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BEST FITTING GEODESIC GOING THROUGH THE RIEMANNIAN MEAN

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Abstract: We address the problem of finding the geodesic that best fits a given finite set of points on connected and compact Lie groups and on Grassmannians. By constraining the geodesic to pass through the Riemannian mean of the data, a property that is shared by their Euclidean counterparts, we derive an implicit equation defining the velocity of the best fitting geodesic and give a geometric interpretation of the result.

Keywords: Riemannian manifolds, matrix Lie groups, Lie algebras, geodesics, least squares problems.

1. Introduction

The role of Riemannian geometry is vital in computer vision and machine learning. In these areas, most data is intrinsically non-Euclidean and naturally manifold-valued. Using Riemannian computations based on the geometry of the underlying manifolds is faster and more accurate than their classical Euclidean counterparts [21], [18].

Riemannian regression is a technique that allows to estimate a continuous evolution of a process from a discrete set of points q_i, corresponding to observed data (typically images) at time t_i, within a certain time interval [10], [9]. But unlike the linear setting, these problems will typically not yield an analytic solution. The most simple method of regression on a Riemannian manifold is the geodesic regression, which is an extension of linear regression [3]. It allows to best fit a geodesic to a time-series of data points, typically images obtained over a period of time. In machine learning, a typical procedure is to infer a functional relationship between a set of attribute variables and associated target variables in order to predict the response for any set of attributes. This is frequently used in medical imaging, for instance, to study ageing processes, brain development, or disease progression [22], [7]. Riemannian regression, for instance, has been used in [6] to analyze shape changes in the brain as a function of age.

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In this context, Grassmann manifolds, which are manifolds of linear subspaces play an important role. Also, most of the transformations used in computer vision have matrix Lie group structure [5].

The literature about geodesic regression in already quite vast, but mainly dedicated to the implementation of methods to find approximated solutions and present simulation studies to validate the proposed algorithms. This is very important from the viewpoint of real applications, but we are primarily interested in the mathematical aspects of these problems. The derivation of the normal equations for the geodesic that best fits data on manifolds appeared, for instance, in [17] for Lie groups and spheres and more recently in [4] for the Grassmann manifold. Here we simplify those equations by constraining the solution to pass through the Riemannian mean of the data, a property which is shared by their Euclidean counterparts. We also present a geometric interpretation of the implicit equation defining the velocity of the best fitting geodesic. This may facilitate the implementation of more suitable algorithms for solving these least square problems. The organization of the paper is the following. The general problem on a Riemannian manifold is formulated in Section 2. In Section 3 we briefly review the normal equations for the regression line in Euclidean spaces. Section 4 contains the main results for connected and compact Lie groups and is followed by an adaptation to the Grassmannian case in Section 5. We end with some final remarks.

2. Formulation of the problem

Let M be a Riemannian manifold with Riemannian metric $\langle .,. \rangle$, and assume that M is geodesically complete. We first review some concepts that are important to understand the formulation of the problem. For more details we refer to [14].

Geodesic distance between two points p and q in M, denoted by $d(p,q)$, is defined as the length of the shortest geodesic in M that joins those points. *Weighted Riemannian mean* of a given set of N distinct points in M, $q_1, \ldots, q_N$, each q_i having attached a positive weight ω_i, is defined to be a point $\overline{q}$ in M that yields the minimum value for the functional

$$\Phi(q) = \tfrac{1}{2} \sum_{i=1}^{N} \omega_i \, d^2(q, q_i).$$

We now formulate the general problem consisting in finding the geodesic that best fits a given finite set of data, consisting of N distinct points in M and N instants of time.

Let $q_1, \ldots, q_N$, be N distinct points in M and $t_1, \ldots, t_N$ a set of instants of time. Denote by $\overline{q}$ the Riemannian mean of the given points and let $\overline{t} := \frac{1}{N}\sum_{i=1}^{N} t_i$. Furthermore, let C denote the class of geodesic curves $t \mapsto \gamma(t)$ in M, satisfying $\gamma(\overline{t}) = \overline{q}$.

