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# DIFFERENCE–RESTRICTION ALGEBRAS OF PARTIAL FUNCTIONS: DISCRETE DUALITY AND COMPLETION

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ABSTRACT: We exhibit an adjunction between a category of abstract algebras of partial functions and a category of set quotients. The algebras are those atomic algebras representable as a collection of partial functions closed under relative complement and domain restriction; the morphisms are the complete homomorphisms. This generalises the discrete adjunction between the atomic Boolean algebras and the category of sets. We define the compatible completion of a representable algebra, and show that the monad induced by our adjunction yields the compatible completion of any atomic representable algebra. As a corollary, the adjunction restricts to a duality on the compatibly complete atomic representable algebras, generalising the discrete duality between complete atomic Boolean algebras and sets. We then extend these adjunction, duality, and completion results to representable algebras equipped with arbitrary additional completely additive and compatibility preserving operators.

KEYWORDS: Partial function, complete representation, duality, compatible completion, completely additive operators.

## 1. Introduction

The study of algebras of partial functions is an active area of research that investigates collections of partial functions and their interrelationships from an algebraic perspective. The partial functions are treated as abstract elements that may be combined algebraically using various natural operations such as composition, domain restriction, 'override', or 'update'. In pure mathematics, algebras of partial functions arise naturally as structures such as inverse semigroups [41], pseudogroups [31], and skew lattices [32]. In theoretical computer science, they appear in the theories of finite state transducers [10], computable functions [21], deterministic propositional dynamic logics [20], and separation logic [18]. Many different selections of operations have been

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considered, each leading to a different class/category of abstract algebras [38, 11, 40, 39, 6, 19, 20, 17, 3, 22]. (See [34, §3.2] for a guide to this literature.) Recently, dualities for some of these categories have started to appear [28, 29, 27, 30, 31, 35, 2, 26], opening the way for these algebras to be studied via their duals, as has been done successfully for the algebraisations of classical and many non-classical propositional logics [14, 9, 7, 13].

In [5], we initiated a project to develop a general and modular framework for producing and understanding dualities for such categories. For this we are inspired strongly by Jónsson and Tarski's theory of Boolean algebras with operators [24] and the duality between them and descriptive general frames [4, Chapter 5: Algebras and General Frames]. Our central thesis is that in our case the appropriate base class—the analogue of Boolean algebras—must be more than just a class of *ordered* structures but must record additional *compatibility* data. This reflects the fact that the union of two partial functions is not always a function (and this determination can not be made solely from the inclusion/extension ordering).

In [5], we investigated algebras of partial functions for a signature that we believe provides the necessary order and compatibility structure. The signature has two operations, both binary: the standard set-theoretic *relative complement* operation and a *domain restriction* operation. We gave and proved a finite equational axiomatisation for the class of isomorphs of such algebras of partial functions [5, Theorem 5.7].

In the present paper we continue our project with an investigation of 'discrete' duality—a term used for dualities requiring no topological information on the duals. A secondary component of our thesis relates specifically to such dualities. Recall the prototypical discrete duality between complete atomic Boolean algebras and sets, and that this extends to a duality between complete atomic Boolean algebras with completely additive operators and Kripke frames. A first observation is that these dualities are just specialisations of (contravariant) adjunctions, where we drop the completeness requirement on the algebra side. A second observation is that for Boolean algebras, the 'atomic' condition that remains is an intrinsic/first-order characterisation of the more extrinsic/semantic condition of being *completely representable* (arbitrary cardinality joins/meets become unions/intersections, respectively). Hence we posit the general principle that the correct class to use for a discrete adjunction is always the class of completely representable algebras.

Following this reasoning, in [5] we identified the completely representable algebras of our class. Just as for Boolean algebras, they are exactly the atomic ones [5, Theorem 6.16]. The main results of the present paper are the elaboration of a discrete adjunction between this class of atomic representable algebras and a certain class of set quotients (Theorem 3.9) and the extension of that theorem to algebras with additional operators (Theorem 5.6). We also show that, as for Boolean algebras, the monad induced by the adjunction gives an appropriate form of completion (the *compatible completion*) of algebras (Theorem 4.12/Corollary 5.14) and that the adjunction restricts to a duality on the compatibly complete algebras (Corollary 4.13/Corollary 5.15).

*Structure of paper.* Section 2 contains preliminaries, including formal definitions of the classes of representable and of completely representable algebras. We recall the axiomatisations of these two classes as presented in [5].

In Section 3, we present and prove our central result: the adjunction between the atomic representable algebras and a category of set quotients (Theorem 3.9).

Section 4 concerns completion. We define the notions of compatibly complete (Definition 4.3) and of a compatible completion (Definition 4.7), and we prove that compatible completions are unique up to isomorphism (Proposition 4.11). We prove that the monad induced by our adjunction yields the compatible completion on atomic representable algebras (Theorem 4.12), and we conclude that the adjunction restricts to a duality on the compatibly complete atomic representable algebras (Corollary 4.13).

Section 5 concerns additional operations. We define the notion of a compatibility preserving completely additive operator (Definition 5.1) and extend the adjunction (Theorem 5.6), completion (Corollary 5.14), and duality (Corollary 5.15) results of the previous two sections to representable algebras equipped with such operators.

## 2. Algebras of functions

Given an algebra $\mathfrak{A}$, when we write $a \in \mathfrak{A}$ or say that $a$ is an element of $\mathfrak{A}$, we mean that $a$ is an element of the domain of $\mathfrak{A}$. Similarly for the notation $S \subseteq \mathfrak{A}$ or saying that $S$ is a subset of $\mathfrak{A}$. We follow the convention that algebras are always nonempty. If $S$ is a subset of the domain of a map $\theta$ then $\theta[S]$ denotes the set $\{\theta(s) \mid s \in S\}$. We use $\sum$ and $\prod$ respectively as our default notations for joins (suprema) and meets (infima).

We begin by making precise what is meant by partial functions and algebras of partial functions.

**Definition 2.1.** *Let $X$ and $Y$ be sets. A* ***partial function*** *from $X$ to $Y$ is a subset $f$ of $X \times Y$ validating*

$$(x, y) \in f \text{ and } (x, z) \in f \implies y = z.$$

*If $X = Y$ then $f$ is called simply a partial function on $X$. For a partial function $f \subseteq X \times Y$, if $(x, y)$ belongs to $f$ then we may write $y = f(x)$. Given such a partial function, its* ***domain*** *is the set*

$$\mathrm{dom}(f) := \{x \in X \mid \exists\, y \in Y : (x, y) \in f\}.$$

*For any binary relation $R \subseteq X \times Y$, we write $R^{-1}$ for the relation $\{(y, x) \mid (x, y) \in R\}$. Notice that a partial function $f \subseteq X \times Y$ is injective if and only if $f^{-1}$ is also a partial function. Finally, for any binary relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, we denote by $S \circ R$ (or simply $SR$) the composition of $R$ and $S$:*

$$S \circ R := \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in R \text{ and } (y, z) \in S\}.$$

*When $R$ and $S$ are partial functions, this is their usual composition.*

**Definition 2.2.** *An* ***algebra of partial functions*** *of the signature $\{-, \triangleright\}$ is a universal algebra $\mathfrak{A} = (A, -, \triangleright)$ where the elements of the universe $A$ are partial functions from some (common) set $X$ to some (common) set $Y$ and the interpretations of the symbols are given as follows:*

- *The binary operation $-$ is* ***relative complement****:*

$$f - g := \{(x, y) \in X \times Y \mid (x, y) \in f \text{ and } (x, y) \notin g\}.$$

- *The binary operation $\triangleright$ is* ***domain restriction****.* * *It is the restriction of the second argument to the domain of the first; that is:*

$$f \triangleright g := \{(x, y) \in X \times Y \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}.$$

Note that in algebras of partial functions of the signature $\{-, \triangleright\}$, the set-theoretic intersection of two elements $f$ and $g$ can be expressed as $f - (f - g)$. We use the symbol $\cdot$ for this derived operation.

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*This operation has historically been called *restrictive multiplication*, where *multiplication* is the historical term for *composition*. But we do not wish to emphasise this operation as a form of composition.

We also observe that, without loss of generality, we may assume $X = Y$ (a common stipulation for algebras of partial functions). Indeed, if $\mathfrak{A}$ is a $\{-, \triangleright\}$-algebra of partial functions from $X$ to $Y$, then it is also a $\{-, \triangleright\}$-algebra of partial functions from $X \cup Y$ to $X \cup Y$. In this case, this non-uniquely-determined single set is called 'the' **base**. However, certain properties may not be preserved by changing the base. For instance, while a partial function is injective as a function on $X$ if and only if it is injective as a function on $X'$, this is not the case for surjectivity.

**Definition 2.3.** *An algebra $\mathfrak{A}$ of the signature $\{-, \triangleright\}$ is **representable** (by partial functions) if it is isomorphic to an algebra of partial functions. An isomorphism from $\mathfrak{A}$ to an algebra of partial functions is a **representation** of $\mathfrak{A}$.*

Just as for algebras of partial functions, for any $\{-, \triangleright\}$-algebra $\mathfrak{A}$, we will consider the derived operation $\cdot$ defined by

$$a \cdot b \coloneqq a - (a - b).$$

In [5] it was shown that the class of $\{-, \triangleright\}$-algebras that is representable by partial functions is axiomatised by the following set of equations.

(Ax.1) $a - (b - a) = a$
(Ax.2) $a \cdot b = b \cdot a$
(Ax.3) $(a - b) - c = (a - c) - b$
(Ax.4) $(a \triangleright c) \cdot (b \triangleright c) = (a \triangleright b) \triangleright c$
(Ax.5) $(a \cdot b) \triangleright a = a \cdot b$

**Theorem 2.4** ([5, Theorem 5.7])**.** *The class of $\{-, \triangleright\}$-algebras representable by partial functions is a variety, axiomatised by the finite set of equations (Ax.1) – (Ax.5).*

Algebras satisfying axioms (Ax.1) – (Ax.3) are called **subtraction algebras** and it is known that in those algebras the $\cdot$ operation gives a semilattice structure (which we view as a meet-semilattice), and the downsets of the form $a^{\downarrow} \coloneqq \{x \mid x \leq a\}$ are Boolean algebras [39]. In particular, the same holds for representable $\{-, \triangleright\}$-algebras.

It has long been known that in a representable algebra the operation $\triangleright$ is associative (see, for example, [43]). Moreover, the inequality $a \triangleright b \leq b$ is valid (an algebraic proof appears in [5]) and will be often used without further mention.

The next two definitions apply to any function between posets $\mathfrak{P}$ and $\mathfrak{Q}$. So in particular, these definitions apply to homomorphisms of Boolean algebras and homomorphisms of representable $\{-, \rhd\}$-algebras. We denote meets and joins in $\mathfrak{P}$ by $\prod$ and $\sum$ respectively, and meets and joins in $\mathfrak{Q}$ by $\bigcap$ and $\bigcup$ respectively.

**Definition 2.5.** *A function $h\colon \mathfrak{P} \to \mathfrak{Q}$ is **meet complete** if, for every nonempty subset $S$ of $\mathfrak{P}$, if $\prod S$ exists, then so does $\bigcap h[S]$ and*

$$h(\prod S) = \bigcap h[S].$$

**Definition 2.6.** *A function $h\colon \mathfrak{P} \to \mathfrak{Q}$ is **join complete** if, for every subset $S$ of $\mathfrak{P}$, if $\sum S$ exists, then so does $\bigcup h[S]$ and*

$$h(\sum S) = \bigcup h[S].$$

Note that $S$ is required to be nonempty in Definition 2.5, but not in Definition 2.6. For homomorphisms of Boolean algebras, being meet complete is equivalent to being join complete, and in [5] we showed that the same is true for homomorphisms of representable $\{-, \rhd\}$-algebras (Corollary 6.5 and Corollary 6.6 there). So in these cases we may simply describe such a homomorphism using the adjective **complete**.[†]

**Definition 2.7.** *A **complete representation** of an algebra of the signature $\{-, \rhd\}$ is a representation $\theta$, with base $X$ say, such that $\theta$ forms a complete homomorphism when viewed as an embedding into the algebra of* all *partial functions on $X$. An algebra is **completely representable** if it has a complete representation.*

Complete representations have been studied previously in the context of various different forms of representability: by sets [8], by binary or higher-order relations [15, 16], or by partial functions [33].

