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A POINTFREE THEORY OF PERVIN SPACES

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Abstract: We lay down the foundations for a pointfree theory of Pervin spaces. A Pervin space is a set equipped with a bounded sublattice of its powerset, and it is known that these objects characterize those quasi-uniform spaces that are transitive and totally bounded. The pointfree notion of a Pervin space, which we call Frith frame, consists of a frame equipped with a generating bounded sublattice. In this paper we introduce and study the category of Frith frames and show that the classical dual adjunction between topological spaces and frames extends to a dual adjunction between Pervin spaces and Frith frames. Unlike what happens for Pervin spaces, we do not have an equivalence between the categories of transitive and totally bounded quasi-uniform frames and of Frith frames, but we show that the latter is a full coreflective subcategory of the former. We also explore the notion of completeness of Frith frames inherited from quasi-uniform frames, providing a characterization of those Frith frames that are complete and a description of the completion of an arbitrary Frith frame.

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1. Introduction

Pervin quasi-uniformities were introduced to answer the question of whether a given topology is induced by some quasi-uniformity. Pervin [13] showed that, if τ is a topology on X, then τ is the topology induced by the quasi-uniformity generated by the *entourages* of the form

$$E_U := (X \times U) \cup (U^c \times X),$$

for $U \in \tau$. Abstracting this idea, we may consider on a set X the quasi-uniformity $\mathcal{E}_\mathcal{F}$ generated by $\{E_U \mid U \in \mathcal{F}\}$, for an arbitrary family $\mathcal{F} \subseteq \mathcal{P}(X)$, and the quasi-uniform spaces obtained in this way are known as *Pervin spaces.* In [4, Proposition 2.1] it is shown that the quasi-uniform spaces $(X, \mathcal{E}_\tau)$ when τ is a topology on X are transitive and totally bounded, but the same argument works for every space $(X, \mathcal{E}_\mathcal{F})$. Actually, more can be said: *every* transitive and totally bounded quasi-uniformity on a set X is of the form $\mathcal{E}_\mathcal{F}$ for some family $\mathcal{F} \subseteq \mathcal{P}(X)$. To the best of our knowledge, this result is due to Gehrke, Grigorieff, and Pin in unpublished work (see also [18]), and at the present date, a proof can be found in the Goubault-Larrecq's blog [8]. Another interesting aspect of Pervin spaces is that the bounded sublattice of $\mathcal{P}(X)$ generated by some family $\mathcal{F}$ can be recovered from the quasi-uniform space $(X, \mathcal{E}_\mathcal{F})$: it consists of the subsets $U \subseteq \mathcal{P}(X)$ such that $E_U \in \mathcal{E}_\mathcal{F}$ (see [7, Theorem 5.1] for a proof). For that reason, Pervin spaces may be elegantly represented by pairs $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a bounded sublattice of $\mathcal{P}(X)$.

The main contribution of this paper is the development of a pointfree theory of Pervin spaces. The central object of study are the pairs of the form (L, S), where L is a frame and S is a join-dense bounded sublattice. We name such pairs of *Frith frames.* This choice is justified by the fact that the pointfree version of Pervin's construction is known as the *Frith quasi-uniformity* [5] on a frame. The correctness of our notion is evidenced by the existence of a Pervin-Frith dual adjunction extending the classical dual adjunction between topological spaces and frames (cf. Proposition 4.3). One major difference in the pointfree setting is the lack of an equivalence between the categories of Frith frames and of transitive and totally bounded quasi-uniform frames, although Frith frames do form a full coreflective subcategory of transitive and totally bounded quasi-uniform frames (cf. Theorem 5.12). The picture is different when we restrict to *symmetric Frith frames* (that is, those Frith frames (L, S) where S is a Boolean algebra) on the one hand, and to transitive and totally bounded uniform frames on the other (cf. Corollary 6.2). We show that every Frith frame admits a *symmetrization*, which defines a reflection of Frith frames onto the symmetric ones (cf. Proposition 6.5). Moreover, the symmetrization of a Frith frame is shown to be, on the one hand, a restriction of the usual uniform reflection of quasi-uniform frames (cf. Proposition 6.6) and, on the other hand, a pointfree version of the *symmetrization* of a Pervin space (cf. Proposition 3.4 and Theorem 6.8). Finally, we explore the notion of *complete* Frith frame, which is naturally inherited from the homonymous notion for quasi-uniform frames (cf. Proposition 7.2), both from the point of view of *dense extremal epimorphisms* and of *Cauchy maps*. In particular, we characterize the complete Frith frames as those whose frame component is *coherent* (cf. Theorem 7.7).

The paper is organized as follows. Section 2 is a preliminary section, where we present the background needed and establish the notation used in the rest of the paper. In Section 3 we give an overview of the theory of Pervin spaces. While we did not intend to go very deep in our exposition, we tried to provide enough details to allow the reader to compare the known results with our pointfree approach. In Section 4 we introduce the category of Frith frames and discuss some of its general properties. In particular, we show the existence of a dual adjunction between Pervin spaces and Frith frames (Section 4.1), we discuss compactness, coherence and zero-dimensionality of Frith frames (Section 4.2), we show that the category of Frith frames is complete and cocomplete (Section 4.3), and we characterize some special morphisms (Sections 4.4 and 4.5). In Section 5 we show how to assign a transitive and totally bounded quasi-uniform frame to each Frith frame and show that this assignment defines a full coreflective embedding. In Section 6 we consider the special case of symmetric Frith frames and of (transitive and totally bounded) uniform frames. In particular, we show that the category of symmetric Frith frames is equivalent to the category of transitive and totally bounded uniform frames, and we show that the symmetric Frith frames form a full reflective and coreflective subcategory of the category of Frith frames. Finally, Section 7 is devoted to the characterization of complete Frith frames and of the completion of a Frith frame. More precisely, in Section 7.1 we discuss completion using *dense extremal epimorphisms*, while in Section 7.2 we characterize complete Frith frames using *Cauchy maps*.

2. Preliminaries

In this section we state the basic results and set up the notation that will be used in the rest of the paper. The reader is assumed to have basic knowledge of category theory, as well as some acquaintance with general topics from pointfree topology. For more on category theory, the reader is referred to [12]. For further reading on frame theory, including uniform and quasi-uniform frames, we refer to [16]. Although not strictly needed, some background on quasi-uniform spaces may also be useful. For this topic, we refer to [4].

2.1. The category of frames and the Top - Frm dual adjunction. A *frame* is a complete lattice $(L, \bigvee, \bigwedge, 0, 1)$ such that for every $a \in L$ and $\{b_i\}_{i\in I} \subseteq L$ the following equality holds:

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I} (a \wedge b_i).$$

The frame distributive law guarantees that, for every element $a \in L$, there is a greatest element a^*, called the *pseudocomplement* of a, satisfying $a \wedge a^* = 0$. That is, a^* is defined by the following property:

$$\forall x \in L,\ x \wedge a = 0 \iff x \leq a^*. \tag{1}$$

When $a \vee a^* = 1$, we have that a is *complemented* and a^* is the *complement* of a. Moreover, if a is complemented then $a^{**} = a$.

A *frame homomorphism* is a map $h : L \to M$ that preserves finite meets and arbitrary joins (including empty meets and empty joins). A frame homomorphism need not preserve the pseudocomplement operation, but whenever a is complemented, we have that $h(a)$ is complemented, too, and $h(a^*) = h(a)^*$.

We denote by **Frm** the category of frames and frame homomorphisms. Because the two finitary distributive laws are duals of each other, every frame is a bounded distributive lattice. The category of bounded distributive lattices and lattice homomorphisms will be denoted by **DLat**. We will only consider lattices that are distributive and have empty meets and empty joins (that is, a top and a bottom element). Therefore we will sometimes omit the words "bounded" and "distributive" when referring to a lattice. For a frame L and a subset $R \subseteq L$, we denote as $\langle R \rangle_{\mathbf{DLat}}$ the sublattice of L generated by R. The expression $\langle R \rangle_{\mathbf{Frm}}$ denotes the subframe of L generated by R.

It is well-known that in **Frm** the one-one homomorphisms are precisely the monomorphisms, while the onto homomorphisms are the extremal epimorphisms.* Extremal epimorphisms of frames will be simply called *surjections*.

A frame homomorphism $h : L \to M$ is *dense* provided $h(a) = 0$ implies $a = 0$. We already observed that frame homomorphisms need not preserve the pseudocomplement operation. However, we have the following:

Lemma 2.1. *Let $h : L \to M$ be a dense extremal epimorphism. Then, for every $a \in L$, we have $h(a^*) = h(a)^*$.*

Proof: We may compute:

$$\begin{aligned} h(a^*) &= h(\bigvee\{x \in L \mid x \wedge a = 0\}) \\ &= h(\bigvee\{x \in L \mid h(x \wedge a) = 0\}) \quad (h \text{ is dense}) \\ &= \bigvee\{h(x) \mid x \in L,\, h(x) \wedge h(a) = 0\} \quad (h \text{ is a frame homomorphism}) \\ &= \bigvee\{y \in M \mid y \wedge h(a) = 0\} \quad (h \text{ is onto}) \\ &= h(a)^*. \quad \blacksquare \end{aligned}$$

At the base of pointfree topology is the classical adjunction between the category of topological spaces and (the dual of the) category of frames [10, 16]. For the reader's convenience, we present it now. We denote by **Top** the category of topological spaces and continuous functions. The adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{\mathrm{op}} : \mathsf{pt}$ is defined as follows. If (X, τ), or simply X, is a topological space, then the topology τ is a frame, and we set $\Omega(X) := \tau$. If $f : X \to Y$ is a continuous function, then taking preimages defines a frame homomorphism $\Omega f := f^{-1} : \Omega(Y) \to \Omega(X)$. This is usually called the *open set functor*. Conversely, the *spectrum functor* pt assigns to each frame L its set of *points* $\mathsf{pt}(L)$, that is, the set of all frame homomorphisms $p : L \to \mathbf{2}$

*In a category $\mathcal{C}$, an epimorphism e is *extremal* if whenever $e = m \circ g$ with m a monomorphism we have that m is an isomorphism.

where $\mathbf{2} := \{0 < 1\}$ denotes the two-element lattice. The topology on $\mathsf{pt}(L)$ is generated by the sets of the form

$$\widehat{a} := \{p \in \mathsf{pt}(L) \mid p(a) = 1\},$$

for $a \in L$. At the level of morphisms, pt is given by precomposition: for a frame homomorphism $h : L \to M$ and a point $q : M \to \mathbf{2}$, we have $\mathsf{pt}(h)(q) := q \circ h$. The adjunction $\Omega \dashv \mathsf{pt}$ is idempotent, that is, its fixpoints in **Top** are precisely those spaces that are isomorphic to $\mathsf{pt}(L)$ for some frame L, and its fixpoints in $\mathbf{Frm}^{\mathrm{op}}$ are those frames that are isomorphic to $\Omega(X)$ for some space X. Spaces of the form $\mathsf{pt}(L)$ are the *sober* spaces – that is, those spaces where every irreducible closed set is the closure of a unique point. Frames of the form $\Omega(X)$ are the *spatial* frames – that is, those frames L such that for $a, b \in L$ we have that $a \nleq b$ implies that there is some frame morphism $h : L \to \mathbf{2}$ such that $h(a) = 1$ and $h(b) = 0$. The adjunction $\Omega \dashv \mathsf{pt}$ restricts to a duality between sober spaces and spatial frames. The component of the unit of $\Omega \dashv \mathsf{pt}$ at a frame L is the frame homomorphism $\varphi_L : L \to \Omega(\mathsf{pt}(L))$, also called the *spatialization map* of L, which maps an element $a \in L$ to the open subset $\widehat{a} \subseteq \mathsf{pt}(L)$. The component of the counit of $\Omega \dashv \mathsf{pt}$ at a space X is the continuous function $\psi_X : X \to \mathsf{pt}(\Omega(X))$, also called the *sobrification map* of X, which sends a point $x \in X$ to its neighborhood map, that is, to the frame homomorphism $\Omega(X) \to \mathbf{2}$ mapping an open subset $U \subseteq X$ to 1 if and only if $x \in U$.

2.2. The congruence frame. Frame surjections may be characterized, up to isomorphism, via frame congruences: if $h : L \twoheadrightarrow M$ is a surjection, then its kernel $\ker(h) := \{(x, y) \mid h(x) = h(y)\}$ is a frame congruence, and conversely, every frame congruence θ induces a frame surjection $L \twoheadrightarrow L/\theta$. These two assignments are mutually inverse. Congruences are closed under arbitrary intersections, and because of this they form a closure system within the poset of all binary relations on a frame. For a frame L, we say that the congruence *generated* by a relation $\rho \subseteq L \times L$ is the smallest congruence containing ρ, and we denote it by $\overline{\rho}$. The set $\mathcal{C}L$ of all frame congruences of a frame L is naturally ordered by inclusion, which endows $\mathcal{C}L$ with a frame structure given by

$$\bigwedge_{i \in I} \theta_i = \bigcap_{i \in I} \theta_i \qquad \text{and} \qquad \bigvee_{i \in I} \theta_i = \overline{\bigcup_{i \in I} \theta_i} = \bigcap \{\theta \in \mathcal{C}L \mid \bigcup_{i \in I} \theta_i \subseteq \theta\}$$

for every family of congruences $\{\theta_i\}_{i\in I}$. There are two types of distinguished congruences: the so-called *open* ones, and the so-called *closed* ones. These are, respectively, the congruences ∇_a generated by an element of the form $(0,a)$, and the congruences Δ_a generated by an element of the form $(a,1)$, for some $a \in L$. It is not hard to see that

$$\nabla_a = \{(x,y) \mid x \vee a = y \vee a\} \qquad \text{and} \qquad \Delta_a = \{(x,y) \mid x \wedge a = y \wedge a\}.$$

We have the following basic results about open and closed congruences, where id_L denotes the identity congruence $\{(a,a) \mid a \in L\}$ on L.

Lemma 2.2. *For a frame L, and elements $a, b, a_i \in L$ $(i \in I)$, the following holds:*

- $\Delta_0 = L \times L$ *and* $\Delta_1 = \mathsf{id}_L$*;*
- $\nabla_0 = \mathsf{id}_L$ *and* $\nabla_1 = L \times L$*;*
- $\Delta_a \vee \Delta_b = \Delta_{a\wedge b}$ *and* $\bigcap_{i\in I} \Delta_{a_i} = \Delta_{\bigvee_{i\in I} a_i}$*;*
- $\bigvee_{i\in I} \nabla_{a_i} = \nabla_{\bigvee_{i\in I} a_i}$ *and* $\nabla_a \wedge \nabla_b = \nabla_{a\wedge b}$*.*

Open and closed congruences suffice to generate $\mathcal{C}L$, meaning that if we close the collection $\{\Delta_a, \nabla_a \mid a \in L\}$ under finite meets and the resulting set under arbitrary joins we get the whole congruence frame $\mathcal{C}L$. In this paper, some subframes of $\mathcal{C}L$ will be particularly relevant, namely those generated by a set of the form $\{\nabla_a \mid a \in L\} \cup \{\Delta_s \mid s \in S\}$, and denoted $\mathcal{C}_S L$, for some subset $S \subseteq L$. We note that the assignment $a \mapsto \nabla_a$ defines a frame embedding $\nabla : L \hookrightarrow \mathcal{C}_S L$. For that reason, we will often treat L as a subframe of $\mathcal{C}_S L$. The following generalizes a well-known property of the congruence frame:

Proposition 2.3 ([19, Theorem 16.2])**.** *Let L and M be frames and $S \subseteq L$ be a subset. Then, every frame homomorphism $h : L \to M$ such that $h(s)$ is complemented for all $s \in S$ may be uniquely extended to a frame homomorphism $\widetilde{h} : \mathcal{C}_S L \to M$, so that the following diagram commutes:*

$$\begin{tikzcd} L \arrow[r, hook, "\nabla"] \arrow[dr, "h"'] & \mathcal{C}_S L \arrow[d, dashed, "\widetilde{h}"] \\ & M \end{tikzcd}$$

In particular, we have the following:

Corollary 2.4. *Let $h : L \to M$ be a frame homomorphism, and let $S \subseteq L$ and $T \subseteq M$ be subsets such that $h[S] \subseteq T$. Then, h may be uniquely extended to a frame homomorphism $\overline{h} : \mathcal{C}_S L \to \mathcal{C}_T M$. Moreover, for every $a \in L$ and $s \in S$, the following equalities hold:*

$$\overline{h}(\nabla_a) = \nabla_{h(a)} \qquad \text{and} \qquad \overline{h}(\Delta_s) = \Delta_{h(s)}.$$

Proof: While the first equality holds simply because $\overline{h}$ is an extension of h, the second one is a consequence of complemented elements being preserved by frame homomorphisms. ■

2.3. Compact, coherent, and zero-dimensional frames. Let L be a frame. We say that an element $a \in L$ is *compact* if whenever $a \leq \bigvee_{i \in I} a_i$ there exists a finite subset $F \subseteq I$ such that $a \leq \bigvee_{i \in F} a_i$ (or equivalently, if $a = \bigvee_{i \in I} a_i$ then $a = \bigvee_{i \in F} a_i$ for some finite subset $F \subseteq I$). A frame L is *compact* if its top element is compact. Notice that a topological space is compact if and only if its frame of opens is. We denote by $K(L)$ the set of compact elements of a frame L. Clearly, $K(L)$ is a join-subsemilattice of L. If we further have that $K(L)$ is a sublattice of L which is join-dense, then we say that L is *coherent*. We denote by **CohFrm** the subcategory of **Frm** whose objects are the coherent frames, and morphisms are those frame homomorphisms that preserve compact elements. Coherent frames will be a central concept in this paper.

A typical example of a coherent frame is the *ideal completion* $\mathrm{Idl}(S)$ of any bounded distributive lattice S. Indeed, it is easy to see that the compact elements of $\mathrm{Idl}(S)$ are precisely the principal ideals of S, and these form a sublattice isomorphic to S, which is clearly join-dense in $\mathrm{Idl}(S)$. We will often see S as a sublattice of $\mathrm{Idl}(S)$, by identifying an element $s \in S$ with the principal ideal $\downarrow s$ it generates. It turns out that every coherent frame is the ideal completion of its sublattice of compact elements: the assignment $S \mapsto \mathrm{Idl}(S)$ is part of a functor $\mathrm{Idl}(-) : \mathbf{DLat} \to \mathbf{Frm}$ which is an equivalence of categories when co-restricted to **CohFrm**.

