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UNIFORM CONTINUITY OF POINTFREE REAL FUNCTIONS VIA FARNESS AND RELATED GALOIS CONNECTIONS

ANA BELÉN AVILEZ AND JORGE PICADO

ABSTRACT: This paper concerns uniform continuity of real-valued functions on a (pre-)uniform frame. The aim of the paper is to characterize uniform continuity of such frame homomorphisms in terms of a farness relation between elements in the frame, and then to derive from it a separation and an extension theorem for real-valued uniform maps on uniform frames. The approach, purely order-theoretic, uses properties of the Galois maps associated with the farness relation. As a byproduct, we identify sufficient conditions under which a (continuous) scale in a frame with a preuniformity generates a real-valued uniform map.

KEYWORDS: Frame, locale, sublocale, uniform cover, proximally far elements, Galois connection, uniform frame, uniform homomorphism, uniformly continuous real function, uniform extension.

MATH. SUBJECT CLASSIFICATION (2020): 06D22, 06A15, 18F70, 54C30, 54E15.

Introduction

There is a quite extensive literature in the last 15 years devoted to extension and insertion results for continuous real-valued functions in pointfree topology (good sources of references are [6, 7]). The research work that led to the present article was prompted by the question about uniform versions of those results in the setting of uniform frames and uniform frame homomorphisms, namely by the quest for pointfree counterparts to the insertion theorems of D. Preiss and J. Vilimovský in uniform spaces [17]. This paper should be considered as a first step towards that goal.

In the first part of the paper we need to fill a gap in the literature and deal with characterizations of uniform continuity of real functions on preuniform frames and with methods for generating these functions from specific types of

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scales (here referred to as *uniform scales*). This is done via a pointfree extension of the proximal relation of farness between sets due to V.A. Efremovič and J.M. Smirnov ([19]).

In the second part of the paper, we describe the relation of farness by Galois adjunctions. To put this in perspective, let L be a frame and let P be the *pseudocomplement operator*

$$(x \mapsto x^* = \bigvee\{y \in L \mid y \wedge x = 0\})\colon L \to L.$$

Further, for each cover U of L, let S_U be the *star operator* on U

$$(x \mapsto Ux = \bigvee\{u \in U \mid u \wedge x \neq 0\})\colon L \to L.$$

While P is a self-adjoint Galois map, i.e., the pair (P, P) is a dual Galois adjunction, the star operator S_U is a left adjoint Galois map with right adjoint $\widetilde{\mathsf{S}}_U$ given by $\widetilde{\mathsf{S}}_U(y) = \bigvee\{x \in L \mid Ux \leq y\}$ (i.e. the pair $(\mathsf{S}_U, \widetilde{\mathsf{S}}_U)$ is a Galois adjunction) and we have the following diagram of adjunctions

$$\begin{tikzcd} L \arrow[r, bend left, "\mathsf{S}_U"] \arrow[r, phantom, "\perp"] & L \arrow[l, bend left, "\widetilde{\mathsf{S}}_U"] \arrow[r, bend left, "\mathsf{P}"] \arrow[r, phantom, "\perp^{op}"] & L \arrow[l, bend left, "\mathsf{P}"] \end{tikzcd}$$

Denoting by F_U the composite $\mathsf{P}\mathsf{S}_U = \widetilde{\mathsf{S}}_U\mathsf{P}$ (see Section 5 below), elements $a, b \in L$ are *U-far* if $a \leq \mathsf{F}_U(b)$ (or, equivalently, $b \leq \mathsf{F}_U(a)$).

We then apply this to prove a Urysohn-type separation result that provides a functional separation of far elements in (pre-)uniform frames with a purely algebraic (order-theoretic) construction and finish with a Tietze-type extension theorem for uniform homomorphisms. The latter result provides a uniform extension of any uniformly continuous real function on a dense sublocale S of L to the whole of L and is based on a well known general result for lattices (Lemma 7.2 below, known as the Katětov Lemma) that extends the original basic lemma of Katětov, formulated for power sets in his 1951 celebrated paper (corrected in 1953 [10]).

The paper is organized as follows: after the first section which reviews all the required background material, the second section introduces a pointfree counterpart of the proximal farness relation between sets and studies it in connection with continuous real functions in frames. Sections 3 and 4 provide a systematic treatment of uniform continuity of continuous real functions in terms of farness. In particular, Section 4 identifies the *uniform scales*, that is, the scales in a preuniform frame $(L,\mathcal{U})$ that generate real-valued uniform maps on $(L,\mathcal{U})$. Section 5 introduces some Galois maps related with farness, and their properties, which are then applied, in Section 6, in the proof of a functional separation result for far elements in a preuniform frame. Finally, Section 7 provides an extension theorem for real-valued uniform maps in dense sublocales.

1. Preliminaries

We refer to Banaschewski (respectively [2] and [3]) for general information on continuous real functions and uniform structures in the pointfree setting. For the origin of the latter subject see Pultr [18]. As general references for the pointfree setting of frames and locales, we refer to Picado-Pultr ([15], [16]).

1.1. Galois connections. Recall that a *Galois adjunction* [5] between posets A and B is a pair (f,g) of maps $f\colon A\to B$ and $g\colon B\to A$ such that

$$f(a)\le b \;\Leftrightarrow\; a\le g(b) \quad \text{for all } a\in A \text{ and } b\in B, \tag{1.1.1}$$

or, equivalently, a pair of order-preserving maps $f\colon A\to B$ and $g\colon B\to A$ satisfying

$$a\le g(f(a)) \text{ for all } a\in A \text{ and } f(g(b))\le b \text{ for all } b\in B.$$

Maps f and g uniquely determine each other. If A and B are complete lattices, left adjoints f in Galois adjunctions (f,g) are precisely the complete join-homomorphisms, that is, the maps such that $f(\bigvee S)=\bigvee f[S]$ for every $S\subseteq A$.

Originally, *Galois connections* were expressed in a contravariant form with maps that reverse order ([4, 13]); these are dual adjunctions between posets A and B, that is, pairs (f,g) of maps $f\colon A\to B$ and $g\colon B\to A$ such that

$$b\le f(a) \;\Leftrightarrow\; a\le g(b) \quad \text{for all } a\in A \text{ and } b\in B, \tag{1.1.2}$$

or, equivalently, pairs of order-reversing maps $f\colon A\to B$ and $g\colon B\to A$ satisfying

$$a\le g(f(a)) \text{ for all } a\in A \text{ and } b\le f(g(b)) \text{ for all } b\in B.$$

If A and B are complete lattices, f is a complete join-homomorphism from A to B^{op} (the dual of B) while g is a complete join-homomorphism $B\to A^{op}$.

Clearly, (f, g) is a Galois connections iff (g, f) is one. Both composites of the partners of a Galois connection are closure operators, and their ranges are dually isomorphic.

1.2. Frames and locales. Our notation and terminology for frames and locales is that of [15].

A *frame* (or *locale*) L is a complete lattice in which

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\} \quad \text{for any } a \in L \text{ and } S \subseteq L.$$

A *frame homomorphism* preserves all joins (in particular, the bottom element 0 of the lattice) and all finite meets (in particular, the top element 1). The category of locales and localic maps is the dual category of the category of frames and frame homomorphisms (see [15] for a concrete description of localic maps).

In a frame L the mappings $(x \mapsto (a \wedge x)) \colon L \to L$ preserve suprema and hence they have right Galois adjoints $(y \mapsto (a \to y)) \colon L \to L$, satisfying

$$a \wedge x \leq y \quad \text{iff} \quad x \leq a \to y$$

and making L a complete Heyting algebra. The *pseudocomplement* of $a \in L$ is the element $a^* = a \to 0 = \bigvee \{x \mid x \wedge a = 0\}$ with properties $a \leq a^{**}$, $a^{***} = a^*$ and $a \leq b \Rightarrow b^* \leq a^*$.

The *rather below relation* $\prec$ in a frame L is defined by $b \prec a$ iff $b^* \vee a = 1$.

1.3. The frame of reals. Recall the frame of reals $\mathfrak{L}(\mathbb{R})$ from [2]. Here we define it, equivalently, as the frame presented by generators $(p, \text{—})$ and $(\text{—}, p)$ for all rationals p, and relations

(r1) $(p, \text{—}) \wedge (\text{—}, q) = 0$ if $q \leq p$,
(r2) $(p, \text{—}) \vee (\text{—}, q) = 1$ if $p < q$,
(r3) $(p, \text{—}) = \bigvee_{r > p} (r, \text{—})$,
(r4) $(\text{—}, q) = \bigvee_{s < q} (\text{—}, s)$,
(r5) $\bigvee_{p \in \mathbb{Q}} (p, \text{—}) = 1$,
(r6) $\bigvee_{q \in \mathbb{Q}} (\text{—}, q) = 1$.

Note that $(\text{—}, q)^* = (q, \text{—})$ and $(p, \text{—})^* = (\text{—}, p)$. For each $p < q$ in $\mathbb{Q}$, the element $(p, \text{—}) \wedge (\text{—}, q)$ in $\mathfrak{L}(\mathbb{R})$ is denoted by (p, q).

1.4. Continuous real functions. The ℓ-ring $\mathcal{R}(L)$ of *continuous real-valued functions* [2] on a frame L is the set of all frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to L$, partially ordered by

$$f \leq g \quad \text{iff} \quad f(p,\text{—}) \leq g(p,\text{—}) \quad \text{for all } p \in \mathbb{Q}$$

(equivalently, $g(\text{—},q) \leq f(\text{—},q)$ for all $q \in \mathbb{Q}$). Each element of $\mathcal{R}(L)$ is uniquely determined by a map defined on the generators of $\mathfrak{L}(\mathbb{R})$ that turns relations (r1)-(r6) into identities in L.

For each $r \in \mathbb{Q}$ we have the *constant function* $\mathbf{r}\colon \mathfrak{L}(\mathbb{R}) \to L$ defined by

$$\mathbf{r}(p,\text{—}) = [\![p < r]\!] \quad \text{and} \quad \mathbf{r}(\text{—},q) = [\![r < q]\!]$$

where $[\![\cdots]\!]$ denotes the truth value, interpreted in L, of the stated condition.

The *bounded functions* are the $f \in \mathcal{R}(L)$ such that $f(\text{—},0) \vee f(1,\text{—}) = 0$ (that is, $\mathbf{0} \leq f \leq \mathbf{1}$).

Scales are a useful tool to define continuous real functions on a frame L. A *descending scale* (resp. *ascending scale*) in L is a family $(a_p)_{p\in\mathbb{Q}} \subseteq L$ such that

(S1) $p < q \;\Rightarrow\; a_q \prec a_p$ (resp. $a_p \prec a_q$).
(S2) $\bigvee_{p\in\mathbb{Q}} a_p = 1 = \bigvee_{p\in\mathbb{Q}} a_p^*$.

