Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 22–17

A COUNTEREXAMPLE TO A CONJECTURE OF M. ISMAIL

K. CASTILLO AND D. MBOUNA

Abstract: In an earlier work [K. Castillo et al., J. Math. Anal. Appl. **514** (2022) 126358], we give positive answer to the first part of a conjecture posed by M. Ismail concerning the characterization of the continuous q-Jacobi polynomials, Al-Salam-Chihara polynomials or special or limiting cases of them. In this note we present a counterexample to the second part of such a conjecture, and so this issue is definitively closed.

Keywords: Askey-Wilson operator, continuous dual q-Hahn polynomials.
Math. Subject Classification (2000): 15A15.

1. Introduction

The Askey-Wilson divided difference operator is defined by

$$\mathcal{D}_q\, f(x) = \frac{\breve{f}\big(q^{1/2}e^{i\theta}\big) - \breve{f}\big(q^{-1/2}e^{i\theta}\big)}{\breve{e}\big(q^{1/2}e^{i\theta}\big) - \breve{e}\big(q^{-1/2}e^{i\theta}\big)}, \tag{1}$$

where, for each polynomial f, $\breve{f}(e^{i\theta}) = f(\cos\theta)$ and $e(x) = x$ (see [3, Section 12.1]). In [1], we give positive answer to the first part of the following Ismail's conjecture (see [3, Conjecture 24.7.8]):

Conjecture 1. *Let $(p_n)_{n\geq 0}$ be a sequence of orthogonal polynomials and let π be a polynomial which does not depend on n. If*

$$\pi(x)\mathcal{D}_q\, p_n(x) = \sum_{k=-1}^{1} c_{n,k}p_{n+k}(x),$$

then $(p_n)_{n\geq 0}$ are continuous q-Jacobi polynomials or Al-Salam-Chihara polynomials, or special or limiting cases of them. The same conclusion follows

Received June 13, 2022.
This work is supported by the Centre for Mathematics of the University of Coimbra, funded by the Portuguese Government through FCT/ MCTES. DM is partially supported by ERDF and Consejería de Economía, Conocimiento, Empresas y Universidad de la Junta de Andalucía (grant UAL18-FQM-B025-A) and by the Research Group FQM-0229 (belonging to Campus of International Excellence CEIMAR)..

if

$$\pi(x)\mathcal{D}_q\, p_n(x) = \sum_{k=-r}^{s} c_{n,k} p_{n+k}(x), \tag{2}$$

for positive integers r *and* s*.*

The second part of this conjecture is certainly a much more complex problem than the first one. However, after much manipulation of similar structural relations in a number of recent works, we found the second part of the conjecture less and less convincing. Now in the next section we propose a counterexample.

2. Counterexample

Throughout this section we assume that $0 < q < 1$. Set $x(s) = (q^s + q^{-s})/2$. Taking $e^{i\theta} = q^s$ in (1), $\mathcal{D}_q$ reads

$$\mathcal{D}_q f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) - f\big(x(s-\frac{1}{2})\big)}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}.$$

Define

$$\mathcal{S}_q f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) + f\big(x(s-\frac{1}{2})\big)}{2},$$

and

$$\alpha_n = \frac{q^{n/2} + q^{-n/2}}{2}, \quad \gamma_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad (n = 0, 1, \dots),$$

and $\alpha = \alpha_1$. Recall that

$$\mathcal{D}_q\big(fg\big) = \big(\mathcal{D}_q f\big)\big(\mathcal{S}_q g\big) + \big(\mathcal{S}_q f\big)\big(\mathcal{D}_q g\big), \tag{3}$$

$$\mathcal{S}_q\big(fg\big) = \big(\mathcal{D}_q f\big)\big(\mathcal{D}_q g\big)\mathtt{U}_2 + \big(\mathcal{S}_q f\big)\big(\mathcal{S}_q g\big), \tag{4}$$

where $\mathtt{U}_2(x) = (\alpha^2 - 1)(x^2 - 1)$. All these properties and definitions, even the notation, can be found, for instance, in [2]. The Askey-Wilson polynomials are defined by

