ominus x'\neq(y\ominus x')+(y\ominus\overline{y})=y-\overline{y},$$

or $\overline{y}>y$, and then

$$y-x=y\ominus x\neq(\overline{y}\ominus x)+(y\ominus\overline{y})=\overline{y}-x.$$

(2) If $X=[x_0,x_1[$ and $y>x_1$, then there exists $x\in X$ with $\overline{y}<x<y$, and so

$$y-x=y\ominus x\neq(\overline{y}\ominus x)+(y\ominus\overline{y})=y-\overline{y}.$$

An analogous argument works for $X=]x_0,x_1]$. ■

7. On algebras for submonads of $\mathbb{P}$ and their morphisms

In the following $\mathbb{T} = (T, \mu, \eta)$ is a submonad of the presheaf monad $\mathbb{P} = (P, m, y)$ in V-$\mathbf{Cat}_{\mathrm{sep}}$ For simplicity we will assume that the injective and fully faithful components of the monad morphism $\sigma : T \to P$ are inclusions. Theorem 5.1 gives immediately that:

Proposition 7.1. *Let (X, a) be a V-category and $\alpha : TX \to X$ be a V-functor. The following are equivalent:*

(1) (X, α) is a $\mathbb{T}$-algebra;
(2) $\forall\, x \in X : \alpha(x^*) = x$.

We would like to identify the $\mathbb{T}$-algebras directly, as we did for $\mathbb{B}_\bullet$ or $\mathbb{B}$ in Proposition 6.4. First of all, we point out that a $\mathbb{T}$-algebra structure $\alpha\colon TX \to X$ must satisfy, for every $\varphi \in TX$ and $x \in X$,

$$X(\alpha(\varphi), x) = TX(\varphi, x^*),$$

and so, in particular,

$$\alpha(\varphi) \leq x \;\Leftrightarrow\; \varphi \leq x^*;$$

hence α must assign to each $\varphi \in TX$ an $x_\varphi \in X$ so that

$$x_\varphi = \min\{x \in X\,;\, \varphi \leq x^*\}.$$

Moreover, for such map $\alpha\colon TX \to X$, α is a V-functor if, and only if,

$$\begin{aligned}
&(\forall \varphi, \rho \in TX)\;\; TX(\varphi, \rho) \leq X(x_\varphi, x_\rho) = TX(X(-, x_\varphi), X(-, x_\rho))\\
\Leftrightarrow\; &(\forall \varphi, \rho \in TX)\;\; TX(\varphi, \rho) \leq \bigwedge_{x \in X} \hom(X(x, x_\varphi), X(x, x_\rho))\\
\Leftrightarrow\; &(\forall x \in X)\; (\forall \varphi, \rho \in TX)\;\; X(x, x_\varphi) \otimes TX(\varphi, \rho) \leq X(x, x_\rho).
\end{aligned}$$

Proposition 7.2. *A V-category X is a $\mathbb{T}$-algebra if, and only if:*

(1) for all $\varphi \in TX$ there exists $\min\{x \in X\,;\, \varphi \leq x^*\}$*;*
(2) for all $\varphi, \rho \in TX$ and for all $x \in X$, $X(x, x_\varphi) \otimes TX(\varphi, \rho) \leq X(x, x_\rho)$.

We remark that condition (2) can be equivalently stated as:

(2') for each $\rho \in TX$, the distributor $\rho_1 = \displaystyle\bigvee_{\varphi \in TX} X(-, x_\varphi) \otimes TX(\varphi, \rho)$ satisfies $x_{\rho_1} = x_\rho$,

which is the condition corresponding to condition (2) of Corollary 6.7.

Finally, as for the formal ball monad, Theorem 5.1 gives the following characterisation of $\mathbb{T}$-algebra morphisms.