It is easy to see that, for any $f \in \mathcal{R}(L)$, the family $(f(p,\text{—}))_{p\in\mathbb{Q}}$ resp. $(f(\text{—},q))_{q\in\mathbb{Q}}$, is a descending resp. ascending scale in L. Conversely, we have:

Proposition 1.4.1. ([15, XIV.5.2.2])

(a) *Let $(a_p)_{p\in\mathbb{Q}}$ be a descending scale in L. Then the formulas*

$$f(p,\text{—}) = \bigvee_{r>p} a_r \quad \textit{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s^*$$

determine an $f \in \mathcal{R}(L)$.

(b) *Let $(a_p)_{p\in\mathbb{Q}}$ be an ascending scale in L. Then the formulas*

$$g(p,\text{—}) = \bigvee_{r>p} a_r^* \quad \textit{and} \quad g(\text{—},q) = \bigvee_{s<q} a_s$$

determine a $g \in \mathcal{R}(L)$.

1.5. Uniform frames. A *cover* of a frame L is a subset $U \subseteq L$ such that $\bigvee U = 1$. A cover U *refines* (or is a *refinement* of) a cover V, written, $U \leq V$, if for any $u \in U$ there is some $v \in V$ such that $u \leq v$. For covers U, V we have the largest common refinement $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$.

For any $U \subseteq L$ and any $x \in L$ the *star* of x in U is the element

$$U \cdot x = \bigvee \{u \in U \mid u \wedge x \neq 0\}.$$

For any $U, V \subseteq L$, set

$$U \cdot V = \{Uv \mid v \in V\}.$$

If U and V are covers, then $U \cdot V$ is also a cover. One usually writes Ux and UV instead of $U \cdot x$ and $U \cdot V$. Since this operator is neither commutative nor associative, we will use parenthesis when needed.

The following proposition lists some of the basic properties of these operators (cf. [15] or [18]).

Proposition 1.5.1. *For any covers $U, V \subseteq L$ and any frame homomorphism $h \colon L \to M$, we have:*

(1) $x \leq Ux$.
(2) $Ux \leq y$ *implies* $x \prec y$.
(3) $U \leq UU$.
(4) $U \leq V$ *and* $x \leq y$ *imply* $Ux \leq Vy$.
(5) $U(Vx) \leq (UV)x = U(V(Ux))$.
(6) $U\left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} Ux_i$.
(7) $h[U]\, h(x) \leq h(Ux)$.

For a cover U, define a cover U^n for $n \geq 1$ inductively by setting

$$U^1 = U \quad \text{ and } \quad U^{n+1} = U \cdot U^n.$$

Hence

$$U^{n+1} = \{U \cdot x \mid x \in U^n\}, \quad n = 1, 2, \ldots.$$

Clearly, for any $n \geq 1$, $U \leq V$ implies $U^n \leq V^n$.

From now on we shall always assume that $1 \neq 0$ in L (that is, $|L| \geq 2$). A (covering) *uniformity* on L is a nonempty system $\mathcal{U}$ of covers of L such that

(U1) $U \in \mathcal{U}$ and $U \leq V$ implies $V \in \mathcal{U}$,
(U2) $U, V \in \mathcal{U}$ implies $U \wedge V \in \mathcal{U}$,
(U3) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$, and
(U4) for every $a \in L$, $a = \bigvee \{b \mid b \triangleleft_{\mathcal{U}} a\}$

where we write $b \triangleleft_{\mathcal{U}} a$ if $Ub \leq a$ for some $U \in \mathcal{U}$.

Without (U4) one speaks of a pre-uniformity, without (U1) one speaks of a *basis of a (pre-)uniformity* (in the latter case one obtains, of course, a (pre-)uniformity adding all the V with $V \geq U \in \mathcal{U}$).

A *uniform frame* (resp. *preuniform frame*) is a pair $(L,\mathcal{U})$ where $\mathcal{U}$ is a uniformity (resp. preuniformity) on L. A frame homomorphism $h\colon L \to M$ is a *uniform homomorphism* $(L,\mathcal{U}) \to (M,\mathcal{V})$ if $h[U] \in \mathcal{V}$ for every $U \in \mathcal{U}$ (if $\mathcal{U}$,$\mathcal{V}$ are bases of (pre-)uniformities this condition is replaced by the existence of some $V \in \mathcal{V}$ such that $h[U] \geq V$).

1.6. The metric uniformity of $\mathfrak{L}(\mathbb{R})$. The frame of reals carries a natural uniformity, its *metric uniformity* [2], generated by covers

$$C_n = \left\{(p,q) \in \mathfrak{L}(\mathbb{R}) \mid 0 < q-p < \tfrac{1}{n}\right\}, \quad n = 1,2,\ldots$$

Alternatively, we may consider the covers

$$D_n = \left\{(r,s) \in \mathfrak{L}(\mathbb{R}) \mid s-r = \tfrac{1}{n}\right\}, \quad n = 1,2,\ldots.$$

Clearly, for $n \leq m$, $D_m \leq D_n$ and $C_m \subseteq C_n$. Moreover, for every $n \in \mathbb{N}$, $C_n \leq D_n$ and $D_{n+1} \subseteq C_n$. Hence these covers also constitute a basis for the metric uniformity on $\mathfrak{L}(\mathbb{R})$.

We will consider, more generally, the covers

$$D_\delta = \left\{(p,q) \in \mathfrak{L}(\mathbb{R}) \mid \ q-p = \tfrac{1}{\delta}\right\}, \quad \delta \in \mathbb{Q}^+$$

(where $\mathbb{Q}^+$ denotes the set of positive rational numbers).

Proposition 1.6.1. *For any $\gamma,\delta \in \mathbb{Q}^+$, $D_\gamma \cdot D_\delta = D_{\frac{\gamma\delta}{\gamma+2\delta}}$.*

Proof: By definition $D_\gamma \cdot D_\delta = \left\{D_\gamma \cdot (p-\frac{1}{2\delta}, p+\frac{1}{2\delta}) \mid p \in \mathbb{Q}\right\}$ and

$$D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) = \bigvee\{(r,s) \mid s-r=\tfrac{1}{\gamma}, (r,s) \wedge (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \neq 0\}.$$

From $(r,s) \wedge (p-\frac{1}{2\delta}, p+\frac{1}{2\delta}) \neq 0$, it follows that

$$(r,s) \leq \left(p-\tfrac{1}{2\delta}-\tfrac{1}{\gamma}, p+\tfrac{1}{2\delta}+\tfrac{1}{\gamma}\right) = \left(p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}\right).$$

Hence

$$D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \leq (p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}).$$

Now,

$$D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \geq \bigvee_{\alpha<\frac{\gamma+2\delta}{2\delta\gamma}} (p-\alpha, p+\alpha) = (p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}),$$

hence $D_\gamma \cdot (p - \frac{1}{2\delta}, p + \frac{1}{2\delta}) = (p - \frac{\gamma+2\delta}{2\delta\gamma}, p + \frac{\gamma+2\delta}{2\delta\gamma})$. Therefore

$$\begin{aligned} D_\gamma \cdot D_\delta &= \left\{ D_\gamma \cdot (p - \tfrac{1}{2\delta}, p + \tfrac{1}{2\delta}) \mid p \in \mathbb{Q} \right\} \\ &= \left\{ (p - \tfrac{\gamma+2\delta}{2\delta\gamma}, p + \tfrac{\gamma+2\delta}{2\delta\gamma}) \mid p \in \mathbb{Q} \right\} = D_{\frac{\gamma\delta}{2\delta+\gamma}}. \end{aligned} \qquad \blacksquare$$

Proposition 1.6.2. *For every $n \in \mathbb{N}$ and $\delta \in \mathbb{Q}^+$, $D_\delta^n = D_{\frac{\delta}{2n-1}}$.*

Proof: Let $\delta \in \mathbb{Q}^+$. We show the result by induction over n. The case $n = 1$ is trivial. Assume it holds for n. Using 1.6.1, we get

$$D_\delta^{n+1} = D_\delta \cdot D_\delta^n = D_\delta \cdot D_{\frac{\delta}{2n-1}} = D_{\frac{\delta^2}{2\delta+\delta(2n-1)}} = D_{\frac{\delta}{2n+1}} = D_{\frac{\delta}{2(n+1)-1}}. \qquad \blacksquare$$

2. Covering farness and continuous real functions

We introduce now the central concept of the paper. Let U be a cover of a frame L. We say that elements a, b in L are *U-far* if

$$Ua \wedge b = 0 \quad \text{or, equivalently,} \quad a \wedge Ub = 0.$$

In other words, a and b are U-far iff $Ua \leq b^*$ iff $a \leq (Ub)^*$ iff $Ub \leq a^*$ iff $b \leq (Ua)^*$. Hence a and b are U-far iff

$$U \leq \{a^*, b^*\}. \tag{2.1.1}$$

In particular,

$$a^* \vee b^* = 1, \text{ that is, } a \prec b^*. \tag{2.1.2}$$

Equivalently, a and b in L are U-far iff

$$\forall u \in U, \quad a \wedge u \neq 0 \Rightarrow b \wedge u = 0. \tag{2.1.3}$$

Remarks 2.1. (1) Let $a, b \in L$ be U-far. If $c \leq a$ and $d \leq b$, then c and d are U-far. Further, for any $V \leq U$, a and b are also V-far.

(2) a and b are U-far iff a^{**} and b^{**} are U-far.

(3) There is an obvious link between the farness relation and the uniform strong relation $\triangleleft_\mathcal{U}$ for a preuniformity $\mathcal{U}$: a and b are U-far for some $U \in \mathcal{U}$ if and only if $a \triangleleft_\mathcal{U} b^*$. Since $a \triangleleft_\mathcal{U} b$ implies $a \prec b$ ([15, VIII.2.3.2]), (2.1.2) also follows from this.

On the other hand, if $a \triangleleft_\mathcal{U} b$ then $a \triangleleft_\mathcal{U} b \leq b^{**}$ hence a and b^* are U-far for some $U \in \mathcal{U}$.

(4) Let $\mathcal{U}$ be a preuniformity on L and let $a, b \in L$. It follows immediately from (U3) that if a and b are U-far for some $U \in \mathcal{U}$ then, for each $m \in \mathbb{N}$, there is some $V \in \mathcal{U}$ such that a and b are V^m-far.

