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COMPARISON OF HIGHER DEGREE STOP-LOSS TRANSFORMS

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Abstract: The higher degree stop-loss transforms provide a way to measure the risk of cutting off the lower-income population. By interpreting these measures as iterated equilibrium distributions, we prove that the monotonicity of the failure rate function allows for an ordering between stop-loss transforms with respect to their degree. We also show that, without relying on this monotonicity property, it is still possible to establish a pointwise comparison between compactly supported iterated equilibrium distributions.

Keywords: Star-shape order, stop-loss transforms, iterated distributions, probabilities of exceedance.
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1. Introduction

Stop-loss transforms are a popular tool in economics and finance when measuring the concentration of wealth (see Goovaerts et al. [11], Heerwaarden [12] or Sundt [16]). They can also be interpreted as risk measures, since the risk associated to the decision of cutting off the lower-income population can be measured by higher degree stop-loss transforms. Different degrees of stop-loss transforms represent different risk measures, thus making it interesting to find criteria under which the comparison between these measures can be established.

Analytically, stop-loss transforms may be viewed as iterated distributions introduced by Averous and Meste [5], and later studied by Fagiuoli and Pellerey [10] and Nanda et al. [14]. Following this approach, stop-loss transforms are usually designated equilibrium distributions, a notion that is also of interest in ageing problems (see Chatterjee and Mukherjee [8]) or renewal theory (see Cox [9]). Intuitively, higher degree equilibrium distributions become more and more asymmetric. Hence, it is of interest to study the asymmetry

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behaviour with respect to their degree. Comparison of the asymmetry is adequately described by the star-shape transform order, introduced by Barlow and Proschan [6]. A recursive relationship between survival functions has been explored by Arab and Oliveira [1] and Arab et al. [2] to give a complete description of the ordering within the iterated gamma and Weibull distributions. As far as the ordering of stop-loss transforms with respect to their degree is concerned, not much work seems to have been done. In this direction, Arab et al. [3] obtained some asymptotic behaviour results. However, results on star-shape transform comparison seem not to be available.

This paper is structured as follows. In Section 2, we recall some definitions and intermediary results. In Section 3, a characterisation of the iterated equilibrium distributions, based on a monotonicity property of the failure rate function, is established. We also present a result that allows us to keep the comparison between iterated distributions, through the star-shape order, even if the mentioned monotonicity property fails. In the last section, we establish some inequalities, known as probabilities of exceedance for compactly supported distributions.

2. Preliminaries

Let $\mathcal{F}$ denote the family of distribution functions vanishing at 0. In the sequel, let X be a nonnegative random variable, with distribution function $F_X \in \mathcal{F}$, density function f_X, and survival function $\overline{F}_X$. Recall that the stop-loss transform of degree $s \geq 1$ of X is, for each $x \geq 0$, given by $\mathbb{E}(X-x)_+^s$, where $t_+ = \max(t,0)$.

Definition 1. *For each $x \geq 0$, define $\overline{T}_{X,0}(x) = f_X(x)$ and $\tilde{\mu}_{X,0} = 1$. For each $s \geq 1$, we define the s-iterated distribution $T_{X,s}$, by their tails, $\overline{T}_{X,s}(x) = 1 - T_{X,s}(x)$, as follows,*

$$\overline{T}_{X,s}(x) = \frac{1}{\tilde{\mu}_{X,s-1}} \int_x^\infty \overline{T}_{X,s-1}(t)\,dt, \ \ \textit{where} \ \ \tilde{\mu}_{X,s-1} = \int_0^\infty \overline{T}_{X,s-1}(t)\,dt,$$

assuming that the integrals above are finite.

Remark 2. *Following Lemma 2 and Remark 3 in* [1], *it is easily verified that $\overline{T}_{X,s}(x) = \frac{1}{\mathbb{E}X^{s-1}}\mathbb{E}(X-x)_+^{s-1}$. Thus, the s-iterated distribution may be interpreted as the normalized stop-loss transform of order s–1.*

In reliability theory, the distribution $T_{X,2}$ is known as the equilibrium distribution of the random variable X. Hence, the iteration process above defines, for each $s \geq 2$, $T_{X,s}$ as the equilibrium distribution of a random variable with survival function $\overline{T}_{X,s-1}$.

We recall a monotonicity notion of the failure rate, that will allow proving a stronger ordering relationship.