**Definition 2.8.** *Let $\mathfrak{P}$ be a poset with a least element, $0$. An **atom** of $\mathfrak{P}$ is a minimal nonzero element of $\mathfrak{P}$. We write $\mathrm{At}(\mathfrak{P})$ for the set of atoms of $\mathfrak{P}$. We say that $\mathfrak{P}$ is **atomic** if every nonzero element is greater than or equal to an atom.*

In [5], it was shown that a $\{-, \rhd\}$-algebra is completely representable if and only if it is both representable and atomic.

[†]In the case of representations of Boolean algebras as fields of sets, the adjectives *strong* and *regular* have also been used.

**Theorem 2.9** ([5, Theorem 6.16]). *The class of $\{-, \triangleright\}$-algebras that are completely representable by partial functions is axiomatised by the finite set of equations (Ax.1) – (Ax.5) together with the $\forall\exists\forall$ first-order formula stating that the algebra is atomic.*

This theorem justifies our interest in the atomic representable algebras, and also indicates that complete homomorphism is the appropriate notion of morphism between these algebras.

This is a good place to define two further terms that we will use later.

**Definition 2.10.** *Let $\mathfrak{P}$ be a poset. A subset $S$ of $\mathfrak{P}$ is* ***join dense*** *(in $\mathfrak{P}$) if each $p \in \mathfrak{P}$ is the join $\sum T$ of some subset $T$ of $S$. The poset $\mathfrak{P}$ is* ***atomistic*** *if $\mathrm{At}(\mathfrak{P})$ is join dense in $\mathfrak{P}$.*

Of course, atomic implies atomistic, for any poset. For representable $\{-, \triangleright\}$-algebras, the converse is also true [5, Lemma 6.13], generalising the same statement for Boolean algebras.

## 3. Discrete adjunction for atomic representable algebras

In this section we exhibit a contravariant adjunction between the atomic representable algebras, **AtRepAlg**, and a certain category $\mathbf{Set_q}$ whose objects are quotients of sets.

We use $\twoheadrightarrow$ to indicate a (total) surjective function between sets, and we use $\hookrightarrow$ to indicate an embedding of algebras. The notation $\rightharpoonup$ indicates a partial function. We may at times use a bracket-free notation for applications of functors to morphisms, for example, $Fh$ in place of $F(h)$.

Before giving the claimed adjunction, we define the two categories involved.

**Definition 3.1.** *We denote by* **AtRepAlg** *the category whose objects are atomic $\{-, \triangleright\}$-algebras representable by partial functions, and whose morphisms are complete homomorphisms of $\{-, \triangleright\}$-algebras.*

**Definition 3.2.** *We denote by $\mathbf{Set_q}$ the category whose objects are set quotients (that is, surjective functions between sets) $\pi\colon X \twoheadrightarrow X_0$, and where a morphism from $\pi\colon X \twoheadrightarrow X_0$ to $\rho\colon Y \twoheadrightarrow Y_0$ is a partial function $\varphi\colon X \rightharpoonup Y$ satisfying the following conditions:*

*(Q.1) $\varphi$* ***preserves equivalence****: if both $\varphi(x)$ and $\varphi(x')$ are defined, then*

$$\pi(x) = \pi(x') \implies \rho(\varphi(x)) = \rho(\varphi(x')).$$

*In particular, $\varphi$ induces a partial function $\widetilde{\varphi}\colon X_0 \rightharpoonup Y_0$ given by*

$$\widetilde{\varphi} \coloneqq \{(\pi(x), \rho(\varphi(x))) \mid x \in \mathrm{dom}(\varphi)\}.$$

*(Q.2) $\varphi$ is **fibrewise injective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the restriction and co-restriction of $\varphi$ induces an injective partial map*

$$\varphi_{(x_0,y_0)}\colon \pi^{-1}(x_0) \rightharpoonup \rho^{-1}(y_0),$$

*(Q.3) $\varphi$ is **fibrewise surjective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the induced partial map $\varphi_{(x_0,y_0)}$ is surjective (that is, the image of $\varphi_{(x_0,y_0)}$ is the whole of $\rho^{-1}(y_0)$).*[‡]

In what follows, we define two functors $F\colon \mathbf{AtRepAlg} \to {\mathbf{Set_q}}^{\mathrm{op}}$ and $G\colon {\mathbf{Set_q}}^{\mathrm{op}} \to \mathbf{AtRepAlg}$, which we then show to form an adjunction (Theorem 3.9).

Before defining $F$, we first recall some notation from [5].

**Definition 3.3.** *Given a representable $\{-, \rhd\}$-algebra $\mathfrak{A}$, the relation $\preceq_{\mathfrak{A}}$ on $\mathfrak{A}$ is defined by*

$$a \preceq_{\mathfrak{A}} b \iff a \leq b \rhd a$$

*and is a preorder. We denote by $\sim_{\mathfrak{A}}$ the equivalence relation induced by $\preceq_{\mathfrak{A}}$, and for a given $a \in \mathfrak{A}$ we use $[a]$ to denote the equivalence class of $a$.*

In the case that $\mathfrak{A}$ is an actual $\{-, \rhd\}$-algebra of partial functions, the relation $\preceq_{\mathfrak{A}}$ is the *domain inclusion* relation $f \preceq_{\mathfrak{A}} g \iff \mathrm{dom}(f) \subseteq \mathrm{dom}(g)$.[§]

The next result summarises some facts about $\preceq_{\mathfrak{A}}$ that were proved in [5] and will be used later.

**Proposition 3.4.** *The following statements hold for a representable $\{-, \rhd\}$-algebra $\mathfrak{A}$.*

*(a) The relation $\leq$ is contained in $\preceq_{\mathfrak{A}}$, and $[0] = \{0\}$.*
*(b) The poset $\mathfrak{A}/{\sim_{\mathfrak{A}}}$ is a meet-semilattice (actually a subtraction algebra) with meet given by $[a] \wedge [b] = [a \rhd b]$.*
*(c) The relations $\leq$ and $\preceq_{\mathfrak{A}}$ coincide in each downset $a^{\downarrow}$. Moreover, the assignment $b \mapsto [b]$ provides an isomorphism between the Boolean algebras $a^{\downarrow}$ and $[a]^{\downarrow}$.*

[‡]Note that in the context of partial maps, the conjunction of 'injective' and 'surjective' is not 'bijective' in the sense of a one-to-one correspondence.

[§]See [37] for axiomatisations for various signatures containing domain inclusion (as a fundamental relation).

**3.1. The functor $F\colon \mathbf{AtRepAlg} \to {\mathbf{Set_q}}^{\mathrm{op}}$.**

We let $F\colon \mathbf{AtRepAlg} \to {\mathbf{Set_q}}^{\mathrm{op}}$ be defined as follows. Given an atomic representable algebra $\mathfrak{A}$, the set quotient $F(\mathfrak{A})$ is the canonical projection $\pi_{\mathfrak{A}}\colon \mathrm{At}(\mathfrak{A}) \twoheadrightarrow \mathrm{At}(\mathfrak{A})/{\sim_{\mathfrak{A}}}$. This defines $F$ on the objects. By Proposition 3.4(c), for a Boolean algebra $\mathfrak{B}$, the relation $\sim_{\mathfrak{B}}$ is the identity and thus the restriction of $F$ to atomic Boolean algebras is simply $\mathrm{At}(\_)$. For defining $F$ on the morphisms, we let $h\colon \mathfrak{A} \to \mathfrak{B}$ be a complete homomorphism of atomic representable algebras. By [5, Lemma 6.3], for each $a \in \mathfrak{A}$, the restriction of $h$ induces a complete homomorphism of (atomic) Boolean algebras $h_a\colon a^{\downarrow} \to h(a)^{\downarrow}$. We denote by $\varphi_a : \mathrm{At}(h(a)^{\downarrow}) \to \mathrm{At}(a^{\downarrow})$ its discrete dual. Recall that $\varphi_a$ is completely determined by the following Galois connection.

$$\forall\ a' \in a^{\downarrow},\ y \in \mathrm{At}(h(a)^{\downarrow}) \quad (\varphi_a(y) \leq a' \iff y \leq h(a')) \tag{1}$$

In particular, for every $a \in \mathfrak{A}$, the map $h \circ \varphi_a$ is a closure operator.

Then $Fh\colon \mathrm{At}(\mathfrak{B}) \rightharpoonup \mathrm{At}(\mathfrak{A})$ has domain $\mathfrak{B}_0 = \bigcup_{a\in\mathfrak{A}} \mathrm{At}(h(a)^{\downarrow})$, and for $y \in \mathrm{At}(h(a)^{\downarrow})$, we set $Fh(y) = \varphi_a(y)$. We observe that this is a well-defined map, that is, if $a_1, a_2 \in \mathfrak{A}$ are such that $y \in \mathrm{At}(\mathfrak{B})$ is below $h(a_1)$ and $h(a_2)$, then $\varphi_{a_1}(y) = \varphi_{a_2}(y)$. Indeed, since $h$ is a homomorphism, we have $y \leq h(a_1 \cdot a_2)$, and by (1) this yields $\varphi_{a_2}(y) \leq a_1 \cdot a_2$. Using again (1) twice, we have

$$\varphi_{a_2}(y) \leq \varphi_{a_2}(y) \iff y \leq h(\varphi_{a_2}(y)) \iff \varphi_{a_1}(y) \leq \varphi_{a_2}(y).$$

Similarly, we can prove the inequality $\varphi_{a_2}(y) \leq \varphi_{a_1}(y)$, thereby concluding that $Fh$ is well-defined. As a consequence of $h \circ \varphi_a$ being a closure operator for each $a$, we have the following.

$$\forall\ y \in \mathfrak{B}_0 \quad y \leq (h \circ Fh)(y) \tag{2}$$

We now prove that $Fh$ defines a morphism in $\mathbf{Set_q}$ from $\pi_{\mathfrak{B}}$ to $\pi_{\mathfrak{A}}$. Let $y_1, y_2 \in \mathfrak{B}_0$ be $\sim_{\mathfrak{B}}$-equivalent, that is, $y_1 = y_2 \rhd y_1$ and $y_2 = y_1 \rhd y_2$. Since $y_i \leq (h \circ Fh)(y_i)$, for $i = 1, 2$, using the fact that $\rhd$ is order preserving in both coordinates and the fact that $h$ is a homomorphism, we have

$$y_1 = y_2 \rhd y_1 \leq h(Fh(y_2)) \rhd h(Fh(y_1)) = h(Fh(y_2) \rhd Fh(y_1)).$$

We may apply (1) to obtain

$$Fh(y_1) \leq Fh(y_2) \rhd Fh(y_1),$$

that is, $Fh(y_1) \preceq_{\mathfrak{A}} Fh(y_2)$. Likewise, we can show that $Fh(y_2) \preceq_{\mathfrak{A}} Fh(y_1)$, and thus $Fh(y_1) \sim_{\mathfrak{A}} Fh(y_2)$. This proves (Q.1).

Finally, to conclude that $Fh$ defines a morphism in $\mathbf{Set_q}$, we only need to show that for every $(y_0, x_0) \in \widetilde{Fh}$, the partial map $Fh_{(y_0,x_0)} \colon \pi_{\mathfrak{B}}^{-1}(y_0) \rightharpoonup \pi_{\mathfrak{A}}^{-1}(x_0)$ is both injective and surjective. (Recall that the subscript $(y_0, x_0)$ indicates restricting and co-restricting to the fibres of $y_0$ and $x_0$ respectively; see Definition 3.2(Q.2).) Injectivity of $Fh_{(y_0,x_0)}$ is a consequence of Proposition 3.4(c) together with the observation that if $y_1, y_2 \in \mathfrak{B}_0$ are such that $x \coloneqq Fh(y_1) = Fh(y_2)$, then $y_1$ and $y_2$ have the common upper bound $h(x)$.