Let $f, g \in \mathcal{R}(L)$. Since

$$g(s, \text{—}) \leq g(\text{—}, s)^* \leq g(s', \text{—}) \quad \text{for any } s' < s, \tag{2.1.4}$$

and

$$f(\text{—}, r) \leq f(r, \text{—})^* \leq f(\text{—}, r') \quad \text{for any } r' > r \tag{2.1.5}$$

we have immediately:

Lemma 2.2. *Let U be a cover of L, $f, g \in \mathcal{R}(L)$ and $\delta \in \mathbb{Q}^+$. The following are equivalent:*

(i) *$f(\text{—}, r)$ and $g(s, \text{—})$ are U-far for all $s - r > \frac{1}{\delta}$.*
(ii) *$f(\text{—}, r)$ and $g(\text{—}, s)^*$ are U-far for all $s - r > \frac{1}{\delta}$.*
(iii) *$f(r, \text{—})^*$ and $g(s, \text{—})$ are U-far for all $s - r > \frac{1}{\delta}$.*
(iv) *$f(r, \text{—})^*$ and $g(\text{—}, s)^*$ are U-far for all $s - r > \frac{1}{\delta}$.*

Let $f, g \in \mathcal{R}(L)$ such that $f \geq g$. For each $\delta \in \mathbb{Q}^+$,

$$D_\delta^{f,g} := \{f(r, \text{—}) \wedge g(\text{—}, s) \mid (r, s) \in D_\delta\}$$

is a cover of L.

Indeed:

$$\begin{aligned}\bigvee_{(r,s)\in D_\delta} (f(r, \text{—}) \wedge g(\text{—}, s)) &\geq \bigvee_{(r,s)\in D_\delta} (g(r, \text{—}) \wedge g(\text{—}, s)) \\ &= g\Big(\bigvee_{(r,s)\in D_\delta} (r, s)\Big) = g(1) = 1.\end{aligned}$$

Note that $D_\delta^{f,g}$ is a refinement of both covers $\{f(r, \text{—}) \mid r \in \mathbb{Q}\}$ and $\{g(\text{—}, r) \mid r \in \mathbb{Q}\}$. When $f = g$, $D_\delta^{f,g} = f[D_\delta]$. We will denote this cover just by D_δ^f.

Proposition 2.3. *Let $(a_p)_{p\in\mathbb{Q}}, (b_q)_{q\in\mathbb{Q}} \subseteq L$ such that*

$$a_q \leq a_p \text{ and } b_p \leq b_q \text{ for every } p \leq q.$$

If $U_\delta = \{a_r \wedge b_s \mid (r, s) \in D_\delta\}$ is a cover for $\delta \in \mathbb{Q}^+$, then a_r^* and b_s^* are U_δ-far for every $s - r > \frac{1}{\delta}$.

Proof: Let $s - r > \frac{1}{\delta}$. By (2.1.1), we need to show that $U_\delta \le \{a_r^{**}, b_s^{**}\}$. Let $(p,q) \in D_\delta$. Then $s > q$ or $r < p$. In the former case we have $b_s \ge b_q$ and thus $a_p \wedge b_q \le b_s \le b_s^{**}$; otherwise, in the latter case, $a_r \ge a_p$ hence $a_p \wedge b_q \le a_r \le a_r^{**}$. ■

Corollary 2.4. *Let $f, g \in \mathcal{R}(L)$ such that $f \ge g$. For each $\delta \in \mathbb{Q}^+$ and every $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$, the elements $f(\text{—}, r)$ and $g(s, \text{—})$ are $D_\delta^{f,g}$-far.*

Proof: As noted above, $D_\delta^{f,g}$ is a cover for every $\delta \in \mathbb{Q}^+$ whenever $f \ge g$. From 2.3, taking $a_r = f(r, \text{—})$ and $b_s = g(\text{—}, s)$ we have that $f(r, \text{—})^*$ and $g(\text{—}, s)^*$ are $D_\delta^{f,g}$-far. Then, by 2.2, $f(\text{—}, r)$ and $g(s, \text{—})$ are also $D_\delta^{f,g}$-far. ■

Proposition 2.5. *Let $(a_r)_{r\in\mathbb{Q}}, (b_s)_{s\in\mathbb{Q}} \subseteq L$ satisfy the following conditions:*

(1) *$a_r \le a_p$ for every $p < r$.*
(2) *$b_s \le b_q$ for every $s < q$.*
(3) $\bigvee_{r\in\mathbb{Q}} a_r^* = 1$.
(4) $\bigvee_{s\in\mathbb{Q}} b_s^* = 1$.
(5) *$a_r^{**} \le a_p$ for $p < r$.*
(6) *$b_s^{**} \le b_q$ for $s < q$.*

If U is a cover of L such that a_r^* and b_s^* are U-far for all $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$, then

$$U \le \{a_r \wedge b_s \mid (r,s) \in D_\gamma\} \qquad \textit{for every } \gamma < \delta \textit{ in } \mathbb{Q}^+.$$

In particular, $\{a_r \wedge b_s \mid (r,s) \in D_\gamma\}$ is a cover of L.

Proof: For each $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$, $U \le \{a_r^{**}, b_s^{**}\}$. Therefore, for every $u \in U$,

$$u \wedge a_r^* \ne 0 \;\Rightarrow\; u \wedge b_s^* = 0. \tag{$*$}$$

Let $u \ne 0$ in U. Since $\{b_s^* \mid s \in \mathbb{Q}\}$ is a cover of L, there exists $s_0 \in \mathbb{Q}$ such that $u \wedge b_{s_0}^* \ne 0$. By $(*)$, $u \wedge a^*_{s_0 - \frac{2}{\delta}} = 0$. Thus, $u \le a^{**}_{s_0-\frac{2}{\delta}} \le a_{s_0 - \frac{3}{\delta}}$ and the set $\{r \in \mathbb{Q} \mid u \le a_r\}$ is nonempty. It should be also noted that $u \not\le a_r$ for some $r \in \mathbb{Q}$ (and therefore $u \not\le a_{r'}$ for any $r' \ge r$), otherwise

$$u \le \bigwedge_{r\in\mathbb{Q}} a_r \le \bigwedge_{r\in\mathbb{Q}} a_r^{**} = \big(\bigvee_{r\in\mathbb{Q}} a_r^*\big)^* = 0,$$

a contradiction. Hence,

$$r_1 = sup\,\{r \in \mathbb{Q} \mid u \le a_r\} \in \mathbb{R}.$$

Now, let $\gamma \in \mathbb{Q}^+$ with $\gamma < \delta$. Give $\varepsilon = \frac{\delta-\gamma}{\delta\gamma} > 0$. Take $r \in \mathbb{Q}$ such that $0 < r_1 - r < \frac{\varepsilon}{5}$ and $p \in \mathbb{Q}$ such that $0 < p - r_1 < \frac{\varepsilon}{5}$. Then $u \leq a_r$ and $u \not\leq a_p$. Since $a_p \geq a^{**}_{p+\frac{\varepsilon}{5}}$, we have further that $u \not\leq a^{**}_{p+\frac{\varepsilon}{5}}$, that is, $u \wedge a^{*}_{p+\frac{\varepsilon}{5}} \neq 0$ and, by $(*)$, $u \wedge b^{*}_{p+\frac{2\varepsilon}{5}+\frac{1}{\delta}} = 0$, that is,

$$u \leq b^{**}_{p+\frac{2\varepsilon}{5}+\frac{1}{\delta}} \leq b_{p+\frac{3\varepsilon}{5}+\frac{1}{\delta}}.$$

In conclusion,

$$u \leq a_r \wedge b_{p+\frac{3\varepsilon}{5}+\frac{1}{\delta}} \leq a_r \wedge b_{r+\frac{1}{\delta}+\varepsilon} \in \{a_r \wedge b_s \mid (r,s) \in D_\gamma\},$$

since

$$p + \tfrac{3\varepsilon}{5} + \tfrac{1}{\delta} - r < \tfrac{\varepsilon}{5} + r_1 + \tfrac{3\varepsilon}{5} + \tfrac{1}{\delta} + \tfrac{\varepsilon}{5} - r_1 = \tfrac{1}{\delta} + \varepsilon$$

and

$$\tfrac{1}{\delta} + \varepsilon = \tfrac{1}{\delta} + \tfrac{\delta-\gamma}{\gamma\delta} = \tfrac{1}{\gamma}. \qquad \blacksquare$$

Corollary 2.6. *Let $f, g \in \mathcal{R}(L)$ such that $f \geq g$ and $\delta \in \mathbb{Q}^+$. If U is a cover of L such that $f(\text{—}, r)$ and $g(s, \text{—})$ are U-far for all $r, s \in \mathbb{Q}$ with $s - r > \frac{1}{\delta}$, then*

$$U \leq D^{f,g}_\gamma \quad \textit{for every } \gamma < \delta \textit{ in } \mathbb{Q}^+.$$

Proof: We will use 2.5. Let $a_r = f(r, \text{—})$ and $b_s = g(\text{—}, s)$. Since $f, g \in \mathcal{R}(L)$, conditions (1)-(6) hold. Furthermore, by assumption and 2.2, a^*_r and b^*_s are U-far for every $s - r > \frac{1}{\delta}$. Thus,

$$U \leq \{a_r \wedge b_s \mid (r,s) \in D_\gamma\} = D^{f,g}_\gamma. \qquad \blacksquare$$

Corollary 2.7. *The following are equivalent for any $f, g \in \mathcal{R}(L)$ such that $f \geq g$:*

(i) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $U \leq D^{f,g}_\delta$.*
(ii) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $f(\text{—}, r)$ and $g(s, \text{—})$ are U-far for any $s - r > \frac{1}{\delta}$.*

Proof: (i)⇒(ii): Let $\delta \in \mathbb{Q}^+$ and U such that $U \leq D^{f,g}_\delta$. By 2.4 and 2.1(1), $f(\text{—}, r)$ and $g(s, \text{—})$ are U-far for every $s - r > \frac{1}{\delta}$.
(ii)⇒(i): Let $\delta \in \mathbb{Q}^+$. By assumption, there is a cover U such that $f(\text{—}, r)$ and $g(s, \text{—})$ are U-far for any $s - r > \frac{1}{\delta+1}$. Then, by 2.6, $U \leq D^{f,g}_\delta$. $\blacksquare$

More generally, we have:

Proposition 2.8. *The following are equivalent for any $f, g \in \mathcal{R}(L)$ such that $f \geq g$:*

(i) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $U^n \leq D^{f,g}_{\frac{\delta}{2n-1}}$ for every $n \in \mathbb{N}$.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $f(—, r)$ and $g(s, —)$ are U^n-far for every $s - r > \frac{n}{\delta}$ and $n \in \mathbb{N}$.*

Proof: (i)⇒(ii): Let $\delta \in \mathbb{Q}^+$ and consider $\varepsilon = 2\delta$. By assumption, there is some U such that $U^n \leq D^{f,g}_{\frac{\varepsilon}{2n-1}}$ for every $n \in \mathbb{N}$. Let $s - r > \frac{n}{\delta} = \frac{2n}{\varepsilon} > \frac{2n-1}{\varepsilon}$. By 2.4, $f(—, r)$ and $g(s, —)$ are $D^{f,g}_{\frac{\varepsilon}{2n-1}}$-far. In particular, they are U^n-far, as required.