Definition 3. *Let X be a nonnegative random variable with distribution function $F_X \in \mathcal{F}$. The random variable is said to be IFR (resp., DFR), if $\frac{f_X(x)}{\overline{F}_X(x)}$ is increasing (resp., decreasing) with $x > 0$.*

Finally, we present below the definition of the star-shape transform order, following Shaked and Shantikumar [15].

Definition 4. *Let X and Y be two nonnegative random variables with distribution functions $F_X, F_Y \in \mathcal{F}$, respectively. We say that X is less asymmetric or smaller than Y in the star-shape order, denoted by $X \leq_* Y$, if $\overline{F}_Y^{-1}(\overline{F}_X(x))$ is star-shaped, that is, $\frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x))$ is increasing with $x > 0$.*

We may also write $F_X \leq_* F_Y$, meaning that $X \leq_* Y$, as this ordering only depends on the distributions of the variables.

In many cases, the survival functions do not have explicit closed representations or manageable ones, so the verification of the star-shapedness of $\overline{F}_Y^{-1}(\overline{F}_X(x))$ often relies on a sign variation technique, as described by Shaked and Shantikumar [15] or Marshall and Olkin [13].

Proposition 5. *Let X and Y be nonnegative random variables with distribution functions $F_X, F_Y \in \mathcal{F}$, respectively. Then $X \leq_* Y$ if and only if for any $c > 0$, $V(x) = \overline{F}_Y(x) - \overline{F}_X(cx)$ changes sign at most once, and if the change of signs occurs, it is in the order "$-, +$", as x traverses from 0 to $+\infty$.*

3. Comparison of iterated distributions

In the following result, it is shown that the monotonicity of the failure rate implies a pointwise majorization between iterated tails, known as first stochastic dominance (cf. Shanked and Shantikumar [15]).

Theorem 6. *Let X be an IFR (DFR) nonnegative random variable with distribution function $F_X \in \mathcal{F}$. Then $\overline{T}_{X,s-1}(x) \geq (\leq)\overline{T}_{X,s}(x)$, for every $s \geq 2$ and $x \geq 0$.*

Proof: Since X is IFR, applying Lemma 8 in Arab and Oliveira [1], we may conclude that $\frac{\overline{T}_{X,s-1}}{\overline{T}_{X,s}}$ is increasing. Given that $\frac{\overline{T}_{X,s-1}(0)}{\overline{T}_{X,s}(0)} = 1$, it follows that, for every $s \geq 2$ and $x \geq 0$, $\overline{T}_{X,s-1}(x) \geq \overline{T}_{X,s}(x)$. ∎

Remark 7. *Under the conditions of Theorem 6, if for every $s \geq 2$, we have that $EX^s \leq EX^{s-1}$, we obtain that for every $x > 0$, $E(X-x)_+^s \leq E(X-x)_+^{s-1}$. If X represents income, this means that the risk of cutting off the population with income smaller than $x > 0$ is decreasing w.r.t its degree. For a given risk ℓ, we may conclude that having for every $s \geq 2$, $E(X-x_s)_+^s = \ell$, and $E(X-x_{s-1})_+^{s-1} = \ell$ implies that $x_s \leq x_{s-1}$, meaning that the percentage of the population that is cut off becomes smaller when s increases.*

The following theorem gives a sufficient condition for the comparison of stop-loss transforms w.r.t. the star-shape order under a mild condition, characterized by the bathtub behaviour of the ratio $Q(x) = \frac{f(cx)}{\overline{F}(x)}$. We will say that Q is at most upside down bathtub, that is Q changes monotonicity at most once, and when it does, it starts strictly increasing and then it becomes strictly decreasing. Similarly, we say that Q is at most bathtub if -Q is at most upside down bathtub.

Theorem 8. *Let X be a nonnegative random variable with distribution function $F_X \in \mathcal{F}$. If for every $c > 0$, $Q(x) = \frac{f_X(cx)}{\overline{F}_X(x)}$ is at most upside down bathtub, then for every $s \geq 2$, $T_{X,s-1} \leq_* T_{X,s}$.*

Proof: According to Proposition 5, we need to prove that, for every $c > 0$, $V(x) = \overline{T}_{X,s}(x) - \overline{T}_{X,s-1}(cx)$ changes sign at most once, and if it occurs, it is in the order "$-,+$", as x goes from 0 to $+\infty$. Differentiating $k \leq s-1$ times, we have that

$$V^{(k)}(x) = (-1)^k \left(\frac{1}{\prod_{j=1}^{k} \tilde{\mu}_{X,s-j}} \overline{T}_{X,s-k}(x) - \frac{c^k}{\prod_{j=1}^{k} \tilde{\mu}_{X,s-j-1}} \overline{T}_{X,s-k-1}(cx) \right).$$