Find the solution of the following minimization problem:

$$\min_{\gamma \in C} \tfrac{1}{2} \sum_{i=1}^{N} d^2(q_i, \gamma(t_i)). \tag{1}$$

3. Best fitting straight line on Euclidean spaces

The solution of the above problem, when M is the Euclidean space $\mathbb{R}^n$, is well known [15]. In this case, geodesics are straight lines and the best fitting geodesic can be parametrized by $\gamma(t) = \overline{q} + (t - \overline{t})X$, where $t \in [t_1, t_N]$. The solution of the following minimization problem

$$\min_{X \in \mathbb{R}^n} \tfrac{1}{2} \sum_{i=1}^{N} \left\| q_i - \overline{q} - (t_i - \overline{t})X \right\|^2,$$

where $\| \cdot \|$ denotes the Euclidean norm, is given explicitly by

$$X = \frac{\sum_{i=1}^{N}(t_i - \overline{t})(q_i - \overline{q})}{\sum_{i=1}^{N}(t_i - \overline{t})^2}. \tag{2}$$

Moreover, it can be seen that the curve γ satisfies the so called normal equations

$$\left\{ \begin{array}{l} \sum_{i=1}^{N}(\gamma(t_i) - q_i) = 0 \\ \sum_{i=1}^{N} t_i(\gamma(t_i) - q_i) = 0 \end{array} \right. . \tag{3}$$

4. Best fitting geodesic on connected and compact matrix Lie groups

In this section, we assume that M is a connected and compact matrix Lie group equipped with its bi-invariant Riemannian metric, and look for solutions of the problem formulated in Section 2. To do so, one needs to introduce some facts about the geometry of these groups.

4.1. Geometric properties of connected and compact Lie groups. Let G be a connected and compact real $n-$dimensional matrix Lie group, equipped with the bi-invariant Riemannian metric, and $\mathfrak{g}$ its Lie algebra. Given $X \in \mathfrak{g}$, the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, defined by $\mathrm{ad}_X(Y) = [X, Y]$, where $[\cdot, \cdot]$ is the Lie bracket in $\mathfrak{g}$, is called the adjoint operator. The exponential map $\exp : \mathfrak{g} \to G$ is surjective, but only a local diffeomorphism whose inverse is called logarithm map. When $\sigma(B) \cap \mathbb{R}_0^- = \emptyset$, where $\sigma(B)$ denotes the spectrum of B, there exists a unique real logarithm of B whose spectrum lies in the infinite horizontal strip $\{z \in \mathbb{C} : -\pi < Im(z) < \pi\}$ of the complex plane. This logarithm is called the *principal logarithm* of B and hereafter will be denoted by $\log(B)$. When $\|B - I\| < 1$, $\log(B)$ is uniquely defined by the following convergent power series:

$$\log(B) = \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{(B-I)^k}{k}.$$

When the previous norm restriction is not satisfied, one can still define a unique real logarithm of B using the following identity

$$\log(B) := 2^k \log\bigl(B^{\frac{1}{2^k}}\bigr), \quad k \in \mathbb{Z}.$$

This is the so called *inverse scaling and squaring method* whose details may be found in [12]. This allows to define $\log(B)$ for any $B \in G$ that has no eigenvalues in $\mathbb{R}_0^-$. This assumption is imposed throughout the paper.

Given p, $q \in G$, the geodesic that joins p to q can be parametrized in the interval $[0, 1]$ by $\beta(s) = p \exp\bigl(s \log(p^{-1}q)\bigr)$. Thus, the geodesic distance between p and q is given by

$$d(p, q) = \bigl\|\log(p^{-1}q)\bigr\|. \tag{4}$$

Next, we list some properties that are required to deduce the main results.