Proposition 2.5 ([11, pages 59 and 65])**.** *The category* **Frm** *of frames is a reflective subcategory of the category* **DLat** *of bounded distributive lattices. The reflector* $\mathrm{Idl}(-) : \mathbf{DLat} \to \mathbf{Frm}$ *sends a lattice S to its ideal completion* $\mathrm{Idl}(S)$ *and a lattice homomorphism $h : S \to T$ to*

$$\mathrm{Idl}(h) : \mathrm{Idl}(S) \to \mathrm{Idl}(T),\ J \mapsto \langle h[J] \rangle_{\mathrm{Idl}} = \downarrow h[J].$$

Moreover, the corestriction of $\mathrm{Idl}(-)$ *to* **CohFrm** *induces an equivalence of categories whose inverse* $K(-)$: **CohFrm** $\to$ **DLat** *sends a coherent frame to its sublattice of compact elements, and a morphism to its suitable restriction and co-restriction.*

Corollary 2.6. *Every lattice homomorphism* $h : S \to L$*, with* L *a frame, uniquely extends to a frame homomorphism* $\widehat{h} : \mathrm{Idl}(S) \to L$ *defined by* $\widehat{h}(J) := \bigvee h[J]$.

By definition of coherent frame homomorphism, it is not hard to verify that $K(-)$: **CohFrm** $\to$ **DLat** is both a right and a left adjoint of $\mathrm{Idl}(-)$: **DLat** $\to$ **CohFrm**. By suitably composing the adjunctions **CohFrm** $\leftrightarrows$ **DLat** and **DLat** $\leftrightarrows$ **Frm**, and using the equivalence **CohFrm** $\cong$ **DLat**, we obtain the following (see [1, Proposition 1] for a direct proof):

Proposition 2.7. *The category* **CohFrm** *of coherent frames and coherent frame homomorphisms is a coreflective subcategory of the category* **Frm** *of frames and frame homomorphisms. The coreflector is the restriction and co-restriction* $\mathrm{Idl}(-)$: **Frm** $\to$ **CohFrm** *of the functor* $\mathrm{Idl}(-)$.

Finally, we say that a frame L is *zero-dimensional* if the sublattice $B(L)$ of its complemented elements join-generates the frame.

Proposition 2.8. *Let* L *be a frame. Then,*

(a) *if* L *is compact, then every complemented element is compact;*
(b) *if* L *is zero-dimensional, then every compact element is complemented.*

In particular, every compact and zero-dimensional frame is coherent.

Proof: For (a), suppose that L is compact and let $a \in L$ be a complemented element. If $a \leq \bigvee_{i\in I} a_i$ for some family $\{a_i\}_{i\in I}$, then $1 = a^* \vee a \leq a^* \vee \bigvee_{i\in I} a_i$. By compactness of L there is a finite subset $F \subseteq I$ satisfying $1 \leq a^* \vee \bigvee_{i\in F} a_i$, and thus, $a = a \wedge 1 \leq a \wedge (a^* \vee \bigvee_{i\in F} a_i) = \bigvee_{i\in F} a_i$. For ($b$), suppose that L is zero-dimensional and let $a \in L$ be a compact element. Since a is a join of complemented elements, by compactness, it is also a finite join of complemented elements. Therefore, it is complemented as well. ■

2.4. Uniform and quasi-uniform frames. We will introduce (quasi-)uniformities on a frame L using *entourages*. Entourages on L are nothing but certain elements of the coproduct $L \oplus L$, which we describe now. A description of more general frame coproducts may be found, for instance, in [16].

Definition 2.9. *Let L be a frame. A* C-ideal *(or* coproduct-ideal*) of L is a subset $A \subseteq L \times L$ that satisfies the following properties:*
(J.1) A is a down-set;
(J.2) if $\{(a_i, b)\}_{i \in I} \subseteq A$, then $(\bigvee_{i \in I} a_i, b) \in A$;
(J.3) if $\{(a, b_i)\}_{i \in I} \subseteq A$, then $(a, \bigvee_{i \in I} b_i) \in A$.

We observe that, by taking $I = \emptyset$ in (J.2) and (J.3), we obtain that every C-ideal contains both $(\{0\} \times L)$ and $(L \times \{0\})$. One can also show that the set of all C-ideals ordered by inclusion is a frame. More precisely, if $\{A_i\}_{i \in I}$ is a family of C-ideals, then the meet $\bigwedge_{i \in I} A_i$ is given by the intersection $\bigcap_{i \in I} A_i$, and the join $\bigvee_{i \in I} A_i$ is the C-ideal generated by $\bigcup_{i \in I} A_i$. Moreover, the greatest C-ideal is $L \times L$, and the smallest one is $\mathbf{n} := (\{0\} \times L) \cup (L \times \{0\})$. Given $(a, b) \in L \times L$, we denote by $a \oplus b$ the smallest C-ideal containing (a, b). Explicitly, we have $a \oplus b = {\downarrow}(a, b) \cup \mathbf{n}$. The following properties will be needed later:

Lemma 2.10. *For every $a, a', b, b', a_i, b_i \in L$ $(i \in I)$, the following equalities hold:*
- $(a \oplus b) \wedge (a' \oplus b') = (a \wedge a') \oplus (b \wedge b')$;
- $\bigvee_{i \in I} (a_i \oplus b) = (\bigvee_{i \in I} a_i) \oplus b$;
- $\bigvee_{i \in I} (a \oplus b_i) = a \oplus (\bigvee_{i \in I} b_i)$.

Proposition 2.11. *The coproduct $L \oplus L$ is isomorphic to the set of all C-ideals of L ordered by inclusion. The coproduct maps are injections, and they are given by*

$$(\iota_1 : L \hookrightarrow L \oplus L, \quad a \mapsto a \oplus 1) \qquad \textit{and} \qquad (\iota_2 : L \hookrightarrow L \oplus L, \quad b \mapsto 1 \oplus b).$$

Let L and M be frames. If $f_i : L \to M$ $(i = 1, 2)$ are frame homomorphisms, then we denote by $f_1 \oplus f_2$ the unique frame homomorphism $L \oplus L \to M \oplus M$ given by the universal property of coproducts. Explicitly, for each C-ideal $A \in L \oplus L$, we have

$$(f_1 \oplus f_2)(A) = \bigvee \{f_1(a) \oplus f_2(b) \mid (a, b) \in A\}. \tag{2}$$

Any pair of C-ideals $A, B \in L \oplus L$ may be composed as follows:

$$A \circ B := \bigvee \{a \oplus b \mid \exists c \neq 0 \colon (a, c) \in A, \ (c, b) \in B\},$$

and for every set $\mathcal{J} \subseteq L \oplus L$ of C-ideals, we have the following relations on L:

$$b \lhd_1^{\mathcal{J}} a \iff A \circ (b \oplus b) \subseteq (a \oplus a) \text{ for some } A \in \mathcal{J},$$

and

$$b \lhd_2^{\mathcal{J}} a \iff (b \oplus b) \circ A \subseteq (a \oplus a) \text{ for some } A \in \mathcal{J}.$$

Shall $\mathcal{J}$ be clear from the context, we may simply write $\lhd_i$ instead of $\lhd_i^{\mathcal{J}}$. Given $i \in \{1, 2\}$, we denote

$$\mathcal{L}_i(\mathcal{J}) := \{a \in L \mid a = \bigvee \{b \mid b \lhd_i a\}\}.$$

It is well-known that, if $\mathcal{J}$ is a filter basis, then each $\mathcal{L}_i(\mathcal{J})$ is a subframe of L.

We can now introduce quasi-uniform frames. We follow the approach in [14] and [15]. A *(Weil) entourage* on L is a C-ideal $E \subseteq L \oplus L$ that satisfies

$$\bigvee \{a \mid (a, a) \in E\} = 1.$$

It can be shown that every entourage E is contained in $E \circ E$. If the equality $E = E \circ E$ holds, then E is said to be *transitive*. An entourage E is *finite* if it contains some finite join $\bigvee_{i=1}^{n} a_i \oplus a_i$, with $a_1 \vee \cdots \vee a_n = 1$. Notice that every finite intersection of transitive and finite entourages is again transitive and finite. Also notice that, if E is an entourage, then so is $E^{-1} := \{(b, a) \mid (a, b) \in E\}$. An entourage E is *symmetric* if $E = E^{-1}$.

Definition 2.12. *A* quasi-uniformity *on L is a subset $\mathcal{E} \subseteq L \oplus L$ of entourages on L such that*

(QU.1) $\mathcal{E}$ is a filter,
(QU.2) for every $E \in \mathcal{E}$, there exists some $F \in \mathcal{E}$ such that $F \circ F \subseteq E$,
(QU.3) L is the frame generated by $\mathcal{L}_1(\mathcal{E}) \cup \mathcal{L}_2(\mathcal{E})$.

The set $\mathcal{E}$ is called a uniformity *if it further satisfies*

(QU.4) if $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$.

A (quasi-)uniform frame *is a pair $(L, \mathcal{E})$, where L is a frame and $\mathcal{E}$ is a (quasi-)uniformity on L.*

If $\mathcal{E}' \subseteq L \oplus L$ is a filter (sub)basis of entourages that satisfies (QU.2) and (QU.3), then we say that $\mathcal{E}'$ is a *(sub)basis for a quasi-uniformity* on L. In that case, the filter generated by $\mathcal{E}'$ is a quasi-uniformity. If $\mathcal{E}'$ consists of symmetric entourages, then the quasi-uniformity it generates is actually a uniformity. For every quasi-uniform frame $(L, \mathcal{E})$, the set $\{E \cap E^{-1} \mid E \in \mathcal{E}\}$ is a basis consisting of symmetric entourages, and it generates the coarsest uniformity $\overline{\mathcal{E}}$ on L that contains $\mathcal{E}$. Finally, a quasi-uniform frame $(L, \mathcal{E})$ is *transitive* if it has a basis of transitive entourages, and *totally bounded* if it has a basis of finite entourages.

A *(quasi-)uniform homomorphism* $h : (L, \mathcal{E}) \to (M, \mathcal{F})$ is a frame homomorphism $h : L \to M$ such that for every $E \in \mathcal{E}$ we have $(h \oplus h)(E) \in \mathcal{F}$. We denote by **QUniFrm** the category of quasi-uniform frames and quasi-uniform homomorphisms, and by **UniFrm** the category of uniform frames and uniform frame morphisms. The following proposition is proven in [3], in which the authors use an alternative definition of quasi-uniform frame. This definition is shown in [14] to be equivalent to the one we use.

Proposition 2.13 ([3, Corollary 4.7]). *Uniform frames form a full reflective subcategory of quasi-uniform frames. The reflector* $\mathbf{Sym}_{\mathrm{QUniFrm}} : \mathbf{QUniFrm} \to \mathbf{UniFrm}$ *maps a quasi-uniform frame* $(L, \mathcal{E})$ *to* $(L, \overline{\mathcal{E}})$*, and a morphism to itself.*

A (quasi-)uniform homomorphism $h : (L, \mathcal{E}) \to (M, \mathcal{F})$ is an extremal epimorphism if and only if it is onto and $\mathcal{F}$ is the quasi-uniformity generated by $(h \oplus h)[\mathcal{E}]$. A (quasi-)uniform frame $(M, \mathcal{F})$ is said to be *complete* provided every dense extremal epimorphism $(L, \mathcal{E}) \twoheadrightarrow (M, \mathcal{F})$, with $(L, \mathcal{E})$ a (quasi-)uniform frame, is an isomorphism. A *completion* of a (quasi-)uniform frame $(L, \mathcal{E})$ is a complete (quasi-)uniform frame $(M, \mathcal{F})$ together with a dense extremal epimorphism $(M, \mathcal{F}) \twoheadrightarrow (L, \mathcal{E})$. The next two results will be needed in the sequel.

Proposition 2.14 ([6, Proposition 3.3]). *A quasi-uniform frame* $(L, \mathcal{E})$ *is complete if and only if so is its uniform reflection* $(L, \overline{\mathcal{E}})$*.*

Proposition 2.15 ([16, Chapter VII, Proposition 2.2.2]). *Let* $h : (L, \mathcal{E}) \twoheadrightarrow (M, \mathcal{F})$ *be a dense extremal epimorphism of uniform frames. If* M *is compact, then* h *is an isomorphism.*

Completions of quasi-uniform frames may also be characterized in terms of the so-called *Cauchy maps*. These can be thought of as analogues of Cauchy filters for quasi-uniform spaces. If L and M are semilattices with top and bottom elements, then a map $\phi : L \to M$ is called a *bounded meet homomorphism* provided it preserves the bottom element and all finite meets (including the empty one).

Definition 2.16. *Let* $(L, \mathcal{E})$ *be a quasi-uniform frame and* M *be any frame. A* Cauchy map $\phi : (L, \mathcal{E}) \to M$ *is a function* $\phi : L \to M$ *such that*

(*a*) ϕ *is a bounded meet homomorphism,*
(*b*) *for every* $a \in L$*,* $\phi(a) \leq \bigvee\{\phi(b) \mid b \lhd_1 a \text{ or } b \lhd_2 a\}$*,*
(*c*) *for every* $E \in \mathcal{E}$*,* $1 = \bigvee\{\phi(a) \mid (a,a) \in E\}$*.*

Theorem 2.17 ([17, Theorem 6.5])**.** *Let* $(L, \mathcal{E})$ *be a quasi-uniform frame. Then* $(L, \mathcal{E})$ *is complete if and only if each Cauchy map* $(L, \mathcal{E}) \to M$ *is a frame homomorphism.*

3. Pervin spaces

In what follows, a *Pervin space* will be a pair $(X, \mathcal{S})$ such that X is a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ is a bounded sublattice. A morphism of Pervin spaces $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is a function $f : X \to Y$ such that whenever $T \in \mathcal{T}$ we also have $f^{-1}(T) \in \mathcal{S}$. The category of Pervin spaces and corresponding morphisms will be denoted by **Pervin**. As mentioned in the introduction, each Pervin space uniquely determines a *transitive and totally bounded quasi-uniform space*, and every such space arises from a Pervin space. Actually, one can show that there is an equivalence between the categories of Pervin spaces and of transitive and totally bounded quasi-uniform spaces. However, since quasi-uniform spaces are not the subject of this paper, but rather its pointfree counterpart, we will not provide further details on this matter. We refer the reader to [13, 4, 18] for further reading.

We start by noticing that **Top** may be seen as a full subcategory of **Pervin**, with a topological space (X, τ) being identified with the Pervin space (X, τ). Conversely, if $(X, \mathcal{S})$ is a Pervin space, then we may consider on X the topology $\Omega_{\mathcal{S}}(X)$ generated by $\mathcal{S}$.† It is easily seen that this assignment is part of a forgetful functor U : **Pervin** $\to$ **Top** which is right adjoint to the embedding **Top** $\hookrightarrow$ **Pervin**. Thus, **Top** a full coreflective subcategory of **Pervin**.

Next, we will characterize the *extremal monomorphisms* of Pervin spaces. For that, we first need to describe the epimorphisms.

Lemma 3.1. *Let* $e : (X, \mathcal{S}) \to (Y, \mathcal{T})$ *be a morphism of Pervin spaces. Then,* e *is an epimorphism if and only if the set map* $e : X \to Y$ *is surjective.*

Proof: The argument to show that every surjection is an epimorphism is analogous to the set-theoretical one. For the converse, let $e : (X, \mathcal{S}) \to$

†The reader interested in the connections between Pervin and quasi-uniform spaces may notice that, if $(X, \mathcal{E}_{\mathcal{S}})$ is the quasi-uniform space defined by $(X, \mathcal{S})$, then $\Omega_{\mathcal{S}}(X)$ is precisely the topology induced by $\mathcal{E}_{\mathcal{S}}$.

$(Y, \mathcal{T})$ be an epimorphism. We consider the two-point Pervin space $(Z, \mathcal{U}) := (\{0,1\}, \{\emptyset, \{0,1\}\})$, and we let $f_1, f_2 : Y \to Z$ be defined by $f_1(y) = 1$ if and only if $y \in e[X]$, and by $f_2(y) = 1$ for all $y \in Y$. Then, f_1 and f_2 induce morphisms of Pervin spaces $(Y, \mathcal{T}) \to (Z, \mathcal{U})$ satisfying $f_1 \circ e = f_2 \circ e$. Since e is an epimorphism, we must have $f_1 = f_2$. But this implies $e[X] = Y$, that is, e is a surjection. ∎

Proposition 3.2. *A map $m : (X, \mathcal{S}) \to (Y, \mathcal{T})$ of Pervin spaces is an extremal monomorphism if and only if the set map $m : X \to Y$ is injective and every $S \in \mathcal{S}$ is of the form $m^{-1}(T)$ for some $T \in \mathcal{T}$.*

Proof: Given an extremal monomorphism $m : (X, \mathcal{S}) \to (Y, \mathcal{T})$, we let $Z := m[X]$, and $\mathcal{U}$ be the bounded sublattice of $\mathcal{P}(Z)$ consisting of the subsets of the form $T \cap m[X]$ for some $T \in \mathcal{T}$, so that we have a Pervin space $(Z, \mathcal{U})$. Then, the direct image factorization of $m : X \to Y$ induces morphisms of Pervin spaces $e : (X, \mathcal{S}) \twoheadrightarrow (Z, \mathcal{U})$ and $g : (Z, \mathcal{U}) \hookrightarrow (Y, \mathcal{T})$ satisfying $m = g \circ e$. By Lemma 3.1, e is an epimorphism and thus, an isomorphism. In particular, the underlying set-theoretical map of e is a bijection, and since $m = g \circ e$, it follows that m is injective. It remains to show that $m^{-1}(\mathcal{T}) = \mathcal{S}$. For that, we let n be the inverse of e and we pick $S \in \mathcal{S}$. Since n is a morphism of Pervin spaces, we have that $n^{-1}[S] = e[S] = m[S]$ belongs to $\mathcal{U}$. Therefore, there exists $T \in \mathcal{T}$ such that $m[S] = T \cap m[X]$. But since m is injective, this implies $S = m^{-1}(T)$ as required.

Conversely, suppose that $m : (X, \mathcal{S}) \hookrightarrow (Y, \mathcal{T})$ is injective and satisfies $\mathcal{S} = m^{-1}(\mathcal{T})$. Consider a factorization $m = g \circ e$, where $e : (Z, \mathcal{U}) \to (Y, \mathcal{T})$ is an epimorphism. By Lemma 3.1, e is a surjection, and it is also injective by injectivity of m. We let n be the set-theoretical inverse of e and we show that n is a morphism of Pervin spaces. Given $S \in \mathcal{S}$, there is, by hypothesis, an element $T \in \mathcal{T}$ such that $S = m^{-1}(T)$. Then, since g is a morphism of Pervin spaces, we have that

$$n^{-1}(S) = (mn)^{-1}(T) = g^{-1}(T)$$

belongs to $\mathcal{U}$. Thus, n is an isomorphism in **Pervin**. ∎

We will say that $(Y, \mathcal{T})$ is a *subspace* of $(X, \mathcal{S})$ if there exists an extremal monomorphism $(Y, \mathcal{T}) \hookrightarrow (X, \mathcal{S})$. We remark that subspaces of $(X, \mathcal{S})$ are, up to isomorphism, in a bijection with the subsets of X. We also note that every extremal monomorphism $m : (X, \mathcal{S}) \hookrightarrow (Y, \mathcal{T})$ is regular. To see this, we consider the two-point Pervin space $(Z, \mathcal{U}) := (\{0,1\}, \{\emptyset, \{0,1\}\})$. Then,

m is the equalizer of the morphisms of Pervin spaces $f_1, f_2 : (Y, \mathcal{T}) \to (Z, \mathcal{U})$ defined by $f_1(y) = 1$ for every $y \in Y$ and by $f_2(y) = 1$ if and only if $y \in m[X]$.