$$p_n(x; a, b, c, d \,|\, q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3\left(\begin{matrix} q^{-n},\ abcbdq^{n-1},\ ae^{i\theta},\ ae^{-i\theta} \\ ab,\ ac,\ ad \end{matrix}\ \middle|\ q,\ q\right),$$

where $x = \cos\theta$. If we take $a = q^{1/2\alpha+1/4}$, $b = q^{1/2\alpha+3/4}$, $c = -a$, and $d = -b$, we get the continuous q-Jacobi polynomials. If we take $d = 0$, we get the continuous dual q-Hahn polynomials. If we take $c = d = 0$, we get the Al-Salam-Chihara polynomials. The sequence of monic continuous dual q-Hahn polynomials, $(H_n(x; a, b|q))_{n\geq 0}$, satisfies

$$xH_n(x; a, b, c|q) = H_{n+1}(x; a, b, c|q) + a_n H_n(x; a, b, c|q) + b_n H_{n-1}(x; a, b, c|q)\ , \tag{5}$$

where $H_{-n-1}(\cdot; a, b, c|q) = 0$ and

$$a_n = \frac{1}{2}(a + a^{-1} - a(1-q^n)(1-bcq^{n-1}) - a^{-1}(1-abq^n)(1-acq^n)),$$

$$b_n = \frac{1}{4}(1-abq^{n-1})(1-acq^{n-1})(1-bcq^{n-1})(1-q^n).$$

Define $P_n = H_n(\cdot; 1, -1, q^{1/4}|q^{1/2})$. Clearly, P_n is not a continuous q-Jacobi polynomial or Al-Salam-Chihara polynomial or, much less, special or limiting cases of them. These polynomials satisfy, among other relations, a relation of type (2) with $r = 2$ and $s = 1$.

PROPOSITION 2.1. *Let $P_n(x) = H_n(x; 1, -1, q^{1/4}|q^{1/2})$. The sequence $(P_n)_{n\geq 0}$ satisfies the following relations:*

$$\mathcal{S}_q P_n(x) = \alpha_n P_n(x) + c_n P_{n-1}(x), \tag{6}$$

$$\begin{aligned} \mathcal{U}_2(x)\mathcal{D}_q P_n(x) = (\alpha^2-1)\gamma_n P_{n+1}(x) + \big(c_{n+1} - \alpha c_n + (1-\alpha)\alpha_n B_n\big) P_n(x) \\ + \big((B_n - \alpha B_{n-1})c_n + (1-\alpha^2)\gamma_n C_n\big) P_{n-1}(x) \\ + (c_{n-1}C_n - \alpha c_n C_{n-1}) P_{n-2}(x), \end{aligned} \tag{7}$$

where

$$B_n = \frac{1}{2}\Big((1+q^{-1/2})q^{n/2} + 1 - q^{-1/2}\Big) q^{(2n+1)/4},$$

$$C_n = \frac{1}{4}(1+q^{(n-1)/2})(1-q^{n/2})(1-q^{n-1/2}),$$

$$c_n = \frac{1}{4}(1-q^{n/2})(1-q^{n-1/2})(1+q^{(n-1)/2})q^{-(2n-1)/4}.$$

Proof: (5) makes it obvious that

$$xP_n(x) = P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) \quad (n = 0, 1, \dots). \tag{8}$$

The proof is by complete mathematical induction on n. Note that $c_0 = 0$ and $\alpha_0 = 1$, and so

$$c_1 - \alpha c_0 + (1-\alpha)\alpha_0 B_0 = 0.$$

Hence, for $n = 0$ we have

$$\mathcal{S}_q P_0 = 1 = \alpha_0 P_0 + c_0 P_{-1} = 1,$$

$$\mathtt{U}_2(x)\mathcal{D}_q P_0(x) = 0 = (\alpha^2 - 1)\gamma_0 P_1 + 0P_0 = 0.$$