Proposition 4.1. ([20]) *Let $t \mapsto X(t)$ and $t \mapsto p(t)$ be two differentiable mappings taking values in $\mathfrak{g}$ and G, respectively, and let $\frac{\mathrm{e}^u - 1}{u}$ denote the sum of the series $\sum_{m=0}^{+\infty} \frac{u^m}{(m+1)!}$ and $\frac{u}{\mathrm{e}^u - 1}$ the sum of the series $\sum_{m=0}^{+\infty} (-1)^m \frac{(\mathrm{e}^u - 1)^m}{m+1}$. Then,*

1. $\dfrac{d \exp(X(t))}{dt} = \frac{\mathrm{e}^u - 1}{u}\Big|_{u = \mathrm{ad}_{X(t)}} \bigl(\dot{X}(t)\bigr) \exp(X(t)),$
2. $\dfrac{d \log\bigl(p(t)\bigr)}{dt} = \frac{u}{\mathrm{e}^u - 1}\Big|_{u = \mathrm{ad}_{\log(p(t))}} \bigl(\dot{p}(t) p^{-1}(t)\bigr),$

Proposition 4.2. ([17]) *The operators that appear in Proposition 4.1 are skew-adjoint with respect to the bi-invariant Riemannian metric in G, i.e., for all X, Y, $Z \in \mathfrak{g}$,*

1. $\left\langle \frac{e^u-1}{u}\Big|_{u=\mathrm{ad}_X}(Y), Z \right\rangle = \left\langle Y, \frac{1-e^{-u}}{u}\Big|_{u=\mathrm{ad}_X}(Z) \right\rangle$,
2. $\left\langle \frac{u}{e^u-1}\Big|_{u=\mathrm{ad}_X}(Y), Z \right\rangle = \left\langle Y, \frac{u}{1-e^{-u}}\Big|_{u=\mathrm{ad}_X}(Z) \right\rangle$.

Due to the differentiable structure of G, right translations on G, $R_g : h \mapsto hg$,
are diffeomorphisms for all $g \in G$. Taking into account that $T_gG \simeq \mathfrak{g}$, the tangent map (or differential) of R_g at the identity $e \in G$

$$\begin{array}{rccl} \big(dR_g\big)_e : & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & Z & \longmapsto & \big(dR_g\big)_e(Z) = Zg \end{array},$$

is an isomorphism in $\mathfrak{g}$, whose inverse is defined by $\big(dR_g\big)_e^{-1}(Z) = Zg^{-1}$, $\forall Z \in \mathfrak{g}$.

Next, one gives a characterization for the left operator appearing in the expression of the derivative of the exponential (1. of Proposition 4.1) which is the counterpart of the result given in [11] (Theorem 1.7, Chapter II) for the right operator.

Proposition 4.3. *Let $X \in \mathfrak{g}$. Then*

$$\frac{e^u - 1}{u}\bigg|_{u=\mathrm{ad}_X} = \big(dR_{\exp(-X)}\big)_e^{-1} \circ (d\exp)_X, \tag{5}$$

where $(d\exp)_X$ denotes the tangent map of the exponential map at X.

The following result gives an implicit characterization of the weighted Riemannian mean of a given data set of points in G with attached weights.

Proposition 4.4. ([17]) *Let $q_1, \ldots, q_N$ be N distinct points in G, having attached the weight ω_i at each point q_i. The weighted Riemannian mean of (q_i, ω_i), $i = 1, \ldots, N$, is the set of points $\overline{q}$ satisfying the equation*

$$\sum_{i=1}^{N} \omega_i \log(q_i^{-1}\overline{q}) = 0. \tag{6}$$

Remark 4.1. *When all the weights are equal, the previous equation reduces to*

$$\sum_{i=1}^{N} \log(q_i^{-1}\overline{q}) = 0, \tag{7}$$

which characterizes the Riemannian mean of the points $q_1, \dots, q_N$. The uniqueness of the Riemannian mean can only be ensured if all the data points lie inside a geodesic ball of appropriate radius [13].