(ii)⇒(i): Let $\delta \in \mathbb{Q}^+$. By assumption, there is some U such that $f(—, r)$ and $g(s, —)$ are U^n-far for every $s - r > \frac{n}{\delta+1}$ and $n \in \mathbb{N}$. Then, by 2.6, since $\frac{\delta}{2n-1} < \frac{\delta+1}{n}$, we have $U^n \leq D^{f,g}_{\frac{\delta}{2n-1}}$ as required. ■

Remark 2.9. Note that assumption $f \geq g$ is crucial here. For instance, the condition $f \leq g$ does not imply that

$$\{f(r, —) \wedge g(—, s) \mid (r, s) \in D_\delta\}$$

is even a cover of L. Moreover, if $f \leq g$ and, for every $\delta \in \mathbb{Q}^+$, there is some U such that $f(—, r)$ and $g(s, —)$ are U-far for any $s - r > \frac{1}{\delta}$, then $f = g$. Indeed, for every pair $r < s$, $f(—, r) \wedge g(s, —) = 0$ (consequence of the farness), thus,

$$g(s, —) \leq \bigwedge_{r<s} f(—, r)^* = f(—, s)^*.$$

This means that, for every $s \in \mathbb{Q}$, $g(s, —) \leq f(—, s)^*$ and then, for any rational q,

$$g(q, —) = \bigvee_{s>q} g(s, —) \leq \bigvee_{s>q} f(—, s)^* \leq \bigvee_{s'>q} f(s', —) = f(q, —),$$

which shows that $f = g$.

3. Uniform continuity via covering farness

From now on, we consider continuous real functions $f \in \mathcal{R}(L)$ for a preuniform frame $(L, \mathcal{U})$ and say that f is *uniformly continuous* on L whenever it is a uniform frame homomorphism $\mathfrak{L}(\mathbb{R}) \to (L, \mathcal{U})$ with respect to the metric uniformity on $\mathfrak{L}(\mathbb{R})$.

Theorem 3.1. *Let $(L,\mathcal{U})$ be a preuniform frame. The following are equivalent for any $f \in \mathcal{R}(L)$:*

(i) *f is uniformly continuous.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that $f(-,r)$ and $f(s,-)$ are U-far for all $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$.*

(iii) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that $f(-,r)$ and $f(s,-)$ are U^n-far for every natural n and every $r, s \in \mathbb{Q}$ such that $s - r > \frac{n}{\delta}$.*

(iv) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that*

$$U \leq D^f_\delta = \{f(r,s) \mid (r,s) \in D_\delta\}.$$

(v) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that $U^n \leq D^f_{\frac{\delta}{2n-1}}$ for every $n \in \mathbb{N}$.*

Proof: (i)⇒(iii): Let $\delta \in \mathbb{Q}^+$ and consider a natural m such that $\frac{1}{m} \leq \frac{1}{\delta}$. By assumption, there is a uniform cover $U \in \mathcal{U}$ such that $U \leq f[D_{2m}] = D^f_{2m}$. We claim this is the cover we are looking for.

Let $n \in \mathbb{N}$ and $r, s \in \mathbb{Q}$ such that $s - r > \frac{n}{\delta}$. If $n = 1$ then, $s - r > \frac{1}{m} > \frac{1}{2m}$. By 2.4, $f(-,r)$ and $f(s,-)$ are D^f_{2m}-far, and since $U \leq D^f_{2m}$, they are U-far. For $n \geq 2$, suppose $f(-,r)$ and $f(s,-)$ are not U^n-far. Since $U \leq D^f_{2m}$, using 1.5.1(7) and 1.6.2 we obtain

$$U^n \leq (D^f_{2m})^n = f[D_{2m}]^n \leq f[(D_{2m})^n] = f[D_{\frac{2m}{2n-1}}] = D^f_{\frac{2m}{2n-1}}.$$

Thus, $f(-,r)$ and $f(s,-)$ are not $D^f_{\frac{2m}{2n-1}}$-far. This means that there is some pair $(p,q) \in D_{\frac{2m}{2n-1}}$ such that

$$f(-,r) \wedge f(p,q) \neq 0 \quad \text{and} \quad f(s,-) \wedge f(p,q) \neq 0.$$

It then follows that $p < r$ and $s < q$ and therefore that

$$\tfrac{n}{\delta} < s - r < q - p = \tfrac{2n-1}{2m} < \tfrac{n}{m} \leq \tfrac{n}{\delta},$$

a contradiction. Hence $f(-,r)$ and $f(s,-)$ have to be U^n-far.

(iii)⇒(ii) is obvious.

(ii)⇒(i): Let $n \in \mathbb{N}$ and $m = n + 1$. By assumption, there is some $U \in \mathcal{U}$ such that the elements $f(-,r)$ and $f(s,-)$ are U-far for all $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{m}$. Since $m > n$, by 2.6 we get $U \leq D^f_n = f[D_n]$.

(iv)⇔(ii): By 2.7.

(v)⇔(iii): By 2.8. ■

Recall Proposition 1.4.1. In the present situation, we have:

Corollary 3.2. *Let* $(L,\mathcal{U})$ *be a preuniform frame and let* $f \in \mathcal{R}(L)$ *be given by a descending (resp. ascending) scale* $(a_p)_{p\in\mathbb{Q}}$*. Then* f *is uniformly continuous if and only if for every* $\delta \in \mathbb{Q}^+$ *there is a* $U \in \mathcal{U}$ *such that* a_r^* *and* a_s *(resp.* a_r *and* a_s^**) are* U*-far for any* $s - r > \frac{1}{\delta}$*.*

Proof: Assume f is uniformly continuous and let $\delta \in \mathbb{Q}^+$. By 3.1, there exists $U \in \mathcal{U}$ such that $f(\text{—}, r) = \bigvee_{p<r} a_p^*$ and $f(s, \text{—}) = \bigvee_{q>s} a_q$ are U-far for any $s - r > \frac{1}{\delta}$. Take $q - p > \frac{1}{\delta}$ and $s, r \in \mathbb{Q}$ such that $p < r < s < q$ and $s - r > \frac{1}{\delta}$. Then $a_q \le f(s, \text{—})$ and $a_p^* \le f(\text{—}, r)$. Hence, a_q and a_p^* are also U-far.

Conversely, let $\delta \in \mathbb{Q}^+$ and $s - r > \frac{1}{\delta}$. Take the U provided by the hypothesis and consider $u \in U$ such that $u \wedge f(s, \text{—}) \neq 0$. Then there exists $q > s$ such that $u \wedge a_q \neq 0$. By the farness hypothesis, $a_p^* \wedge u = 0$ for every $p < r$. Hence, $u \wedge f(\text{—}, r) = 0$. Therefore, $f(s, \text{—})$ and $f(\text{—}, r)$ are U-far and we may use 3.1 to conclude that f is uniformly continuous. ■

4. Uniform scales

We now identify sufficient conditions on a scale on a preuniform frame $(L,\mathcal{U})$ under which it generates a uniformly continuous real function on L.

We say that a family $(a_p)_{p\in\mathbb{Q}} \subseteq L$ is an *ascending uniform scale* whenever the formulas

$$h(\text{—}, r) = \bigvee_{p<r} a_p \quad \text{and} \quad h(s, \text{—}) = \bigvee_{q>s} a_q^* \tag{4.1.1}$$

induce a uniformly continuous $h \in \mathcal{R}(L)$. Similarly, a family $(b_p)_{p\in\mathbb{Q}} \subseteq L$ is a *descending uniform scale* if the formulas $h(r, \text{—}) = \bigvee_{r<p} b_p$ and $h(\text{—}, s) = \bigvee_{q<s} b_q^*$ yield a uniformly continuous $h \in \mathcal{R}(L)$.

We know already, from 3.2 (recall also 1.4), that any $(a_p)_{p\in\mathbb{Q}} \subseteq L$ such that

(S1) $a_p \prec a_q$ for every $p < q$,

(S2) $\bigvee_{p\in\mathbb{Q}} a_p = 1 = \bigvee_{p\in Q} a_p^*$,

(Far) for every $\delta \in \mathbb{Q}^+$ there is a $V \in \mathcal{U}$ such that a_r and a_s^* are V-far for any $s - r > \frac{1}{\delta}$,

is an (ascending) uniform scale.

Note 4.1. Condition (S1) can be replaced by the weaker

(wS1) $a_p^{**} \le a_q$ for every $p < q$.

Indeed, condition (wS1) together with (Far) imply (S1): if $p < q$ then there is $s \in \mathbb{Q}$ such that $p < s < q$ and, by (Far), there is $U \in \mathcal{U}$ such that a_p and a_s^* are U-far. In particular, this means that $a_p^* \vee a_s^{**} = 1$. Hence, using (wS1), we get $1 = a_p^* \vee a_s^{**} \leq a_p^* \vee a_q$, which means that $a_p \prec a_q$.

Hence, by 3.2 we have:

Proposition 4.2. *Let $(L, \mathcal{U})$ be a preuniform frame. If $(a_p)_{p\in\mathbb{Q}} \subseteq L$ satisfies (wS1), (S2) and (Far), then it is an ascending uniform scale.*

Inspired by condition (iv) of 3.1 we can define uniform scales in a different way. For that, consider the following property, clearly stronger than (S2):

(C) For each $\delta \in \mathbb{Q}^+$ there is a $V \in \mathcal{U}$ such that

$$V \leq \{a_p^* \wedge a_q \mid (p, q) \in D_\delta\}.$$

And, instead of (S1), consider

(s1) $a_p \leq a_q$ for every $p < q$.