Since every epimorphism which is also an extremal monomorphisms is an isomorphism (cf. [2, Proposition 4.3.7]), we obtain the following:

Corollary 3.3. *Isomorphisms in* **Pervin** *are the bijections $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ such that $f[S]$ belongs to $\mathcal{T}$, for every $S \in \mathcal{S}$.*

Proof: By Lemma 3.1 and Proposition 3.2, we know that a morphism of Pervin spaces $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is an isomorphism if and only if it is bijective and satisfies $\mathcal{S} = f^{-1}(\mathcal{T})$. Thus, the claim follows from having that, if f is a bijection, then $f[\mathcal{S}] \subseteq \mathcal{T} \iff \mathcal{S} \subseteq f^{-1}[\mathcal{T}]$. ■

We finally consider the full subcategory $\mathbf{Pervin}_{\mathrm{sym}}$ of **Pervin** whose objects are those Pervin spaces $(X, \mathcal{B})$ such that $\mathcal{B}$ a Boolean algebra. In the setting of Pervin quasi-uniform spaces, this is a relevant subcategory because it corresponds to the uniform spaces. In our paper, it will play a role when the pointfree version of this fact is discussed (cf. Section 6, namely Theorem 6.8).

An important property of $\mathbf{Pervin}_{\mathrm{sym}}$ is that of being coreflective in **Pervin** as we will see now (see Proposition 5.9 for the pointfree version of this result). We define a functor $\mathbf{Sym}_{\mathrm{Perv}} : \mathbf{Pervin} \to \mathbf{Pervin}_{\mathrm{sym}}$ as follows. For an object $(X, \mathcal{S})$, we let $\mathbf{Sym}_{\mathrm{Perv}}(X, \mathcal{S})$ be the Pervin space $(X, \overline{\mathcal{S}})$, where $\overline{\mathcal{S}}$ is the Boolean subalgebra of the powerset $\mathcal{P}(X)$ generated by the elements of $\mathcal{S}$. On morphisms, we simply map a function to itself. Notice that, since a map of Pervin spaces $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is such that the preimage of a complement of an element in $\mathcal{T}$ is the complement of an element in $\mathcal{S}$, this assignment is well-defined on morphisms. Clearly, $\mathbf{Sym}_{\mathrm{Perv}}$ is a functor. We show now that $\mathbf{Sym}_{\mathrm{Perv}}$ is right adjoint to the embedding $\mathbf{Pervin}_{\mathrm{sym}} \hookrightarrow \mathbf{Pervin}$.

Proposition 3.4. *The category of symmetric Pervin spaces is a full coreflective subcategory of that of Pervin spaces. More precisely, the functor $\mathbf{Sym}_{\mathrm{Perv}}$ is right adjoint to the embedding $\mathbf{Pervin}_{\mathrm{sym}} \hookrightarrow \mathbf{Pervin}$.*

Proof: We first notice that, for every Pervin space $(X, \mathcal{S})$, the identity function on X induces a morphism of Pervin spaces $\mathsf{id}_X : (X, \overline{\mathcal{S}}) \to (X, \mathcal{S})$. Thus, we only need to show that, for every morphism of Pervin spaces $f : (Y, \mathcal{B}) \to (X, \mathcal{S})$, with $\mathcal{B}$ a Boolean algebra, there is a unique Pervin map $\widetilde{f} : (Y, \mathcal{B}) \to (X, \overline{\mathcal{S}})$ satisfying $f = \mathsf{id}_X \circ \widetilde{f}$. Of course, this holds if and only if f also induces a morphism of Pervin spaces $f : (Y, \mathcal{B}) \to (X, \overline{\mathcal{S}})$. This is indeed the case because $\overline{\mathcal{S}}$ is generated, as a lattice, by $\mathcal{S}$ together with the complements in $\mathcal{P}(X)$ of the elements of $\mathcal{S}$ and, since $\mathcal{B}$ is a Boolean subalgebra of $\mathcal{P}(Y)$, it follows that $f^{-1}(S^c) = (f^{-1}(S))^c$ belongs to $\mathcal{B}$ for every $S \in \mathcal{S}$. ■

Recall that the *Skula topology* on a given topological space (X, τ) is the topology generated by τ together with the complements of its elements. Therefore, if $(X, \mathcal{S})$ is a Pervin space, then the topology $\Omega_{\overline{\mathcal{S}}}(X)$ on X defined by its symmetrization is precisely the Skula topology on the topological space $(X, \Omega_{\mathcal{S}}(X))$ defined by $(X, \mathcal{S})$. For that reason, we will say that $\Omega_{\overline{\mathcal{S}}}(X)$ is the *Skula topology* on X induced by $(X, \mathcal{S})$.

4. Frith frames as a pointfree version of Pervin spaces

4.1. The category of Frith frames and the Pervin - Frith dual adjunction. A *Frith frame* is a pair (L, S) where L is a frame and $S \subseteq L$ is a bounded sublattice such that all elements in L are joins of elements in S. A morphism $h : (L, S) \to (M, T)$ of Frith frames is a frame homomorphism $h : L \to M$ such that whenever $s \in S$ we have $h(s) \in T$. We denote the category of Frith frames and corresponding morphisms by **Frith**. By identifying a frame L with the Frith frame (L, L), we may see **Frm** as a full reflective subcategory of **Frith**. Indeed, we have the following:

Proposition 4.1. *The category of frames is a full reflective subcategory of that of Frith frames. More precisely, the forgetful functor* **Frith** $\to$ **Frm** *is left adjoint to the embedding* **Frm** $\hookrightarrow$ **Frith**.

Proof: Let U : **Frith** $\to$ **Frm** denote the forgetful functor and F : **Frm** $\hookrightarrow$ **Frith** the embedding identifying a frame L with the Frith frame (L, L). We only need to observe that for every Frith frame (L, S) and every frame M, every frame homomorphism $L \to M$ induces a morphism of Frith frames $(L, S) \to (M, M)$, and thus, we have a natural isomorphism $\mathbf{Frm}(U-, -) \cong \mathbf{Frith}(-, F-)$. ■

Next, we will see that Frith frames may indeed be considered the pointfree analogues of Pervin spaces, by showing that the classical adjunction Ω : $\mathbf{Top} \leftrightarrows \mathbf{Frm}^{\mathrm{op}}$: pt extends to an adjunction Ω : $\mathbf{Pervin} \leftrightarrows \mathbf{Frith}^{\mathrm{op}}$: pt between the categories of Pervin frames and of Frith frames, so that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, hook] \arrow[d, shift right, "\Omega"'] & \mathbf{Pervin} \arrow[d, shift right, "\Omega"'] \\ \mathbf{Frm}^{\mathrm{op}} \arrow[r, hook] \arrow[u, shift right, "\mathsf{pt}"'] & \mathbf{Frith}^{\mathrm{op}} \arrow[u, shift right, "\mathsf{pt}"'] \end{tikzcd}$$

Let us define the open set functor $\Omega : \mathbf{Pervin} \to \mathbf{Frith}$. Given a Pervin space $(X, \mathcal{S})$ we set $\Omega(X, \mathcal{S}) := (\Omega_{\mathcal{S}}(X), \mathcal{S})$, where $\Omega_{\mathcal{S}}(X)$ denotes the topology on X generated by $\mathcal{S}$ (recall Section 3). If $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is a morphism of Pervin spaces then taking preimages under f defines a morphism of Frith frames $\Omega(f) := f^{-1} : (\Omega_{\mathcal{T}}(Y), \mathcal{T}) \to (\Omega_{\mathcal{S}}(X), \mathcal{S})$. It is easily seen that this assignment yields a functor $\Omega : \mathbf{Pervin} \to \mathbf{Frith}^{\mathrm{op}}$. In turn, the spectrum functor $\mathsf{pt} : \mathbf{Frith}^{\mathrm{op}} \to \mathbf{Pervin}$ is defined on objects by $\mathsf{pt}(L, S) := (\mathsf{pt}(L), \widehat{S})$ for every Frith frame (L, S), where $\widehat{S} := \{\widehat{s} \mid s \in S\}$. Finally, if $h : (L, S) \to (M, T)$ is a morphism of Frith frames, then $\mathsf{pt}(h) := (- \circ h)$ is given by precomposition with h. The following lemma shows that $\mathsf{pt}(h)$ defines a morphism between the Pervin spaces $(\mathsf{pt}(M), \widehat{T})$ and $(\mathsf{pt}(L), \widehat{S})$.

Lemma 4.2. *Let $h : (L, S) \to (M, T)$ be a morphism of Frith frames. Then, for every $s \in S$, the equality $(\mathsf{pt}(h))^{-1}(\widehat{s}) = \widehat{h(s)}$ holds. In particular, $\mathsf{pt}(h)$ induces a morphism of Pervin spaces $\mathsf{pt}(h) : (\mathsf{pt}(M), \widehat{T}) \to (\mathsf{pt}(L), \widehat{S})$.*

Proof: Let $s \in S$ and $p : M \to \mathbf{2}$ be a point. The claim is a consequence of the following computation:

$$p \in (\mathsf{pt}(h))^{-1}(\widehat{s}) \iff \mathsf{pt}(h)(p) \in \widehat{s} \iff p(h(s)) = 1 \iff p \in \widehat{h(s)}. \quad \blacksquare$$

We may now prove that the functors just defined form an adjunction.

Proposition 4.3. *There is an adjunction $\Omega : \mathbf{Pervin} \leftrightarrows \mathbf{Frith}^{\mathrm{op}} : \mathsf{pt}$, which extends the classical adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{\mathrm{op}} : \mathsf{pt}$.*

Proof: By definition of $\Omega : \mathbf{Pervin} \leftrightarrows \mathbf{Frith}^{\mathrm{op}} : \mathsf{pt}$, we only need to show that the unit and counit of the adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{\mathrm{op}} : \mathsf{pt}$ define, respectively, a morphism $\varphi_L : (L, S) \to (\Omega_{\widehat{S}}(\mathsf{pt}(L)), \widehat{S})$ of Frith frames and a morphism $\psi_X : (X, \mathcal{S}) \to (\mathsf{pt}(\Omega_{\mathcal{S}}(X)), \widehat{\mathcal{S}})$ of Pervin spaces.

By definition of φ_L, we have $\varphi_L(s) = \widehat{s} \in \widehat{S}$, for every $s \in S$. So φ_L does define a morphism of Frith frames. It remains to show that $\psi_X^{-1}(\widehat{S}) \in \mathcal{S}$, for every $S \in \mathcal{S}$. That is indeed the case because, for every $x \in X$, we have

$$x \in \psi_X^{-1}(\widehat{S}) \iff \psi_X(x) \in \widehat{S} \iff \psi_X(x)(S) = 1 \iff x \in S. \quad \blacksquare$$

4.2. Compact, coherent, and zero-dimensional Frith frames. In this section we discuss the appropriate notions of compactness, coherence, and zero-dimensionality for Frith frames. Let L be a frame and $S \subseteq L$. We say that an element $a \in L$ is *S-compact* if whenever $a \leq \bigvee_{i \in I} s_i$ for some $\{s_i\}_{i \in I} \subseteq S$, there exists $F \subseteq I$ finite so that $a \leq \bigvee_{i \in F} s_i$, and we say that L is *S-compact* if its top element is S-compact. Clearly, every compact element of L is also S-compact. If we further assume that S is join-dense in L (which is the case when (L, S) is a Frith frame), then we also have the converse:

Lemma 4.4. *Let L be a frame, $S \subseteq L$ be a join-dense subset, and $a \in L$. Then, a is S-compact if and only if a is compact. In particular, L is S-compact if and only if it is compact.*

Proof: Let a be an S-compact element and suppose that $a \leq \bigvee_{i \in I} a_i$ for some $i \in I$. Since S is join-dense in L, we may write each a_i as a join of elements in S. Thus, since a is S-compact, there exists a finite subset $F \subseteq I$ satisfying $a \leq \bigvee_{i \in F} a_i$. ∎

We will say that a Frith frame (L, S) is *compact* if its frame component L is compact, and we say that (L, S) is *coherent* if S consists of compact elements of L. We call **CohFrith** the full subcategory of **Frith** determined by the coherent Frith frames. Since S is, by definition of Frith frame, a bounded sublattice of L, we have that every coherent Frith frame is compact. Also, every coherent frame L gives rise to a coherent Frith frame $(L, K(L))$. We now show that every coherent Frith frame is of this form.

Lemma 4.5. *Let (L, S) be a Frith frame. Then, the following are equivalent:*
(a) *S consists of compact elements;*
(b) *S is the set of all compact elements of L.*

In particular, (L, S) is a coherent Frith frame if and only if L is coherent and $S = K(L)$.

Proof: The implication $(b) \implies (a)$ is trivial. Conversely, suppose that $S \subseteq K(L)$, and let $a \in K(L)$. Since (L, S) is a Frith frame, we have that S is join-dense in L, and thus we may write $a = \bigvee_{i \in I} s_i$ for some subset $\{s_i\}_{i \in I} \subseteq S$. But compactness of a yields the existence of a finite subset $F \subseteq I$ satisfying $a = \bigvee_{i \in F} s_i$. Since S is closed under finite joins, it follows that a belongs to S. ∎

A consequence of Lemma 4.5 is that, if (L, S) and (M, T) are coherent Frith frames, then a frame homomorphism $h : L \to M$ induces a morphism between the corresponding Frith frames if and only if it is coherent. Thus, the categories **CohFrm** of coherent frames and **CohFrith** of coherent Frith frames are isomorphic. In particular, since **DLat** and **CohFrm** are equivalent categories, we also have and equivalence **DLat** $\cong$ **CohFrith**. More generally, we have the following analogue of Proposition 2.5:

Proposition 4.6. *There is an adjunction* $\mathrm{Idl}(-) \dashv U$*, where* U : **Frith** $\to$ **DLat** *is the forgetful functor and* $\mathrm{Idl}(-)$: **DLat** $\to$ **Frith** *is defined by*

$$\mathrm{Idl}(S) := (\mathrm{Idl}(S), S)$$

and

$$\mathrm{Idl}(h : S \to T) := (\mathrm{Idl}(h) : (\mathrm{Idl}(S), S) \to (\mathrm{Idl}(T), T)).$$

Moreover, the corestriction of $\mathrm{Idl}(-)$ *to* **CohFrith** *induces an equivalence of categories whose inverse is the suitable restriction of* U*.*

Proof: It is clear that $\mathrm{Idl}(-)$ is a well-defined functor. The fact that $\mathrm{Idl}(-)$ is left adjoint to U follows from the existence of a natural isomorphism **Frith**$(\mathrm{Idl}(-), -) \cong$ **DLat**$(-, U-)$, which is a straightforward consequence of Proposition 2.5.

Now, again by Proposition 2.5, the functors $\mathrm{Idl}(-)$: **DLat** $\to$ **CohFrm** and $K(-)$: **CohFrm** $\to$ **DLat** are mutually inverse, up to natural isomorphism. Composing these with the isomorphism **CohFrm** $\equiv$ **CohFrith**, we obtain the equivalence described in the last statement. ∎

Just like for frames, we also have in this setting that U : **CohFrm** $\to$ **DLat** is both a left and a right adjoint of $\mathrm{Idl}(-)$: **DLat** $\to$ **CohFrm**. Therefore, we have the following version of Proposition 2.7:

Proposition 4.7. *The category* **CohFrith** *is a full coreflective subcategory of* **Frith**. *The coreflector is the functor* $\mathrm{Idl}(-) \circ U$: **Frith** $\to$ **CohFrith**.

On the other hand, unlike what happens for the frame ideal completion $\mathrm{Idl}(-)$: **Frm** $\to$ **CohFrm**, the functor $\mathrm{Idl}(-) \circ U$: **Frith** $\to$ **CohFrm** is idempotent. Indeed, that may be seen as a consequence of having a full embedding **CohFrm** $\hookrightarrow$ **Frm**:

Proposition 4.8 ([2, Proposition 3.4.1])**.** *Let* $F : \mathcal{D} \to \mathcal{C}$ *and* $G : \mathcal{C} \to \mathcal{D}$ *be two functors, and suppose that* F *is the left adjoint of* G*. Then,* F *is full and faithful if and only if the unit of* $F \dashv G$ *is a natural isomorphism.*

We finally define what is a zero-dimensional Frith frame. First observe that, if $S \subseteq B(L)$ and (L, S) is a Frith frame, then the frame L is zero-dimensional. Thus, we will say that (L, S) is *zero-dimensional* provided S consists of complemented elements, and we have that L is a zero-dimensional frame if and only if $(L, B(L))$ is a zero-dimensional Frith frame. Let us analyze the relationship between compactness, coherence, and zero-dimensionality of Frith frames. If (L, S) is compact and zero-dimensional then, L is compact and zero-dimensional, too, and, by Proposition 2.8, we have $B(L) = K(L)$. Therefore, S consists of compact elements and thus, (L, S) is coherent. Hence, we have the following analogue of Proposition 2.8:

Lemma 4.9. *Let (L, S) be a Frith frame. If (L, S) is compact and zero-dimensional, then it is coherent.*

4.3. Limits and colimits. In this section we show that the category of Frith frames has all (co)products and (co)equalizers, and provide their description. In particular, it follows that **Frith** is a complete and cocomplete category.

We start by computing the products and the equalizers. By Proposition 4.6, the forgetful functor U : **Frith** $\to$ **DLat** is a right adjoint. Since right adjoints preserve limits, we automatically know how to compute the lattice component of every existing limit in **Frith**.