Assuming (6) and (7) hold for all $n \leq k$, we will prove it for $n = k+1$. Set

$$\begin{aligned} c_{n,1} &= (\alpha^2 - 1)\gamma_n, \\ c_{n,2} &= c_{n+1} - \alpha c_n + (1-\alpha)\alpha_n B_n, \\ c_{n,3} &= (B_n - \alpha B_{n-1})c_n + (1-\alpha^2)\gamma_n C_n, \\ c_{n,4} &= c_{n-1}C_n - \alpha c_n C_{n-1}. \end{aligned}$$

Now (7) reads as

$$\mathtt{U}_2(x)\mathcal{D}_q P_n(x) = c_{n,1}P_{n+1}(x) + c_{n,2}P_n(x) + c_{n,3}P_{n-1}(x) + c_{n,4}P_{n-2}(x). \tag{9}$$

Applying $\mathcal{S}_q$ to (8), and using (4), we get

$$\begin{aligned} \mathcal{S}_q P_{n+1}(x) &= \mathtt{U}_2(x)\mathcal{D}_q\, x\mathcal{D}_q P_n(x) + \mathcal{S}_q\, x\mathcal{S}_q P_n(x) - B_n\mathcal{S}_q P_n(x) + C_n\mathcal{S}_q P_{n-1}(x) \\ &= \mathtt{U}_2(x)\mathcal{D}_q P_n(x) - B_n\mathcal{S}_q P_n(x) - C_n\mathcal{S}_q P_{n-1}(x) + \alpha x\mathcal{S}_q P_n(x). \end{aligned} \tag{10}$$

From (10), and using (9) for $n = k$ and (6) for $n = k-1$ and $n = k$, we obtain

$$\begin{aligned} \mathcal{S}_q P_{k+1}(x) &= c_{k,1}P_{k+1}(x) + c_{k,2}P_k(x) + c_{k,3}P_{k-1}(x) + c_{k,4}P_{k-2}(x) \\ &\quad - B_k(\alpha_k P_k(x) + c_k P_{k-1}(x)) - C_k(\alpha_{k-1}P_{k-1}(x) + c_{k-1}P_{k-2}(x)) \\ &\quad + \alpha x(\alpha_k P_k(x) + c_k P_{k-1}(x)). \end{aligned}$$

From (8) we have

$$\begin{aligned} \alpha x(\alpha_k P_k(x) + c_k P_{k-1}(x)) &= \alpha\alpha_k(P_{k+1}(x) + B_k P_k(x) + C_k P_{k-1}(x)) \\ &\quad + \alpha c_k(P_k(x) + B_{k-1}P_{k-1}(x) + C_{k-1}P_{k-2}(x)). \end{aligned}$$

We leave to the reader the verification that

$$c_{k,3} + (\alpha\alpha_k - \alpha_{k-1})C_k + (\alpha B_{k-1} - B_k)c_k = c_{k,4} + \alpha c_k C_{k-1} - c_{k-1}C_k = 0,$$

and

$$\alpha_{k+1} = c_{k,1} + \alpha\,\alpha_k, \quad c_{k+1} = c_{k,2} + \alpha c_k + (\alpha - 1)\alpha_k B_k.$$

We thus get

$$\begin{aligned}\mathcal{S}_q P_{k+1}(x) &= (c_{k,1} + \alpha\,\alpha_k)P_{k+1}(x) + (c_{k,2} + \alpha c_k + (\alpha-1)\alpha_k B_k)P_k(x)\\&\quad + (c_{k,3} + (\alpha\alpha_k - \alpha_{k-1})C_k + (\alpha B_{k-1} - B_k)c_k)P_{k-1}(x)\\&\quad + (c_{k,4} + \alpha c_k C_{k-1} - c_{k-1}C_k)P_{k-2}(x)\\&= \alpha_{k+1}P_{k+1}(x) + c_{k+1}P_k(x),\end{aligned}$$

and (6) holds for $n = k+1$.