Notes 4.3. (1) (C) & (s1) $\Rightarrow$ (S1): Let $p < q$ and consider $\delta \in \mathbb{Q}^+$ such that $q - p > \frac{1}{\delta}$. For each $r \in \mathbb{Q}$, $r \geq p$ or $r + \frac{1}{\delta} \leq q$. In the former case,

$$a_r^* \wedge a_{r+\frac{1}{\delta}} \leq a_r^* \leq a_p^* \leq a_p^* \vee a_q.$$

Otherwise, if $r + \frac{1}{\delta} \leq q$ then $a_r^* \wedge a_{r+\frac{1}{\delta}} \leq a_{r+\frac{1}{\delta}} \leq a_q \leq a_p^* \vee a_q$. Thus, $1 = \bigvee\{a_r^* \wedge a_s \mid (r, s) \in D_\delta\} \leq a_p^* \vee a_q$, that is, $a_p \prec a_q$.

(2) (Far) & (wS1) & (S2) $\Rightarrow$ (C): Let $\delta \in \mathbb{Q}^+$, $d_r = a_r^*$ and $e_s = a_s$ for $r, s \in \mathbb{Q}$. Then $(d_r)_{r\in\mathbb{Q}}$ is descending and $(e_s)_{s\in\mathbb{Q}}$ ascending. Further, $d_r^{**} = a_r^* \leq a_p^* = d_p$ for $p < r$, and, by (wS1), $e_s^{**} = a_s^{**} \leq a_q = e_q$ for $s < q$. From (S2) we have

$$\bigvee_{r\in\mathbb{Q}} d_r^* = \bigvee_{r\in\mathbb{Q}} a_r^{**} \geq \bigvee_{r\in\mathbb{Q}} a_r = 1 \quad \text{and} \quad \bigvee_{s\in\mathbb{Q}} e_s = \bigvee_{s\in\mathbb{Q}} a_s^* = 1.$$

Note that by Remark 2.1(2) the condition (Far) implies that for every $\delta \in \mathbb{Q}^+$ there is a $V \in U$ such that a_r^{**} and a_s^* are V-far for any $s - r > \frac{1}{\delta}$. Then for $\delta + 1$ there is a cover $U \in \mathcal{U}$ such that d_r^* and e_s^* are U-far for every $s - r > \frac{1}{\delta+1}$. Finally, from 2.5 we get

$$U \leq \{d_r \wedge e_s \mid (r, s) \in D_\delta\} = \{a_r^* \wedge a_s \mid (r, s) \in D_\delta\}.$$

(3) (s1) & (C) $\Rightarrow$ (Far): Let $\delta \in \mathbb{Q}^+$. By assumption there is a $V \in \mathcal{U}$ such that $V \leq \{a_r^* \wedge a_s \mid (r, s) \in D_\delta\}$. In particular, $\{a_r^* \wedge a_s \mid (r, s) \in D_\delta\}$ is a cover in $\mathcal{U}$. Furthermore, $(a_r)_{r\in\mathbb{Q}}$ is ascending while $(a_r^*)_{r\in\mathbb{Q}}$ is descending.

Thus, by 2.3, a_r^{**} and a_s^* are $\{a_r^* \wedge a_s \mid (r,s) \in D_\delta\}$-far for every $s-r > \frac{1}{\delta}$. In particular, a_r and a_s^* are also V-far.

It follows immediately from 4.2 and 4.3 that

Proposition 4.4. *Let $(L,\mathcal{U})$ be a preuniform frame. If $(a_p)_{p\in\mathbb{Q}} \subseteq L$ satisfies (s1) and (C), then it is an ascending uniform scale.*

It seems natural to think also on uniform scales given by some condition defined in terms of the uniform strong relation $\lhd_{\mathcal{U}}$. Thus, consider the following property:

(U) For every $\delta \in \mathbb{Q}^+$ there is some $U \in \mathcal{U}$ such that $Ua_p \leq a_q$ for every $q - p > \frac{1}{\delta}$.

Notes 4.5. (1) (U) $\Rightarrow$ (S1) is obvious since (U) implies that $a_r \lhd_{\mathcal{U}} a_s$ for every $r < s$.

(2) (Far) & (wS1) $\Leftrightarrow$ (U): Indeed, assume (U) and let $\delta \in \mathbb{Q}^+$. There is a cover $U \in \mathcal{U}$ such that $Ua_r \leq a_s$ whenever $s - r > \frac{1}{\delta}$. Hence, $Ua_r \wedge a_s^* = 0$, that is, a_r and a_s^* are U-far. Conversely, assume (Far) and (wS1) and let $\delta \in \mathbb{Q}^+$ and $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$. Consider the cover $U \in \mathcal{U}$ given by (Far) and take $s' \in \mathbb{Q}$ such that $s' - r > \frac{1}{\delta}$ and $s' < s$. Then a_r and $a_{s'}^*$ are U-far. Thus, by farness and (wS1), $Ua_r \leq a_{s'}^{**} \leq a_s$, showing that (U) holds.

Now, it follows immediately from 4.2 and 4.5 that

Proposition 4.6. *Let $(L,\mathcal{U})$ be a preuniform frame. If $(a_p)_{p\in\mathbb{Q}} \subseteq L$ satisfies (U) and (S2), then it is an ascending uniform scale.*

Remark 4.7. The results above can be easily adapted for descending uniform scales $(b_p)_{p\in\mathbb{Q}}$, by changing the conditions accordingly:

(S1') $b_p \prec b_q$ for every $q < p$.

(Far') For every $\delta \in \mathbb{Q}^+$ there is $V \in \mathcal{U}$ such that b_r^* and b_s are V-far for any $s - r > \frac{1}{\delta}$.

(wS1') $b_p^{**} \leq b_q$ for every $q < p$.

(s1') $b_p \leq b_q$ for every $q < p$.

(C') For every $\delta \in \mathbb{Q}^+$ there is $V \in \mathcal{U}$ such that $V \leq \{b_p \wedge b_q^* \mid (p,q) \in D_\delta\}$.

(U') For every $\delta \in \mathbb{Q}^+$ there is some $U \in \mathcal{U}$ such that $Ub_p \leq b_q$ for every $p - q > \frac{1}{\delta}$.

5. Farness and Galois connections

For a given cover U define the *star operator*

$$\mathsf{S}_U\colon L \to L, \quad x \mapsto \mathsf{S}_U(x) = Ux.$$

By 1.5.1(6), this is a Galois left adjoint with right adjoint

$$\widetilde{\mathsf{S}}_U\colon L \to L, \quad x \mapsto x/U = \bigvee\{y \in L \mid Uy \leq x\}.$$

Since $Ua \wedge b = 0$ iff $a \wedge Ub = 0$, the operator S_U has also the following "adjoint-like" property:

$$\mathsf{S}_U(a) \leq b^* \;\Leftrightarrow\; a \leq (\mathsf{S}_U(b))^*. \tag{5.1.1}$$

The operators S_{U^n} and $\mathsf{S}_U^n = (\mathsf{S}_U)^n$ are closely related ([9, 2.4]):

$$\mathsf{S}_{U^n} = \mathsf{S}_U^{2n-1} \quad n = 1, 2, \ldots. \tag{5.1.2}$$

In particular, for each n,

$$\mathsf{S}_U^n(x) = \begin{cases} \mathsf{S}_U^{2k-1}(x) = \mathsf{S}_{U^k}(x) & \text{if } n = 2k-1 \\ \mathsf{S}_U^{2k}(x) = \mathsf{S}_U^{2k-1}\mathsf{S}_U(x) = \mathsf{S}_{U^k}\mathsf{S}_U(x) & \text{if } n = 2k. \end{cases}$$

Fact 5.1. *For $n \geq 1$, $y \in U^{n+1}$ if and only if $y = \mathsf{S}_U^n(x)$ for some $x \in U$.*

Proof: For $n = 1$ it is clear. Assume it holds for some k. If $y \in U^{k+1} = UU^k$, then $y = \mathsf{S}_U(a)$ for some $a \in U^k$. By inductive hypothesis, $a \in U^k$ iff $a = \mathsf{S}_U^{k-1}(x)$ for some $x \in U$. Hence $y = \mathsf{S}_U(a) = \mathsf{S}_U(\mathsf{S}_U^{k-1}(x)) = \mathsf{S}_U^k(x)$. ∎

Hence

$$U^{n+1} = \{\mathsf{S}_U^n(x) \mid x \in U\} = \mathsf{S}_U^n[U] \quad n = 1, 2 \ldots.$$

Now, let P denote the pseudocomplement operator $L \to L$ $(x \mapsto x^*)$. This is a *self-adjoint Galois map*, that is, the pair (P, P) is a Galois connection. Then, by (5.1.1), the composite

$$\mathsf{F}_U = \mathsf{P}\mathsf{S}_U$$

satisfies

$$\mathsf{S}_U(a) \leq \mathsf{P}(b) \quad \text{iff} \quad a \leq \mathsf{F}_U(b) \tag{5.1.3}$$

and since $\mathsf{S}_U(a) \leq \mathsf{P}(b)$ iff $\mathsf{S}_U(b) \leq \mathsf{P}(a)$, then

$$b \leq \mathsf{F}_U(a) \quad \text{iff} \quad a \leq \mathsf{F}_U(b) \tag{5.1.4}$$

and F_U is also a self-adjoint Galois map (and by the uniqueness of adjoints, $\mathsf{F}_U = \widetilde{\mathsf{S}}_U\mathsf{P}$):

$$\begin{tikzcd} L \arrow[r, shift left, "\mathsf{S}_U"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left, "\mathsf{F}_U"] & L \arrow[l, shift left, "\widetilde{\mathsf{S}}_U"] \arrow[r, shift left, "\mathsf{P}"] \arrow[r, phantom, "\perp^{op}"] & L \arrow[l, shift left, "\mathsf{P}"] \arrow[ll, bend left, "\mathsf{F}_U"] \end{tikzcd}$$

The pair $(\mathsf{F}_U, \mathsf{F}_U)$ being a Galois connection, we have immediately the following:

(F1) $\mathsf{F}_U(\bigvee a_i) = \bigwedge \mathsf{F}_U(a_i)$ (in particular, $\mathsf{F}_U(0) = 1$). In this case, we have also $\mathsf{F}_U(1) = 0$.
(F2) $\mathsf{F}_U^2 \geq \mathrm{id}_L$.
(F3) $\mathsf{F}_U^3 = \mathsf{F}_U$.

Now, we can use this in association with the farness relation. First, elements a, b in L are *U-far* iff $\mathsf{S}_U(a) \leq \mathsf{P}(b)$. Hence, by (5.1.3) and (5.1.4),

$$a \text{ and } b \text{ are } U\text{-far iff } a \leq \mathsf{F}_U(b) \text{ iff } b \leq \mathsf{F}_U(a),$$

and it follows from property (F2) that $\mathsf{F}_U(a)$ is the largest element in L that is U-far from a.