Let $\{(L_i, S_i)\}_{i\in I}$ be a family of Frith frames. As just mentioned, if the product $\prod_{i\in I}(L_i, S_i)$ exists in the category of Frith frames, then it is of the form $(L, \prod_{i\in I} S_i)$, for some frame L. Moreover, each of the product maps $\pi_i : (L, \prod_{i\in I} S_i) \to (L_i, S_i)$ is such that its restriction to $\prod_{i\in I} S_i$ is the i-th projection to S_i. Note that we may see $\prod_{i\in I} S_i$ as a sublattice of $\prod_{i\in I} L_i$. The natural candidate for L is then the subframe of $\prod_{i\in I} L_i$ generated by $\prod_{i\in I} S_i$. Since each (L_i, S_i) is a Frith frame, this is the whole frame $\prod_{i\in I} L_i$ and π_i defined on $\prod_{i\in I} L_i$ is simply the i-th projection to L_i. We show that this is indeed the product of the family $\{(L_i, S_i)\}_{i\in I}$:

Proposition 4.10. *Let $\{(L_i, S_i)\}_{i\in I}$ be a family of Frith frames. Then, the product $\prod_{i\in I}(L_i, S_i)$ in the category of Frith frames is $(\prod_{i\in I} L_i, \prod_{i\in I} S_i)$, and the product map $\pi_i : (\prod_{i\in I} L_i, \prod_{i\in I} S_i) \to (L_i, S_i)$ is the i-th projection.*

Proof: Let $\{h_i : (M, T) \to (L_i, S_i)\}_{i\in I}$ be a collection of morphisms of Frith frames, and $h : M \to \prod_{i\in I} L_i$ be the unique frame homomorphism such that $\pi_i \circ h = h_i$, for every $i \in I$, and whose existence is guaranteed by the universal property of products in **Frm**. We only need to check that this is a map of

Frith frames. Indeed, for every $t \in T$ and $i \in I$, we have $h_i(t) \in S_i$ and so, the tuple $h(t) = (h_i(t))_{i\in I}$ belongs to $\prod_{i\in I} S_i$. ■

We now let $h_1, h_2 : (L, S) \to (M, T)$ be a pair of morphisms of Frith frames. Following a similar reasoning, we know that, if the equalizer of h_1 and h_2 exists, then it is of the form $e : (K, S_{h_1=h_2}) \to (L, S)$, where $S_{h_1=h_2} := \{s \in S \mid h_1(s) = h_2(s)\}$ is the equalizer of h_1, h_2 in **DLat**, and e restricted to $S_{h_1=h_2}$ is the inclusion map. Thus, we take for K the subframe of L generated by $S_{h_1=h_2}$, for e the inclusion map, and we show that the morphism of Frith frames $e : (K, S_{h_1=h_2}) \to (L, S)$ satisfies the universal property of equalizers in **Frith**.

Proposition 4.11. *Given two morphisms $h_1, h_2 : (L, S) \rightrightarrows (M, T)$ in the category of Frith frames, let $S_{h_1=h_2}$ denote the sublattice $\{s \in S \mid h_1(s) = h_2(s)\}$ of S, and K be the subframe of L generated by $S_{h_1=h_2}$. Then, the equalizer of h_1 and h_2 is the subframe inclusion $e : (K, S_{h_1=h_2}) \hookrightarrow (L, S)$.*

Proof: Since $S_{h_1=h_2}$ is, by definition, join-dense in K, we have $h_1 \circ e = h_2 \circ e$. If $e' : (L', S') \to (L, S)$ is a morphism of Frith frames satisfying $h_1 \circ e' = h_2 \circ e'$, then we must have $e'[S'] \subseteq S_{h_1=h_2}$, and since L' is generated, as a frame, by S', we also have $e'[L] \subseteq K$. Therefore, the co-restriction of e' to K is the unique morphism $u : (L', S') \to (K, S_{h_1=h_2})$ such that $e' = e \circ u$. ■

In order to compute coproducts and coequalizers in **Frith**, we follow a similar strategy. By Proposition 4.1, the forgetful functor **Frith** $\to$ **Frm** is a left adjoint. Thus, it preserves existing colimits.

Let $\{(L_i, S_i)\}_{i\in I}$ be a family of Frith frames. If its coproduct exists, then it has to be a Frith frame of the form $(\bigoplus_{i\in I} L_i, S)$, for some join-dense sublattice S of $\bigoplus_{i\in I} L_i$, and the coproduct maps must be given by the coproduct injections $\iota_i : (L_i, S_i) \to (\bigoplus_{i\in I} L_i, S)$ in **Frm**. In particular, S must contain the sublattice $\iota_i[S_i]$, for every $i \in I$. We have the following:

Proposition 4.12. *Let $\{(L_i, S_i)\}_{i\in I}$ be a family of Frith frames, and $\{\iota_i : L_i \hookrightarrow \bigoplus_{i\in I} L_i\}_{i\in I}$ be the coproduct injections in* **Frm**. *The coproduct $\bigoplus_{i\in I}(L_i, S_i)$ of Frith frames in* **Frith** *is $(\bigoplus_{i\in I} L_i, S)$, where S is the sublattice of the coproduct $\bigoplus_{i\in I} L_i$ generated by $\bigcup_i \iota_i[S_i]$. Moreover, the i-th coproduct map is the morphism of Frith frames $\iota_i : (L_i, S_i) \to (\bigoplus_{i\in I} L_i, S)$ defined by ι_i.*

Proof: We first note that $(\bigoplus_{i\in I} L_i, S)$ is a Frith frame. Indeed, this is because $\bigoplus_{i\in I} L_i$ is generated, as a frame, by $\bigcup_{i\in I} \iota_i[L_i]$, and each $\iota_i[L_i]$ is generated, as a frame, by $\iota_i[S_i]$. Suppose that we have a collection of maps of Frith frames $h_i : (L_i, S_i) \to (M, T)$. We let $h : \bigoplus_{i\in I} L_i \to M$ be the unique frame homomorphism such that $h \circ \iota_i = h_i$, for every $i \in I$, as given by the universal property of coproducts in **Frm**. It suffices to show that h defines a morphism of Frith frames $h : (\bigoplus_{i\in I} L_i, S) \to (M, T)$. But this is a straightforward consequence of having $h \circ \iota_i[S_i] = h_i[S_i] \subseteq T$, for every $i \in I$. ■

Finally, it remains to compute the coequalizers in **Frith**. Given a pair of parallel morphisms $h_1, h_2 : (L, S) \to (M, T)$, we let $q : M \twoheadrightarrow K$ be the coequalizer in **Frm** of the frame homomorphisms h_1 and h_2. Explicitly, K is the quotient of M by the congruence generated by the subframe $\{(h_1(a), h_2(a)) \mid a \in L\}$ of $M \times M$. Once again, we know that, if the coequalizer of h_1, h_2 in **Frith** exists, then it is the morphism $q : (M, T) \twoheadrightarrow (K, R)$ induced by q, for a suitable sublattice R of K. It is easy to see that, if we take $R = q[T]$, then (K, R) is a Frith frame and q is a morphism of Frith frames that satisfies the universal property of coequalizers in **Frith**. Therefore, we have the following:

Proposition 4.13. *Let $h_1, h_2 : (L, S) \to (M, T)$ be two morphisms of Frith frames. Then, their coequalizer is $q : (M, T) \to (K, R)$, where $q : M \twoheadrightarrow K$ is the coequalizer of the frame homomorphisms h_1, h_2 in* **Frm** *and $R = q[T]$.*

As a consequence of Propositions 4.10, 4.11, 4.12, and 4.13, we have:

Corollary 4.14. *The category* **Frith** *is complete and cocomplete.*

4.4. Special morphisms. This section is devoted to the study of some special morphisms in the category of Frith frames. More precisely, we will start by characterizing the monomorphisms, the extremal epimorphisms, and the isomorphisms. In the setting of frames, extremal epimorphisms are relevant because they are the pointfree notion of *subspace embedding*. We will show a Pervin-Frith analogue of this. We will also see that, unlike what happens for frames, not every extremal epimorphism is regular.

The proof of the first result is analogous to the usual proof for posets.

Lemma 4.15. *A morphism $m : (L, S) \to (M, T)$ of Frith frames is a monomorphism if and only if the map $m : L \to M$ is injective.*

Proposition 4.16. *A morphism $e : (L, S) \to (M, T)$ of Frith frames is an extremal epimorphism if and only if it satisfies $e[S] = T$. In particular, all extremal epimorphisms are surjective.*

Proof: Let $e : (L, S) \to (M, T)$ be a morphism satisfying $e[S] = T$ and $m : (K, R) \hookrightarrow (M, T)$ be a monomorphism such that $e = m \circ g$ for some other morphism $g : (L, S) \to (K, R)$. In particular, we have

$$T = e[S] = m \circ g[S] \subseteq m[R].$$

Since M is generated by T, it follows that $m : K \to M$ is surjective and thus, a frame isomorphism. Let $f : M \to K$ be its inverse. We claim that f induces a morphism of Frith frames $f : (M, T) \to (K, R)$. Indeed, that is a consequence of having

$$f[T] = f \circ e[S] = g[S] \subseteq R.$$

Clearly, as morphisms of Frith frames, m and f are also mutually inverse. Therefore, m is an isomorphism. For the converse, suppose that $e : (L, S) \to (M, T)$ is an extremal epimorphism. Then, $(e[L], e[S])$ is a Frith frame to which e corestricts and, by Lemma 4.15, we have a monomorphism $m : (e[L], e[S]) \hookrightarrow (M, T)$. Thus, m has to be an isomorphism, and so the map e satisfies $e[S] = T$. Finally, since T generates M, the map e is a frame surjection. ∎

We will say that (M, T) is a *quotient* of (L, S) if there is an extremal epimorphism $e : (L, S) \twoheadrightarrow (M, T)$. Notice that quotients of (L, S) are, up to isomorphism, in a one-one correspondence with the congruences on L. The characterization of Proposition 4.16, together with that of the extremal monomorphisms in **Pervin** made in Proposition 3.2, allows us to conclude that the extremal epimorphisms in **Frith** are indeed the pointfree version of *embeddings of Pervin spaces*. We will say that a Pervin space is T_0 provided so is the topological space it defines.

Corollary 4.17. *Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be Pervin spaces. If $m : (X, \mathcal{S}) \hookrightarrow (Y, \mathcal{T})$ is an extremal monomorphism in* **Pervin** *then $\Omega(m) : (\Omega_{\mathcal{T}}(Y), \mathcal{T}) \twoheadrightarrow (\Omega_{\mathcal{S}}(X), \mathcal{S})$ is an extremal epimorphism in* **Frith**. *The converse holds provided $(X, \mathcal{S})$ is T_0.*

Proof: By Propositions 3.2 and 4.16, the only non-trivial part is to show that if $\Omega(m)$ is an extremal epimorphism then m is injective. Let $x_1, x_2 \in X$ be two distinct points. Since $(X, \mathcal{S})$ is T_0, there exists some $U \in \Omega_{\mathcal{S}}(X)$ such that $x_1 \in U$ and $x_2 \notin U$. By Proposition 4.16, there exists some $V \in \Omega_{\mathcal{T}}(Y)$ such that $\Omega(m)(V) = m^{-1}(V) = U$. But then, $m(x_1) \in V$ and $m(x_2) \notin V$ and thus, $m(x_1) \neq m(x_2)$ as we intended to show. ■

Since every morphism which is both a monomorphism and an extremal epimorphism is an isomorphism (cf. [2, Proposition 4.3.7]), we may also conclude the following:

Corollary 4.18. *Let $h : (L, S) \to (M, T)$ be a morphism of Frith frames. Then, h is an isomorphism if and only if h is one-one and satisfies $h[S] = T$.*

We finish this section by exploring the relationship between extremal and regular epimorphisms. Recall that every regular epimorphism is extremal (cf. [2, Proposition 4.3.3]). In order to state the conditions under which the converse holds, we will need the following notion of *Frith congruence*:

Definition 4.19. *A* Frith congruence *on a Frith frame (L, S) is a frame congruence on L generated by a relation $\rho \subseteq S \times S$.*

Lemma 4.20. *Let (L, S) be a Frith frame. Then, a congruence on L is a Frith congruence if and only if it is generated by its restriction to $S \times S$.*

Proof: The backwards direction of the implication is trivial. For the converse, suppose that θ is a Frith congruence. Let $\rho \subseteq S \times S$ be such that $\overline{\rho} = \theta$. We have that $\theta \cap (S \times S) \subseteq \theta$ and since closure operators are monotone and idempotent this implies $\overline{\theta \cap (S \times S)} \subseteq \theta$. For the reverse inclusion, since θ extends ρ we have $\rho \subseteq \theta \cap (S \times S)$, and since closures are monotone we have $\theta = \overline{\rho} \subseteq \overline{\theta \cap (S \times S)}$. ■

In the example below we show that not every congruence on L is a Frith congruence.

Example 4.21. Let L be the frame whose underlying poset is the ordinal $\omega + 2$, and let S be the sublattice $L \setminus \{\omega\}$ of L. Since $\omega = \bigvee \{n \mid n \in \omega\}$ is the join of elements in S, the pair (L, S) is a Frith frame. Then, the open congruence Δ_ω is not a Frith congruence. Indeed, since $\Delta_\omega = \{(x, y) \in L \times L \mid x \wedge \omega = y \wedge \omega\} = \{(x, y) \in L \times L \mid x = y \text{ or } x, y \geq \omega\}$ does not identify any two distinct elements of S, if it were a Frith congruence and thus generated by its restriction to $S \times S$, then it had to be the identity. But that is not the case as it contains, for instance, the element $(\omega, \omega + 1)$.

In fact, we have the following:

Proposition 4.22. *Let (L, S) be a Frith frame and $\theta \subseteq L \times L$ be a congruence on L. Then, θ is a Frith congruence if and only if it belongs to $\mathcal{C}_S L$.*

Proof: Note that the congruence generated by a relation $\rho \subseteq L \times L$ is $\bigvee\{\nabla_a \wedge \Delta_b \mid (a, b) \in \rho\}$. Thus, by definition of Frith congruence, it suffices to observe that the fact that ∇ preserves arbitrary joins and S is join-dense in L implies that $\mathcal{C}_S L$ is generated by the set $\{\nabla_s,\ \Delta_s \mid s \in S\}$. ■

We may now characterize those extremal epimorphisms that are regular.

Proposition 4.23. *Let $q : (L, S) \twoheadrightarrow (M, T)$ be an extremal epimorphism of Frith frames. Then, q is a regular epimorphism if and only if* $\ker(q)$ *is a Frith congruence.*

Proof: Suppose that $q : (L, S) \twoheadrightarrow (M, T)$ is a regular epimorphism, let us say that q is the coequalizer of $h_1, h_2 : (K, R) \to (L, S)$. It follows from Proposition 4.13 that $\ker(q)$ is the congruence generated by the subframe $\{(h_1(a), h_2(a)) \mid a \in K\}$ of $L \times L$. Since R is join dense in K, this subframe is generated by $\{(h_1(r), h_2(r)) \mid r \in R\}$ and thus, $\ker(q)$ is generated by $\{(h_1(r), h_2(r)) \mid r \in R\}$. Since h_1 and h_2 are morphisms of Frith frames, the set $\{(h_1(r), h_2(r)) \mid r \in R\}$ is a relation on S. Thus, $\ker(q)$ is a Frith congruence.

Conversely, suppose that $\ker(q)$ is a Frith congruence, and let K be the subframe of $L \times L$ generated by $R := \ker(q) \cap (S \times S)$. Clearly, the pair (K, R) is a Frith frame, and the two projection maps $K \to L$ induce morphisms of Frith frames $\pi_1, \pi_2 : (K, R) \to (L, S)$. We claim that q is the coequalizer of π_1 and π_2. By Proposition 4.13 and using the fact that q is an extremal epimorphism, it suffices to show that $\ker(q) = \overline{\rho}$, where $\rho := \{(\pi_1(x, y), \pi_2(x, y)) \mid (x, y) \in K\}$. We observe that $\rho = K$ and, since R generates K as a frame, it follows that ρ and R generate the same congruence on L. Finally, since $\ker(q)$ is a Frith congruence, by Lemma 4.20, we may then conclude that

$$\ker(q) = \overline{\ker(q) \cap (S \times S)} = \overline{R} = \overline{\rho},$$

as required. ■

A *Frith quotient* of (L, S) is then a Frith frame (M, T) for which there is a regular epimorphism $q : (L, S) \twoheadrightarrow (M, T)$. By Propositions 4.22 and 4.23, there is a bijection between Frith quotients of (L, S), up to isomorphism, and the congruences of $\mathcal{C}_S L$. We finally show that an analogue of Corollary 4.17

does not hold with respect to regular morphisms, even if we restrict to sober spaces.

Example 4.24. Consider the set $X := \omega + 1$, equipped with the lattice $\mathcal{S} \subseteq \mathcal{P}(X)$ consisting of the downsets of X. Since the topology $\Omega_{\mathcal{S}}(X)$ on X has as open subsets the elements of $\mathcal{S}$ together with ω, we have $\Omega(X, \mathcal{S}) = (\omega{+}2, (\omega{+}2)\backslash\{\omega\})$. We let $(Y, \mathcal{T})$ be the Pervin subspace of $(X, \mathcal{S})$ defined by the subset $Y := \omega \subseteq X$ and we let $m : (Y, \mathcal{T}) \hookrightarrow (X, \mathcal{S})$ be the corresponding subspace embedding. Then,

$$\begin{aligned}\ker(\Omega(m)) &= \{(U_1, U_2) \mid U_1, U_2 \in \Omega_{\mathcal{S}}(X),\ m^{-1}(U_1) = m^{-1}(U_2)\} \\ &= \{(x, y) \mid x, y \in \omega + 2,\ x \wedge \omega = y \wedge \omega\} = \Delta_\omega\end{aligned}$$

is the congruence described in Example 4.21, which is not a Frith congruence. Therefore, by Proposition 4.23, $\Omega(m)$ is not a regular epimorphism. Finally, we argue that the topological space $(X, \Omega_{\mathcal{S}}(X))$ defined by $(X, \mathcal{S})$ is sober. For a subset $Q \subseteq X$, we let $p_Q : \omega + 2 \to \mathbf{2}$ be the unique function satisfying $p_Q^{-1}(1) = Q$. Clearly, if p_Q is a point of $\omega{+}2$, that is, a frame homomorphism, then Q must be an upset. Also, it is not hard to verify that, among the upsets of $\omega{+}2$, all but $\{\omega, \omega{+}1\}$ give rise to a point ($p_{\{\omega,\omega+1\}}$ is not a point because $\omega = \bigvee_{n\in\omega} n$). Therefore, the points of $\omega + 2$ may be identified with the elements of $\omega + 1$, via the correspondence $p_{\uparrow n} \mapsto n$ and $p_{\{\omega+2\}} \mapsto \omega$. Under this correspondence the open subsets of $\mathsf{pt}(\omega + 2)$ are precisely the downsets of $\omega + 1$, that is, $(X, \Omega_{\mathcal{S}}(X))$ is isomorphic to the space $\mathsf{pt}(\omega + 2)$ and thus, it is sober.

4.5. Frith quotients and the T_D axiom for Pervin spaces. Recall that a topological space X is T_D if for every $x \in X$ there exists an open subset $U \subseteq X$ containing x such that $U\backslash\{x\}$ is open. In the classical topological setting, every subspace of X induces, via Ω, a quotient of $\Omega(X)$, and we have that X is T_D if and only if *different subspaces* of X induce *different quotients* of $\Omega(X)$. For Pervin spaces, this may be translated as follows: by Corollary 4.17, every subspace of a Pervin space $(X, \mathcal{S})$ induces, via Ω, a quotient of $(\Omega_{\mathcal{S}}(X), \mathcal{S})$, and we have that *different subspaces* of $(X, \mathcal{S})$ induce *different quotients* on $(\Omega_{\mathcal{S}}(X), \mathcal{S})$ if and only if the topology $\Omega_{\mathcal{S}}(X)$ on X is T_D. In this section we will state and prove a version of this result where *quotient* is replaced by *Frith quotient*. Actually, we will show a version of the following:

Proposition 4.25 ([16, Chapter I, Section 1]). *Let X be a topological space. The following are equivalent:*

(a) The space X is T_D.
(b) Different subspaces of X induce different quotients of $\Omega(X)$.
(c) For no $x \in X$ does the subspace inclusion $X \backslash \{x\} \hookrightarrow X$ induce an isomorphism $\Omega(X\backslash\{x\}) \cong \Omega(X)$.
(d) The Skula topology on X is discrete.