To see that $Fh_{(y_0,x_0)}$ is surjective, take $x \in \pi_{\mathfrak{A}}^{-1}(x_0)$. As $\widetilde{Fh}$ is defined on $y_0$, by the definition of $\widetilde{Fh}$ from $Fh$ we know $Fh$ is defined on at least one element $y$ of $\pi_{\mathfrak{B}}^{-1}(y_0)$, and that $Fh(y) \sim_{\mathfrak{A}} x$. We show that $Fh_{(y_0,x_0)}(y \rhd h(x)) = x$. Using (2) and then applying the fact that $h$ is a homomorphism to $Fh(y) \sim_{\mathfrak{A}} x$, we obtain $y \leq (h \circ Fh)(y) \sim_{\mathfrak{B}} h(x)$, from which we obtain $y \preceq_{\mathfrak{B}} h(x)$. By [5, Lemma 6.14(b)], $y \rhd h(x)$ is an atom of $\mathfrak{B}$ (hence of $h(x)^{\downarrow}$), which is $\sim_{\mathfrak{B}}$-equivalent to $y$. From $y \rhd h(x) \leq h(x)$ we obtain $\varphi_x(y \rhd h(x)) \leq x$ and hence $Fh(y \rhd h(x)) = x$. Since $y \rhd h(x) \sim_{\mathfrak{B}} y$ we have $Fh_{(y_0,x_0)}(y \rhd h(x)) = x$, as required.

To summarise, we have proved the following.

**Proposition 3.5.** *There is a functor*

$$F \colon \mathbf{AtRepAlg} \to \mathbf{Set_q}^{\mathrm{op}}$$

*that maps an atomic representable* $\{-, \rhd\}$*-algebra* $\mathfrak{A}$ *to the canonical projection* $\pi_{\mathfrak{A}} \colon \mathrm{At}(\mathfrak{A}) \twoheadrightarrow \mathrm{At}(\mathfrak{A})/{\sim_{\mathfrak{A}}}$.

### 3.2. The functor $G \colon \mathbf{Set_q}^{\mathrm{op}} \to \mathbf{AtRepAlg}$.

We now define the functor $G \colon \mathbf{Set_q}^{\mathrm{op}} \to \mathbf{AtRepAlg}$. For a set quotient $\pi \colon X \twoheadrightarrow X_0$, we let $G(\pi) = \mathfrak{A}_\pi$ be the algebra of partial functions consisting of all partial functions $f \colon X_0 \rightharpoonup X$ that are a subset of $\pi^{-1} = \{(\pi(x), x) \mid x \in X\}$. Since for every $f, g \in \mathfrak{A}_\pi$, both $f - g$ and $g \rhd f$ are subsets of $f$, and $\mathfrak{A}_\pi$ is closed under subsets, $f - g$ and $g \rhd f$ belong to $\mathfrak{A}_\pi$.

**Lemma 3.6.** *For every quotient* $\pi \colon X \twoheadrightarrow X_0$*, the* $\{-, \rhd\}$*-algebra* $\mathfrak{A}_\pi$ *is atomic and representable.*

*Proof*: The algebra $\mathfrak{A}_\pi$ is an actual algebra of partial functions, hence $\mathfrak{A}_\pi$ is, trivially, representable. The minimal element of $\mathfrak{A}_\pi$ is of course the empty function. The minimal nonzero elements are then precisely the singletons $\{(\pi(x), x)\}$, for $x \in X$. Every nonzero element of $\mathfrak{A}_\pi$ includes one of these singletons, and hence $\mathfrak{A}_\pi$ is atomic. ■

Let us define $G$ on morphisms. Given a morphism $\varphi$ from $(\pi\colon X \twoheadrightarrow X_0)$ to $(\rho\colon Y \twoheadrightarrow Y_0)$ in $\mathbf{Set_q}$ (that is, a morphism from $\rho$ to $\pi$ in ${\mathbf{Set_q}}^{\mathrm{op}}$), we let $G\varphi\colon \mathfrak{A}_\rho \to \mathfrak{A}_\pi$ assign to each partial function $g \in \mathfrak{A}_\rho$, the partial function $G\varphi(g)\colon X_0 \rightharpoonup X$ given by

$$G\varphi(g) = \{(\pi(x), x) \in X_0 \times X \mid \exists y \in Y\colon (x, y) \in \varphi \text{ and } (\rho(y), y) \in g\}.$$

Using the fact that $\varphi$ is injective on each fibre of $\pi$, it is easy to check that $G\varphi(g)$ is indeed a partial function and thus an element of $\mathfrak{A}_\pi$. We remark that since $\varphi$ is a partial function, when $(\pi(x), x)$ belongs to $G\varphi(g)$ there is in fact *a unique* $y \in Y$, namely $y = \varphi(x)$, so that $(x, y) \in \varphi$ and $(\rho(y), y) \in g$. Thus, $G\varphi$ may be alternatively described as

$$G\varphi(g) = \{(\pi(x), x) \mid x \in \mathrm{dom}(\varphi) \text{ and } (\rho(\varphi(x)), \varphi(x)) \in g\}. \tag{3}$$

**Lemma 3.7.** *For every morphism $\varphi$ as above, the function $G\varphi$ defines a complete homomorphism of $\{-, \rhd\}$-algebras.*

*Proof*: The fact that $G\varphi$ preserves $-$ is a trivial consequence of (3). To see that $G\varphi$ preserves $\rhd$, it suffices to show that, for every $x \in \mathrm{dom}(\varphi)$ and $g \in \mathfrak{A}_\rho$, we have

$$\rho(\varphi(x)) \in \mathrm{dom}(g) \iff \pi(x) \in \mathrm{dom}(G\varphi(g)). \tag{4}$$

Indeed, given any $f, g \in \mathfrak{A}_\rho$, the two partial-function images $G\varphi(g \rhd f)[X_0]$ and $(G\varphi(g) \rhd G\varphi(f))[X_0]$ are subsets of $\varphi^{-1}(f[Y_0])$, and for all $x \in \varphi^{-1}(f[Y_0])$ (that is, for all $x \in \mathrm{dom}(\varphi)$ such that $(\rho(\varphi(x)), \varphi(x)) \in f$) we have $x \in G\varphi(g \rhd f)[X_0] \iff \rho(\varphi(x)) \in \mathrm{dom}(g)$ and $x \in (G\varphi(g) \rhd G\varphi(f))[X_0] \iff \pi(x) \in \mathrm{dom}(G\varphi(g))$. Thus the images of the injective partial functions $G\varphi(g \rhd f)$ and $G\varphi(g) \rhd G\varphi(f)$ are equal provided (4) holds.

To prove (4), first suppose that $\rho(\varphi(x)) \in \mathrm{dom}(g)$. Then there exists $y \in Y$ so that $(\rho(\varphi(x)), y) \in g \subseteq \rho^{-1}$, and since $\varphi$ is fibrewise surjective by (Q.3), there exists $x' \in \mathrm{dom}(\varphi)$ so that $\pi(x') = \pi(x)$ and $\varphi(x') = y$. Then $(\pi(x'), x') \in G\varphi(g)$, so $\pi(x) = \pi(x')$ belongs to the domain of $\mathrm{dom}(G\varphi(g))$. Conversely, if $\pi(x) \in \mathrm{dom}(G\varphi(g))$, then there exists some $x' \in \mathrm{dom}\,\varphi$ such that $\pi(x) = \pi(x')$ and $(\rho(\varphi(x')), \varphi(x')) \in g$. Using (Q.1), we have $\rho(\varphi(x')) = \rho(\varphi(x))$, implying that $\rho(\varphi(x)) \in \mathrm{dom}(g)$ as required.

Finally, since infima in $\mathfrak{A}_\pi$ and $\mathfrak{A}_\rho$ are given by intersections, by the definition of $G\varphi$ we easily conclude that $G\varphi$ is a meet-complete, hence complete, homomorphism. ■

We have proved the following.

**Proposition 3.8.** *There is a functor*

$$G\colon \mathbf{Set_q}^{\mathrm{op}} \to \mathbf{AtRepAlg}$$

*that sends a quotient $\pi\colon X \twoheadrightarrow X_0$ to the $\{-, \triangleright\}$-algebra $\mathfrak{A}_\pi$ consisting of all partial functions included in $\pi^{-1} = \{(\pi(x), x) \mid x \in X\}$.*

We can now prove that we have the claimed adjunction.

**Theorem 3.9.** *The functors $F\colon \mathbf{AtRepAlg} \to \mathbf{Set_q}^{\mathrm{op}}$ and $G\colon \mathbf{Set_q}^{\mathrm{op}} \to \mathbf{AtRepAlg}$ form an adjunction. That is*

$$F \dashv G.$$

*Proof*: We only need to define natural transformations $\eta\colon \mathrm{Id}_{\mathbf{AtRepAlg}} \implies G \circ F$ and $\lambda\colon \mathrm{Id}_{\mathbf{Set_q}} \implies F \circ G$ satisfying

$$F\eta \circ \lambda_F = \mathrm{Id}_F \qquad \text{and} \qquad G\lambda \circ \eta_G = \mathrm{Id}_G. \tag{5}$$

Let us first define $\eta$. Notice that, if $\mathfrak{A}$ is an atomic and representable algebra, the algebra $(G \circ F)(\mathfrak{A})$ consists of all partial functions contained in $\{([x], x) \mid x \in \mathrm{At}(\mathfrak{A})\}$. Thus we take for $\eta_{\mathfrak{A}}$ the representation map of [5, Corollary 6.17], that is,

$$\eta_{\mathfrak{A}}(a) := \{([x], x) \mid x \in \mathrm{At}(\mathfrak{A}) \text{ and } x \leq a\}.$$

In order to define $\lambda$, we observe that for a set quotient $\pi\colon X \twoheadrightarrow X_0$, since the atoms of $G(\pi) = \mathfrak{A}_\pi$ are precisely the singletons $a_x := \{(\pi(x), x)\}$, for $x \in X$, we have ($a_{x_1} \sim_{\mathfrak{A}_\pi} a_{x_2} \iff \pi(x_1) = \pi(x_2)$), and thus $(F \circ G)(\pi)$ is a quotient isomorphic to $\pi$. Therefore $\lambda_\pi\colon X \rightharpoonup (F \circ G)(\pi)$ is simply the identity (total) function on $X$ (or, strictly speaking, the function that maps $x \in X$ to $a_x \in (F \circ G)(\pi)$).

Finally, checking that $\eta$ and $\lambda$ as defined are indeed natural transformations satisfying (5) amounts to a tedious but routine computation. ∎

## 4. Compatible completion and discrete duality

In this section we first define the appropriate analogues of completeness and completion for algebras with compatibility information. Then we show that, similarly to the contravariant adjunction between atomic Boolean algebras and sets, the monad induced by $F\colon \mathbf{AtRepAlg} \dashv \mathbf{Set_q}^{\mathrm{op}} : G$ gives precisely the ‘completion’ of the algebra. In fact, our contravariant adjunction is a generalisation of the one between atomic Boolean algebras and sets. Just like that adjunction, ours restricts to a duality of the full subcategory of (compatibly) complete algebras.

In order to define analogues of completeness and completion in the presence of compatibility information, we first need to formalise this idea of compatibility.

**Definition 4.1.** *Let $\mathfrak{P}$ be a poset. A binary relation $C$ on $\mathfrak{P}$ is a* ***compatibility relation*** *if it is reflexive, symmetric, and downward closed in $\mathfrak{P} \times \mathfrak{P}$. We say that two elements $a_1, a_2 \in \mathfrak{P}$ are* ***compatible*** *if $a_1 C a_2$.*

One can show that 'reflexive, symmetric, and downward closed' is an axiomatisation of a rather general conception of compatibility in the following sense.

**Proposition 4.2.** *Let $(P, \leq, C)$ be a poset equipped with a binary relation $C$. Then $(P, \leq, C)$ is isomorphic to a poset $(P', \subseteq, C')$ of partial functions ordered by inclusion and equipped with the relation 'agree on the intersection of their domains' if and only if $C$ is reflexive, symmetric, and downward closed.*¶

*Proof*: It is clearly necessary that $C$ be reflexive, symmetric, and downward closed. Showing these conditions are sufficient can be proved with the map sending each $p \in P$ to the partial function

$$\theta(p) := \{(\{p'\}, p') \mid p' \leq p\} \cup \{(\{p', q\}, p') \mid p' \leq p \text{ and } p' \not{C}\, q\}.$$

These are indeed partial functions: a pair of type $(\{p', q\}, p')$ cannot also be of type $(\{p'\}, p')$, for $p' = q$ contradicts $p' \not{C}\, q$; so we only need check that $\theta(p)$ never contains both the pairs $(\{p', q\}, p')$ and $(\{p', q\}, q)$, for $p' \neq q$. But containing both would imply $p' \leq p$ and $q \leq p$, which by reflexivity and downward closure of $C$ contradicts $p' \not{C}\, q$.