4.2. Best fitting geodesic. The main goal is to find solutions for the problem (1) formulated in Section 2 when M is a connected and compact matrix Lie group G with bi-invariant metric.

Let $q_1, \dots, q_N$, be N points in G and $t_1, \dots, t_N$, a set of N instants of time. Denote by $\overline{q}$ the Riemannian mean of points $q_1, \dots, q_N$, and let $\overline{t} = \frac{1}{N}\sum_{i=1}^{N} t_i$.

Geodesic curves γ satisfying $\gamma(\overline{t}) = \overline{q}$ can be parametrized explicitly by $\gamma(t) = \overline{q}\exp\big((t-\overline{t})X\big)$, where $X \in \mathfrak{g}$. We denote by C the class of such geodesic curves.

We now state and prove the main result.

Theorem 4.5. *A necessary condition for the geodesic curve γ, defined by $\gamma(t) = \overline{q}\exp\big((t-\overline{t})X\big)$, to be a solution of the problem (1), is that it satisfies the equation*

$$\sum_{i=1}^{N} (t_i - \overline{t})\, \frac{\mathrm{e}^u - 1}{u}\bigg|_{u=\mathrm{ad}_{(t_i-\overline{t})X}} \Big(\log\big(q_i^{-1}\gamma(t_i)\big)\Big) = 0. \tag{8}$$

Proof: In order to find the critical points for the problem (1), one needs to compute the tangent map of the function Φ, defined in $\mathfrak{g}$, by

$$\Phi(X) = \frac{1}{2}\sum_{i=1}^{N}\Big\langle \log\big(q_i^{-1}\overline{q}\exp\big((t_i-\overline{t})X\big)\big), \log\big(q_i^{-1}\overline{q}\exp\big((t_i-\overline{t})X\big)\big)\Big\rangle.$$

Let $Y \in \mathfrak{g}$ be arbitrary. The tangent map of Φ at X is therefore obtained by

$$(d\Phi)_X(Y) = \frac{d}{ds}\bigg|_{s=0} \Phi(X+sY)$$

$$= \sum_{i=1}^{N}\Big\langle \frac{d}{ds}\bigg|_{s=0} \log\big(q_i^{-1}\overline{q}\exp\big((t_i-\overline{t})(X+sY)\big)\big), \log\big(q_i^{-1}\overline{q}\exp\big((t_i-\overline{t})(X)\big)\big)\Big\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \left\langle \frac{u}{\mathrm{e}^u - 1}\Big|_{u=\mathrm{ad}_{\log(q_i^{-1}\gamma(t_i))}} \left(q_i^{-1}\overline{q}\, \frac{\mathrm{e}^u-1}{u}\Big|_{u=\mathrm{ad}_{(t_i-\overline{t})X}} \big((t_i-\overline{t})Y\big)\overline{q}^{-1}q_i\right), \log\big(q_i^{-1}\gamma(t_i)\big)\right\rangle \\
&= \sum_{i=1}^{N} \left\langle q_i^{-1}\overline{q}\, \frac{\mathrm{e}^u-1}{u}\Big|_{u=\mathrm{ad}_{(t_i-\overline{t})X}} \big((t_i-\overline{t})Y\big)\overline{q}^{-1}q_i,\ \frac{u}{1-\mathrm{e}^{-u}}\Big|_{u=\mathrm{ad}_{\log(q_i^{-1}\gamma(t_i))}} \Big(\log\big(q_i^{-1}\gamma(t_i)\big)\Big)\right\rangle \\
&= \sum_{i=1}^{N} \left\langle \frac{\mathrm{e}^u-1}{u}\Big|_{u=\mathrm{ad}_{(t_i-\overline{t})X}} \big((t_i-\overline{t})Y\big), \overline{q}^{-1}q_i \log\big(q_i^{-1}\gamma(t_i)\big)q_i^{-1}\overline{q}\right\rangle \\
&= \sum_{i=1}^{N} \left\langle (t_i-\overline{t})Y,\ \frac{1-\mathrm{e}^{-u}}{u}\Big|_{u=\mathrm{ad}_{(t_i-\overline{t})X}} \Big(\log\big(\exp((t_i-\overline{t})X)q_i^{-1}\overline{q}\big)\Big)\right\rangle \\
&= \left\langle Y, \sum_{i=1}^{N}(t_i-\overline{t})\ \frac{\mathrm{e}^u-1}{u}\Big|_{u=\mathrm{ad}_{(t_i-\overline{t})X}} \Big(\log\big(q_i^{-1}\gamma(t_i)\big)\Big)\right\rangle.
\end{aligned}
$$