Proposition 5.2. *Elements a and b are U^n-far if and only if $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are U^{n-k}-far for every $1 \leq k < n$.*

Proof: $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are U^{n-k}-far iff

$$\mathsf{S}_U^k(b) \leq \mathsf{F}_{U^{n-k}}(\mathsf{S}_U^k(a)) = \mathsf{P}\mathsf{S}_{U^{n-k}}\mathsf{S}_U^k(a).$$

By (5.1.2),

$$\mathsf{P}\mathsf{S}_{U^{n-k}}\mathsf{S}_U^k(a) = \mathsf{P}\mathsf{S}_U^{2n-2k-1}\mathsf{S}_U^k(a) = \mathsf{P}\mathsf{S}_U^{2n-k-1}(a).$$

Hence, by (5.1.1) and using (5.1.2) again we may conclude that $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are U^{n-k}-far iff $b \leq \mathsf{P}\mathsf{S}_U^{2n-1}(a) = \mathsf{P}\mathsf{S}_{U^n}(a) = \mathsf{F}_{U^n}(a)$. ■

In particular, a and b are U^n-far if and only if $\mathsf{S}_U(a)$ and $\mathsf{S}_U(b)$ are U^{n-1}-far. (2.1.2) can now be extended to

Corollary 5.3. *If a and b are U^n-far then $(\mathsf{S}_U^j(a))^* \vee (\mathsf{S}_U^k(b))^* = 1$ for every $1 \leq j \leq k < n$.*

Proof: Clearly, $(\mathsf{S}_U^j(a))^* \vee (\mathsf{S}_U^k(b))^* \geq (\mathsf{S}_U^k(a))^* \vee (\mathsf{S}_U^k(b))^*$. By the proposition, $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are V-far for some V. Hence, by Remark 3 in 2.1,

$$(\mathsf{S}_U^k(a))^* \vee (\mathsf{S}_U^k(b))^* = 1. \qquad \blacksquare$$

It may be worth pointing out that, by (5.1.2), $(\mathsf{S}_U^n(x))^*$ is given by

$$\begin{cases} \mathsf{PS}_U^{2k-1}(x) = \mathsf{PS}_{U^k}(x) = \mathsf{F}_{U^k}(x) & \text{if } n = 2k-1 \\ \mathsf{PS}_U^{2k}(x) = \mathsf{PS}_U^{2k-1}\mathsf{S}_U(x) = \mathsf{PS}_{U^k}\mathsf{S}_U(x) = \mathsf{F}_{U^k}(\mathsf{S}_U(x)) & \text{if } n = 2k. \end{cases}$$

6. A uniform functional separation theorem

As an application of our previous results, we now give a method for constructing a uniformly continuous real function that separates far elements in a preuniform frame. This result is a pointfree counterpart to Smirnov functional separation result [8, p. 292].

Denote by $\mathbb{D}$ the set of dyadic rationals in the closed unit interval $[0,1] \subseteq \mathbb{R}$:

$$\mathbb{D} = \left\{ \tfrac{m}{2^n} \mid n \in \mathbb{N},\ m = 0, 1, \ldots, 2^n \right\} = \{0,1\} \,\dot{\cup} \bigcup_{n\in\mathbb{N}} \left\{ \tfrac{2k-1}{2^n} \mid k = 1, 2, \ldots, 2^{n-1} \right\}.$$

Given a preuniformity $\mathcal{U}$ in L, let $a, b \in L$ be U-far for some $U \in \mathcal{U}$ and consider a chain of uniform covers

$$\cdots \leq U_3 \leq U_2 \leq U_1 \leq U_0 = U$$

such that $U_{n+1}^2 \leq U_n$ for every n.

Recall the Galois operator F from Section 5. By Corollary 5.3, if x and y are U_m-far then

$$\mathsf{F}_{U_n}(x) \vee \mathsf{F}_{U_n}(y) = 1 \tag{6.1.1}$$

for every $n > m$.

Now, define, recursively, two families $(a_d)_{d\in\mathbb{D}}$ and $(b_d)_{d\in\mathbb{D}}$, in the following way:

Definition 6.1. For $n = 0$,

$$a_0 = a, \quad a_1 = 1 \quad \text{and} \quad b_0 = 1, \quad b_1 = b.$$

For each $n \geq 1$,

$$a_{\frac{2k-1}{2^n}} = \mathsf{F}_{U_n}(b_{\frac{k}{2^{n-1}}}) \quad \text{and} \quad b_{\frac{2k-1}{2^n}} = \mathsf{F}_{U_n}(a_{\frac{k-1}{2^{n-1}}})$$

(cf. Table 1).

0	1	2	3	4
$a_1 = 1$				
				$a_{\frac{15}{16}} = \mathsf{F}_{U_4}(b)$
			$a_{\frac{7}{8}} = \mathsf{F}_{U_3}(b)$	
				$a_{\frac{13}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}(b)$
		$a_{\frac{3}{4}} = \mathsf{F}_{U_2}(b)$		
				$a_{\frac{11}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$
			$a_{\frac{5}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$	
				$a_{\frac{9}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_1}(b)$
	$a_{\frac{1}{2}} = \mathsf{F}_{U_1}(b)$			
				$a_{\frac{7}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_1}(a)$
			$a_{\frac{3}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_1}(a)$	
				$a_{\frac{5}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$
		$a_{\frac{1}{4}} = \mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$		
				$a_{\frac{3}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}(a)$
			$a_{\frac{1}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}(a)$	
				$a_{\frac{1}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}(a)$
$a_0 = a$				
$b_1 = b$				
				$b_{\frac{15}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}(b)$
			$b_{\frac{7}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}(b)$	
				$b_{\frac{13}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}(b)$
		$b_{\frac{3}{4}} = \mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$		
				$b_{\frac{11}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$
			$b_{\frac{5}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_1}(b)$	
				$b_{\frac{9}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_1}(b)$
	$b_{\frac{1}{2}} = \mathsf{F}_{U_1}(a)$			
				$b_{\frac{7}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_1}(a)$
			$b_{\frac{3}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$	
				$b_{\frac{5}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$
		$b_{\frac{1}{4}} = \mathsf{F}_{U_2}(a)$		
				$b_{\frac{3}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}(a)$
			$b_{\frac{1}{8}} = \mathsf{F}_{U_3}(a)$	
				$b_{\frac{1}{16}} = \mathsf{F}_{U_4}(a)$
$b_0 = 1$				

TABLE 1. Definition of $(a_d)_{d\in\mathbb{D}}$ and $(b_d)_{d\in\mathbb{D}}$ for $n = 0, 1, 2, 3, 4$.

Lemma 6.2. *$a_{\frac{m-1}{2^n}}$ and $b_{\frac{m}{2^n}}$ are U_n-far for every $n \in \mathbb{N}$ and $m = 1, 2, \ldots, 2^n$.*

Proof: We proceed by induction. The fact that $a_0 = a$ and $b_1 = b$ are U_0-far is obvious. Assuming that the fact holds for $1, 2, \ldots, n$ we need to show that it also holds for $n+1$, that is, that $a_{\frac{m-1}{2^{n+1}}}$ and $b_{\frac{m}{2^{n+1}}}$ are U_{n+1}-far for $m = 1, 2, \ldots 2^{n+1}$. There are two cases:

(1) $m = 2k$ for $1 \leq k \leq 2^n$: Then

$$a_{\frac{m-1}{2^{n+1}}} = a_{\frac{2k-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) \quad \text{and} \quad b_{\frac{m}{2^{n+1}}} = b_{\frac{k}{2^n}},$$

that is,

$$a_{\frac{m-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{m}{2^{n+1}}}),$$

which implies that $a_{\frac{m-1}{2^{n+1}}}$ and $b_{\frac{m}{2^{n+1}}}$ are U_{n+1}-far.

(2) $m = 2k-1$ for $1 \leq k \leq 2^n$: In this case, $a_{\frac{m-1}{2^{n+1}}} = a_{\frac{2k-2}{2^{n+1}}} = a_{\frac{k-1}{2^n}}$, and thus

$$b_{\frac{m}{2^{n+1}}} = b_{\frac{2k-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(a_{\frac{k-1}{2^n}}) = \mathsf{F}_{U_{n+1}}(a_{\frac{m-1}{2^{n+1}}}). \qquad \blacksquare$$

Lemma 6.3. *$a_{\frac{m}{2^n}} \vee b_{\frac{m}{2^n}} = 1$ for every $n \in \mathbb{N}$ and $m = 0, 1, \ldots, 2^n$.*

Proof: We proceed by induction. For $n = 0$ we clearly have

$$a_0 \vee b_0 = a \vee 1 = 1 \quad \text{and} \quad a_1 \vee b_1 = 1 \vee b = 1.$$

Assume it holds for n, and consider $a_{\frac{m}{2^{n+1}}}$ and $b_{\frac{m}{2^{n+1}}}$. Again, if $m = 2k$ for some $0 \leq k \leq 2^n$, then by the inductive hypothesis we have

$$a_{\frac{m}{2^{n+1}}} \vee b_{\frac{m}{2^{n+1}}} = a_{\frac{k}{2^n}} \vee b_{\frac{k}{2^n}} = 1.$$

Otherwise, if $m = 2k-1$ for some $1 \leq k \leq 2^n$, then

$$a_{\frac{m}{2^{n+1}}} \vee b_{\frac{m}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) \vee \mathsf{F}_{U_{n+1}}(a_{\frac{k-1}{2^n}}) = 1$$

where the last equality follows from (6.1.1) and the fact that, by 6.2, $a_{\frac{k-1}{2^n}}$ and $b_{\frac{k}{2^n}}$ are U_n-far. $\blacksquare$

Lemma 6.4. *$(a_d)_{d \in \mathbb{D}}$ is an ascending family while $(b_d)_{d \in \mathbb{D}}$ is a descending family.*

Proof: It suffices to show that $a_0 \leq a_1$ (which is obvious) and that

$$a_{\frac{k-1}{2^n}} \leq a_{\frac{2k-1}{2^{n+1}}} \leq a_{\frac{k}{2^n}} \quad \text{for} \quad n \in \mathbb{N},\ k = 1, 2, \ldots, 2^n.$$

By 6.2, $a_{\frac{k-1}{2^n}}$ and $b_{\frac{k}{2^n}}$ are U_n-far. Since $U_{n+1} \leq U_n$, they are also U_{n+1}- far, hence

$$a_{\frac{k-1}{2^n}} \leq \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) = a_{\frac{2k-1}{2^{n+1}}}.$$

Furthermore, by 6.3,

$$1 = b_{\frac{k}{2^n}} \vee a_{\frac{k}{2^n}} \leq \mathsf{S}_{U_{n+1}}(b_{\frac{k}{2^n}}) \vee a_{\frac{k}{2^n}},$$

which implies

$$a_{\frac{2k-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) \leq a_{\frac{k}{2^n}}.$$