We have seen in the previous section that, unlike what happens for frames, the extremal epimorphisms in **Frith** do not coincide with the regular ones (cf. Proposition 4.23), and Example 4.24 shows that, in general, the functor Ω does not map subspaces of a Pervin space $(X, \mathcal{S})$ to Frith quotients of $(\Omega_{\mathcal{S}}(X), \mathcal{S})$. There is however a natural way to assign to each subspace of $(X, \mathcal{S})$ a Frith quotient on $(\Omega_{\mathcal{S}}(X), \mathcal{S})$: given a subset $Y \subseteq X$, we consider the Frith quotient defined by the Frith congruence θ_Y generated by $\ker(\Omega(m)) \cap (\mathcal{S} \times \mathcal{S}) = \{(S_1, S_2) \in \mathcal{S} \times \mathcal{S} \mid S_1 \cap Y = S_2 \cap Y\}$, where $m : (Y, \mathcal{T}) \hookrightarrow (X, \mathcal{S})$ is the subspace embedding determined by Y. Note that, since $\ker(\Omega(m))$ is a frame congruence on $\Omega_{\mathcal{S}}(X)$, we have $\theta_Y \cap (\mathcal{S} \times \mathcal{S}) = \ker(\Omega(m)) \cap (\mathcal{S} \times \mathcal{S})$. The following property will be useful later:

Lemma 4.26. *Let $(X, \mathcal{S})$ be a Pervin space and $x \in X$. If θ_X and $\theta_{X\backslash\{x\}}$ are distinct, then there are $S_1, S_2 \in \mathcal{S}$ such that $\{x\} = S_1 \backslash S_2$.*

Proof: Clearly, θ_X is the identity relation on $\Omega_{\mathcal{S}}(X)$. Therefore, since $\theta_{X\backslash\{x\}}$ is, by definition, a Frith congruence, there exists some $(S_1, S_2) \in \theta_{X\backslash\{x\}} \cap (\mathcal{S} \times \mathcal{S})$ such that $S_1 \neq S_2$, say $S_1 \nsubseteq S_2$ without loss of generality. By definition of $\theta_{X\backslash\{x\}}$, we have $S_1\backslash\{x\} = S_2\backslash\{x\}$. Thus, since $S_1 \nsubseteq S_2$, it follows that $\{x\} = S_1\backslash S_2$ as required. ■

We say that a Pervin space $(X, \mathcal{S})$ is *Pervin-T_D* if for every $x \in X$ there is some $S \in \mathcal{S}$ that contains x and such that $S\backslash\{x\} \in \mathcal{S}$. We first show that in a Pervin-T_D space $(X, \mathcal{S})$ any two different subspaces of $(X, \mathcal{S})$ induce different Frith quotients on $(\Omega_{\mathcal{S}}(X), \mathcal{S})$.

Lemma 4.27. *If $(X, \mathcal{S})$ is a Pervin-T_D space, then different subspaces of $(X, \mathcal{S})$ induce different Frith quotients on $(\Omega_{\mathcal{S}}(X), \mathcal{S})$.*

Proof: Let $(X, \mathcal{S})$ be a Pervin-T_D space, and let $Y, Z \subseteq X$ be two distinct subsets of X. We need to show that $\theta_Y \neq \theta_Z$. Without loss of generality, we may assume that there exists some $x \in Y\backslash Z$. Since $(X, \mathcal{S})$ is Pervin-T_D, we can take $S \in \mathcal{S}$ such that $x \in S$ and $S\backslash\{x\} \in \mathcal{S}$. Then, as x belongs to Y but not to Z, we have

$$S \cap Y \neq (S\backslash\{x\}) \cap Y \quad \text{and} \quad S \cap Z = (S\backslash\{x\}) \cap Z.$$

Finally, using that $(S, S\backslash\{x\}) \in \mathcal{S} \times \mathcal{S}$ and $\theta_Y \cap (\mathcal{S} \times \mathcal{S}) = \{(S_1, S_2) \in \mathcal{S} \times \mathcal{S} \mid S_1 \cap Y = S_2 \cap Y\}$, we may conclude that $(S, S\backslash\{x\}) \in \theta_Z \backslash \theta_Y$. Thus, $\theta_Y \neq \theta_Z$ as required. ■

We have now all the ingredients to show the main result of this section.

Proposition 4.28. *Let $(X, \mathcal{S})$ be a Pervin space. The following are equivalent.*

(a) *The space $(X, \mathcal{S})$ is Pervin-T_D.*
(b) *Different subspaces of $(X, \mathcal{S})$ induce different Frith quotients of $(\Omega_{\mathcal{S}}(X), \mathcal{S})$.*
(c) *For no $x \in X$ do we have that $\theta_{X\backslash\{x\}}$ is trivial.*
(d) *The Skula topology on X induced by $(X, \mathcal{S})$ is discrete.*

Proof: The fact that (a) implies (b) is the content of Lemma 4.27, that (b) implies (c) is trivial, and that (c) implies (d) follows easily from Lemma 4.26. It remains to show that (d) implies (a). Let $x \in X$. Since the Skula topology on X induced by $(X, \mathcal{S})$ is, by definition, generated by the elements of $\mathcal{S}$ and their complements, there exist $S_1, S_2 \in \mathcal{S}$ such that $\{x\} = S_1 \backslash S_2$. In turn, this implies that S_1 is the disjoint union of $\{x\}$ and $S_1 \cap S_2$. Therefore, S_1 is such that $x \in S_1$ and $S_1\backslash\{x\} = S_1 \cap S_2$ belongs to $\mathcal{S}$. This shows that $(X, \mathcal{S})$ is Pervin-T_D as intended. ■

5. Frith frames as quasi-uniform frames

In this section we show that the category of Frith frames may be seen as a coreflective full subcategory of the category of transitive and totally bounded quasi-uniform frames.

If K is a frame and $r \in K$, then we denote

$$E_r := (r \oplus 1) \vee (1 \oplus r^*).$$

Note that the set $(r \oplus 1) \cup (1 \oplus r^*)$ is already a C-ideal, thus it equals E_r. It is shown in [9, Proposition 5.2] that $E_r \circ E_r = E_r$, and in [9, Proposition 5.3] that E_r is an entourage if and only if r is complemented. Moreover, if r is complemented, then $\{r, r^*\}$ is a cover of K and, since $(r \oplus r) \vee (r^* \oplus r^*) \subseteq E_r$, it follows that E_r is a finite entourage.

For a subset $R \subseteq K$ of complemented elements, we denote $R^* := \{r^* \mid r \in R\}$, and we let $\mathcal{E}_R$ be the filter generated by $\{E_r \mid r \in R\}$. We start by establishing an important property of the relations $\lhd_1$ and $\lhd_2$ for the filter $\mathcal{E}_R$.

Lemma 5.1. *Let K be a frame and $R \subseteq K$ be a subset of complemented elements. For every $x, a \in K$,*

(a) if $x \lhd_1 a$, then there exists some $r \in \langle R \rangle_{\mathbf{DLat}}$ such that $x \leq r \leq a$,
(b) if $x \lhd_2 a$, then there exists some $r \in \langle R^* \rangle_{\mathbf{DLat}}$ such that $x \leq r \leq a$,

Proof: We will only prove (a), as the proof of (b) is similar. By definition, if $x \lhd_1 a$, then there are some $r_1, \ldots, r_n \in R$ such that $(\bigcap_{i=1}^n E_{r_i}) \circ (x \oplus x) \subseteq (a \oplus a)$. We let $[n]$ denote the set $\{1, \ldots, n\}$, and for every $P \subseteq [n]$ we set $r_P := \bigwedge_{i \in P} r_i$ and $\overline{r}_P := \bigwedge_{i \notin P} r_i^*$. Since each r_i is complemented, we have

$$1 = \bigwedge_{i=1}^{n} (r_i \vee r_i^*) = \bigvee \{r_P \wedge \overline{r}_P \mid P \subseteq [n]\}.$$

Therefore,

$$x = \bigvee \{r_P \wedge \overline{r}_P \wedge x \mid P \subseteq [n]\} \leq \bigvee \{r_P \mid P \subseteq [n], \overline{r}_P \wedge x \neq 0\}.$$

Thus, it suffices to show that the element $\bigvee \{r_P \mid P \subseteq [n], \overline{r}_P \wedge x \neq 0\}$, which belongs to the sublattice of K generated by R, is smaller than or equal to a. We let $P \subseteq [n]$ satisfy $\overline{r}_P \wedge x \neq 0$. Then, we have $(r_P, \overline{r}_P \wedge x) \in \bigcap_{i=1}^n E_{r_i}$ and $(\overline{r}_P \wedge x, x) \in (x \oplus x)$. Since $\overline{r}_P \wedge x \neq 0$, it follows that (r_P, x) belongs to $(\bigcap_{i=1}^n E_{r_i}) \circ (x \oplus x)$ which, by hypothesis, is contained in $(a \oplus a)$. In particular, it follows that $r_P \leq a$, as required. ∎

The following is a slight variation of [9, Theorem 5.5] and the proof in there may be easily adapted. For the sake of self-containment, we will use the previous lemma to provide an alternative proof.

Theorem 5.2. *Let K be a frame and $R \subseteq K$ be a subset of complemented elements. Then, the filter $\mathcal{E}_R$ generated by the set of entourages $\{E_r \mid r \in R\}$ is such that $\mathcal{L}_1(\mathcal{E}_R) = \langle R \rangle_{\mathbf{Frm}}$ and $\mathcal{L}_2(\mathcal{E}_R) = \langle R^* \rangle_{\mathbf{Frm}}$. In particular, if K is generated by $R \cup R^*$, then $\mathcal{E}_R$ is a transitive and totally bounded quasi-uniformity on K.*

Proof: We first show that $\mathcal{L}_1(\mathcal{E}_R) \subseteq \langle R \rangle_{\mathbf{Frm}}$. We fix an element $a \in \mathcal{L}_1(\mathcal{E}_R)$, and for each $x \in K$ satisfying $x \lhd_1 a$, we let $r_x \in \langle R \rangle_{\mathbf{DLat}}$ be such that

$x \leq r_x \leq a$, as given by Lemma 5.1(a). Then, we have

$$a = \bigvee\{x \in K \mid x \lhd_1 a\} \leq \bigvee\{r_x \mid x \in K,\, x \lhd_1 a\} \leq a.$$

Thus, $a = \bigvee\{r_x \mid x \in K,\, x \lhd_1 a\}$ is a frame combination of elements of R. Conversely, since $\mathcal{E}_R$ is a filter and hence, $\mathcal{L}_1(\mathcal{E}_R)$ is a frame, it suffices to show that $R \subseteq \mathcal{L}_1(\mathcal{E}_R)$. But that is a consequence of the inclusion $E_r \circ (r \oplus r) \subseteq (r \oplus r)$ which holds for every complemented element $r \in K$. ∎

Let L be a frame. Then, for each $a \in L$, the congruence ∇_a is complemented in $\mathcal{C}L$, and $\mathcal{C}L$ is generated, as a frame, by these congruences together with their complements. Thus, by Theorem 5.2, the set $\{E_{\nabla_a} \mid a \in L\}$ is a subbasis for a transitive and totally bounded quasi-uniformity $\mathcal{E}_L$ on $\mathcal{C}L$. This is called the *Frith quasi-uniformity*, and it is the pointfree counterpart of the *Pervin quasi-uniformity*, in the sense that, for every frame L, the frame $\mathcal{L}_1(\mathcal{E}_L)$ is isomorphic to L. More generally, for every Frith frame (L, S), the set $\{E_{\nabla_s} \mid s \in S\}$ is a subbasis for a transitive and totally bounded quasi-uniformity $\mathcal{E}_S$ on $\mathcal{C}_S L$. We will now see that the assignment $(L, S) \mapsto (\mathcal{C}_S L, \mathcal{E}_S)$ extends to a full embedding $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$, where $\mathbf{QUniFrm}_{\text{trans, tot bd}}$ denotes the full subcategory of $\mathbf{QUniFrm}$ consisting of those quasi-uniform frames that are transitive and totally bounded.

Before proceeding, we show a couple of technical results concerning C-ideals of the form E_r.

Lemma 5.3. *Let $h : K_1 \to K_2$ be a frame homomorphism. Then, for every $r \in K_1$, we have $(h \oplus h)(E_r) \subseteq E_{h(r)}$. The converse inclusion holds if r is complemented.*

Proof: Given $r \in K_1$, we may compute:

$$\begin{aligned}(h \oplus h)(E_r) &\overset{(2)}{=} \bigvee\{h(x) \oplus h(y) \mid x \leq r \text{ or } y \leq r^*\} = (h(r) \oplus 1) \vee (1 \oplus h(r^*)) \\ &\subseteq (h(r) \oplus 1) \vee (1 \oplus h(r)^*) = E_{h(r)},\end{aligned}$$

where the only inclusion follows from the inequality $h(r^*) \leq h(r)^*$ (recall (1)). Now, if r is complemented, we have $h(r^*) = h(r)^*$, and the above inclusion becomes an equality. ∎

Lemma 5.4. *Let K be a frame and $r, r_1, \ldots, r_n \in K$ be complemented elements. If $\bigcap_{i=1}^n E_{r_i} \subseteq E_r$, then r is a lattice combination of the elements of $\{r_1, \ldots, r_n\}$.*

Proof: As before, we use $[n]$ to denote the set $\{1, \ldots, n\}$ and, for every $P \subseteq [n]$ we let $r_P := \bigwedge_{i \in P} r_i$ and $\overline{r}_P := \bigwedge_{i \notin P} r_i^*$. Since, for every $P \subseteq [n]$, we have $(r_P, \overline{r}_P) \in \bigcap_{i=1}^n E_{r_i}$, it follows from the hypothesis that $(r_P, \overline{r}_P) \in E_r$, that is, $r_P \leq r$ or $\overline{r}_P \leq r^*$. Thus, by (1), for every $P \subseteq [n]$, we have

$$r_P \leq r \qquad \text{or} \qquad \overline{r}_P \wedge r = 0. \tag{3}$$

On the other hand, since each r_i is complemented, we have

$$r = r \wedge \bigwedge_{i=1}^{n} (r_i \vee r_i^*) = r \wedge \bigvee \{r_P \wedge \overline{r}_P \mid P \subseteq [n]\} = \bigvee \{r \wedge r_P \wedge \overline{r}_P \mid P \subseteq [n]\}.$$

Using (3), we may then conclude that $r = \bigvee \{r_P \mid P \subseteq [n],\, r \wedge \overline{r}_P \neq 0\}$, that is, r is a lattice combination of the elements of $\{r_1, \ldots, r_n\}$. ∎

Let us now define the functor E on the morphisms. If $h : (L, S) \to (M, T)$ is a morphism of Frith frames then, by Corollary 2.4, h uniquely extends to a frame homomorphism $\overline{h} : \mathcal{C}_S L \to \mathcal{C}_T M$. The fact that $\overline{h}$ defines a quasi-uniform homomorphism $\overline{h} : (\mathcal{C}_S L, \mathcal{E}_S) \to (\mathcal{C}_T M, \mathcal{E}_T)$ is a consequence of the following more general result:

Proposition 5.5. *Let K, N be two frames, and $R \subseteq K$ and $U \subseteq N$ be sublattices of complemented elements. Let also $g : K \to N$ be a frame homomorphism. Then,*

(*a*) *$g[R] \subseteq U$ if and only if $(g \oplus g)[\mathcal{E}_R] \subseteq \mathcal{E}_U$;*
(*b*) *$U \subseteq g[R]$ if and only if $\mathcal{E}_U$ is contained in the filter generated by $(g \oplus g)[\mathcal{E}_R]$.*

Proof: We start by proving (a). Suppose that $g[R] \subseteq U$ and let $r \in R$. Since r is complemented, by Lemma 5.3, we have $(g \oplus g)(E_r) = E_{g(r)}$, which belongs to $\mathcal{E}_U$. Conversely, given $r \in R$, we have $E_{g(r)} = (g \oplus g)(E_r) \in \mathcal{E}_U$ and so, there exist $u_1, \ldots, u_n \in U$ such that $E_{g(r)} \supseteq E_{u_1} \cap \cdots \cap E_{u_n}$. By Lemma 5.4, and since U is a sublattice of N, it follows that $g(r) \in U$.

For proving (b), we first assume that $U \subseteq g[R]$. Then, given $u \in U$, there exists some $r \in R$ such that $g(r) = u$ and, for such an r, we have $E_u = E_{g(r)} = (g \oplus g)(E_r)$. Thus, $\mathcal{E}_U$ is contained in the filter generated by $(g \oplus g)[\mathcal{E}_R]$. Conversely, if $\mathcal{E}_U$ is contained in the filter generated by $(g \oplus g)[\mathcal{E}_R]$ then, for every $u \in U$, there are some $r_1, \ldots, r_n \in R$ satisfying

$$E_u \supseteq (g \oplus g)(E_{r_1}) \cap \cdots \cap (g \oplus g)(E_{r_n}).$$

Then, by Lemma 5.3 we have

$$E_u \supseteq E_{g(r_1)} \cap \cdots \cap E_{g(r_n)},$$

and by Lemma 5.4, u is a lattice combination of the elements of $\{g(r_1), \ldots, g(r_n)\}$ and thus, it belongs to $g[R]$ as required. ■

As a consequence, we have an embedding $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$ defined by $E(L, S) = (\mathcal{C}_S L,\ \mathcal{E}_S)$ and $E(h) = \overline{h}$. In fact, Proposition 5.5 also implies that E is full.

Proposition 5.6. *There is a full embedding* $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$.