That $\theta$ is injective is ensured by pairs of the first type, for if $p_1 \neq p_2$ then without loss of generality $p_1 \not\leq p_2$, so that $(\{p_2\}, p_2)$ is in $\theta(p_2)$ but not $\theta(p_1)$. It is clear that $p_1 \leq p_2$ implies $\theta(p_1) \subseteq \theta(p_2)$.

Lastly, we show that $p_1 \not{C}\, p_2$ if and only if $\theta(p_1) \not{C}'\, \theta(p_2)$. Suppose $p_1 \not{C}\, p_2$. Then in particular $p_1 \neq p_2$. But by the supposition (and symmetry of the $C$ relation) $(\{p_1, p_2\}, p_1) \in \theta(p_1)$ and $(\{p_1, p_2\}, p_2) \in \theta(p_2)$. So $\theta(p_1)$ and $\theta(p_2)$ disagree at the point $\{p_1, p_2\}$. Conversely, suppose $\theta(p_1) \not{C}'\, \theta(p_2)$. Then the disagreement can only be at points of cardinality two. That is, for some $p'_1, p'_2$

¶The relation $C'$ was introduced in [42] for semigroups of partial transformations, where it was called *semi-compatibility*.

we have $(\{p_1', p_2'\}, p_1') \in \theta(p_1)$ and $(\{p_1', p_2'\}, p_2') \in \theta(p_2)$. But that implies $p_1' \not\mathrel{C} p_2'$, which by downward-closure of compatibility gives $p_1 \not\mathrel{C} p_2$. ∎

**Definition 4.3.** *A poset $\mathfrak{P}$ equipped with a compatibility relation is said to be **compatibly complete** provided it has joins of all subsets of pairwise-compatible elements. We say $\mathfrak{P}$ is **meet complete** if it has meets of all* nonempty *subsets.*

**Definition 4.4.** *When speaking about compatibility for representable $\{-, \rhd\}$-algebras, we mean the relation that makes two elements compatible precisely when*

$$a_1 \rhd a_2 = a_2 \rhd a_1.$$

It is clear that for an actual $\{-, \rhd\}$-algebra of partial functions, two elements are compatible exactly when they agree on their shared domain. This is the easiest way to see that Definition 4.4 does indeed define a compatibility relation.

Note that in the case that all pairs of elements are compatible, *compatibly complete* is equivalent to *complete*. Boolean algebras provide examples of representable $\{-, \rhd\}$-algebras where every pair is compatible, if $-$ is the Boolean complement and we use $\rhd$ as the meet symbol. In the same way, generalised Boolean algebras provide a more general class of examples.[‖] Thus 'compatibly complete' is a coherent generalisation of 'complete' from situations where there is no compatibility information, to those where there is.

In a poset equipped with a compatibility relation, if a subset $S$ has an upper bound $u$, then by reflexivity and downward closure of compatibility, $S$ is pairwise compatible. Thus *compatibly complete* $\Longrightarrow$ *bounded complete*. For a similar reason, *compatibly complete* $\Longrightarrow$ *directed complete*. So 'compatibly complete' subsumes these two preexisting order-theoretic concepts.

From now on, we focus exclusively on our specific case of interest: representable $\{-, \rhd\}$-algebras.

**Lemma 4.5.** *Let $\mathfrak{A}$ be a representable $\{-, \rhd\}$-algebra. If $\mathfrak{A}$ is compatibly complete, then it is meet complete. The converse is false.*

*Proof*: Suppose $\mathfrak{A}$ is compatibly complete, and take a nonempty subset $S$ of $\mathfrak{A}$. Let $S^l$ denote the set of lower bounds for $S$. Since $S$ is nonempty, we have

[‖]A *generalised Boolean algebra* is a 'Boolean algebra without a top', that is, a distributive lattice with a bottom and a relative complement operation, $-$, validating $a \wedge (b - a) = \bot$ and $a \vee (b - a) = a \vee b$.

an $s \in S$. Then $s$ is an upper bound for $S^l$, and so $S^l$ is pairwise compatible. Then by the supposition, $S^l$ has a join. It is then straightforward to show that this join is the meet of $S$.

The three-element $\{-, \rhd\}$-algebra consisting of the partial functions $\emptyset$, $\{(1,1)\}$, and $\{(2,2)\}$ provides a counterexample for the converse statement. ∎

**Lemma 4.6.** *Let $\mathfrak{A}$ be a representable $\{-, \rhd\}$-algebra. Then $\mathfrak{A}$ is meet complete if and only if for every $a \in \mathfrak{A}$ the Boolean algebra $a^{\downarrow}$ is complete.*

*Proof*: The forward direction holds simply because for every nonempty subset $S \subseteq a^{\downarrow}$ we have $\prod_{a^{\downarrow}} S = \prod_{\mathfrak{A}} S$, and for the empty subset we have $\prod_{a^{\downarrow}} \emptyset = a$. For the converse, given a nonempty subset $S \subseteq \mathfrak{A}$, fix $a \in S$. Then $\prod_{\mathfrak{A}} S = \prod_{a^{\downarrow}} \{a \cdot s \mid s \in S\}$. ∎

Next we define what a compatible completion of a representable $\{-, \rhd\}$-algebra is. We are guided by completions of Boolean algebras. Just as for Boolean algebras, completions will be unique up to isomorphism (Proposition 4.11).

**Definition 4.7.** *A* ***compatible completion*** *of a representable $\{-, \rhd\}$-algebra $\mathfrak{A}$ is an embedding $\iota \colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ of $\{-, \rhd\}$-algebras such that $\mathfrak{C}$ is representable and compatibly complete and $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$.*

The next lemma tells us in particular that compatible completions are complete homomorphisms.

**Lemma 4.8.** *Let $\iota \colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ be an embedding of representable $\{-, \rhd\}$-algebras. If $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{B}$ then $\iota$ is complete.*

*Proof*: Suppose the join of $S \subseteq \mathfrak{A}$ exists. Then the restriction $\iota \colon (\sum S)^{\downarrow} \to \iota(\sum S)^{\downarrow}$ is an embedding of Boolean algebras with join dense image, so just apply the corresponding known result for Boolean algebras, yielding $\iota(\sum S) = \sum_{\iota(\sum S)^{\downarrow}} \iota[S] = \sum_{\mathfrak{B}} \iota[S]$. ∎

The following technical results will be used, in particular, for the proof that compatible completions are unique up to unique isomorphism. Given an $n$-ary operation $\Omega$ on $\mathfrak{A}$ and subsets $S_1, \dots, S_n \subseteq \mathfrak{A}$, we shall use $\Omega(S_1, \dots, S_n)$ to denote the set $\{\Omega(s_1, \dots, s_n) \mid s_1 \in S_1, \dots, s_n \in S_n\}$, and if one of the sets is a singleton $\{a\}$, we may simply write $a$.

**Lemma 4.9.** *Let $\mathfrak{A}$ be a compatibly complete and representable $\{-, \rhd\}$-algebra and $S, T \subseteq \mathfrak{A}$ two subsets of pairwise-compatible elements.*

*(a) The set $S \rhd T$ consists of pairwise-compatible elements, and*

$$\sum S \rhd \sum T = \sum (S \rhd T).$$

*(b) Suppose $S$ is nonempty. Then for every $a \in \mathfrak{A}$ we have*

$$\prod (a - S) = a - \sum S.$$

*Proof*: For (a): the fact that $S \rhd T$ is a set whose elements are pairwise compatible follows from a repeated use of the equality $a \rhd b \rhd c = b \rhd a \rhd c$, which is a consequence of (Ax.4). Fix an arbitrary element $a \in \mathfrak{A}$ be arbitrary and, for the sake of readability, define $s_0 := \sum S$ and $t_0 := \sum T$. We prove the asserted equation in two steps: first we show that $a \rhd t_0 = \sum (a \rhd T)$ and then that $s_0 \rhd a = \sum (S \rhd a)$. From there we have

$$\begin{aligned}\sum S \rhd \sum T = s_0 \rhd t_0 = \sum (s_0 \rhd T) &= \sum_{t \in T} (s_0 \rhd t)\\ &= \sum_{t \in T} \sum (S \rhd t) = \sum_{t \in T} \sum_{s \in S} s \rhd t = \sum S \rhd T\end{aligned}$$

as required. (The last equality is a true property of suprema for any doubly indexed set, when all the suprema exist.)

For $a \rhd t_0 = \sum (a \rhd T)$, Proposition 3.4(c) states that for any $c \in \mathfrak{A}$, the map $b \mapsto [b]$ provides an order isomorphism $(c^{\downarrow}, \leq)$ to $([c]^{\downarrow}, \preceq)$. So given that $t_0$ is an upper bound for both sides of $a \rhd t_0 = \sum (a \rhd T)$, and $a$ is an upper bound for both sides of $s_0 \rhd a = \sum (S \rhd a)$, it is actually sufficient to show that $[a \rhd t_0] = [\sum (a \rhd T)]$ and that $[s_0 \rhd a] = [\sum (S \rhd a)]$.

By Lemmas 4.5 and 4.6, the poset $t_0^{\downarrow}$ is a complete Boolean algebra, and so $[t_0]^{\downarrow}$ is as well. The meets/joins in $[t_0]^{\downarrow}$ are written in $\wedge/\vee$ notation; in $[t_0]^{\downarrow}$ we have that $[b] \wedge [c] = [b \rhd c]$ is valid (Proposition 3.4(b)). Being careful that all terms rest in $[t_0]^{\downarrow}$, we compute

$$\begin{aligned}[a \rhd t_0] = [a \rhd t_0] \wedge [\sum T] &= [a \rhd t_0] \wedge \bigvee_{t \in T} [t]\\ = \bigvee_{t \in T} ([a \rhd t_0] \wedge [t]) &= \bigvee_{t \in T} [a \rhd t] = [\sum (a \rhd T)],\end{aligned}$$

with the central distributive equality being a valid property of any Boolean algebra (provided the join on the left side exists) [25].

Proving the equality $s_0 \rhd a = \sum(S \rhd a)$ is very similar. In $[a]^{\downarrow}$ we have

$$[\sum(S \rhd a)] = \bigvee_{s \in S}[s \rhd a],$$

and then in $[s_0]^{\downarrow}$ we compute

$$\bigvee_{s\in S}[s \rhd a] = \bigvee_{s\in S}([s] \wedge [a \rhd s_0]) = (\bigvee_{s\in S}[s]) \wedge [a \rhd s_0] = [s_0] \wedge [a \rhd s_0] = [s_0 \rhd a].$$

For (b): we first observe that in the complete Boolean algebra $a^{\downarrow}$, we have

$$\prod(a - S) = \prod(a - a \cdot S) = a - \sum(a \cdot S),$$

where $a \cdot S$ denotes the set $\{a \cdot s \mid s \in S\}$, and the first equality holds because $a - (a \cdot b) = a - b$ is a valid equation in any representable algebra. In turn, in the complete Boolean algebra $s_0^{\downarrow}$, we may compute

$$\sum(a \cdot S) = \sum(a \cdot s_0 \cdot S) = a \cdot s_0 \cdot \sum S = a \cdot \sum S.$$

Thus again by the validity of $a - (a \cdot b) = a - b$ the desired equality follows. ■

**Lemma 4.10.** *Let $\iota \colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ and $\iota' \colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ be complete embeddings of representable $\{-, \rhd\}$-algebras, and suppose that $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{B}$, and that $\mathfrak{C}$ is compatibly complete. Then $\theta \colon b \mapsto \sum\{\iota'(a) \mid \iota(a) \leq b\}$ is a well-defined complete embedding of $\{-, \rhd\}$-algebras from $\mathfrak{B}$ to $\mathfrak{C}$, and $\iota' = \theta \circ \iota$.*

*Proof*: First we check $\theta$ is well-defined. The set $\{\iota(a) \mid \iota(a) \leq b\}$ is pairwise compatible, as it is bounded above. Hence, by injectivity of $\iota$, the set $\{a \mid \iota(a) \leq b\}$ is pairwise compatible, and so $\{\iota'(a) \mid \iota(a) \leq b\}$ is too. By compatible completeness of $\mathfrak{C}$, the value $\sum\{\iota'(a) \mid \iota(a) \leq b\}$ exists. It is clear that $\theta$ is order preserving and that $\theta \circ \iota = \iota'$.