The proof for $(b_d)_{d\in\mathbb{D}}$ is similar. ∎

Theorem 6.5. *Let $\mathcal{U}$ be a preuniformity on a frame L. If a and b are U-far, for some $U \in \mathcal{U}$, then there is a uniformly continuous $f \in \mathcal{R}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $f(0,\text{--}) \leq a^*$ and $f(\text{--},1) \leq b^*$.*

Proof: Let $(a_d)_{d\in\mathbb{D}}$ and $(b_d)_{d\in\mathbb{D}}$ be the families defined in 6.1. We extend $(a_d)_{d\in\mathbb{D}}$ to $\mathbb{Q}$ using the procedure of Banaschewski in the proof of pointfree Urysohn's Lemma [2, Prop. 5]: for every $r, s \in \mathbb{Q}$ let

$$c_r = \begin{cases} 0 & \text{if } r < 0 \\ \bigvee\{a_{\frac{m}{2^n}} \mid \frac{m}{2^n} \leq r\} & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } 1 < r \end{cases}$$

We will show that $(c_r)_{r\in\mathbb{Q}}$ is an ascending uniform scale. By Lemma 6.4, observe that $(c_r)_{r\in\mathbb{Q}}$ is ascending and, trivially, (S2) holds:

$$\bigvee_{r\in\mathbb{Q}} c_r = 1 = \bigvee_{r\in\mathbb{Q}} c_r^*.$$

Now, we claim that property (U) also holds. Indeed, let $1 \leq \delta \in \mathbb{Q}$ (because of how we defined the $c_r's$ notice that the case $\delta < 1$ is trivial). Take $n \in \mathbb{N}$ such that $\delta \leq 2^n$ we will show U_{n+1} is the cover we are looking for. Let $s, r \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$. Clearly, for $r < 0$ or $1 < s$, we have $U_{n+1}c_r \leq c_s$. Consider $0 \leq r < s \leq 1$, since $s - r > \frac{1}{\delta} \geq \frac{1}{2^n}$, there is $0 \leq m \leq 2^n$ such that $r \leq \frac{m}{2^n} \leq s$. Let

$$m_0 = \max\{m \mid 0 \leq m \leq 2^n \text{ and } r \leq \tfrac{m}{2^n} \leq s\}.$$

Then $r < \frac{m_0}{2^n} \leq s$ and $r \leq \frac{2m_0-1}{2^{n+1}} < \frac{2m_0}{2^{n+1}} \leq s$. By Lemma 6.2, $a_{\frac{2m_0-1}{2^{n+1}}}$ and $b_{\frac{2m_0}{2^{n+1}}}$ are U_{n+1}-far. Thus, $U_{n+1} \cdot a_{\frac{2m_0-1}{2^{n+1}}} \leq b^*_{\frac{2m_0}{2^{n+1}}}$. Since $(a_r)_{r\in\mathbb{D}}$ is ascending,

$$U_{n+1} \cdot c_r = U_{n+1} \cdot \Big(\bigvee_{\frac{m}{2^n}\leq r} a_{\frac{m}{2^n}}\Big) \leq U_{n+1} \cdot a_{\frac{2m_0-1}{2^{n+1}}} \leq b^*_{\frac{2m_0-1}{2^{n+1}}}.$$

By Lemma 6.3, $b^*_{\frac{2m_0-1}{2^{n+1}}} \leq a_{\frac{2m_0-1}{2^{n+1}}} \leq c_s$. Hence, $U_{n+1} \cdot c_r \leq c_s$, as required. By 4.6, $(c_p)_{p\in\mathbb{Q}}$ is a uniform scale and thus the formulas

$$f(\text{—}, r) = \bigvee_{p<r} c_p \quad \text{and} \quad f(s, \text{—}) = \bigvee_{q>s} c_q^*$$

define a uniformly continuous $f \in \mathcal{R}(L)$.

Notice that $a \leq f(\text{—}, r)$ for every $r > 0$. Indeed, $f(\text{—}, r) = \bigvee_{q<r} c_q \geq c_0 = a_0 = a$. Moreover, $b \leq f(s, \text{—})$ for every $s < 1$. Since $s < 1$, there is some $n \in \mathbb{N}$ such $s < \frac{2^n-1}{2^n} < 1$, and $a_{\frac{2^n-1}{2^n}}$ and b_1 are U_n-far (by Lemma 6.2). Hence,

$$f(s, \text{—}) = \bigvee_{q>s} c_q^* \geq c^*_{\frac{2^n-1}{2^n}} = a^*_{\frac{2^n-1}{2^n}} \geq b_1 = b.$$

Moreover, $f(0, \text{—}) \leq a^*$:

$$a \wedge f(0, \text{—}) = a_0 \wedge \bigvee_{q>0} c_q^* \leq a_0 \wedge \bigvee_{n\in\mathbb{N}} a^*_{\frac{1}{2^n}} = \bigvee_{n\in\mathbb{N}} a_0 \wedge a^*_{\frac{1}{2^n}} \leq \bigvee_{n\in\mathbb{N}} a_0 \wedge b_{\frac{1}{2^n}} = 0$$

(where the last inequality follows from 6.3 and the last equality from 6.2).

Similarly, $f(\text{—}, 1) \leq b^*$:

$$b \wedge f(\text{—}, 1) = b_1 \wedge \bigvee_{q<1} c_q \leq b_1 \wedge \bigvee_{n\in\mathbb{N}} a_{\frac{2^n-1}{2^n}} = \bigvee_{n\in\mathbb{N}} b_1 \wedge a_{\frac{2^n-1}{2^n}} = 0$$

(where the last equality holds by 6.2).

Finally, it is obvious from the definition of f that $f(\text{—}, 0) \vee f(1, \text{—}) = 0$. Hence, $\mathbf{0} \leq f \leq \mathbf{1}$ and f is bounded. ∎

Corollary 6.6. *For each preuniformity $\mathcal{U}$ on a frame L, elements $a, b \in L$ are U-far for some $U \in \mathcal{U}$ if and only if there is a uniformly continuous $f \in \mathcal{R}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $f(0, \text{—}) \leq a^*$ and $f(\text{—}, 1) \leq b^*$.*

Proof: If $f(0, \text{—}) \leq a^*$ and $f(\text{—}, 1) \leq b^*$ for some uniformly continuous $f \in \mathcal{R}(L)$, then $a \leq f(0, \text{—})^*$ and $b \leq f(\text{—}, 1)^*$ and thus, by 3.1 (recall also 2.2 and 2.1(1)), a and b are U-far for some $U \in \mathcal{U}$. ∎

7. A uniform extension theorem

We end with a second application that uses a modern version (Lemma 7.2 below) of the original basic lemma of Katětov [10, Lemma 1] to provide a construction of uniform extensions for real-valued uniform homomorphisms on dense uniform sublocales.

We need first to recall some basic background on sublocales, that is, the subobjects in the category of locales. A *sublocale* of a locale L is a subset $S \subseteq L$ closed under arbitrary meets such that

$$\forall x \in L \;\; \forall s \in S \;\; (x \to s \in S).$$

Then, each sublocale is closed under the Heyting operation and meets and hence it is a complete Heyting algebra, and therefore a locale, with the same meets and the same Heyting operation as in L but with a different join operation (that we shall denote by $\bigsqcup$). In particular, the bottom element of S (the meet $\bigwedge S$) may differ from 0 (the condition $\bigwedge S = 0$ characterizes precisely the *dense sublocales*, that is, the sublocales whose closure coincides with L), hence the pseudocomplement in S of an $a \in S$, that we shall denote by a^{*_S}, may differ from the pseudocomplement a^* of a in L. In general, we only have the inequality $a^{*_S} \geq a^*$.

The sublocales of L are precisely the subsets of L for which the embedding $S \hookrightarrow L$ is a morphism in the category of locales. For alternative representations of sublocales in the literature (namely, *frame quotients* or *nuclei*) see [15, III.5].

Let S be a sublocale of L and let $\mathfrak{c}_S \colon L \to S$ be the corresponding morphism in the category of frames, that is, the frame homomorphism given by

$$\mathfrak{c}_S \colon x \mapsto \bigwedge\{s \in S \mid s \geq x\}.$$

An $\widetilde{f} \in \mathcal{R}(L)$ is a *continuous extension* of $f \in \mathcal{R}(S)$ if the following diagram commutes ([1]):

$$\begin{tikzcd} \mathfrak{L}(\mathbb{R}) \arrow[r, "\widetilde{f}"] \arrow[dr, "f"'] & L \arrow[d, "\mathfrak{c}_S"] \\ & S \end{tikzcd} \tag{7.1.1}$$

Note 7.1. If $\mathcal{U}$ is a (pre-)uniformity in L then

$$\mathcal{U}_S := \{\mathfrak{c}_S[U] \mid U \in \mathcal{U}\}$$

is a (pre-)uniformity in S ([3, Lemma 2.2]) such that $a \lhd_{\mathcal{U}} b \Rightarrow \mathfrak{c}_S(a) \lhd_{\mathcal{U}_S} \mathfrak{c}_S(b)$ for any $a, b \in L$.

Note that moreover $\mathcal{U}_S \subseteq \mathcal{U}$ since $U \leq \mathfrak{c}_S[U]$ for every $U \in \mathcal{U}$. In case S is dense, since meets in S are computed as in L and $0_S = 0_L$, then, for any $a, b \in S$, if a and b are U-far in S for some $U \in \mathcal{U}_S$, they are also U-far in L.

We also need to recall that a binary relation $\Subset$ on a lattice L is a *Katětov relation* ([10, 11]) if it satisfies the following conditions for all $a, b, a', b' \in L$:

(K1) $a \Subset b \Rightarrow a \leq b$.
(K2) $a' \leq a \Subset b \leq b' \Rightarrow a' \Subset b'$.
(K3) $a \Subset b$ and $a' \Subset b \Rightarrow (a \vee a') \Subset b$.
(K4) $a \Subset b$ and $a \Subset b' \Rightarrow a \Subset (b \wedge b')$.
(K5) $a \Subset b \Rightarrow \exists c \in L \colon a \Subset c \Subset b$.

The strong relation $\lhd_{\mathcal{U}}$ induced by a preuniformity on a frame L is an example of a Katětov relation.

The last ingredient we will need is a general result, known as the Katětov Lemma ([11, 12]), that extends the original basic lemma of Katětov [10, Lemma 1] from power sets to general lattices.