Our next goal is to show that the embedding $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$ is a coreflection (cf. Theorem 5.12). We will need the following technical result:

Lemma 5.7. *Let K be a frame, and $R_1, R_2 \subseteq K$ be subsets of complemented elements. Then, $\mathcal{E}_{R_1} = \mathcal{E}_{R_2}$ if and only if $\langle R_1 \rangle_{\mathbf{DLat}} = \langle R_2 \rangle_{\mathbf{DLat}}$.*

Proof: For $i = 1, 2$, we let R_i' denote the lattice generated by R_i. We first argue that $\mathcal{E}_{R_i} = \mathcal{E}_{R_i'}$, which implies that $\mathcal{E}_{R_1} = \mathcal{E}_{R_2}$ whenever R_1 and R_2 generate the same sublattice of K. Since $R_i \subseteq R_i'$, we clearly have $\mathcal{E}_{R_i} \subseteq \mathcal{E}_{R_i'}$. For the reverse inclusion, it suffices to observe that for every $r, r' \in R_i$, the entourage $E_r \cap E_{r'}$ is contained both in $E_{r \wedge r'}$ and in $E_{r \vee r'}$. Let us prove the converse implication. We suppose that $\mathcal{E}_{R_1} = \mathcal{E}_{R_2}$ and we let $r \in R_1$. Then, $E_r \in \mathcal{E}_{R_1}$ and so, there are some $r_1, \ldots, r_n \in R_2$ such that $E_r \supseteq E_{r_1} \cap \cdots \cap E_{r_n}$. By Lemma 5.4, this yields $r \in \langle R_2 \rangle_{\mathbf{DLat}}$ and thus, $\langle R_1 \rangle_{\mathbf{DLat}} \subseteq \langle R_2 \rangle_{\mathbf{DLat}}$. By symmetry, we have $\langle R_1 \rangle_{\mathbf{DLat}} = \langle R_2 \rangle_{\mathbf{DLat}}$ as required. ■

We now show that every transitive and totally bounded quasi-uniformity $\mathcal{E}$ on a frame K is of the form $\mathcal{E}_R$ for a suitable bounded sublattice R of K. We introduce the following notation:

Definition 5.8. *Given a transitive and totally bounded quasi-uniform frame $(K, \mathcal{E})$, we will denote by $R_{(K,\mathcal{E})}$ the set of all complemented elements of K such that $E_r \in \mathcal{E}$.*

The proof of the next result is inspired by the proof of an unpublished result for Pervin spaces, which is due to Gehrke, Grigorieff, and Pin.

Proposition 5.9. *Let $(K, \mathcal{E})$ be a transitive and totally bounded quasi-uniform frame. Then, $R_{(K,\mathcal{E})}$ is a sublattice of K satisfying $\mathcal{E} = \mathcal{E}_{R_{(K,\mathcal{E})}}$.*

Proof: Let $(K, \mathcal{E})$ be a transitive and totally bounded quasi-uniform frame. By Lemma 5.7, we have that $R_{(K,\mathcal{E})}$ is a sublattice of K, and by definition of $R_{(K,\mathcal{E})}$, we also have $\mathcal{E}_{R_{(K,\mathcal{E})}} \subseteq \mathcal{E}$. Since $\mathcal{E}$ is a transitive quasi-uniformity, in order to show the converse inclusion, it suffices to show that $\mathcal{E}_{R_{(K,\mathcal{E})}}$ contains every transitive entourage of $\mathcal{E}$.

Let us fix a transitive entourage $E \in \mathcal{E}$. Since $\mathcal{E}$ is totally bounded, there exists a finite cover C of K such that $\bigvee_{x \in C} x \oplus x \subseteq E$. Moreover, since E is transitive, we may assume without loss of generality that C is a partition. Indeed, suppose otherwise, say $x \wedge y \neq 0$ for some distinct $x, y \in C$. Then, since $(x, x), (y, y) \in E$ implies, by (J.1), that $(x, x \wedge y), (x \wedge y, y) \in E$, by transitivity of E, we have that (x, y) belongs to E, and by (J.3), it follows that $(x, x \vee y) \in E$. Similarly, we can show that $(x \vee y, x) \in E$. Using again transitivity of E, we may conclude that $(x \vee y, x \vee y)$ belongs to E. Now, for each $x \in C$, let us denote

$$r_x := \bigvee\{y \in C \mid (x, y) \notin E\}. \tag{4}$$

Since C is a partition of M, each r_x is complemented with complement given by

$$r_x^* = \bigvee\{z \in C \mid (x, z) \in E\}. \tag{5}$$

We now show the following equality:

$$E = \bigcap\{E_{r_x} \mid x \in C\} \tag{6}$$

Let $(a, b) \in K \times K$ be such that a and b are both non-zero, or else, (a, b) belongs to both sides of (6). Suppose that $(a, b) \in E$, let $x \in C$, and assume that $a \not\leq r_x$. To show that $(a, b) \in E_{r_x}$, we need to show that $b \leq r_x^*$. Since r_x is complemented, by (1), having $a \not\leq r_x$ is equivalent to having $a \wedge r_x^* \neq 0$. Therefore, by (5), there exists some $z \in C$ such that $a \wedge z \neq 0$ and $(x, z) \in E$. On the other hand, again by (1), $b \leq r_x^*$ if and only if $b \wedge r_x = 0$ and, by (4), $b \wedge r_x = 0$ if and only if $(x, y) \in E$ for every $y \in C$ satisfying $b \wedge y \neq 0$. So, we let $y \in C$ be such that $b \wedge y \neq 0$. Since (x, z), (a, b), and (y, y) belong to E and E is a downset, we have that $(x, a \wedge z)$, $(a \wedge z, b \wedge y)$, and $(b \wedge y, y)$ also belong to E. By transitivity of E, and since $a \wedge z, b \wedge y \neq 0$, it follows that $(x, y) \in E$. This shows that $b \leq r_x^*$ as required. Conversely, suppose that (a, b) belongs to E_{r_x}, for every $x \in C$. Since C is a partition of K, we have $a = \bigvee\{a \wedge x \mid x \in C\}$ and $b = \bigvee\{b \wedge y \mid y \in C\}$. Therefore, by (J.2) and (J.3), in order to show that $(a, b) \in E$, it suffices to show that $(a \wedge x, b \wedge y) \in E$ for every $x, y \in C$ satisfying $a \wedge x \neq 0$ and $b \wedge y \neq 0$.

Since $x \leq r_x^*$, if $a \wedge x \neq 0$ then we also have $a \wedge r_x^* \neq 0$, that is, $a \not\leq r_x$. Since $(a, b) \in E_{r_x}$, this implies $b \leq r_x^*$, that is, $b \wedge r_x = 0$. Finally, by (4), if $b \wedge r_x = 0$, then $(x, y) \in E$ whenever $b \wedge y \neq 0$. Since E is a downset, we have $(a \wedge x, b \wedge y) \in E$ and this finishes the proof of (6).

Finally, (6) implies, on the one hand, that each r_x belongs to $R_{(K,\mathcal{E})}$ (because r_x is complemented, $E \in \mathcal{E}$ and $E = \bigcap_{x \in C} E_{r_x} \subseteq E_{r_x}$) and, on the other hand, that $E \in \mathcal{E}_{R_{(K,\mathcal{E})}}$ (because $E = \bigcap_{x \in C} E_{r_x}$ and each r_x belongs to $R_{(K,\mathcal{E})}$). Therefore, $\mathcal{E} = \mathcal{E}_{R_{(K,\mathcal{E})}}$ as required. ∎

In particular, using Theorem 5.2, we have the following:

Corollary 5.10. *Every frame admitting a transitive and totally bounded quasi-uniformity is zero-dimensional.*

Proposition 5.11. *Given a transitive and totally bounded quasi-uniform frame $(K, \mathcal{E})$, we let L be the subframe of K generated by $R_{(K,\mathcal{E})}$ and we let $e : L \hookrightarrow K$ be the corresponding embedding, so that we have a Frith frame (L, S), where S is such that $e[S] = R_{(K,\mathcal{E})}$. Then, the embedding e extends to a dense extremal epimorphism $\gamma_{(K,\mathcal{E})} : (\mathcal{C}_S L, \mathcal{E}_S) \twoheadrightarrow (K, \mathcal{E})$ of quasi-uniform frames.*

Proof: Since the elements of $e[S] = R_{(K,\mathcal{E})}$ are complemented in K, by Proposition 2.3, the embedding $e : L \hookrightarrow K$ uniquely extends to a frame homomorphism $\gamma_{(K,\mathcal{E})} : \mathcal{C}_S L \to K$. Moreover, $\gamma_{(K,\mathcal{E})}$ is surjective: since frame homomorphisms preserve complemented elements, we have $R_{(K,\mathcal{E})} \cup R^*_{(K,\mathcal{E})} = \gamma_{(K,\mathcal{E})}\,[\{\nabla_s, \Delta_s\}_{s \in S}]$ and, by Theorem 5.2 and Proposition 5.9, it follows that

$$K = \langle R_{(K,\mathcal{E})} \cup R^*_{(K,\mathcal{E})} \rangle_{\mathbf{Frm}} = \gamma_{(K,\mathcal{E})}[\mathcal{C}_S L].$$

Since, by Lemma 5.3, we have $(\gamma_{(K,\mathcal{E})} \oplus \gamma_{(K,\mathcal{E})})(E_{\nabla_s}) = E_{\gamma_{(K,\mathcal{E})}(s)} = E_{e(s)}$ for every $s \in S$, and by Proposition 5.9, $\mathcal{E} = \mathcal{E}_{R_{(K,\mathcal{E})}}$, we may conclude that $\gamma_{(K,\mathcal{E})}$ induces a quasi-uniform homomorphism $\gamma_{(K,\mathcal{E})} : (\mathcal{C}_S L, \mathcal{E}_S) \twoheadrightarrow (M, \mathcal{E})$, which is an extremal epimorphism. It remains to show that $\gamma_{(K,\mathcal{E})}$ is dense. Let $s_1, s_2 \in S$ be such that $\gamma_{(K,\mathcal{E})}(\nabla_{s_1} \wedge \Delta_{s_2}) = 0$, or equivalently, $e(s_1) \wedge e(s_2)^* = 0$. Since $e(s_2) \in R_{(K,\mathcal{E})}$ is complemented, by (1), it follows that $e(s_1) \leq e(s_2)$. Since e is an embedding, we have $s_1 \leq s_2$ and we obtain $\nabla_{s_1} \wedge \Delta_{s_2} = 0$ as required. ∎

Finally, we may use Proposition 5.11 to show that

$$E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$$

is a coreflection.

Theorem 5.12. *The category of Frith frames is coreflective in the category of transitive and totally bounded quasi-uniform frames.*

Proof: For a transitive and totally bounded quasi-uniform frame $(K, \mathcal{E})$, we let $\gamma_{(K,\mathcal{E})} : (\mathcal{C}_S L, \mathcal{E}_S) \twoheadrightarrow (K, \mathcal{E})$ be the dense extremal epimorphism of Proposition 5.11, and $e : L \hookrightarrow K$ denote the embedding of L into K. To show that $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$ is a coreflection, it suffices to show that, for every Frith frame (M, T) and every quasi-uniform homomorphism $h : (\mathcal{C}_T M, \mathcal{E}_T) \to (K, \mathcal{E})$, there exists a unique morphism of Frith frames $g : (M, T) \to (L, S)$ such that the following diagram commutes:

$$\begin{tikzcd} & (\mathcal{C}_T M, \mathcal{E}_T) \arrow[dl, dashed, "Eg =: \overline{g}"'] \arrow[d, "h"] \\ (\mathcal{C}_S L, \mathcal{E}_S) \arrow[r, two heads, "\gamma_{(K,\mathcal{E})}"'] & (K, \mathcal{E}) \end{tikzcd}$$

If such a morphism g exists then, in particular, we must have $\gamma_{(K,\mathcal{E})} \circ \overline{g}(\nabla_a) = h(\nabla_a)$ for every $a \in M$. Since $\overline{g}$ and $\gamma_{(K,\mathcal{E})}$ are extensions of g and e, respectively, this amounts to having $e \circ g(a) = h(\nabla_a)$ for every $a \in M$. Since by Proposition 5.5(a), we have $h[T] \subseteq R_{(K,\mathcal{E})}$ and thus, $h[M] \subseteq \langle R_{(K,\mathcal{E})} \rangle_{\mathbf{Frm}} = e[L]$ and e is an embedding, there is a unique morphism of Frith frames $g : (M, T) \to (L, S)$ satisfying $e \circ g(a) = h(\nabla_a)$ for every $a \in M$. Finally, to see that $\overline{g}$ makes the above diagram commute, it suffices to observe that, for every $t \in T$, the following equalities hold:

$$\gamma_{(K,\mathcal{E})} \circ \overline{g}(\Delta_t) = \gamma_{(K,\mathcal{E})}(\Delta_{g(t)}) \stackrel{(*)}{=} (e \circ g(t))^* = h(\nabla_t)^* = h(\Delta_t),$$

where the equality marked by $(*)$ holds because $g(t)$ belongs to S. ■

Note that, unlike what happens in the point-set framework, where we have an equivalence between Pervin spaces and transitive and totally bounded quasi-uniform spaces (see [18]), the categories of Frith frames and of transitive and totally bounded quasi-uniform frames are not equivalent. This is because, in general, the dense extremal epimorphism $\gamma_{(K,\mathcal{E})} : (\mathcal{C}_S L, \mathcal{E}_S) \twoheadrightarrow (K, \mathcal{E})$ from Proposition 5.11 is not an isomorphism. Intuitively, the reason behind this phenomenon is that there are several ways in which $L := \langle R_{(K,\mathcal{E})} \rangle_{\mathbf{Frm}}$ may be extended so that every element of $S := R_{(K,\mathcal{E})}$ becomes complemented, and $\mathcal{C}_S L$ is just one of them that may not reflect how the complements in K of the elements of $R_{(K,\mathcal{E})}$ behave. Accordingly, if $(K,\mathcal{E})$ is such that the elements of $S = R_{(K,\mathcal{E})}$ are already complemented in L, then $\gamma_{(K,\mathcal{E})}$ is an isomorphism. As we shall see in the next section, this happens if $\mathcal{E}$ is a uniformity (cf. Proposition 6.1) and so, the category of transitive and totally bounded uniform frames may be nicely represented by a suitable full subcategory of Frith frames (cf. Corollary 6.2). A similar conclusion may be taken if we restrict to those transitive and totally bounded quasi-uniform frames that are complete (cf. Corollary 7.3): if $(K,\mathcal{E})$ is complete, then $\gamma_{(K,\mathcal{E})}$ has to be an isomorphism.

6. Symmetric Frith frames and uniform frames

We say that a Frith frame (L,B) is *symmetric* if B is a Boolean algebra, and we let $\mathbf{Frith}_{\mathrm{sym}}$ denote the full subcategory of **Frith** whose objects are the symmetric Frith frames. The relevance of symmetric Frith frames lies in the fact that they exactly capture those transitive and totally bounded quasi-uniform frames that are actually uniform.

Proposition 6.1. *Let $(K,\mathcal{E})$ be a transitive and totally bounded quasi-uniform frame. Then, $\mathcal{E}$ is a uniformity if and only if $R_{(K,\mathcal{E})}$ is a Boolean algebra. In particular, for every Frith frame (L,S), we have that (L,S) is symmetric if and only if $(\mathcal{C}_S L,\mathcal{E}_S)$ is a uniform frame.*

Proof: The first claim is a straightforward consequence of Proposition 5.9 and of the equality $E_r^{-1} = E_{r^*}$, which holds whenever r is complemented. For the second statement, we only need to observe that, by Proposition 5.9, we have $\mathcal{E}_S = \mathcal{E}_{R_{(\mathcal{C}_S L,\mathcal{E}_S)}}$ and thus, by Lemma 5.7, we have $R_{(\mathcal{C}_S L,\mathcal{E}_S)} = \{\nabla_s \mid s \in S\}$ which, by Lemma 2.2, is a lattice isomorphic to S. Therefore, by the first claim, $(\mathcal{C}_S L,\mathcal{E}_S)$ is a uniform frame if and only if S is a Boolean algebra. ■

In the following, we let $\mathbf{UniFrm}_{\mathrm{trans,\,tot\,bd}}$ denote the full subcategory of **QUniFrm** formed by the transitive and totally bounded uniform frames.

Corollary 6.2. *The coreflection $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\mathrm{trans,\,tot\,bd}}$ restricts and co-restricts to an isomorphism $E' : \mathbf{Frith}_{\mathrm{sym}} \hookrightarrow \mathbf{UniFrm}_{\mathrm{trans,\,tot\,bd}}$.*

Proof: By Proposition 6.1, the functor E restricts and co-restricts to a functor $E' : \mathbf{Frith}_{\mathrm{sym}} \hookrightarrow \mathbf{UniFrm}_{\mathrm{trans,\,tot\,bd}}$. Let $(K,\mathcal{E})$ be a transitive and totally bounded uniform frame. By Proposition 6.1, $R_{(K,\mathcal{E})}$ is a Boolean algebra and so, by Theorem 5.2 and Proposition 5.9, K is generated, as a frame, by

$R_{(K,\mathcal{E})}$. Therefore, the pair $(K, R_{(K,\mathcal{E})})$ is a symmetric Frith frame. Moreover, by Propositions 5.5(a) and 5.9, if $h : (K,\mathcal{E}) \to (M,\mathcal{F})$ is a homomorphism of transitive and totally bounded uniform frames, then h induces a morphism of (symmetric) Frith frames $h : (K, R_{(K,\mathcal{E})}) \to (M, R_{(M,\mathcal{F})})$. Hence, the assignment $(K,\mathcal{E}) \mapsto (K, R_{(K,\mathcal{E})})$ yields a well-defined functor $\gamma' : \mathbf{UniFrm}_{\text{trans, tot bd}} \to \mathbf{Frith}_{\text{sym}}$. Finally, one can easily show that E' and γ' are mutually inverse. ∎

We will now show that the category $\mathbf{Frith}_{\text{sym}}$ is both reflective and coreflective in **Frith**. Let us start by coreflectivity.

Proposition 6.3. *The category* $\mathbf{Frith}_{\text{sym}}$ *is a full coreflective subcategory of* **Frith**.

Proof: Let (L,S) be a Frith frame. We consider the subset $C \subseteq S$ consisting of those elements having a complement in S, and we let N be the subframe of L generated by C. Observe that C is a Boolean subalgebra of N, so that we have a symmetric Frith frame $b(L,S) := (N,C)$. Moreover, the embedding $N \hookrightarrow L$ defines a homomorphism of Frith frames $e_{(L,S)} : b(L,S) \hookrightarrow (L,S)$. To complete the proof, we only need to show that, for every symmetric Frith frame (M,B) and for every morphism $h : (M,B) \to (L,S)$ there is a unique morphism $\widetilde{h} : (M,B) \to b(L,S)$ making the following diagram commute:

$$\begin{tikzcd} & (M,B) \arrow[dl, dashed, "\widetilde{h}"'] \arrow[d, "h"] \\ b(L,S) \arrow[r, hook, "e_{(L,S)}"'] & (L,S) \end{tikzcd}$$

That is, we need to show that h co-restricts to a morphism of Frith frames $\widetilde{h} : (M,B) \to b(L,S)$. This is indeed the case because, since frame homomorphisms preserve pairs of complemented elements, $h[B] \subseteq S$, and B is a Boolean algebra, every element of $h[B]$ is complemented in S, that is, we have $h[B] \subseteq C$. ∎

Combining Theorem 5.12, Corollary 6.2, and Proposition 6.3, we also have the following:

Corollary 6.4. *The category* $\mathbf{UniFrm}_{\text{trans, tot bd}}$ *is coreflective in the category* $\mathbf{QUniFrm}_{\text{trans, tot bd}}$.