Next we show that $\theta$ is join complete, as a map between posets. For simplicity, we treat $\mathfrak{A}$ as a subset of $\mathfrak{B}$. Suppose $S \subseteq \mathfrak{B}$ and that $\sum S \in \mathfrak{B}$ exists. Then since $\mathfrak{C}$ is compatibly complete and $\theta(\sum S)$ is an upper bound for $\theta[S]$, the join $\sum \theta[S]$ exists, and $\theta(\sum S) \geq \sum \theta[S]$. For the reverse inequality, let $a \in \mathfrak{A}$ be bounded above by $\sum S$. Then $a = a \cdot \sum \bigcup_{s \in S}\{a_1 \in \mathfrak{A} \mid a_1 \leq s\} = \sum \bigcup_{s\in S}\{a \cdot a_1 \in \mathfrak{A} \mid a_1 \leq s\}$. So the latter join exists not only in $\mathfrak{B}$ but also in $\mathfrak{A}$, since $a \in \mathfrak{A}$. Hence $\iota'(a) = \sum \bigcup_{s\in S}\{\iota'(a) \cdot \iota'(a_1) \in \mathfrak{A} \mid a_1 \leq s\}$, as $\iota'$ is a complete homomorphism. Thus $\sum \theta[S] \geq \iota'(a)$. As $\theta(\sum S)$ is by definition the join of these $\iota'(a)$, we are done.

Now we show that $\theta$ preserves $-$. Since $\theta$ is order preserving, it is clear that $\theta(b_1 \cdot b_2) \leq \theta(b_1) \cdot \theta(b_2)$. As $b_1$ is the join of $\{b_1 \cdot b_2, b_1 - b_2\}$, and $\theta$ preserves joins, we deduce that $\theta(b_1) \leq \theta(b_1) \cdot \theta(b_2) + \theta(b_1 - b_2)$. Thus, in the Boolean algebra $\theta(b_1)^{\downarrow}$, we have $\theta(b_1) - \theta(b_2) = \theta(b_1) - \theta(b_1) \cdot \theta(b_2) \leq \theta(b_1 - b_2)$. For the reverse inequality, suppose $a \in \mathfrak{A}$ with $a \leq b_1 - b_2$. Then using Lemma 4.9(b) we have

$$\begin{aligned}
\theta(b_1) - \theta(b_2) = \theta(b_1) - \sum_{\substack{a_2 \in \mathfrak{A} \\ a_2 \leq b_2}} \iota'(a_2) &= \prod_{\substack{a_2 \in \mathfrak{A} \\ a_2 \leq b_2}} (\theta(b_1) - \iota'(a_2)) \\
&\geq \prod_{\substack{a_2 \in \mathfrak{A} \\ a_2 \leq b_2}} (\iota'(a) - \iota'(a_2)) = \prod_{\substack{a_2 \in \mathfrak{A} \\ a_2 \leq b_2}} \iota'(a - a_2).
\end{aligned}$$

But when $a \leq b_1 - b_2$ and $a_2 \leq b_2$, we know $a - a_2 = a$ (by an elementary property of sets/partial functions). Hence $\theta(b_1) - \theta(b_2) \geq \iota'(a)$. As $\theta(b_1 - b_2)$ is by definition the join of these $\iota'(a)$, we are finished.

Let us show that $\theta$ preserves $\triangleright$. We now treat $\mathfrak{A}$ as a subset of both $\mathfrak{B}$ and $\mathfrak{C}$, conflating $\mathfrak{A}$ with its images under $\iota$ and $\iota'$. For $b_1, b_2 \in \mathfrak{B}$ define $A_1 = \{a_1 \in \mathfrak{A} \mid a_1 \leq b_1\}$ and $A_2 = \{a_2 \in \mathfrak{A} \mid a_2 \leq b_2\}$. We calculate

$$\begin{aligned}
\theta(b_1) \triangleright \theta(b_2) &= \sum_{\mathfrak{C}} A_1 \triangleright \sum_{\mathfrak{C}} A_2 && \text{by the definition of } \theta \\
&= \sum_{\mathfrak{C}} (A_1 \triangleright A_2) && \text{by Lemma 4.9(a)} \\
&= \theta(\sum_{\mathfrak{B}} (A_1 \triangleright A_2)) && \text{as } \theta \text{ is join complete} \\
&= \theta(\sum_{\mathfrak{B}} A_1 \triangleright \sum_{\mathfrak{B}} A_2) && \text{by Lemma 4.9(a)} \\
&= \theta(b_1 \triangleright b_2) && \text{by density of } \mathfrak{A} \text{ in } \mathfrak{B}.
\end{aligned}$$

It remains to show that $\theta$ is injective. We claim that since $\theta$ is a homomorphism, injectivity of $\theta$ amounts to having $\theta^{-1}(0) = \{0\}$. Suppose $\theta^{-1}(0) = \{0\}$. Then given $a, b \in \mathfrak{A}$ we have $\theta(a) = \theta(b) \implies \theta(a - b) = 0 = \theta(b - a)$, and hence $a - b = 0 = b - a$. By an elementary property of sets/partial functions, this implies $a = b$, proving the claim. Now to show $\theta^{-1}(0) = \{0\}$, let $b \in \mathfrak{B}$ satisfy $\theta(b) = 0$. By the definition of $\theta$, this means that $\iota'(a) = 0$ whenever $\iota(a) \leq b$. But since $\iota'$ is injective, if $\iota'(a) = 0$ then $a = 0$ and so, $\iota(a) = 0$. Since $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{B}$, we may conclude that $b = 0$, as required. ∎

Now we can show that compatible completions are unique up to unique isomorphism.

**Proposition 4.11.** *If* $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ *and* $\iota'\colon \mathfrak{A} \hookrightarrow \mathfrak{C}'$ *are compatible completions of the representable* $\{-, \rhd\}$*-algebra* $\mathfrak{A}$ *then there is a unique isomorphism* $\theta\colon \mathfrak{C} \to \mathfrak{C}'$ *satisfying the condition* $\theta \circ \iota = \iota'$.

*Proof*: For uniqueness, suppose we have an isomorphism $\theta\colon \mathfrak{C} \to \mathfrak{C}'$ satisfying $\theta \circ \iota = \iota'$. As $\iota'[\mathfrak{A}]$ is both join dense in $\mathfrak{C}'$ and a subset of $\theta[\mathfrak{C}]$, by applying Lemma 4.8 to $\theta$ we see that $\theta$ is complete. Then as each $c \in \mathfrak{C}$ is equal to $\sum\{\iota(a) \mid a \in \mathfrak{A} \text{ and } \iota(a) \leq c\}$, our $\theta$ must be given by $\theta\colon c \mapsto \sum\{\iota'(a) \mid a \in \mathfrak{A} \text{ and } \iota(a) \leq c\}$.

For existence, we argue that $\theta\colon c \mapsto \sum\{\iota'(a) \mid a \in \mathfrak{A} \text{ and } \iota(a) \leq c\}$ indeed works. As compatible completions are complete homomorphisms, we can apply Lemma 4.10 to $\iota$ and $\iota'$. Hence $\theta$ is a well-defined complete embedding of $\{-, \rhd\}$-algebras. By symmetry, there is a complete embedding $\theta'\colon \mathfrak{C}' \hookrightarrow \mathfrak{C}$ with $\theta' \circ \iota' = \iota$. Then $\theta' \circ \theta$ is complete and fixes $\iota[\mathfrak{A}]$. So by join density of $\iota[\mathfrak{A}]$, the homomorphism $\theta' \circ \theta$ is the identity on $\mathfrak{C}$. Similarly, $\theta \circ \theta'$ is the identity on $\mathfrak{C}'$. Thus $\theta$ is an isomorphism. ∎

Hence if an algebra has a compatible completion then that compatible completion is 'unique', for which reason we may refer to a compatible completion as *the* compatible completion. We may also, as is common, refer to $\mathfrak{C}$ itself as the compatible completion of $\mathfrak{A}$, when $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ is a compatible completion.

Next, we show how to explicitly construct the completion of any *atomic* representable algebra, by showing that the monad on atomic representable algebras induced by the adjunction of Theorem 3.9 gives precisely the compatible completion of the algebra. As a corollary, we obtain a duality for compatibly complete atomic representable algebras. Recall from Section 3.1 that, for every atomic representable algebra $\mathfrak{A}$, we use $\pi_{\mathfrak{A}}$ to denote the canonical projection $\mathrm{At}(\mathfrak{A}) \twoheadrightarrow \mathrm{At}(\mathfrak{A})/{\sim_{\mathfrak{A}}}$ (see also Definition 3.3).

**Theorem 4.12.** *For every atomic representable* $\{-, \rhd\}$*-algebra* $\mathfrak{A}$*, the homomorphism*

$$\begin{aligned} \eta_{\mathfrak{A}}\colon \mathfrak{A} &\to (G \circ F)(\mathfrak{A}) = \{f\colon \mathrm{At}(\mathfrak{A})/{\sim_{\mathfrak{A}}} \rightharpoonup \mathrm{At}(\mathfrak{A}) \mid f \subseteq \pi_{\mathfrak{A}}^{-1}\} \\ a &\mapsto \{([x], x) \mid x \in \mathrm{At}(\mathfrak{A}) \text{ and } x \leq a\} \end{aligned}$$

*is the compatible completion of* $\mathfrak{A}$.

*Proof*: For injectivity, suppose $\eta_{\mathfrak{A}}(a) = \eta_{\mathfrak{A}}(b)$. Then as $\mathfrak{A}$ is atomic, it is atomistic. (Recall Definition 2.10 and the following remark.) So we have

$$a = \sum\{x \in \mathrm{At}(\mathfrak{A}) \mid x \leq a\} = \sum\{x \in \mathrm{At}(\mathfrak{A}) \mid x \leq b\} = b.$$

For compatible completeness of $(G \circ F)(\mathfrak{A})$, let $S$ be a pairwise-compatible subset of $(G \circ F)(\mathfrak{A})$. Then as $(G \circ F)(\mathfrak{A})$ is an algebra of partial functions, all pairs of elements of $S$ agree on their shared domains. That is, $\bigcup S$ is a partial function. Given that $f \subseteq \pi_{\mathfrak{A}}^{-1}$ for each $f \in S$, we have $\bigcup S \subseteq \pi_{\mathfrak{A}}^{-1}$. So $\bigcup S \in (G \circ F)(\mathfrak{A})$ and is the least upper bound of $S$.

For join density of $\eta_{\mathfrak{A}}[\mathfrak{A}]$, it suffices to note the join density of $\eta_{\mathfrak{A}}[\mathrm{At}(\mathfrak{A})]$. ∎

As a consequence we have the following corollaries.

**Corollary 4.13.** *There is a duality between* **CAtRepAlg** *and* $\mathbf{Set_q}$*, where* **CAtRepAlg** *is the full subcategory of* **AtRepAlg** *consisting of the compatibly complete algebras.*

*Proof*: Clearly $G$ maps every set quotient $\pi$ to a compatibly complete algebra. Thus, $G$ co-restricts to a functor $\mathbf{Set_q}^{\mathrm{op}} \to \mathbf{CAtRepAlg}$. Moreover, as observed in the proof of Theorem 3.9, the functor $F \circ G$ is naturally isomorphic to the identity functor on $\mathbf{Set_q}$. To conclude that **CAtRepAlg** and $\mathbf{Set_q}$ are dually equivalent, it only remains to argue that $\eta$ restricted to **CAtRepAlg** provides a natural isomorphism from $\mathrm{Id}_{\mathbf{CAtRepAlg}}$ to $G \circ F$. It suffices to show that $\eta_{\mathfrak{A}}$ is an isomorphism (algebraically speaking) for every compatibly complete $\mathfrak{A}$ (given that isomorphisms, and hence their inverses, are complete homomorphisms). For this we just note that if $\mathfrak{A}$ is compatibly complete then the identity map $\mathfrak{A} \hookrightarrow \mathfrak{A}$ is a compatible completion of $\mathfrak{A}$, and then apply Theorem 4.12 and Proposition 4.11. ∎

**Corollary 4.14.** *The category* **CAtRepAlg** *is a reflective subcategory of* **AtRepAlg**.

*Proof*: We saw in the proof of Corollary 4.13 that the restriction of $G \circ F$ to **CAtRepAlg** is naturally isomorphic to the identity. It is a direct consequence that $G \circ F$, viewed as a functor $\mathbf{AtRepAlg} \to \mathbf{CAtRepAlg}$, is left adjoint to the inclusion $\mathbf{CAtRepAlg} \to \mathbf{AtRepAlg}$. ∎

We have now achieved all our main objectives for this section. However, in order to better situate these results, it is worth being precise about which category our definition of compatible completion inhabits. Thus we will specify that Definition 4.7 defines a *compatible completion in* **RepAlg**, where **RepAlg** is the category of representable $\{-, \triangleright\}$-algebras with $\{-, \triangleright\}$-homomorphisms. So Proposition 4.11 says that compatible completions in this category are unique, and Theorem 4.12 shows how to construct them for atomic algebras. If now we replace **RepAlg** with the category $\mathbf{RepAlg}_\infty$ of representable $\{-, \triangleright\}$-algebras with *complete* $\{-, \triangleright\}$-homomorphisms, we obtain the following definition.