Lemma 7.2. *Let L be a lattice, $\Subset$ a Katětov relation on L and $\lhd$ a transitive and irreflexive relation on a countable set D. Further, let $(a_d \mid d \in D)$ and $(b_d \mid d \in D)$ be two families of elements of L such that*

$$d_1 \lhd d_2 \quad \textit{implies} \quad a_{d_2} \leq a_{d_1}, \quad b_{d_2} \leq b_{d_1} \quad \textit{and} \quad a_{d_2} \Subset b_{d_1}.$$

Then there exists a family $(c_d \mid d \in D) \subseteq L$ such that

$$d_1 \lhd d_2 \quad \textit{implies} \quad c_{d_2} \Subset c_{d_1}, \quad a_{d_2} \Subset c_{d_1} \quad \textit{and} \quad c_{d_2} \Subset b_{d_1}.$$

We are now ready to prove the counterpart in the uniform setting of the standard Tietze-type extension theorem for closed sublocales ([20, 14]).

Theorem 7.3. *Let $\mathcal{U}$ be a preuniformity on a frame L and let S be a dense sublocale of L. Any uniformly continuous $f \colon \mathfrak{L}(\mathbb{R}) \to (S, \mathcal{U}_S)$ has a uniformly continuous extension $\widetilde{f} \colon \mathfrak{L}(\mathbb{R}) \to (L, \mathcal{U})$.*

Proof: Let $f \in \mathcal{R}(S)$ be uniformly continuous. By 3.1 and 2.2 we know that for each $\delta \in \mathbb{Q}^+$ there is some $U_\delta \in \mathcal{U}_S$ such that $f(p, \text{—})^{*_S}$ and $f(q, \text{—})$ are U_δ-far in S for every $p, q \in \mathbb{Q}$ such that $q - p > \frac{1}{\delta}$. By Note 7.1 we have

$$\forall \delta \in \mathbb{Q}^+ \; \exists U_\delta \in \mathcal{U} \text{ such that } f(p, \text{—})^{*_S} \text{ and } f(q, \text{—}) \text{ are } U_\delta\text{-far in } L. \quad (7.2.1)$$

Equivalently, by Remark 2.1 (4),

$$\forall \delta \in \mathbb{Q}^+ \ \exists U_\delta \in \mathcal{U} \text{ such that } f(p,\text{—})^{*s**} \text{ and } f(q,\text{—})^{**} \text{ are } U_\delta\text{-far in } L. \tag{7.2.2}$$

By Remark 2.1(2), condition (7.2.1) implies

$$f(q,\text{—}) \lhd_{\mathcal{U}} f(p,\text{—})^{*s*}.$$

Denoting $f(q,\text{—})$ by a_q and $f(p,\text{—})^{*s*}$ by b_p and taking $D = \mathbb{Q}$, $\lhd = <$ and $\Subset = \lhd_{\mathcal{U}}$ we may conclude from Lemma 7.2 that there exists

$$(c_p \mid p \in \mathbb{Q}) \subseteq L$$

such that

$$c_q \Subset c_p, \quad a_q \Subset c_p \quad \text{and} \quad c_q \Subset b_p \tag{7.2.3}$$

for every rationals $p < q$. Note that $a_p^{*s*} = b_p$.

Claim 1: *$(c_p)_{p\in\mathbb{Q}}$ is a descending uniform scale in L.*

To prove the claim, we will show that $(c_p)_{p\in\mathbb{Q}}$ satisfies (wS1'), (S2) and (Far').

(wS1'): Let $p < q$. Then $c_q \lhd_{\mathcal{U}} c_p$. In particular, $c_q \prec c_p$ which implies $c_q^{**} \le c_p$.

(S2): Let $\delta \in \mathbb{Q}^+$. By 3.1(iv), there is a $U \in \mathcal{U}$ such that

$$\mathfrak{c}_S[U] \le \{f(r,s) \mid (r,s) \in D_\delta\} \le \{a_r \mid r \in \mathbb{Q}\}.$$

From (7.2.3), we have $\mathfrak{c}_S[U] \le \{c_p \mid p \in \mathbb{Q}\}$. Then $1 = \bigvee U \le \bigvee_{p\in\mathbb{Q}} c_p$.

On the other hand,

$$\mathfrak{c}_S[U] \le \{f(r,s) \mid (r,s) \in D_\delta\} \le \{f(\text{—},s) \mid s \in \mathbb{Q}\}$$

and therefore, by (2.1.5),

$$\mathfrak{c}_S[U] \le \{f(p,\text{—})^{*s} \mid p \in \mathbb{Q}\} \le \{f(p,\text{—})^{*s**} \mid p \in \mathbb{Q}\} = \{b_p^* \mid p \in \mathbb{Q}\}.$$

It then follows from (7.2.3) that $b_p^* \le c_q^*$ for every $p < q$. Hence

$$\{b_p^* \mid p \in \mathbb{Q}\} \le \{c_p^* \mid p \in \mathbb{Q}\}$$

and therefore also $1 = \bigvee U \le \bigvee_{p\in\mathbb{Q}} c_p^*$.

(Far'): Let $\delta \in \mathbb{Q}^+$. We will show that there is a $U \in \mathcal{U}$ such that c_p^* and c_q are U-far whenever $q - p > \frac{1}{\delta}$. We claim that the cover U_δ given by (7.2.2) satisfies this property. Let $p, q \in \mathbb{Q}$ such that $q - p > \frac{1}{\delta}$, then there exist

$r, s \in \mathbb{Q}$ such that $p < r < s < q$ and $s - r > \frac{1}{\delta}$. Since $p < r$, by (7.2.3), we have

$$a_r \lhd_{\mathcal{U}} c_p \Rightarrow a_r \leq c_p \Rightarrow c_p^* \leq a_r^*$$

and $a_r^* \leq a_r^{*S} \leq a_r^{*S**} = b_r^*$. Hence, $c_p^* \leq b_r^*$. Again, by (7.2.3) we have

$$c_q \lhd_{\mathcal{U}} b_s \Rightarrow c_q \leq b_s \Rightarrow b_s^* \leq c_q^*$$

and $a_s^* \leq a_s^{*S} \leq a_s^{*S**} = b_s^*$. Thus, $c_q \leq c_q^{**} \leq a_s^{**}$. By (7.2.2), b_r^* and a_s^{**} are U_δ-far, and since $c_p^* \leq b_r^*$ and $c_q \leq a_s^{**}$, so are c_p^* and c_q.

From Claim 1 it follows, using 4.2 and 4.7, that the formulas

$$\widetilde{f}(p, -) = \bigvee_{r>p} c_r \quad \text{and} \quad \widetilde{f}(-, q) = \bigvee_{s<q} c_s^*$$

define a uniformly continuous $\widetilde{f} \in \mathcal{R}(L)$.

Claim 2: *$\widetilde{f}$ extends f, that is, $\mathfrak{c}_S \cdot \widetilde{f} = f$.*

By (7.2.3), we know that $\bigsqcup_{r>p} \mathfrak{c}_S(c_r) \geq \bigsqcup_{r>p} \mathfrak{c}_S(a_r)$. Hence,

$$\mathfrak{c}_S \widetilde{f}(p, -) = \bigsqcup_{r>p} \mathfrak{c}_S(c_r) \geq \bigsqcup_{r>p} \mathfrak{c}_S(a_r) = \bigsqcup_{r>p} a_r = \bigsqcup_{r>p} f(r, -) = f(p, -).$$

For the other inequality notice that, from (7.2.3), we have that $\bigsqcup_{r>p} \mathfrak{c}_S(c_r) \leq \bigsqcup_{r>p} \mathfrak{c}_S(b_r)$. Then,

$$\begin{aligned} \mathfrak{c}_S \widetilde{f}(p, -) &= \bigsqcup_{r>p} \mathfrak{c}_S(c_r) \leq \bigsqcup_{r>p} \mathfrak{c}_S(b_r) = \bigsqcup_{r>p} \mathfrak{c}_S(f(r, -)^{*S*}) \\ &\leq \bigsqcup_{r>p} \mathfrak{c}_S(f(r, -)^{*S*S}) = \bigsqcup_{r>p} (f(r, -)^{*S*S}). \end{aligned}$$

Finally, since $f(r, -) \leq f(r, -)^{*S*S} \leq f(t, -)$ for every $t < r$, we obtain $\mathfrak{c}_S \widetilde{f}(p, -) \leq \bigsqcup_{t>p} f(t, -) = f(p, -)$ for every $p \in \mathbb{Q}$. ∎

It may be worth noting the following about the above proof:

1. It shows that the extension map $\widetilde{f}$ is bounded whenever the given f is bounded.
2. It can be shortened a little by replacing Claim 1 just by the claim that $(c_p)_{p \in \mathbb{Q}}$ *is a descending scale in L* and then by applying the following general principle:

   Uniform Extension Principle. *In diagram (7.1.1), if S is dense and the given f is uniform then $\widetilde{f}$ is also uniform.*

Proof: Let $\delta \in \mathbb{Q}^+$. By Theorem 3.1, there is a $U \in \mathcal{U}$ such that $f(\text{—}, r)$ and $f(s, \text{—})$ are $\mathfrak{c}_S[U]$-far in S for every $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$. Since S is dense, the bottom elements of S and L coincide and $f(\text{—}, r)$ and $f(s, \text{—})$ are also $\mathfrak{c}_S[U]$-far in L. Then, since

$$U \leq \mathfrak{c}_S[U], \quad \widetilde{f}(\text{—}, r) \leq \mathfrak{c}_S \widetilde{f}(\text{—}, r) = f(\text{—}, r) \text{ and}$$
$$\widetilde{f}(s, \text{—}) \leq \mathfrak{c}_S \widetilde{f}(s, \text{—}) = f(s, \text{—}),$$

$\widetilde{f}(\text{—}, r)$ and $\widetilde{f}(s, \text{—})$ are U-far in L for every $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$ (by Remark 2.1(1)). Hence, by 3.1, the extension $\widetilde{f}$ is also uniform. ■

Note 7.4. The question about the pointfree counterpart of the uniform insertion theorem of D. Preiss and J. Vilimovský [17], mentioned in the Introduction, is left open. We have tried a direct approach to it based on the Katětov Lemma above, but it has eluded us so far. We point out that even in the standard setting of uniform spaces such a proof seems to be missing (see Subsection 3.4 of the excellent survey [8] on the development of methods of extensions of mappings).

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Ana Belén Avilez
University of Coimbra, CMUC, Department of Mathematics, 3001-501 Coimbra, Portugal
E-mail address: anabelenag@ciencias.unam.mx

Jorge Picado
University of Coimbra, CMUC, Department of Mathematics, 3001-501 Coimbra, Portugal
E-mail address: picado@mat.uc.pt