Thus, since X is a critical point of Φ if, and only if, $(d\Phi)_X(Y) = 0,\ \forall Y \in \mathfrak{g}$, it follows that X has to be the solution of equation (8). ■

4.3. Geometric interpretation of equation (8). Regarding the expression of the operator $\frac{\mathrm{e}^u-1}{u}\big|_{u=\mathrm{ad}_X}$ given in Proposition 4.3, equation (8) can be rewritten as

$$
\sum_{i=1}^{N}(t_i-\overline{t})\Big(\big(dR_{\exp((\overline{t}-t_i)X)}\big)_{\mathrm{e}}^{-1} \circ (d\exp)_{(t_i-\overline{t})X}\Big)\Big(\log\big(q_i^{-1}\gamma(t_i)\big)\Big) = 0.
$$

Now, according to the characterization of the weighted Riemannian mean (6), equation (8) can be interpreted as a weighted Riemannian mean of a transformation of the set of the velocity vectors of the minimizing geodesic connecting points q_i to $\gamma(t_i)$ via the operators $\big(dR_{\exp((\overline{t}-t_i)X)}\big)_{\mathrm{e}}^{-1} \circ (d\exp)_{(t_i-\overline{t})X}$, for each $i = 1, \dots, N$.

4.4. Particular case when G is Abelian. In this case, since all matrices in G commute, the Riemannian mean of the points $q_1, \dots, q_N$ is given by $\overline{q} = (q_1 \cdots q_N)^{\frac{1}{N}}$. Regarding equation (8), it assumes the simpler form

$$
\sum_{i=1}^{N}(t_i-\overline{t})\Big(\log\big(q_i^{-1}\gamma(t_i)\big)\Big) = 0. \tag{9}
$$

The velocity vector of the geodesic γ can now be computed explicitly as in the Euclidean case. In fact

$$\sum_{i=1}^{N}(t_i-\bar{t})\big(\log\big(q_i^{-1}\gamma(t_i)\big)\big)=\sum_{i=1}^{N}(t_i-\bar{t})\big(\log\big(\gamma(t_i)\big)-\log(q_i)\big)$$
$$=\sum_{i=1}^{N}(t_i-\bar{t})\big(\log\bar{q}+(t_i-\bar{t})X-\log(q_i)\big),$$

and so, the solution of equation (9) is explicitly given by

$$X=\frac{\sum_{i=1}^{N}(t_i-\bar{t})\big(\log(q_i)-\log\bar{q}\big)}{\sum_{i=1}^{N}(t_i-\bar{t})^2},$$

which is equivalent to equation (2) for the velocity vector in the Euclidean case.

Next proposition shows that the best fitting geodesic γ shares a property of the regression line in $\mathbb{R}^n$, namely it passes through the weighted mean of the points q_i, $\bar{\bar{q}}=\big(q_1^{t_1}\cdots q_N^{t_N}\big)^{1/\sum_{i=1}^{N}t_i}$, at the instant of time $\bar{\bar{t}}=\frac{\sum_{i=1}^{N}t_i^2}{\sum_{i=1}^{N}t_i}$.