**Definition 4.15.** *A* ***compatible completion in*** $\mathbf{RepAlg}_\infty$ *of a representable $\{-, \triangleright\}$-algebra $\mathfrak{A}$ is a complete embedding $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ of $\{-, \triangleright\}$-algebras such that $\mathfrak{C}$ is representable and compatibly complete and $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$.*

Given Lemma 4.8 and the fact that isomorphisms are complete homomorphisms, we can also claim that Proposition 4.11 says that compatible completions in $\mathbf{RepAlg}_\infty$ are unique, and that Theorem 4.12 shows how to construct them for atomic algebras.

For Boolean algebras, several equivalent definition of completions are possible [1]. The same is partially true for compatible completions in $\mathbf{RepAlg}_\infty$.

**Proposition 4.16.** *Let $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ be a complete embedding of representable $\{-, \triangleright\}$-algebras. Consider the following statements about $\iota$.*

*(a) $\mathfrak{C}$ is compatibly complete, and the image of $\mathfrak{A}$ is join dense in $\mathfrak{C}$.*

*(b) $\mathfrak{C}$ is the 'smallest' extension of $\mathfrak{A}$ that is compatibly complete. That is, $\mathfrak{C}$ is compatibly complete, and for every other complete embedding $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ into a compatibly complete and representable $\{-, \triangleright\}$-algebra $\mathfrak{B}$, there exists a complete embedding $\widehat{\kappa}\colon \mathfrak{C} \hookrightarrow \mathfrak{B}$ making the following diagram commute.*

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, hook, "\iota"] \arrow[dr, hook, "\kappa"'] & \mathfrak{C} \arrow[d, hook, dashed, "\widehat{\kappa}"] \\
& \mathfrak{B}
\end{tikzcd}$$

*(c) $\mathfrak{C}$ is the 'largest' extension of $\mathfrak{A}$ in which the image of $\mathfrak{A}$ is join dense. That is, $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$, and for every other complete embedding $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ into a representable $\{-, \triangleright\}$-algebra $\mathfrak{B}$ in which the image of $\mathfrak{A}$ is join dense, there exists a complete embedding $\widehat{\kappa}\colon \mathfrak{B} \hookrightarrow \mathfrak{C}$ making the following diagram commute.*

$$\begin{tikzcd} \mathfrak{A} \arrow[r, hook, "\kappa"] \arrow[dr, hook, "\iota"'] & \mathfrak{B} \arrow[d, hook, dashed, "\widehat{\kappa}"] \\ & \mathfrak{C} \end{tikzcd}$$

*Then (a) $\implies$ (b), and (a) $\implies$ (c), and if $\mathfrak{A}$ has a completion then all three conditions are equivalent.*

*Proof*: For (a) $\implies$ (b), apply Lemma 4.10 to $\iota$ and $\kappa$, using join density of $\iota[\mathfrak{A}]$ in $\mathfrak{C}$. For (a) $\implies$ (c), apply Lemma 4.10 to $\kappa$ and $\iota$, using compatible completeness of $\mathfrak{C}$.

For the last part, suppose $\mathfrak{A}$ has a completion $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$. If (b) holds for $\mathfrak{C}$, then it can be applied to $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$. Then $\widehat{\kappa}$ is surjective, since now $\kappa[\mathfrak{A}]$ is join dense in $\mathfrak{B}$ and $\widehat{\kappa}$ is complete. Hence the embedding $\widehat{\kappa}$ is in fact an isomorphism, and so (a) holds.

Similarly, if (c) holds for $\mathfrak{C}$, then apply it to $\kappa\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ to obtain $\widehat{\kappa}\colon \mathfrak{B} \hookrightarrow \mathfrak{C}$. Then as the image of $\mathfrak{A}$ is also join dense in $\mathfrak{B}$ we similarly have $\widehat{\iota}\colon \mathfrak{C} \hookrightarrow \mathfrak{B}$ commuting with the embedding of $\mathfrak{A}$. Both compositions $\widehat{\kappa} \circ \widehat{\iota}$ and $\widehat{\iota} \circ \widehat{\kappa}$ are complete homomorphisms fixing the embedded copies of $\mathfrak{A}$ (which are join dense), hence both compositions are the identity. So $\widehat{\kappa}$ is an isomorphism and (a) holds. ■

In light of Proposition 4.16, it would be interesting to know which algebras in $\mathbf{RepAlg}_\infty$ (beyond the atomic ones) have compatible completions and how to construct those completions. We leave this as an open problem.

**Problem 4.17.** *Which representable $\{-, \triangleright\}$-algebras have a compatible completion in $\mathbf{RepAlg}_\infty$? Describe a general method to construct these completions.*

A set of implications similar to those in Proposition 4.16 is not possible for compatible completions in **RepAlg**, as the following example shows.

**Example 4.18.** *We will show that (a) $\not\Longrightarrow$ (b) in Proposition 4.16, if we drop the assumption of completeness of the homomorphisms. Let $2 \coloneqq \{0, 1\}$, and let $\mathfrak{F}$ be the $\{-, \triangleright\}$-algebra consisting of the following partial functions $\mathbb{N} \rightharpoonup 2$:*

- *those with finite domain,*
- *those such that the inverse image of $0$ is a cofinite set.*

*It is straightforward to check that* $\mathfrak{F}$ *is closed under* $-$ *and* $\rhd$*, so is indeed a* $\{-, \rhd\}$*-algebra of partial functions. This algebra* $\mathfrak{F}$ *is atomic, and its compatible completion* $\mathfrak{F}'$ *consists of all partial functions* $\mathbb{N} \rightharpoonup 2$*. Let* $\{\emptyset, \mathrm{Id}\}$ *be the cardinality 2* $\{-, \rhd\}$*-algebra of both partial endofunctions on some singleton set, and let* $\mathfrak{G} = \mathfrak{F}' \times \{\emptyset, \mathrm{Id}\}$*. Then* $\mathfrak{G}$ *is also compatibly complete (and evidently representable by partial functions). Define* $\kappa\colon \mathfrak{F} \hookrightarrow \mathfrak{G}$ *by* $\kappa\colon f \mapsto (f, \emptyset)$ *for* $f$ *with finite domain and* $\kappa\colon f \mapsto (f, \mathrm{Id})$ *otherwise.*

*There does not exist a homomorphism* $\widehat{\kappa}\colon \mathfrak{F}' \to \mathfrak{G}$ *such that* $\iota \circ \widehat{\kappa} = \kappa$*. For suppose otherwise, and consider the constant functions* $\overline{1}\colon \mathbb{N} \to 2$ *belonging to* $\mathfrak{F}'$ *and* $\overline{0}\colon \mathbb{N} \to 2$ *belonging to both* $\mathfrak{F}$ *and* $\mathfrak{F}'$*. Now*

$$\widehat{\kappa}(\overline{1}) \cdot (\overline{0}, \mathrm{Id}) = \widehat{\kappa}(\overline{1}) \cdot \kappa(\overline{0}) = \widehat{\kappa}(\overline{1}) \cdot \widehat{\kappa}(\overline{0}) = \widehat{\kappa}(\overline{1} \cdot \overline{0}) = \widehat{\kappa}(\emptyset) = \kappa(\emptyset) = (\emptyset, \emptyset),$$

*and hence the second component of* $\widehat{\kappa}(\overline{1})$ *must be* $\emptyset$*. However*

$$\widehat{\kappa}(\overline{1}) \rhd (\overline{0}, \mathrm{Id}) = \widehat{\kappa}(\overline{1}) \rhd \kappa(\overline{0}) = \widehat{\kappa}(\overline{1}) \rhd \widehat{\kappa}(\overline{0}) = \widehat{\kappa}(\overline{1} \rhd \overline{0}) = \widehat{\kappa}(\overline{0}) = \kappa(\overline{0}) = (\overline{0}, \mathrm{Id}),$$

*indicating that the second component of* $\widehat{\kappa}(\overline{1})$ *is* Id*—a contradiction.*

Note that the issue in Example 4.18 cannot be overcome by restricting to the full subcategory of algebras having joins for all *finite* pairwise-compatible sets, as $\mathfrak{F}$ already satisfies this finite compatible completeness condition.

## 5. Discrete duality for compatibly complete algebras with operators

In this section we extend the adjunction, completion, and duality results of the previous two sections to results allowing the algebras to be equipped with arbitrary additional completely additive operators respecting the compatibility structure. Unless specified otherwise, let $\mathfrak{A}$ be an atomic representable $\{-, \rhd\}$-algebra.

First we introduce the class of operations we are interested in.

**Definition 5.1.** *Let* $\Omega$ *be an* $n$*-ary operation on* $\mathfrak{A}$*. Then* $\Omega$ *is* ***compatibility preserving*** *if whenever* $a_i, a_i'$ *are compatible, for all* $i$*, we have that* $\Omega(a_1, \ldots, a_n)$ *and* $\Omega(a_1', \ldots, a_n')$ *are compatible.*

*The operation* $\Omega$ *is* ***completely additive*** *if whenever the supremum* $\sum S$ *exists, for* $S \subseteq \mathfrak{A}$*, we have*

$$\Omega(a_1, \ldots, a_{i-1}, \sum S, a_{i+1}, \ldots, a_n) = \sum \Omega(a_1, \ldots, a_{i-1}, S, a_{i+1}, \ldots, a_n) \quad (6)$$

*for any* $i$ *and any* $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in \mathfrak{A}$*.*

It is worth being aware that in the literature on algebras 'with operators', the term 'operator' is not merely a synonym for 'operation' but means *finitely additive operation*, that is, (6) holds for all finite (possibly empty) $S$.

The operations we will treat are those that are both compatibility preserving and completely additive. Hence the algebraic categories we consider in this section take the following form, for a functional signature $\sigma$ (disjoint from $\{-, \rhd\}$).

**Definition 5.2.** *The category* $\mathbf{AtRepAlg}(\sigma)$ *has*

- *objects: algebras of the signature* $\{-, \rhd\} \cup \sigma$ *whose* $\{-, \rhd\}$*-reduct is atomic and representable, and such that the symbols of* $\sigma$ *are interpreted as compatibility preserving completely additive operations,*
- *morphisms: complete homomorphisms of* $(\{-, \rhd\} \cup \sigma)$*-algebras.*

For reference, we will briefly list concrete operations $\Omega$ on partial functions according to whether or not they are compatibility preserving and completely additive—by which we mean the completely representable $\{-, \rhd, \Omega\}$-algebras are a (necessarily full) subcategory of $\mathbf{AtRepAlg}(\{\Omega\})$.** Verifying that such inclusions hold is a simple matter, achieved by checking the operation is compatibility preserving and completely additive for actual algebras of partial functions in which any joins are given by unions.

We can list the following compatibility preserving and completely additive operations from the literature: *composition* (usually denoted ;), the unary D (*domain*), R (*range*), and F (*fixset*) operations (the identity function restricted, respectively, to the domain, range, and fixed points of the argument [17]), the constant 1 (*identity*), and the binary $\lhd$ (*range restriction*). *Intersection* is of course already expressible in our base signature.

The signature obtained by adding composition to $\{-, \rhd\}$ has been studied by Schein and the representation class axiomatised under the name of *difference semigroups* [39]. The signature obtained by adding composition, domain, and identity is term equivalent to $\{;, \cdot, \mathrm{A}\}$, for which the representation class is axiomatised in [20] and the completely representable algebras axiomatised in [33]. Adding range to this last signature we obtain $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$, and the representation class is axiomatised in [17]. Axiomatising the completely representable algebras for $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$ is currently an open problem.

**Note this is not the same as the more subtle matter of whether 'representable + $\{-, \rhd\}$-reduct is completely representable $\implies$ subcategory' (see, for example, the case of composition in [33]).