Applying now $\mathcal{D}_q$ to (8), and using (3), we get

$$\mathcal{D}_q P_{k+1}(x) = \mathcal{S}_q P_k(x) + (\alpha x - B_k)\mathcal{D}_q P_k(x) - C_k \mathcal{D}_q P_{k-1}(x). \tag{11}$$

From (8) we have

$$\begin{aligned}\mathtt{U}_2(x)P_k(x) &= (\alpha^2-1)P_{k+2}(x) + (\alpha^2-1)(B_k + B_{k+1})P_{k+1}(x)\\&\quad + (\alpha^2-1)(B_k^2 + C_{k+1} + C_k - 1)P_k(x)\\&\quad + (\alpha^2-1)C_k(B_k + B_{k-1})P_{k-1}(x) + (\alpha^2-1)C_k C_{k-1}P_{k-2}(x).\end{aligned}$$

Hence, multiplying (11) by $\mathtt{U}_2$ and using (6) for $n = k$ and (9) for $n = k-1$ and $n = k$, we get

$$\begin{aligned}\mathtt{U}_2(x)\mathcal{D}_q P_{k+1}(x) &= d_{k,1}P_{k+2}(x) + d_{k,2}P_{k+1}(x) + d_{k,3}P_k(x) + d_{k,4}P_{k-1}(x)\\&\quad + d_{k,5}P_{k-2}(x) + d_{k,6}P_{k-3}(x),\end{aligned}$$

where

$$
\begin{aligned}
d_{k,1} &= (\alpha^2-1)\alpha_k + \alpha c_{k,1},\\
d_{k,2} &= (\alpha^2-1)(c_k + \alpha_k(B_k+B_{k+1})) + \alpha c_{k,2} - (B_k - \alpha B_{k+1})c_{k,1},\\
d_{k,3} &= (\alpha^2-1)((B_k+B_{k-1})c_k + \alpha_k(B_k^2 + C_k + C_{k+1} - 1)) + \alpha c_{k,1}C_{k+1}\\
&\quad - c_{k-1,1}C_k + (\alpha-1)c_{k,2}B_n + \alpha c_{k,3},\\
d_{k,4} &= (\alpha^2-1)((B_k+B_{k-1})\alpha_k C_k + (C_k + B_{k-1}^2 + C_{k-1} - 1)c_k)\\
&\quad - (c_{k-1,2} - \alpha c_{k,2})C_k - (B_k - \alpha B_{k-1})c_{k,3} + \alpha c_{k,4},\\
d_{k,5} &= (\alpha^2-1)C_{k-1}(\alpha_k C_k + c_k(B_{k-1}+B_{k-2})) + \alpha c_{k,3}C_{k-1} - c_{k-1,3}C_k\\
&\quad - (B_k - \alpha B_{k-2})c_{k,4},\\
d_{k,6} &= (\alpha^2-1)c_k C_{k-1}C_{k-2} + \alpha c_{k,4}C_{k-2} - c_{k-1,4}C_k.
\end{aligned}
$$

Finally, the reader should satisfy himself that $d_{k,1} = c_{k+1,1}$, $d_{k,2} = c_{k+1,2}$, $d_{k,3} = c_{k+1,3}$, $d_{k,4} = c_{k+1,4}$, $d_{k,5} = 0$, and $d_{k,6} = 0$, and (7) holds for $n = k+1$. That completes the inductive step, and hence the proof. ■

In view of Proposition 2.1, an interesting open problem is to characterize the sequence of orthogonal polynomials such that $\mathcal{S}_q P_n$ can be writen as a linear combination of P_n and P_{n-1}.

References

[1] K. Castillo, D. Mbouna, and J. Petronilho, A characterization of continuous q-Jacobi, Chebyshev of the first kind and Al-Salam Chihara polynomials, J. Math. Anal. Appl., 514 (2022) 126358.

[2] K. Castillo, D. Mbouna and J. Petronilho, On the functional equation for classical orthogonal polynomials on lattices, J. Math. Anal. Appl., 515 (2022) 126390.

[3] M. E. H. Ismail, Classical and quantum orthogonal polynomials in one variable. With two chapters by W. Van Assche. With a foreword by R. Askey., Encyclopedia of Mathematics and its Applications 98. Cambridge University Press, Cambridge, 2005.

K. Castillo
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: kenier@mat.uc.pt

D. Mbouna
University of Almería, Department of Mathematics, Almería, Spain
E-mail address: mbouna@ual.es