Therefore, we may rewrite,

$$V^{(s-1)}(x) = (-1)^s \frac{c^{s-1}}{\prod_{j=1}^{s-1} \tilde{\mu}_{X,s-j-1}} \overline{F}_X(x) P(x),$$

where $P(x) = -L + Q(x)$ and $L = \frac{1}{\prod_{j=1}^{s-1} \tilde{\mu}_{X,s-j}} \prod_{j=1}^{s-1} \tilde{\mu}_{X,s-j-1} c^{s-1} > 0$. We shall separate our analysis into two cases.

s even: Given that Q is at most upside down bathtub, P and thus $V^{(s-1)}$ changes sign at most twice in the order "$-,+,-$". Since, $\lim_{x\to+\infty} V^{(s-2)}(x) = 0$, it follows that $V^{(s-2)}$ changes sign at most once in the order "$+,-,+$". Again, taking into consideration that $\lim_{x\to+\infty} V^{(s-3)} = 0$, it follows that $V^{(s-3)}$ changes sign at most twice in the order "$-,+,-$". Repeating the same argument, we obtain that V' has the same possible sign variations as $V^{(s-1)}$. Taking into account that $V(0) = 0$ and $\lim_{x\to+\infty} V(x) = 0$, we conclude that V changes sign at most once, in $(0,+\infty)$, and if the change occurs, it is in the order "$-,+$".

s odd: In this case, $V^{(s-1)}$ changes sign at most twice in the order "$+,-,+$". Hence, following the arguments of the first case mutatis mutandis, we find that V changes sign at most once, in $(0,+\infty)$, and if the sign change occurs, it is in the order "$-,+$".

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Remark 9. *Observe that for distributions with compact support* $[0,x_1]$, $x_1 > 0$, *the previous result remains true if, for* $c \leq 1$, *we have that* $\lim_{x\to x_1} Q(x) = +\infty$, *meaning that* Q *is strictly increasing. Moreover, if* Q *is at most bathtub and* X *has distribution with support* $[0,+\infty)$, *we may obtain analogously that* $T_{X,s} \leq_* T_{X,s-1}$. *However, if* X *has a distribution with support* $[0,x_1]$, *it may not be possible to establish the star-shape order. In fact, in this case, if* $c > 1$ *we may have that* V' *changes sign at most three times, thus we cannot conclude that* V *changes sign at most once.*

Remark 10. *Theorem 8 states that* $T_{X,s-1}$ *is less asymmetric than* $T_{X,s}$. *Therefore, the higher the degree of the stop-loss transform, the more asymmetric is the corresponding risk measure. Moreover, the monotonicity assumption about* Q *is satisfied by several families of distributions, such as Weibull, Gamma, Beta or Power distributions. In fact, it can be verified that for Weibull and Gamma distributions, with shape parameter* $\alpha > 1 (\alpha < 1)$, *the function* Q *is at most upside down bathtub (bathtub). For Beta distributions, the monotonicity assumption of the function* Q *is also satisfied, if both shape parameters are bigger than* 1.

The following result allows us to conclude that under the conditions of Theorem 8 it is still possible to establish a pointwise comparison between iterated equilibrium distributions when the inducing random variable is compactly supported.

Proposition 11. *Let X and Y be nonnegative random variables, with distribution functions $F_X, F_Y \in \mathcal{F}$ supported in $[0, x_1]$ and $[0, y_1]$, respectively, such that $X \leq_* Y$. If $y_1 \leq x_1$ then $\overline{F}_X(x) \geq \overline{F}_Y(x)$. If $y_1 > x_1$, then $\overline{F}_X(x) \leq \overline{F}_Y(x)$ if $\lim_{x\to 0^+} \frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x)) \geq 1$, and if $\lim_{x\to 0^+} \frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x)) < 1$ and $EY \leq EX$, then $E(Y-x)_+ \geq E(X-x)_+$.*

Proof: If $y_1 \leq x_1$, since $X \leq_* Y$, it follows that $\frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x)) \leq \frac{y_1}{x_1} \leq 1$, implying that $\overline{F}_X(x) \geq \overline{F}_Y(x)$. Assume now that $y_1 > x_1$. If $\lim_{x\to 0^+} \frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x)) \geq 1$, we have that $\frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x)) \geq 1$, and the conclusion follows. Finally, if $\lim_{x\to 0^+} \frac{1}{x}\overline{F}_Y^{-1}(\overline{F}_X(x)) < 1$ and $EY \leq EX$, if follows from Theorem 4.B.4 in [15] that $Y \leq_{icv} X$, which implies that $E(Y-x)_+ \geq E(X-x)_+$. ■

4. Probabilities of exceedance

The probability of a random variable being greater (smaller) than its expected value is monotone with respect to the convex ordering (cf. Shaked and Shantikumar [15]) of their distributions. As noted by [17], this is an immediate consequence of Jensen's inequality. [4] proved that this monotonicity property remains true for any functional satisfying a Jensen-type inequality. If we consider the star-shape order relation, a similar result to Theorem 4 in [4] may be obtained.