Operations that fail to be compatibility preserving and completely additive usually do so because they are not even order-preserving. We can list: *antidomain* (identity function restricted to complement of domain) and its range analogue *antirange*, similar negative versions of $\rhd$ and $\lhd$, which we might call *antidomain restriction* and *antirange restriction* (antidomain restriction is called *minus* in [3]), *override* (also known as *preferential union*) and *update* (see [3]), *maximal iterate* (see [23]) and *opposite* (converse restricted to points with a unique preimage [36]). *Converse* is an operation that *is* completely additive but fails to be compatibility preserving. An interesting future project would be to extend the results of this section in a way that encompasses some of these operations, in particular the several that are either order-preserving or order-reversing in each coordinate. (An analogous extension of the theory of Boolean algebras with operators can be found in [12].)

Starting with a compatibility preserving and completely additive $n$-ary operation $\Omega$, we can define an $(n+1)$-ary relation $R_\Omega$ on the atoms of $\mathfrak{A}$ by

$$R_\Omega x_1 \ldots x_{n+1} \iff \Omega(x_1, \ldots, x_n) \geq x_{n+1}. \tag{7}$$

Next we introduce a class of relations on set quotients, which will turn out to be precisely the relations obtained from operations in the way just described.

**Definition 5.3.** *Take sets $X, X_0$, and a surjection $\pi\colon X \twoheadrightarrow X_0$, and let $R$ be an $(n+1)$-ary relation on $X$. The* ***compatibility relation*** *$C \subseteq X \times X$ is given by $xCy$ if and only if $\pi(x) = \pi(y) \implies x = y$. Then $R$ has the* ***compatibility property*** *(with respect to $\pi$) if given $x_1Cx'_1, \ldots,\ x_nCx'_n$ and $Rx_1 \ldots x_{n+1}$ and $Rx'_1 \ldots x'_{n+1}$, we have $x_{n+1}Cx'_{n+1}$.*

Observe that, in the case where $\pi$ is the canonical projection $\pi_{\mathfrak{A}}\colon \mathrm{At}(\mathfrak{A}) \twoheadrightarrow \mathrm{At}(\mathfrak{A})/{\sim_{\mathfrak{A}}}$ for some atomic representable algebra $\mathfrak{A}$, the compatibility relation $C$ coincides with the compatibility relation introduced in Definition 4.4 (restricted to atoms).

Given an $R$ satisfying the compatibility property, we can define an $n$-ary operation $\Omega_R$ on the dual $\mathfrak{A}_\pi$ of $\pi\colon X \twoheadrightarrow X_0$ by conflating elements of $\mathfrak{A}_\pi$ with their image, and then setting

$$\Omega_R(X_1, \ldots, X_n) = R(X_1, \ldots, X_n, \_), \tag{8}$$

where $R(X_1, \ldots, X_n, \_) \coloneqq \bigcup_{x_1 \in X_1, \ldots, x_n \in X_n} \{x_{n+1} \in X \mid Rx_1 \ldots x_{n+1}\}$. Notice that a subset $X' \subseteq X$ defines a partial function of $\mathfrak{A}_\pi$ exactly when it contains at most one element of each fibre of $\pi$, that is, when $xCx'$ for every $x, x' \in X'$. So suppose that $x_{n+1}, x'_{n+1} \in \Omega_R(X_1, \ldots, X_n)$. Then there exist $x_1 \in X_1, \ldots, x_n \in X_n$ with $Rx_1 \ldots x_{n+1}$, and $x'_1 \in X_1, \ldots, x'_n \in X_n$ with $Rx'_1 \ldots x'_{n+1}$. Since $X_1, \ldots, X_n$ are images of partial sections, we have $x_1 C x'_1, \ldots, x_n C x'_n$. Then by the hypothesis that $R$ has the compatibility property, if $x_{n+1}$ and $x'_{n+1}$ lie in the same fibre they are equal. Therefore, $\Omega_R$ is indeed a well-defined operation on $\mathfrak{A}_\pi$.

Finally, we define conditions that morphisms of set quotients are required to satisfy, when those set quotients are equipped with additional relations.

**Definition 5.4.** *Take a partial function $\varphi\colon X \rightharpoonup Y$ and $(n+1)$-ary relations $R_X$ and $R_Y$ on $X$ and $Y$. Then $\varphi$ satisfies the **reverse forth condition** if whenever $R_X x_1 \ldots x_{n+1}$ and $\varphi(x_1), \ldots, \varphi(x_n)$ are defined, then $\varphi(x_{n+1})$ is defined and $R_Y \varphi(x_1) \ldots \varphi(x_{n+1})$. The partial map $\varphi$ satisfies the **back condition** if whenever $\varphi(x_{n+1})$ is defined and $R_Y y_1 \ldots y_n \varphi(x_{n+1})$, then there exist $x_1, \ldots, x_n \in \mathrm{dom}(\varphi)$ such that $\varphi(x_1) = y_1, \ldots, \varphi(x_n) = y_n$ and $R_X x_1 \ldots x_{n+1}$.*

We are now ready to extend the adjunction $F\colon \mathbf{AtRepAlg} \dashv \mathbf{Set_q}^{\mathrm{op}} : G$ to algebras with operators. We fix a functional signature $\sigma$ (disjoint from $\{-, \triangleright\}$). We have already defined $\mathbf{AtRepAlg}(\sigma)$.

**Definition 5.5.** *The category $\mathbf{Set_q}(\sigma)$ has*

- *objects: the objects of $\mathbf{Set_q}$ equipped with, for each $\Omega \in \sigma$, an $(n+1)$-ary relation $R_\Omega$ that has the compatibility property, where $n$ is the arity of $\Omega$,*
- *morphisms: morphisms of $\mathbf{Set_q}$ that satisfy the reverse forth condition and the back condition with respect to $R_\Omega$, for every $\Omega \in \sigma$.*

We are required to note at this point that both the reverse forth condition and the back condition are preserved by composition of partial maps (and are also satisfied by identity maps); hence $\mathbf{Set_q}(\sigma)$ is indeed a category.

**Theorem 5.6.** *There is an adjunction $F'\colon \mathbf{AtRepAlg}(\sigma) \dashv \mathbf{Set_q}(\sigma)^{\mathrm{op}} : G'$ that extends the adjunction $F \dashv G$ of Section 3 in the sense that the appropriate reducts of $F'(\mathfrak{A})$ and $G'(\pi\colon X \twoheadrightarrow X_0)$ equal $F(\mathfrak{A})$ and $G(\pi\colon X \twoheadrightarrow X_0)$, respectively.*

The proof of this theorem takes up most of the remainder of the paper. The reader is likely to have understood by now how to form $F'$ and $G'$. For $F'$: given an atomic representable $\{-, \triangleright\}$-algebra $\mathfrak{A}$ equipped with compatibility preserving completely additive operators indexed by $\sigma$, take $F(\mathfrak{A})$ and equip it with, for each operation $\Omega$ on $\mathfrak{A}$, the relation $R_\Omega$ defined according to (7). For $G'$: given a set quotient $\pi\colon X \twoheadrightarrow X_0$ equipped with relations, take $G(\pi\colon X \twoheadrightarrow X_0)$ and equip it with, for each relation $R$, the operation $\Omega_R$ defined according to (8). The proof consists therefore of establishing the following facts.

(1) The $F'$ we wish to define is well-defined on objects. That is, the defined relations $R_\Omega$ have the compatibility property (Lemma 5.7).
(2) $F'$ is well-defined on morphisms. That is, for a morphism $h$ in **AtRepAlg** that preserves additional operations, the partial map $Fh$ satisfies the reverse forth and the back conditions with respect to each pair of relations (Lemma 5.8).
(3) $G'$ is well-defined on objects: each defined $\Omega_R$ is a compatibility preserving and completely additive operation (Lemma 5.9).
(4) $G'$ is well-defined on morphisms: given a morphism $H$ in $\mathbf{Set_q}$, the defined operations are preserved by $G\varphi$ (Lemma 5.10).
(5) The unit and counit used in Theorem 3.9 are still permitted families of morphisms. That is, for each algebra $\mathfrak{A}$, the map $\eta_{\mathfrak{A}}$ preserves the additional operations, and for each set quotient $\pi$, the map $\lambda_\pi$ satisfies the reverse forth condition and the back condition (Lemma 5.11).

**Lemma 5.7.** *If an operation $\Omega$ is compatibility preserving, then $R_\Omega$ has the compatibility property.*

*Proof*: Take $x_1Cx'_1, \dots, x_nCx'_n$ and $R_\Omega x_1 \dots x_{n+1}$ and $R_\Omega x'_1 \dots x'_{n+1}$. Then $x_i, x'_i \in \mathfrak{A}$ are compatible, for each $i$. So since $\Omega$ is compatibility preserving, $\Omega(x_1, \dots, x_n)$ and $\Omega(x'_1, \dots, x'_n)$ are compatible. We have, by the hypotheses and the definition of $R_\Omega$, that $\Omega(x_1, \dots, x_n) \geq x_{n+1}$ and $\Omega(x'_1, \dots, x'_n) \geq x'_{n+1}$. Hence $x_{n+1}$ and $x'_{n+1}$ are compatible elements of $\mathfrak{A}$. Now $C$ is just the restriction of compatibility to pairs of atoms, hence $x_{n+1}Cx'_{n+1}$, as required. ■

**Lemma 5.8.** *Let $h\colon \mathfrak{A} \to \mathfrak{B}$ be a complete homomorphism of atomic representable $\{-, \triangleright\}$-algebras, and let $\Omega^{\mathfrak{A}}$ and $\Omega^{\mathfrak{B}}$ be compatibility preserving completely additive $n$-ary operations on $\mathfrak{A}$ and $\mathfrak{B}$ respectively. If $h$ validates*

$$h(\Omega^{\mathfrak{A}}(a_1, \dots, a_n)) = \Omega^{\mathfrak{B}}(h(a_1), \dots, h(a_n)),$$

*then $Fh$ satisfies the reverse forth and the back conditions with respect to $R_{\Omega^{\mathfrak{B}}}$ and $R_{\Omega^{\mathfrak{A}}}$.*

*Proof*: We write $R_{\mathfrak{A}}$ for $R_{\Omega^{\mathfrak{A}}}$, and we write $R_{\mathfrak{B}}$ for $R_{\Omega^{\mathfrak{B}}}$. For the reverse forth condition, suppose $R_{\mathfrak{B}} y_1 \dots y_{n+1}$ holds and that $Fh(y_1), \dots, Fh(y_n)$ are defined. Denote $Fh(y_1), \ldots, Fh(y_n)$ by $x_1, \ldots, x_n$ respectively. By the definition of $Fh$, we have $h(x_i) \geq y_i$ for each $i$. Then $h(\Omega^{\mathfrak{A}}(x_1, \ldots, x_n)) = \Omega^{\mathfrak{B}}(h(x_1), \ldots, h(x_n)) \geq \Omega^{\mathfrak{B}}(y_1, \ldots, y_n)$, as $\Omega^{\mathfrak{B}}$ is order preserving, since it is completely additive. But $\Omega^{\mathfrak{B}}(y_1, \ldots, y_n) \geq y_{n+1}$ by the hypothesis $R_{\mathfrak{B}} y_1 \ldots y_{n+1}$ and the definition of $R_{\mathfrak{B}}$. Since $h(\Omega^{\mathfrak{A}}(x_1, \ldots, x_n)) \geq y_{n+1}$, we have that $Fh$ is defined at $y_{n+1}$ and $\Omega^{\mathfrak{A}}(x_1, \ldots, x_n) \geq Fh(y_{n+1})$. By the definition of $R_{\mathfrak{A}}$, the relation $R_{\mathfrak{A}} x_1 \ldots x_n Fh(y_{n+1})$ holds, and the reverse forth condition is established.