Corollary 7.3. *Let $(X,\alpha),(Y,\beta)$ be $\mathbb{T}$-algebras. Then a V-functor $f : X \to Y$ is a $\mathbb{T}$-algebra morphism if and only if*

$$(\forall \varphi \in TX)\;\; \beta(\varphi \cdot f^*) \geq f(\alpha(\varphi)).$$

Example 7.4. The Lawvere monad. Among the examples presented in [2] there is a special submonad of $\mathbb{P}$ which is inspired by the crucial remark of Lawvere in [17] that Cauchy completeness for metric spaces is a kind of cocompleteness for V-categories. Indeed, the submonad $\mathbb{L}$ of $\mathbb{P}$ induced by

$$\Phi = \{\varphi\colon X {\longrightarrow\!\!\!\!\!\!\!\circ\;\;} Y \;;\; \varphi \text{ is a right adjoint } V\text{-distributor}\}$$

has as $\mathbb{L}$-algebras the *Lawvere complete V-categories.* These were studied also in [3], and in [14] under the name L-complete V-categories. When $V = [0,\infty]_+$, using the usual order in $[0,\infty]$, for distributors $\varphi\colon X {\longrightarrow\!\!\!\!\!\!\!\circ\;\;} E$, $\psi\colon E {\longrightarrow\!\!\!\!\!\!\!\circ\;\;} X$ to be adjoint

$$X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} E \qquad (\top)$$

means that

$$\begin{aligned}(\forall x,x' \in X)\;\; X(x,x') &\leq \varphi(x)+\psi(x'),\\ 0 &\geq \inf_{x\in X}(\psi(x)+\varphi(x)).\end{aligned}$$

This means in particular that

$$(\forall n \in \mathbb{N})\;(\exists x_n \in X)\;\; \psi(x_n)+\varphi(x_n) \leq \frac{1}{n},$$

and, moreover,

$$X(x_n,x_m) \leq \varphi(x_n)+\psi(x_m) \leq \frac{1}{n}+\frac{1}{m}.$$

This defines a *Cauchy sequence* $(x_n)_n$, so that

$$(\forall \varepsilon>0)\,(\exists p \in \mathbb{N})\,(\forall n,m \in \mathbb{N})\, n \geq p \wedge m \geq p \;\Rightarrow\; X(x_n,x_m)+X(x_m,x_n) < \varepsilon.$$

Hence, any such pair induces a (equivalence class of) Cauchy sequence(s) $(x_n)_n$, and a representative for

$$\begin{array}{ccc} X & \xrightarrow{(1_X)_*} & X \\ {\scriptstyle\varphi}\downarrow & \leq & \nearrow {\scriptstyle[\varphi,(1_X)_*]} \\ E & & \end{array}$$

is nothing but a limit point for $(x_n)_n$. Conversely, it is easily checked that every Cauchy sequence $(x_n)_n$ in X gives rise to a pair of adjoint distributors

$$\varphi = \lim_n X(-, x_n) \text{ and } \psi = \lim_n X(x_n, -).$$

We point out that the $\mathbb{L}$-embeddings, i.e. the fully faithful and fully dense V-functors $f\colon X \to Y$ do not coincide with the $\mathbb{L}$-dense ones (so that f_* is a right adjoint). For instance, assuming for simplicity that V is integral, a V-functor $y\colon E \to X$ ($y \in X$) is fully dense if and only if $y \simeq x$ for all $x \in X$, while it is an $\mathbb{L}$-embedding if and only if $y \leq x$ for all $x \in X$. Indeed, $y\colon E \to X$ is $\mathbb{L}$-dense if, and only if,

– there is a distributor $\varphi\colon X \multimap E$, i.e.

$$(\forall x, x' \in X)\ \ X(x, x') \otimes \varphi(x') \leq \varphi(x), \tag{7.i}$$

such that

– $k \geq \varphi \cdot y_*$, which is trivially true, and $a \leq y_* \cdot \varphi$, i.e.