Proposition 4.6. *Assume that $q_1,\ldots,q_N$ belong to a connected, compact and Abelian Lie group G. Then the geodesic γ that best fits the given data at the given times $t_1,\ldots,t_N$ satisfies the condition $\gamma\big(\bar{\bar{t}}\big)=\bar{\bar{q}}$.*

Proof: Note that

$$\bar{\bar{t}}-\bar{t}=\frac{N\sum_{i=1}^{N}t_i^2-(\sum_{i=1}^{N}t_i)^2}{N\sum_{i=1}^{N}t_i}\ \text{and}\ \sum_{i=1}^{N}(t_i-\bar{t})^2=\sum_{i=1}^{N}t_i^2-\frac{1}{N}\Big(\sum_{i=1}^{N}t_i\Big)^2.$$

Moreover,

$$\sum_{i=1}^{N}(t_i-\bar{t})\big(\log(q_i)-\log\bar{q}\big)=\sum_{i=1}^{N}t_i\log(q_i)-\frac{1}{N}\sum_{i=1}^{N}t_i\sum_{i=1}^{N}\log(q_i).$$

Therefore,

$$
\begin{aligned}
\gamma(\bar{\bar{t}}) &= \bar{q}\exp\Big(\tfrac{1}{\sum_{i=1}^N t_i}\sum_{i=1}^N (t_i-\bar{t})(\log(q_i)-\log\bar{q})\Big)\\
&= \bar{q}\exp\Big(\tfrac{1}{\sum_{i=1}^N t_i}\sum_{i=1}^N t_i\log(q_i)\Big)\exp\Big(-\tfrac{1}{N}\sum_{i=1}^N \log(q_i)\Big)\\
&= \exp\Big(\tfrac{1}{N}\sum_{i=1}^N \log(q_i)\Big)\exp\Big(\tfrac{1}{\sum_{i=1}^N t_i}\sum_{i=1}^N t_i\log(q_i)\Big)\exp\Big(-\tfrac{1}{N}\sum_{i=1}^N \log(q_i)\Big)\\
&= \exp\Big(\tfrac{1}{\sum_{i=1}^N t_i}\sum_{i=1}^N t_i\log(q_i)\Big) = \big(q_1^{t_1}\cdots q_N^{t_N}\big)^{1/\sum_{i=1}^N t_i}. \qquad \blacksquare
\end{aligned}
$$

5. Best fitting geodesic on the Grassmann manifold

The objective in this section is to determine the geodesic that best fits a given data set of points in the Grassmann manifold, assuming that it passes through the Riemannian mean of the given data. For details on the differential geometric structure of these manifolds, we refer to [8] and [1].

5.1. Some facts about the geometry of the Grassmann manifold. The Grassmann manifold (or Grassmannian) $G_{n,k}$, consisting of the k-dimensional subspaces of $\mathbb{R}^n$, admits a representation by projection matrices. If $\mathfrak{s}(n)$ denotes the vector space of all $n\times n$ symmetric matrices, then

$$G_{n,k} := \big\{P\in\mathfrak{s}(n)\ :\ P^2=P \text{ and } \operatorname{rank}(P)=k\big\}. \tag{10}$$

Let $\mathfrak{so}(n)$ denote the Lie algebra of all $n\times n$ skew-symmetric matrices, and for each $P\in G_{n,k}$ define the subspace of $\mathfrak{so}(n)$,

$$\mathfrak{so}_P(n)=\{\Omega\in\mathfrak{so}(n):\ \Omega P+P\Omega=\Omega\}.$$

The tangent space of $G_{n,k}$ at a point P is given by

$$T_PG_{n,k}=\big\{[\Omega,P]\ :\ \Omega\in\mathfrak{so}_P(n)\big\}. \tag{11}$$

Consider the Riemannian metric induced by the Frobenius inner product

$$\langle[\Omega_1,P],[\Omega_2,P]\rangle=-\operatorname{tr}\big(\Omega_1\Omega_2\big). \tag{12}$$