For the back condition, suppose that $Fh(y_{n+1})$ is defined and that the relation $R_{\mathfrak{A}} x_1 \ldots x_n Fh(y_{n+1})$ holds. Write $x_{n+1}$ for $Fh(y_{n+1})$. Then by the definition of $R_{\mathfrak{A}}$, the inequality $\Omega^{\mathfrak{A}}(x_1, \ldots, x_n) \geq x_{n+1}$ holds. Hence

$$\Omega^{\mathfrak{B}}(h(x_1), \ldots, h(x_n)) = h(\Omega^{\mathfrak{A}}(x_1, \ldots, x_n)) \geq h(x_{n+1}) \geq y_{n+1}.$$

As $\mathfrak{B}$ is atomic, it is atomistic. Hence by iterative application of the complete additivity of $\Omega^{\mathfrak{B}}$ to each argument, we find

$$\sum \{\Omega^{\mathfrak{B}}(y_1, \ldots, y_n) \mid y_1, \ldots, y_n \in \mathrm{At}(\mathfrak{B}) : y_1 \leq h(x_1), \ldots, y_n \leq h(x_n)\} \geq y_{n+1}.$$

Since $y_{n+1}$ is an atom, there are therefore some $y_1, \ldots, y_n \in \mathrm{At}(\mathfrak{B})$ with $y_i \leq h(x_i)$ for each $i$, such that $\Omega^{\mathfrak{B}}(y_1, \ldots, y_n) \geq y_{n+1}$. Then by the definitions, $Fh(y_i) = x_i$ for each $i$, and $R_{\mathfrak{B}} y_1 \ldots y_{n+1}$ holds; hence the back condition is established. ∎

**Lemma 5.9.** *If a relation $R$ has the compatibility property, then $\Omega_R$ is compatibility preserving and completely additive.*

*Proof*: To see that $\Omega_R$ is compatibility preserving, let $X_i, X_i' \in \mathfrak{A}_\pi$ be compatible, for each $i$. Since $\Omega_R(X_1 \cup X_1', \ldots, X_n \cup X_n')$ is (the image of) a well-defined partial section, of which $\Omega_R(X_1, \ldots, X_n)$ and $\Omega_R(X_1', \ldots, X_n')$ are restrictions, $\Omega_R(X_1, \ldots, X_n)$ and $\Omega_R(X_1', \ldots, X_n')$ are compatible.

To see that $\Omega_R$ is completely additive, let $X_1, \ldots, X_n \in \mathfrak{A}_\pi$ and $i \in \{1, \ldots, n\}$, and suppose $\mathcal{S}$ is a subset of $\mathfrak{A}_\pi$ whose join $\sum \mathcal{S}$ exists. So

$\sum \mathcal{S} = \bigcup \mathcal{S}$. It is clear from the definition of $\Omega_R$ that

$$\begin{aligned}\Omega_R(X_1, \dots, X_{i-1}, \bigcup \mathcal{S}, X_{i+1}, \dots, X_n) &= \bigcup_{x_i \in \bigcup \mathcal{S}} \Omega_R(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_n) \\ &= \bigcup_{T \in \mathcal{S}} \bigcup_{x_i \in T} \Omega_R(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_n) \\ &= \bigcup_{T \in \mathcal{S}} \Omega_R(X_1, \dots, X_{i-1}, T, X_{i+1}, \dots, X_n)\end{aligned}$$

and hence provides a supremum for $\{\bigcup_{T\in\mathcal{S}} \Omega_R(X_1, \dots, X_{i-1}, T, X_{i+1}, \dots, X_n) \mid T \in \mathcal{S}\}$, as required. ■

**Lemma 5.10.** *Let $\varphi\colon X \rightharpoonup Y$ define a morphism in $\mathbf{Set_q}$ from $(\pi\colon X \twoheadrightarrow X_0)$ to $(\rho\colon Y \twoheadrightarrow Y_0)$, and let $R_X$ and $R_Y$ be $(n+1)$-ary relations on $X$ and $Y$ respectively, both having the compatibility property. If $\varphi$ satisfies the reverse forth and the back conditions with respect to $R_X$ and $R_Y$, then the $n$-ary operations $\Omega_{R_X}$ and $\Omega_{R_Y}$ validate*

$$G\varphi(\Omega_{R_Y}(Y_1, \dots, Y_n)) = \Omega_{R_X}(G\varphi(Y_1), \dots, G\varphi(Y_n)).$$

*Proof*: Recall that we are identifying elements of $G(\pi)$ and $G(\rho)$—partial sections—with their images, and note that according to this view, $G\varphi$ is given by $\varphi^{-1}$ (inverse image). Let $Y_1, \dots, Y_n \in G(\rho)$.

First we show that

$$\varphi^{-1}(\Omega_{R_Y}(Y_1, \dots, Y_n)) \subseteq \Omega_{R_X}(\varphi^{-1}(Y_1), \dots, \varphi^{-1}(Y_n)),$$

and later the reverse inclusion. So let $x_{n+1}$ be an arbitrary element of $\varphi^{-1}(\Omega_{R_Y}(Y_1, \dots, Y_n))$. That is, $\varphi$ is defined on $x_{n+1}$, with value $y_{n+1}$ say, and there are some $y_1 \in Y_1, \dots, y_n \in Y_n$ such that $Ry_1 \dots y_{n+1}$. So by the back condition, there exist $x_1, \dots, x_n \in \mathrm{dom}(\varphi)$ such that $\varphi(x_1) = y_1, \dots, \varphi(x_n) = y_n$ and $R_X x_1 \dots x_{n+1}$. Then $x_i \in \varphi^{-1}(Y_i)$ for each $i \leq n$. Hence by the definition of $\Omega_{R_X}$, we have $x_{n+1} \in \Omega_{R_X}(\varphi^{-1}(Y_1), \dots, \varphi^{-1}(Y_n))$, as required.

Now we show that

$$\varphi^{-1}(\Omega_{R_Y}(Y_1, \dots, Y_n)) \supseteq \Omega_{R_X}(\varphi^{-1}(Y_1), \dots, \varphi^{-1}(Y_n)).$$

Let $x_{n+1}$ be an arbitrary element of $\Omega_{R_X}(\varphi^{-1}(Y_1), \dots, \varphi^{-1}(Y_n))$, so there exist $x_1 \in \varphi^{-1}(Y_1), \dots, x_n \in \varphi^{-1}(Y_n)$ with $R_X x_1 \dots x_{n+1}$. (So in particular $\varphi(x_1), \dots, \varphi(x_n)$ are defined.) Then by the reverse forth condition, $\varphi(x_{n+1})$ is defined and $R_Y \varphi(x_1) \dots \varphi(x_{n+1})$. Since $\varphi(x_i) \in Y_i$, for $i \leq n$, we then have $\varphi(x_{n+1}) \in \Omega_{R_Y}(Y_1, \dots, Y_n)$. Hence $x_{n+1} \in \varphi^{-1}(\Omega_{R_Y}(Y_1, \dots, Y_n))$. ■

**Lemma 5.11.** *Let $\mathfrak{A}$ be an atomic representable $\{-, \rhd\}$-algebra and $\Omega$ be a compatibility preserving completely additive $n$-ary operation on $\mathfrak{A}$. Then the map $\eta_{\mathfrak{A}}$ used in Theorem 3.9 validates*

$$\eta_{\mathfrak{A}}(\Omega(a_1, \dots, a_n)) = \Omega_{R_\Omega}(\eta_{\mathfrak{A}}(a_1), \dots, \eta_{\mathfrak{A}}(a_n)).$$

*Let $\pi\colon X \twoheadrightarrow X_0$ be a set quotient, and $R$ be an $(n+1)$-ary relation on $X$ with the compatibility property. The map $\lambda_\pi$ used in Theorem 3.9 satisfies the reverse forth condition and the back condition with respect to $R$ and $R_{\Omega_R}$.*

*Proof*: For the first part, we unwrap the definition of the right-hand side of the equality. Identifying partial functions with their images, we have

$$\Omega_{R_\Omega}(\eta_{\mathfrak{A}}(a_1), \dots, \eta_{\mathfrak{A}}(a_n)) = R_\Omega(\mathrm{At}(a_1^{\downarrow}), \dots, \mathrm{At}(a_n^{\downarrow}), \_),$$

and by the definition of $R_\Omega$, this set consists of all $x_{n+1} \in \mathrm{At}(\mathfrak{A})$ for which there are $x_i \in \mathrm{At}(a_i^{\downarrow})$, $i = 1, \dots, n$, such that $\Omega(x_1, \dots, x_n) \geq x_{n+1}$. Since $\Omega$ is completely additive and $\mathfrak{A}$ is atomistic, such atoms $x_{n+1}$ are precisely those in the downset $\Omega(a_1, \dots, a_n)^{\downarrow}$. Thus

$$\Omega_{R_\Omega}(\eta_{\mathfrak{A}}(a_1), \dots, \eta_{\mathfrak{A}}(a_n)) = \mathrm{At}(\Omega(a_1, \dots, a_n)^{\downarrow}),$$

which is precisely the definition of the left-hand side $\eta_{\mathfrak{A}}(\Omega(a_1, \dots, a_n))$, so we are done.

For the reverse forth and back conditions, since we saw in the proof of Theorem 3.9 that $\pi$ and $(F \circ G)(\pi)$ are isomorphic via the correspondence $x \mapsto \{x\}$, it suffices to show that every relation $R \subseteq X^{n+1}$ coincides with $R_{\Omega_R}$ under this identification. And indeed, by definition, we have

$$\begin{aligned} R_{\Omega_R}\{x_1\} \dots \{x_{n+1}\} &\iff \Omega_R(\{x_1\}, \dots, \{x_n\}) \supseteq \{x_{n+1}\} \\ &\iff Rx_1 \dots x_{n+1}. \quad \blacksquare \end{aligned}$$

This completes the proof that $F' \dashv G'$, and hence the proof of Theorem 5.6.

It is now straightforward to extend the completeness and duality results of the previous section.

**Definition 5.12.** *Let $\mathfrak{A}$ be an algebra in* **AtRepAlg**$(\sigma)$. *A* ***compatible completion*** *of $\mathfrak{A}$ is an embedding $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ of $(\{-, \rhd\} \cup \sigma)$-algebras such that $\mathfrak{C}$ is in* **AtRepAlg**$(\sigma)$ *and compatibly complete, and $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$.*

**Corollary 5.13.** *Let $\mathfrak{A}$ be an algebra in* **AtRepAlg**$(\sigma)$. *If $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ and $\iota'\colon \mathfrak{A} \hookrightarrow \mathfrak{C}'$ are compatible completions of $\mathfrak{A}$ then there is a unique isomorphism $\theta\colon \mathfrak{C} \to \mathfrak{C}'$ satisfying the condition $\theta \circ \iota = \iota'$.*

*Proof*: Use the isomorphism $\theta$ from Proposition 4.11. The fact that $\iota[\mathfrak{A}]$ is join dense in $\mathfrak{C}$ and the complete additivity of the additional operations ensure those additional operations are preserved by $\theta$. ■

**Corollary 5.14.** *For every algebra $\mathfrak{A}$ in $\mathbf{AtRepAlg}(\sigma)$, the embedding $\eta_{\mathfrak{A}} \colon \mathfrak{A} \hookrightarrow (G' \circ F')(\mathfrak{A})$ is the compatible completion of $\mathfrak{A}$.*

**Corollary 5.15.** *There is a duality between $\mathbf{CAtRepAlg}(\sigma)$ and $\mathbf{Set_q}(\sigma)^{\mathrm{op}}$, where $\mathbf{CAtRepAlg}(\sigma)$ is the full subcategory of $\mathbf{AtRepAlg}(\sigma)$ consisting of the compatibly complete algebras.*

*Proof*: Given Corollary 4.13, we only need to check that the families of functions $\eta_{\mathfrak{A}}$ and $\lambda_\pi$ are still isomorphisms in the expanded categories, for which it only remains to show that the functions $\eta_{\mathfrak{A}}^{-1}$ and $\lambda_\pi^{-1}$ are valid morphisms in the expanded categories.

We know $\eta_{\mathfrak{A}}$ is a bijection and preserves additional operations, and it is an elementary algebraic fact that this implies its inverse $\eta_{\mathfrak{A}}^{-1}$ preserves those same additional operations. Thus $\eta_{\mathfrak{A}}^{-1}$ is a morphism.

For $\lambda_\pi^{-1}$, we must check that the reverse forth condition and the back condition are satisfied with respect to additional relations. But we saw in the proof of Lemma 5.11 that $\lambda_\pi$, and therefore $\lambda_\pi^{-1}$, preserves and reflects each additional relation. As $\lambda_\pi^{-1}$ is a bijection, it is then evident that the reverse forth and back conditions are respected. ■

**Corollary 5.16.** *The category $\mathbf{CAtRepAlg}(\sigma)$ is a reflective subcategory of $\mathbf{AtRepAlg}(\sigma)$.*

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