$$(\forall x, x' \in X)\ \ X(x, x') \leq \varphi(x) \otimes X(y, x'). \tag{7.ii}$$

Since (7.i) follows from (7.ii),

$$y \text{ is } \mathbb{L}\text{-dense} \ \ \Leftrightarrow \ \ (\forall x, x' \in X)\ \ X(x, x') \leq \varphi(x) \otimes X(y, x').$$

In particular, when $x = x'$, this gives $k \leq \varphi(x) \otimes X(y, x)$, and so we can conclude that, for all $x \in X$, $y \leq x$ and $\varphi(x) = k$. The converse is also true; that is

$$y \text{ is } \mathbb{L}\text{-dense} \ \ \Leftrightarrow \ \ (\forall x \in X)\ \ y \leq x.$$

Still, it was shown in [14] that injectivity with respect to fully dense and fully faithful V-functors (called L-dense in [14]) characterizes also the $\mathbb{L}$-algebras.

References

[1] F. Borceux, G.M. Kelly, A notion of limit for enriched categories. *Bull. Austral. Math. Soc.* **12** (1975), 49–72.

[2] M.M. Clementino, D. Hofmann, Relative injectivity as cocompleteness for a class of distributors. *Theory Appl. Categ.* **21** (2008), 210–230.

[3] M.M. Clementino, D. Hofmann, Lawvere completeness in topology. *Appl. Categ. Structures* **17** (2009), 175–210.

[4] M.M. Clementino, D. Hofmann, The rise and fall of V-functors. *Fuzzy Sets and Systems* **321** (2017), 29–49.

[5] M.M. Clementino, D. Hofmann, G. Janelidze, The monads of classical algebra are seldom weakly cartesian. *J. Homotopy Relat. Struct.* **9** (2014), 175–197.

[6] M.M. Clementino, I. López Franco, Lax orthogonal factorisations in ordered structures. *Theory Appl. Categ.* **35**, (2020), 1379–1423.

[7] S. Eilenberg and G. Max Kelly. Closed categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 421–562. Springer, New York, 1966.

[8] M. Escardó, Properly injective spaces and function spaces. *Topology Appl.* **89** (1998), 75–120.

[9] M. Escardó, R. Flagg, Semantic domains, injective spaces and monads. *Electr. Notes in Theor. Comp. Science* **20**, electronic paper 15 (1999).

[10] J. Goubault-Larrecq, Formal ball monads. *Topology Appl.* **263** (2019), 372–391.

[11] D. Hofmann, Injective spaces via adjunction. *J. Pure Appl. Algebra* **215** (2011), 283–302.

[12] D. Hofmann, P. Nora, Hausdorff coalgebras. *Appl. Categ. Structures* **28** (2020), 773–806.

[13] D. Hofmann, C.D. Reis, Probabilistic metric spaces as enriched categories, *Fuzzy Sets and Systems* **210** (2013), 1–21.

[14] D. Hofmann, W. Tholen, Lawvere completion and separation via closure, *Appl. Categ. Structures* **18** (2010), 259–287.

[15] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982. Republished in: Reprints in Theory and Applications of Categories. No. 10 (2005), 1–136.

[16] M. Kostanek, P. Waszkiewicz, The formal ball model for $\mathcal{Q}$-categories. *Math. Structures Comput. Sci.* **21** (2011), 41–64.

[17] F.W. Lawvere, Metric spaces, generalized logic, and closed categories. *Rend. Semin. Mat. Fis. Milano*, **43** (1973), 135–166. Republished in: *Reprints in Theory and Applications of Categories*, No. **1** (2002), 1–37.

[18] E. Manes, Taut monads and T0-spaces. *Theoret. Comput. Sci.* **275** (2002), 79–109.

[19] I. Stubbe, Categorical structures enriched in a quantaloid: categories, distributors and functors. *Theory Appl. Categ.* **14** (2005), 1–45.

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