Let us now prove reflexivity. Given a Frith frame (L, S), we let $\overline{S}$ denote the sublattice of $\mathcal{C}_S L$ generated by the elements of the form ∇_s together with their complements. As complemented elements of a frame are closed under finite meets and finite joins, the lattice $\overline{S}$ is a Boolean algebra. Moreover, since L is generated by S, we have that $\mathcal{C}_S L$ is generated by $\overline{S}$ and so $(\mathcal{C}_S L, \overline{S})$ is a Frith frame. We may then define a functor $\mathbf{Sym}_{\mathrm{Frith}} : \mathbf{Frith} \to \mathbf{Frith}_{\mathrm{sym}}$ as follows. For a Frith frame (L, S) we set $\mathbf{Sym}_{\mathrm{Frith}}(L, S) := (\mathcal{C}_S L, \overline{S})$; and for a morphism of Frith frames $h : (L, S) \to (M, T)$ we set $\mathbf{Sym}_{\mathrm{Frith}}(h) := \overline{h}$, where $\overline{h}$ is the unique extension of h to a frame homomorphism $\overline{h} : \mathcal{C}_S L \to \mathcal{C}_T M$ (recall Corollary 2.4). In particular, we have $\overline{h}(\nabla_s) = \nabla_{h(s)}$ and $\overline{h}(\Delta_s) = \Delta_{h(s)}$ for every $s \in S$ and thus, $\overline{h} : (\mathcal{C}_S L, \overline{S}) \to (\mathcal{C}_T M, \overline{T})$ is a morphism of Frith frames and $\mathbf{Sym}_{\mathrm{Frith}}$ is a well-defined functor. We shall refer to $\mathbf{Sym}_{\mathrm{Frith}}(L, S)$ as the *symmetrization* of (L, S).

Proposition 6.5. *The full subcategory of symmetric Frith frames is reflective in* **Frith**.

Proof: We first observe that if (L, S) is a Frith frame, then it embeds in its symmetrization via $\nabla : L \hookrightarrow \mathcal{C}_S L$. Let (M, B) be a symmetric Frith frame, and $h : (L, S) \to (M, B)$ be a morphism in **Frith**. We need to show that there is a unique morphism $\widetilde{h} : (\mathcal{C}_S L, \overline{S}) \to (M, B)$ making the following diagram commute:

$$\begin{tikzcd}
(L, S) \arrow[r, hook, "\nabla"] \arrow[dr, "h"'] & (\mathcal{C}_S L, \overline{S}) \arrow[d, dashed, "\widetilde{h}"] \\
& (M, B)
\end{tikzcd}$$

Since $h[S] \subseteq B$ consists of complemented elements of M, by Proposition 2.3, there is a unique frame map $\widetilde{h} : \mathcal{C}_S L \to M$ making the above triangle commute. Hence, we only need to show that $\widetilde{h}[\,\overline{S}\,] \subseteq B$. This is indeed the case because, for every $s \in S$, we have $\widetilde{h}(\nabla_s) = h(s)$ and $\widetilde{h}(\Delta_s) = h(s)^*$, and B is closed under taking complements. ■

We now argue that $\mathbf{Sym}_{\mathrm{Frith}}$ is a restriction of the usual reflection of **QUniFrm** onto **UniFrm**.

Proposition 6.6. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Frith} \arrow[r, "\mathbf{Sym}_{\mathrm{Frith}}"] \arrow[d, hook, "E"'] & \mathbf{Frith}_{\mathrm{sym}} \arrow[d, hook, "E"] \\
\mathbf{QUniFrm} \arrow[r, "\mathbf{Sym}_{\mathrm{QUniFrm}}"'] & \mathbf{UniFrm}
\end{tikzcd}$$

Proof: Let (L, S) be a Frith frame. By definition, we have

$$E\circ\mathbf{Sym}_{\mathrm{Frith}}(L,S) = (\mathcal{C}_S L, \mathcal{E}_{\overline{S}}) \qquad \text{and} \qquad \mathbf{Sym}_{\mathrm{QUniFrm}}\circ E(L,S) = (\mathcal{C}_S L, \overline{\mathcal{E}_S}),$$

where $\overline{\mathcal{E}_S}$ is the uniformity generated by $\{E_{\nabla_s}, E_{\nabla_s}^{-1} \mid s \in S\}$. Since $E_{\nabla_s}^{-1} = E_{\Delta_s}$, by definition of $\overline{S}$ and by Lemma 5.7, $\mathcal{E}_{\overline{S}}$ and $\overline{\mathcal{E}_S}$ are both the (quasi-)uniformity generated by $\{E_{\nabla_s}, E_{\Delta_s} \mid s \in S\}$. Commutativity at the level of morphisms is trivial. ■

Finally, we will see that $\mathbf{Sym}_{\mathrm{Frith}}$ is the pointfree analogue of $\mathbf{Sym}_{\mathrm{Perv}}$ discussed in Section 3.

Proposition 6.7. *There is an isomorphism $\alpha_{(L,S)} : \mathbf{Sym}_{\mathrm{Perv}} \circ \mathsf{pt}(L,S) \cong \mathsf{pt} \circ \mathbf{Sym}_{\mathrm{Frith}}(L,S)$ for every Frith frame (L,S).*

Proof: Let us define $\alpha_{(L,S)} : \mathsf{pt}(L) \to \mathsf{pt}(\mathcal{C}_S L)$ by $p \mapsto \widetilde{p}$, where for every point $p \in \mathsf{pt}(L)$ the map $\widetilde{p}$ is the unique morphism such that $\widetilde{p} \circ \nabla = p$, as given by Proposition 2.3. Since L is a subframe of $\mathcal{C}_S L$ and thus, every point of $\mathcal{C}_S L$ restricts to a point of L, this assignment is a bijection. Let us show that $\alpha_{(L,S)}$ defines an isomorphism of Pervin spaces $\alpha_{(L,S)} : \mathbf{Sym}_{\mathrm{Perv}} \circ \mathsf{pt}(L,S) \to \mathsf{pt} \circ \mathbf{Sym}_{\mathrm{Frith}}(L,S)$. By Corollary 3.3, we need to show that the preimages of the elements of the lattice component of $\mathsf{pt} \circ \mathbf{Sym}_{\mathrm{Frith}}(L,S)$ are exactly the elements of the lattice component of $\mathbf{Sym}_{\mathrm{Perv}} \circ \mathsf{pt}(L,S)$. Noting that the former lattice is generated by the elements of the form $\widehat{\nabla}_s$ and $\widehat{\Delta}_s$ and the latter by those of the form $\widehat{s}$ and $(\widehat{s})^c$ $(s \in S)$, that is a consequence of having

$$\alpha_{(L,S)}^{-1}(\widehat{\nabla_s}) = \widehat{s} \qquad \text{and} \qquad \alpha_{(L,S)}^{-1}(\widehat{\Delta_s}) = (\widehat{s})^c,$$

for every $s \in S$. ■

Theorem 6.8. *The following diagram commutes up to natural isomorphism.*

$$\begin{tikzcd}
\mathbf{Frith}^{\mathrm{op}} \arrow[r, "\mathbf{Sym}_{\mathrm{Frith}}"] \arrow[d, "\mathsf{pt}"'] & \mathbf{Frith}^{\mathrm{op}}_{\mathrm{sym}} \arrow[d, "\mathsf{pt}"] \\
\mathbf{Pervin} \arrow[r, "\mathbf{Sym}_{\mathrm{Perv}}"'] & \mathbf{Pervin}_{\mathrm{sym}}
\end{tikzcd}$$

Proof: We show that the family of isomorphisms $\{\alpha_{(L,S)} \mid (L,S)$ is a Frith frame$\}$ defined in Proposition 6.7 induces a natural transformation $\mathbf{Sym}_{\mathrm{Perv}} \circ \mathsf{pt} \implies \mathsf{pt} \circ \mathbf{Sym}_{\mathrm{Frith}}$. Suppose that $h : (M,T) \to (L,S)$ is a morphism of Frith frames, that is, h is a morphism $(L,S) \to (M,T)$ in $\mathbf{Frith}^{\mathrm{op}}$. Then, naturality of α amounts to commutativity of the following square:

$$\begin{tikzcd} \mathsf{pt}(L) \arrow[r, "\alpha_{(L,S)}"] \arrow[d, "(-)\circ h"'] & \mathsf{pt}(\mathcal{C}_S L) \arrow[d, "(-)\circ \overline{h}"] \\ \mathsf{pt}(M) \arrow[r, "\alpha_{(M,T)}"'] & \mathsf{pt}(\mathcal{C}_T M) \end{tikzcd}$$

Let $p \in \mathsf{pt}(L)$. Since, by definition, $\alpha_{(M,T)}(p \circ h)$ is the unique point of $\mathsf{pt}(\mathcal{C}_T M)$ extending $p \circ h$, it suffices to show that $\alpha_{(L,S)}(p) \circ \overline{h}$ extends $p \circ h$. That is indeed the case because, for every $a \in M$, we have the following:

$$\alpha_{(L,S)}(p) \circ \overline{h}(\nabla_a) = \alpha_{(L,S)}(p)(\nabla_{h(a)}) \overset{(*)}{=} p \circ h(a),$$

where the equality $(*)$ follows from having that, by definition, $\alpha_{(L,S)}(p)$ is the unique extension of p to a point of $\mathcal{C}_S L$. ■

7. Completion of a Frith frame

As mentioned in Section 2, complete (quasi-)uniform frames may be equivalently characterized via dense extremal epimorphisms or via Cauchy maps. Since the category of Frith frames fully embeds into the category of quasi-uniform frames, there is a natural notion of completion of a Frith frame. In this section we explore it, both from the point of view of dense extremal epimorphisms and of Cauchy maps. As the reader will notice, in this restricted subcategory of quasi-uniform frames, the concepts involved become surprisingly simple.

7.1. Dense extremal epimorphisms. We say that a symmetric Frith frame (L,B) is *complete* if every dense extremal epimorphism $(M,C) \twoheadrightarrow (L,B)$ with (M,C) symmetric is an isomorphism. More generally, a Frith frame (L,S) is *complete* provided its symmetric reflection $\mathbf{Sym}_{\mathrm{Frith}}(L,S)$ is complete. As the reader may expect, completeness of a Frith frame (L,S) is equivalent to completeness of the associated quasi-uniform frame $(\mathcal{C}_S L, \mathcal{E}_S)$ (cf. Proposition 7.2). A *completion* of (L,S) is a complete Frith frame (M,T) together with a dense extremal epimorphism $(M,T) \twoheadrightarrow (L,S)$.

Given a Frith frame (L, S), by Corollary 2.6, there is a unique frame homomorphism $\mathrm{Idl}(S) \to L$ extending the embedding $S \hookrightarrow L$. Clearly, this frame homomorphism induces a dense extremal epimorphism of Frith frames

$$c_{(L,S)} : (\mathrm{Idl}(S), S) \twoheadrightarrow (L, S), \quad J \mapsto \bigvee J.$$

An immediate consequence of the definition of completeness is the following:

Lemma 7.1. *If (L, S) is complete, then $c_{(\mathcal{C}_S L, \overline{S})} : (\mathrm{Idl}(\overline{S}), \overline{S}) \to (\mathcal{C}_S L, \overline{S})$ is an isomorphism.*

Proof: This is simply because, by definition, (L, S) is complete if and only if so is $\mathbf{Sym}_{\mathrm{Frith}}(L, S) = (\mathcal{C}_S L, \overline{S})$ and $c_{(\mathcal{C}_S L, \overline{S})}$ is a dense extremal epimorphism. ■

In particular, if (L, S) is complete, then its symmetric reflection $\mathbf{Sym}_{\mathrm{Frith}}(L, S)$ is coherent. In turn, since $\mathbf{Sym}_{\mathrm{Frith}}(L, S)$ is a zero-dimensional Frith frame, by Lemma 4.9, being coherent is equivalent to being compact, and since L is a subframe of $\mathcal{C}_S L$, we have $K(\mathcal{C}_S L) \cap L \subseteq K(L)$. Now notice that $K(\mathcal{C}_S L) = \overline{S}$ by coherence of $\mathbf{Sym}_{\mathrm{Frith}}(L, S)$ and Lemma 4.5, and thus $S \subseteq K(\mathcal{C}_S L)$. Therefore, $S \subseteq K(\mathcal{C}_S L) \cap L$ and this implies that $S \subseteq K(L)$. This means that (L, S) is coherent, too. We have just proved the following:

$$\begin{aligned}(L, S) \text{ complete } &\implies \mathbf{Sym}_{\mathrm{Frith}}(L, S) \text{ compact} \\ &\iff \mathbf{Sym}_{\mathrm{Frith}}(L, S) \text{ coherent } \implies (L, S) \text{ coherent.}\end{aligned} \tag{7}$$

From this, we may already show that our notion of *completeness* is consistent with usual completeness for (quasi-)uniform frames:

Proposition 7.2. *Let (L, S) be a Frith frame. Then, (L, S) is complete if and only if $(\mathcal{C}_S L, \mathcal{E}_S)$ is complete.*

Proof: We first observe that it suffices to consider the case where (L, S) is symmetric. Indeed, by definition, (L, S) is complete if and only if $\mathbf{Sym}_{\mathrm{Frith}}(L, S) = (\mathcal{C}_S L, \overline{S})$ is complete. On the other hand, by Proposition 2.14, $(\mathcal{C}_S L, \mathcal{E}_S)$ is complete if and only if $(\mathcal{C}_S L, \overline{\mathcal{E}_S})$ is complete and, by Proposition 6.6, we have $(\mathcal{C}_S L, \overline{\mathcal{E}_S}) = \mathbf{Sym}_{\mathrm{QUniFrm}} \circ E(L, S) = E \circ \mathbf{Sym}_{\mathrm{Frith}}(L, S) = (\mathcal{C}_S L, \mathcal{E}_{\overline{S}})$. Therefore, the claim holds if and only if, for every Frith frame (L, S), completeness of $\mathbf{Sym}_{\mathrm{Frith}}(L, S) = (\mathcal{C}_S L, \overline{S})$ and of $E \circ \mathbf{Sym}_{\mathrm{Frith}}(L, S) = (\mathcal{C}_S L, \mathcal{E}_{\overline{S}})$ are equivalent notions.

Now, we let (L, B) be a symmetric Frith frame. Suppose that (L, B) is complete and let $h : (M, \mathcal{E}) \twoheadrightarrow (L, \mathcal{E}_B)$ be a dense extremal epimorphism for some uniform frame $(M, \mathcal{E})$. Since (L, B) is complete, by (7), L is compact. Therefore, by Proposition 2.15, h is an isomorphism. Conversely, suppose that $(L, \mathcal{E}_B)$ is complete. By Proposition 5.5, every dense extremal epimorphism $h : (M, C) \twoheadrightarrow (L, B)$ of symmetric Frith frames induces an extremal epimorphism $h : (M, \mathcal{E}_C) \twoheadrightarrow (L, \mathcal{E}_B)$, which is clearly dense. Since $(L, \mathcal{E}_B)$ is complete, h is one-one and, by Corollary 4.18, it is an isomorphism. Thus, (L, B) is complete as required. ∎

Before proceeding, we remark that a consequence of Proposition 7.2 is that the categories **CFrith** of complete Frith frames and $\mathbf{CQUniFrm}_{\text{trans, tot bd}}$ of complete transitive and totally bounded quasi-uniform frames are equivalent.

Corollary 7.3. *The coreflection $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$ restricts and co-restricts to an equivalence of categories $E'' : \mathbf{CFrith} \to \mathbf{CQUniFrm}_{\text{trans, tot bd}}$.*

Proof: By Proposition 7.2, the functor $E : \mathbf{Frith} \hookrightarrow \mathbf{QUniFrm}_{\text{trans, tot bd}}$ restricts and co-restricts to a functor $E'' : \mathbf{CFrith} \hookrightarrow \mathbf{CQUniFrm}_{\text{trans, tot bd}}$. Since **CFrith** and $\mathbf{CQUniFrm}_{\text{trans, tot bd}}$ are, respectively, full subcategories of **Frith** and of $\mathbf{QUniFrm}_{\text{trans, tot bd}}$, by Proposition 5.6, E'' is a full embedding. Finally, let $(K, \mathcal{E})$ be a transitive and totally bounded quasi-uniform frame. If $(K, \mathcal{E})$ is complete, then the dense extremal epimorphism $\gamma_{(K,\mathcal{E})} : (\mathcal{C}_S L, \mathcal{E}_S) \twoheadrightarrow (K, \mathcal{E})$ of Proposition 5.11 has to be an isomorphism and, again by Proposition 7.2, (L, S) is complete. Therefore, $(K, \mathcal{E}) \cong E''(L, S)$ and E'' is an equivalence of categories. ∎

Our next goal is to show that all the statements in (7) are in fact equivalent. Since by Proposition 4.6 coherent Frith frames are those of the form $(\mathrm{Idl}(S), S)$, for some lattice S, we only need to prove that $(\mathrm{Idl}(S), S)$ is always a complete Frith frame. We will need the following lemma.

Lemma 7.4. *If (L, B) and (M, C) are symmetric Frith frames, then any dense extremal epimorphism $h : (L, B) \to (M, C)$ restricts and co-restricts to a Boolean algebra isomorphism $h' : B \to C$.*

Proof: Suppose that (L, B) and (M, C) are symmetric Frith frames, and let $h : (L, B) \to (M, C)$ be a dense extremal epimorphism. By Proposition 4.16, we have $h[B] = C$. So, we only need to show that the restriction of h to B is injective. Let $b_1, b_2 \in B$ and suppose that $h(b_1) \leq h(b_2)$. This implies that $h(b_1) \wedge h(b_2)^* = 0$ and, since frame morphisms preserve complemented pairs, we have $h(b_1 \wedge b_2^*) = 0$. By density, we may then conclude that $b_1 \wedge b_2^* = 0$ and, since b_2 is complemented, this implies $b_1 \leq b_2$. ∎

Let $\overline{c}_{(L,S)} : \mathbf{Sym}_{\mathrm{Frith}}(\mathrm{Idl}(S), S) \twoheadrightarrow \mathbf{Sym}_{\mathrm{Frith}}(L, S)$ denote the symmetric reflection of $c_{(L,S)}$, that is, $\overline{c}_{(L,S)} : (\mathcal{C}_S\mathrm{Idl}(S), \overline{S}) \twoheadrightarrow (\mathcal{C}_S L, \overline{S})$ is defined by $\overline{c}_{(L,S)}(\nabla_s) = \nabla_s$ and $\overline{c}_{(L,S)}(\Delta_s) = \Delta_s$, for $s \in S$.

Proposition 7.5. *Let (L, S), (M, C) be Frith frames with (M, C) symmetric and let $h : (M, C) \to (\mathcal{C}_S L, \overline{S})$ be a dense extremal epimorphism. Then, there exists a unique morphism $g : (\mathcal{C}_S\mathrm{Idl}(S), \overline{S}) \to (M, C)$ making the following diagram commute:*

$$\begin{tikzcd}
(\mathcal{C}_S\mathrm{Idl}(S), \overline{S}) \arrow[r, dashed, "g"] \arrow[dr, "\overline{c}_{(L,S)}"'] & (M, C) \arrow[d, "h"] \\
& (\mathcal{C}_S L, \overline{S})
\end{tikzcd}$$

Moreover, g is a dense extremal epimorphism.