Proposition 12. *For any interval I, measurable function $\phi : I \to \mathbb{R}$ and random variable X with distribution P supported on I, denote by P_ϕ the distribution of $\phi(X)$. Let $\mathcal{D}$ be a set of continuous probability distributions on intervals in $\mathbb{R}$ and $T : \mathcal{D} \to \mathbb{R}$ a functional satisfying $\phi(T(P)) \leq (\geq) T(P_\phi)$, for all $P \in \mathcal{D}$ and ϕ star-shaped and increasing, with $P_\phi \in \mathcal{D}$. Let X and Y be two random variables with distributions $P, Q \in \mathcal{D}$, respectively, such that $X \leq_* Y$. Then $\mathbb{P}(X \geq T(P)) \geq (\leq) \mathbb{P}(Y \geq T(Q))$.*

Proof: Assume that T satisfies $\phi(T(P)) \leq T(P_\phi)$, for all $P \in \mathcal{D}$ and ϕ is an increasing and star-shaped function, with $P_\phi \in \mathcal{D}$. Let F_X and F_Y be the distribution functions of X and Y, respectively, and $\phi(x) = F_Y^{-1}(F_X(x))$. Then ϕ is increasing and star-shaped, given that $X \leq_* Y$. Therefore, $F_Y^{-1}(F_X(T(P))) = \phi(T(P)) \leq T(P_\phi) = T(Q)$. Since G is increasing, it follows that $F_X(T(P)) \leq F_Y(T(Q))$. If T satisfies $\phi(T(P)) \geq T(P_\phi)$, the conclusion follows by reproducing the same argument with reversed inequality. ■

The following result, due to Barlow et al. [7], gives a criterion for obtaining an inequality similar to Jensen's for star-shaped functions.

Lemma 13. (Barlow et al. [7], Corollary 4.5) *Let ϕ be a star-shaped function with domain $[0,b]$, with $b>0$, such that $\phi(0)=0$. Then, for all increasing functions g, such that $g(0)=0$, $\phi\left(\int_{[0,b]} g(x)dH(x)\right) \leq \int_{[0,b]} \phi(g(x))dH(x)$ if and only if there exists $0 \leq x_1 \leq b$, such that $\overline{H}(u)=\int_{[u,b]} dH(x) \in [0,1]$ and is increasing, for $0 \leq u < x_1$ and $\overline{H}(u)=0$, for $u \geq x_1$.*

Observe that if we consider $\overline{H}(x)=F(x)$, for $x \in [0,x_1)$, where F is the distribution function of a random variable X, and $\overline{H}(x)=0$, for $x \geq x_1$, then by choosing $T(P)=g(x_1)-Eg(X)$, for $X \sim P$, where P has support $[0,x_1]$, $x_1>0$, we conclude that Lemma 13 implies that $\phi(T(P)) \leq T(P_\phi)$ only for $g(x)=x$. Therefore, the following result is an immediate consequence of Proposition 12.

Corollary 14. *Let X and Y be two random variables, whose distributions have a compact support $[0,x_1]$ and $[0,y_1]$, $x_1,y_1>0$, respectively. If $X \leq_* Y$, then $\mathbb{P}(X \leq x_1 - EX) \leq \mathbb{P}(Y \leq y_1 - EY)$.*

Remark 15. *Note that if X and Y represent the losses of two portfolios of size x_1 and y_1, respectively, then $x_1 - X$ and $y_1 - Y$ represent the remainders of the corresponding portfolios, that is the new size (amount of money) of the portfolio after X or Y loss. Since $P(X \leq x_1 - EX) = P(x_1 - X \geq EX)$, this probability may be interpreted as the probability of the remainder of the portfolio being bigger than the average loss. Hence, if $X \leq_* Y$, one may prefer the portfolio with loss Y, as the aforementioned probability is smaller for the portfolio with loss X.*

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