The special orthogonal Lie group $SO(n)$ acts transitively on $G_{n,k}$ via $(\Theta,P)\mapsto\Theta P\Theta^T$ and so, any smooth curves on $G_{n,k}$ can be parametrized explicitly by

$\alpha(t) = \Theta(t)P\Theta(t)^T$, where Θ is a smooth curve on $SO(n)$. In particular, geodesics in the Grassmann manifold are of the form

$$\alpha(t) = \mathrm{e}^{t\Omega}P\mathrm{e}^{-t\Omega},\ \Omega \in \mathfrak{so}(n), \tag{13}$$

and the geodesic connecting the point P (at $t = 0$) to the point Q (at $t = 1$) is of the form (13) (see [4] for details), with

$$\Omega = \frac{1}{2}\log((I - 2Q)(I - 2P)). \tag{14}$$

Therefore, the geodesic distance between P and Q is defined as

$$d(P, Q) = \frac{1}{2}\big\|\log((I - 2Q)(I - 2P))\big\|.$$

According to [4], given a collection of points $P_1, \ldots, P_N$ in the Grassmann manifold $G_{n,k}$, its Riemannian mean is the point $\overline{P}$ that satisfies the equation

$$\sum_{i=1}^{N} \log((I - 2\overline{P})(I - 2P_i)) = 0. \tag{15}$$

5.2. Best fitting geodesic in $G_{n,k}$ through the Riemannian mean. It is clear from the previous considerations that the developments in Section 4 applied to the situation when $G = SO(n)$ will be used here. In [4] we derived the counterpart of the normal equations for the best fitting geodesic in $G_{n,k}$. Following the assumptions in Section 4, we simplify these equations assuming that the geodesic passes through the Riemannian mean of the data. It will be clear that the geometric interpretation also follows immediately from previous discussions. We omit the proof of the main result in this section due to limitation of pages and the fact that it results from obvious adaptations of the proof of Theorem 4.5.

Let $P_1, \ldots, P_N$ be a set of N points in $G_{n,k}$ and $\overline{P}$ its Riemannian mean. Also, let $t_1, \ldots, t_N$ be a set of N instants of time and $\bar{t}$ its arithmetic mean. Consider the family C of geodesics in $G_{n,k}$ that pass through $\overline{P}$ at time $\bar{t}$.

The main goal of this section is to solve the following optimization problem

$$\min_{\gamma \in C} \frac{1}{4} \sum_{i=1}^{N} \big\|\log((I - 2\gamma(t_i))(I - 2P_i))\big\|^2 \tag{16}$$

Arguments similar to those in the proof of Theorem 4.5 can be used to conclude the following.

Theorem 5.1. *A necessary condition for the geodesic curve defined by* $\gamma(t) = \mathrm{e}^{(t-\bar{t})\Omega}\overline{P}\mathrm{e}^{-(t-\bar{t})\Omega}$ *to be a solution of (16) is that it satisfies the equation*

$$\sum_{i=1}^{N}(t_i - \bar{t})\,\frac{1-\mathrm{e}^{-u}}{u}\bigg|_{u=\mathrm{ad}_{(t_i-\bar{t})\Omega}}\Big(\log\big((I-2P_i)(I-2\gamma(t_i))\big)\Big) = 0. \qquad (17)$$

6. Final remarks

In [17] and [4], the normal equations for the best fitting geodesic problem for some Riemannian manifolds have been derived. These coupled equations look rather difficult to solve explicitly. Moreover, it is not clear if the solution of the problem passes through the Riemannian mean, a condition that is fullfiled by the regression line in Euclidean spaces.

In this paper, we assume à priori that the required geodesic goes through the Riemannian mean of the data, to obtain a single equation for its velocity. This reduces the complexity of the problem and hopefully simplifies the search for numerical solutions.

It is important to find out if these two apparently different approaches of the best geodesic fitting problem are equivalent. This will be addressed in the near future.

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