Proof: Since $\overline{S}$ is join-dense in $\mathcal{C}_S\mathrm{Idl}(S)$, frame homomorphisms preserve complemented elements, and C is closed under taking complements, if g exists, then its underlying frame homomorphism is the unique extension of a lattice homomorphism $g' : S \to C$ such that $h \circ g'(s) = \nabla_s$ for every $s \in S$. We first argue that the morphism g' exists. Since h is a dense extremal epimorphism, by Lemma 7.4, it restricts and co-restricts to a lattice isomorphism $h' : C \to \overline{S}$. Let g' be the restriction to S of the inverse of h'. Clearly, g' satisfies $h \circ g'(s) = \nabla_s$ for every $s \in S$. The desired extension of g' exists, too: we may first extend g' to a frame homomorphism $\mathrm{Idl}(S) \to M$, by Corollary 2.6, and then to a frame homomorphism $g : \mathcal{C}_S\mathrm{Idl}(S) \to M$, by Proposition 2.3.

It remains to show that g is a dense extremal epimorphism. Observe that g suitably restricted and co-restricted is the inverse of $h' : C \to \overline{S}$. Therefore $g[\overline{S}] = C$ and g is an extremal epimorphism. Finally, since $\mathcal{C}_S\mathrm{Idl}(S)$ is generated, as a frame, by $\overline{S}$, g is dense provided $\nabla_{s_1} \cap \Delta_{s_2} = 0$ for every $s_1, s_2 \in S$ satisfying $g(\nabla_{s_1} \cap \Delta_{s_2}) = 0$, and this also follows from having that g restricts and co-restricts to a lattice isomorphism $\overline{S} \to C$. ∎

Corollary 7.6. *If (L, S) is a Frith frame, then $(\mathrm{Idl}(S), S)$ is complete and, therefore, $c_{(L,S)} : (\mathrm{Idl}(S), S) \twoheadrightarrow (L, S)$ is a completion of (L, S).*

Proof: By definition, $(\mathrm{Idl}(S), S)$ is complete if and only if so is $\mathbf{Sym}_{\mathrm{Frith}}(\mathrm{Idl}(S), S) = (\mathcal{C}_S\mathrm{Idl}(S), \overline{S})$. Let $h : (M, C) \twoheadrightarrow (\mathcal{C}_S\mathrm{Idl}(S), \overline{S})$ be a dense extremal epimorphism, with (M, C) symmetric. Then, Proposition 7.5 applied to h gives the existence of a dense extremal epimorphism $g : (\mathcal{C}_S\mathrm{Idl}(S), \overline{S}) \twoheadrightarrow (M, C)$ satisfying $h \circ g = \overline{c}_{(\mathrm{Idl}(S),S)}$. Since, $\overline{c}_{(\mathrm{Idl}(S),S)}$ is the identity function, h is one-one, thus an isomorphism. Thus, $\mathbf{Sym}_{\mathrm{Frith}}(\mathrm{Idl}(S), S) = (\mathcal{C}_S\mathrm{Idl}(S), \overline{S})$ is complete as required. ■

We may now state following pointfree analogue of [18, Theorem 4.1], which is a straightforward consequence of (7) and of Corollary 7.6.

Theorem 7.7. *For a Frith frame (L, S) the following are equivalent.*

- (a) *The Frith frame (L, S) is complete.*
- (b) *The Frith frame (L, S) is coherent.*
- (c) *The Frith frame $\mathbf{Sym}_{\mathrm{Frith}}(L, S)$ is coherent.*
- (d) *The Frith frame $\mathbf{Sym}_{\mathrm{Frith}}(L, S)$ is compact.*

We finish this section by showing that completions are unique, up to isomorphism.

Proposition 7.8. *Let (L, S) be a Frith frame. Then, for every morphism $h : (M, T) \to (L, S)$ with (M, T) complete, there exists a unique morphism $\widehat{h} : (M, T) \to (\mathrm{Idl}(S), S)$ such that the following diagram commutes:*

$$\begin{tikzcd} (M,T) \arrow[r, dashed, "\widehat{h}"] \arrow[dr, "h"'] & (\mathrm{Idl}(S), S) \arrow[d, "c_{(L,S)}"] \\ & (L,S) \end{tikzcd}$$

Moreover, if h is dense (respectively, an extremal epimorphism) then $\widehat{h}$ is also dense (respectively, an extremal epimorphism).

Proof: Since T is join-dense in M, if such a homomorphism $\widehat{h}$ exists, then it is completely determined by its restriction to T. In particular, $\widehat{h}$ is unique because we must have $h(t) = c_{(L,S)} \circ \widehat{h}(t) = \widehat{h}(t)$, for every $t \in T$, that is, $\widehat{h}$ must be an extension of the restriction and co-restriction $h' : T \to S$ of h. By Corollary 2.6, h' uniquely extends to a frame homomorphism $\widehat{h} : \mathrm{Idl}(T) \to \mathrm{Idl}(S)$. Since, by Theorem 7.7, (M, T) is coherent and thus, by Proposition 4.6, $M \cong \mathrm{Idl}(T)$, it follows that $\widehat{h}$ is the required homomorphism.

Now, suppose that h is dense. Since T is join-dense in M, $\widehat{h}$ is dense provided, for every $t \in T$, we have $t = 0$ whenever $\widehat{h}(t) = 0$. This is indeed the case because $\widehat{h}(t) = h(t)$ for every $t \in T$. Finally, by the same reason, we also have that $\widehat{h}$ is an extremal epimorphism if so is h. ■

Corollary 7.9. *Each Frith frame has a unique, up to isomorphism, completion.*

Proof: By Corollary 7.6 we already know that every Frith frame (L, S) has a completion $c_{(L,S)} : (\mathrm{Idl}(S), S) \twoheadrightarrow (L, S)$. Let $c : (M, T) \twoheadrightarrow (L, S)$ be another completion of (L, S). Since c is a dense extremal epimorphism, by Proposition 7.8, there exists a dense extremal epimorphism $\widehat{c} : (M, T) \twoheadrightarrow (\mathrm{Idl}(S), S)$ satisfying $c_{(L,S)} \circ \widehat{c} = c$. Since (M, T) is complete, $\widehat{c}$ has to be an isomorphism and so, the two completions of (L, S) are isomorphic. ■

7.2. Cauchy maps. In this section we will show an analogue of Theorem 2.17. We start by proving some properties of a Cauchy map $\phi : (L, \mathcal{E}) \to M$, in the case where $\mathcal{E} = \mathcal{E}_B$, for some Boolean subalgebra $B \subseteq L$, that is, (L, B) is a symmetric Frith frame (recall Theorem 5.2) and $E(L, B) = (L, \mathcal{E}_B)$.

Lemma 7.10. *Let (L, B) be a symmetric Frith frame, M be a frame, and $\phi : (L, \mathcal{E}_B) \to M$ be a Cauchy map. Then, the following statements hold:*

- *ϕ restricts to a lattice homomorphism with domain B,*
- *for every $a \in L$, the equality $\phi(a) = \bigvee\{\phi(b) \mid b \in B,\ b \le a\}$ holds,*
- *for every $b \in B$, $\phi(b) \vee \phi(b)^* = 1$.*

Proof: Let $\phi : (L, \mathcal{E}_B) \to M$ be a Cauchy map, that is, ϕ satisfies conditions (a), (b), and (c) of Definition 2.16. We start by observing that, for every $b \in B$ we have $\phi(b^*) = \phi(b)^*$. Indeed, since $(a, a) \in E_b$ if and only if $a \le b$ or $a \le b^*$, and, by (a), ϕ is order-preserving, by (c), we have $1 = \phi(b) \vee \phi(b^*)$. Since, by (a), $\phi(b) \wedge \phi(b^*) = \phi(b \wedge b^*) = \phi(0) = 0$, it follows that $\phi(b)$ is complemented with complement $\phi(b^*)$, that is, $\phi(b^*) = \phi(b)^*$. In particular, we also have $1 = \phi(b) \vee \phi(b)^*$, which proves the third statement. Now, we let $b_1, b_2 \in B$. By (a), the first statement holds provided $\phi(b_1 \vee b_2) \le \phi(b_1) \vee \phi(b_2)$. Since each $\phi(b_i)$ is complemented, by (1), this is equivalent to the equality $\phi(b_1 \vee b_2) \wedge \phi(b_1)^* \wedge \phi(b_2)^* = 0$, which follows from (a) together with the equality $\phi(b_i^*) = \phi(b_i)^*$ already proved. Finally, let us show that the second statement is valid. We fix some $a \in L$. Since B is a Boolean algebra, by Lemma 5.1, for every $x \in L$ satisfying $x \triangleleft_1 a$ or

$x \lhd_2 a$, there is some $b_x \in B$ such that $x \leq b_x \leq a$. Then, using (b) and the fact that ϕ is order-preserving, we may derive that

$$\phi(a) \leq \bigvee\{\phi(x) \mid x \in L,\ x \lhd_1 a \text{ or } x \lhd_2 a\} \leq \bigvee\{\phi(b_x) \mid x \in L,\ x \lhd_1 a \text{ or } x \lhd_2 a\}$$
$$\leq \bigvee\{\phi(b) \mid b \in B,\ b \leq a\} \leq \phi(a). \quad \blacksquare$$

It is then natural to consider the following definition of *Cauchy map*.

Definition 7.11. *Let (L, S) be a Frith frame and M any frame. A* Cauchy map *$\phi : (L, S) \to M$ is a function $\phi : L \to M$ such that*
(C.1) ϕ restricts to a lattice homomorphism with domain S,
(C.2) for every $a \in L$, the equality $\phi(a) = \bigvee\{\phi(s) \mid s \in S,\ s \leq a\}$ holds,
(C.3) for every $s \in S$, $\phi(s) \vee \phi(s)^* = 1$.

By Lemma 7.10, we have that in the case where (L, B) is a symmetric Frith frame, if $\phi : (L, \mathcal{E}_B) \to M$ is a Cauchy map in the sense of Definition 2.16, then $\phi : (L, B) \to M$ is a Cauchy map in the sense of Definition 7.11. We will now show that the converse is also true, so that our definition of Cauchy map agrees with the classical one for transitive and totally bounded uniform frames (recall Corollary 6.2).

Proposition 7.12. *Let (L, B) be a symmetric Frith frame, M a frame, and $\phi : L \to M$ a function. Then, ϕ defines a Cauchy map $\phi : (L, \mathcal{E}_B) \to M$ if and only if it defines a Cauchy map $\phi : (L, B) \to M$.*

Proof: The forward implication is the content of Lemma 7.10. Conversely, let $\phi : (L, B) \to M$ be a Cauchy map, where (L, B) is a symmetric Frith frame. We first argue that ϕ is a bounded meet homomorphism. Since $\phi|_B$ is a lattice homomorphism, we have $\phi(0) = 0$ and $\phi(1) = 1$. Let $a_1, a_2 \in L$. Then, we may compute

$$\begin{aligned}
\phi(a_1 \wedge a_2) &\overset{(C.2)}{=} \bigvee\{\phi(b) \mid b \in B,\ b \leq a_1 \wedge a_2\} \\
&\overset{(C.1)}{=} \bigvee\{\phi(b_1) \wedge \phi(b_2) \mid b_1, b_2 \in B,\ b_1 \leq a_1,\ b_2 \leq a_2\} \\
&= \left(\bigvee\{\phi(b_1) \mid b_1 \in B,\ b_1 \leq a_1\}\right) \wedge \left(\bigvee\{\phi(b_2) \mid b_2 \in B,\ b_2 \leq a_2\}\right) \\
&\overset{(C.2)}{=} \phi(a_1) \wedge \phi(a_2).
\end{aligned}$$

Thus, ϕ also preserves binary meets and we have (a). Now, using (C.2), in order to show (b), it suffices to observe that, for every $b \in B$, we have $b \lhd_1 a$

whenever $b \leq a$. Indeed, that is a consequence of the inclusion $E_b \circ (b \oplus b) \subseteq (a \oplus a)$ for every $b \leq a$. It remains to show (c). Let $E \in \mathcal{E}_B$, say $E \supseteq \bigcap_{i=1}^n E_{b_i}$ for some $b_1, \ldots, b_n \in B$. Using (C.1) and the fact that each b_i is complemented, we have

$$1 = \phi(\bigwedge_{i=1}^{n}(b_i \vee b_i^*)) = \bigwedge_{i=1}^{n}(\phi(b_i) \vee \phi(b_i^*)) = \bigvee\{\phi(b_P \wedge \bar{b}_P) \mid P \subseteq [n]\},$$

where $[n] := \{1, \ldots, n\}$ and, for $P \subseteq [n]$, we denote $b_P := \bigwedge_{i \in P} b_i$ and $\bar{b}_P := \bigwedge_{i \notin P} b_i^*$. Since, for every $P \subseteq [n]$, we have $(b_P \wedge \bar{b}_P, b_P \wedge \bar{b}_P) \in \bigcap_{i=1}^n E_{b_i} \subseteq E$, it then follows that

$$1 = \bigvee\{\phi(b_P \wedge \bar{b}_P) \mid P \subseteq [n]\} \leq \bigvee\{\phi(x) \mid (x, x) \in E\},$$

which proves (c). ∎

In what follows, we fix a Frith frame (L, S) and a frame M. Recall from the previous section that, for every Frith frame (L, S), there is a dense extremal epimorphism $c : \mathrm{Idl}(S) \twoheadrightarrow L$ defined by $c(J) = \bigvee J$. Since this is a frame homomorphism, it has a right adjoint $c_* : L \hookrightarrow \mathrm{Idl}(S)$ which is determined by the Galois connection

$$\forall a \in L,\ J \in \mathrm{Idl}(S), \quad c(J) \leq a \iff J \subseteq c_*(a).$$

In particular, for every $a \in L$, we have

$$c_*(a) = \bigvee\{J \in \mathrm{Idl}(S) \mid \bigvee J \leq a\}. \tag{8}$$

Lemma 7.13. *For every $a \in L$, we have $c_*(a) = {\downarrow}a \cap S$. In particular, the map c is a right inverse of c_*, that is, $c \circ c_* = \mathsf{id}$.*

Proof: Let $s \in S$. By (8), we have that $s \in c_*(a)$ if and only if there are $J_1, \ldots, J_n \in \mathrm{Idl}(S)$, and $s_i \in J_i$ (for $i = 1, \ldots, n$) such that $\bigvee J_i \leq a$ and $s \leq s_1 \vee \cdots \vee s_n$. Clearly, this holds if and only if $s \leq a$ and thus, we have $c_*(a) = {\downarrow}a \cap S$. Since S is join-dense in L, it then follows that $c \circ c_*(a) = c({\downarrow}a \cap S) = \bigvee({\downarrow}a \cap S) = a$. ∎

We now consider the function $\lambda : L \hookrightarrow \mathcal{C}_S\mathrm{Idl}(S)$ obtained by composing the injections $c_* : L \hookrightarrow \mathrm{Idl}(S)$ and $\nabla : \mathrm{Idl}(S) \hookrightarrow \mathcal{C}_S\mathrm{Idl}(S)$. Explicitly, λ sends the element $a \in L$ to the congruence $\nabla_{{\downarrow}a \cap S}$. Notice that λ defines a Cauchy map $\lambda : (L, S) \to \mathcal{C}_S\mathrm{Idl}(S)$. Indeed, since $c_*(s) = {\downarrow}s \cap S$ for every $s \in S$ and ∇ is a frame homomorphism, we have that λ restricts to a lattice homomorphism with domain S, that is, (C.1) holds. Moreover, since ${\downarrow}a \cap S$ is the ideal generated by $\bigcup\{{\downarrow}s \cap S \mid s \in S,\ s \leq a\}$, we have (C.2). Finally, by definition, each $\lambda(s) = \nabla_s$ is complemented in $\mathcal{C}_S\mathrm{Idl}(S)$, and so, we have (C.3).

Theorem 7.14. *For every Cauchy map $\phi : (L, S) \to M$, there exists a frame homomorphism $g : \mathcal{C}_S\mathrm{Idl}(S) \to M$ such that the following diagram commutes:*

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[dr, "\phi"'] & \mathcal{C}_S\mathrm{Idl}(S) \arrow[d, dashed, "g"] \\ & M \end{tikzcd}$$

Proof: Since $\mathcal{C}_S\mathrm{Idl}(S)$ is generated, as a frame, by the set of congruences $\{\nabla_s, \Delta_s \mid s \in S\}$ and frame homomorphisms preserve pairs of complemented elements, if g exists, then it is completely determined by its restriction to $\{\nabla_s \mid s \in S\}$, which must satisfy $g(\nabla_s) = g \circ \lambda(s) = \phi(s)$, for every $s \in S$. By (C.1), ϕ restricts to a lattice homomorphism $\phi|_S : S \to M$ and, by Corollary 2.6, $\phi|_S$ uniquely extends to frame homomorphism $\widehat{\phi|_S} : \mathrm{Idl}(S) \to M$. Since, by (C.3), each $\phi(s)$, with $s \in S$, is complemented in M, we may then use Proposition 2.3 to derive the existence of a unique frame homomorphism $g : \mathcal{C}_S\mathrm{Idl}(S) \to M$ satisfying $g(\nabla_s) = \phi(s)$, for every $s \in S$. To see that g makes the diagram commute, we may use (C.2) for λ and for ϕ and the fact that g is a frame homomorphism to compute

$$\begin{aligned} g \circ \lambda(a) &= g\left(\bigvee\{\lambda(s) \mid s \in S,\ s \leq a\}\right) = \bigvee\{g \circ \lambda(s) \mid s \in S,\ s \leq a\} \\ &= \bigvee\{\phi(s) \mid s \in S,\ s \leq a\} = \phi(a), \end{aligned}$$

for every $a \in L$. ■

Theorem 7.15. *A Frith frame (L, S) is complete if and only if every Cauchy map $(L, S) \to M$ is a frame homomorphism.*

Proof: If (L, S) is complete, then c is an isomorphism and thus, c_*, hence λ, is a frame homomorphism. Since by Theorem 7.14 every Cauchy map factors through λ via a frame homomorphism, it follows that every Cauchy map is itself a frame homomorphism.

Conversely, if every Cauchy map is a frame homomorphism then λ is a frame homomorphism. In particular, λ induces a morphism of Frith frames $\lambda : (L, S) \to (\mathcal{C}_S\mathrm{Idl}(S), \overline{S})$. On the other hand, by Corollary 7.6 and by definition of complete Frith frame, we have that $(\mathcal{C}_S\mathrm{Idl}(S), \overline{S}) = \mathbf{Sym}_{\mathrm{Frith}}(\mathrm{Idl}(S), S)$ is complete. Therefore, (L, S) is complete provided the symmetric reflection

$\overline{\lambda} : (\mathcal{C}_S L, \overline{S}) \to (\mathcal{C}_S \mathrm{Idl}(S), \overline{S})$ of λ is a dense extremal epimorphism. That is the case because, for every $s \in S$, we have $\overline{\lambda}(\nabla_s) = \nabla_s$ and $\overline{\lambda}(\Delta_s) = \Delta_s$. ■

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