Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 22–18

EQUIDIVISIBILITY AND PROFINITE COPRODUCT

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Abstract: The aim of this work is to investigate the behavior of equidivisibility under coproduct in the category of pro-V semigroups, where V is a pseudovariety of finite semigroups. Exploring the relationship with the two-sided Karnofsky–Rhodes expansion, the notions of KR-cover and strong KR-cover for profinite semigroups are introduced. The former is stronger than equidivisibility and the latter provides a characterization of equidivisible profinite semigroups with an extra mild condition, so-called letter super-cancellativity. Furthermore, under the assumption that V is closed under two-sided Karnofsky–Rhodes expansion, closure of some classes of equidivisible pro-V semigroups under (finite) V-coproduct is established.

Keywords: Profinite semigroup, equidivisible, free product, coproduct, Karnofsky–Rhodes expansion.
Math. Subject Classification (2020): Primary 20M07, 20M05.

1. Introduction

A semigroup is *equidivisible* if any two factorizations of every element have a common refinement. The class of equidivisible semigroups was introduced and studied in [10] as a natural common generalization of free semigroups and completely simple semigroups. More recently, this property has appeared as a useful tool in profinite semigroup theory, beginning with [3, 9], where it was noted, independently, that for several important pseudovarieties of finite semigroups (like that of all finite semigroups, or that of all finite aperiodic semigroups), the corresponding finitely generated relatively free profinite semigroups are equidivisible. Other recent papers where the equidivisibility of relatively free profinite semigroups is applied or deserves some kind of attention include [6, 5, 20]. A

Received June 27, 2022.

The work of J. Almeida, was partially supported by CMUP, which is funded by FCT (Portugal) by national funds through the project UID/MAT/00144/2020. It was developed in part at Masaryk University, whose hospitality is gratefully acknowledged, with the support of the FCT sabbatical scholarship SFRH/BSAB/142872/2018.

The work of A. Costa was carried out in part at Masaryk University and at City College of New York. The hospitality of these institutions is gratefully acknowledged. The first visit had the support of the research grant 19-12790S of the Grant Agency of the Czech Republic, and the second visit had the support of the FCT sabbatical scholarship SFRH/BSAB/150401/2019. His work was also supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

complete characterization of the pseudovarieties for which the corresponding finitely generated relatively free profinite semigroups are equidivisible appears in [4].

As observed in [10], the class of equidivisible semigroups is closed under taking free products, that is, coproducts in the category of semigroups. In this paper, we investigate an analog for profinite semigroups. For that purpose, we introduce V-coproducts of pro-V semigroups with respect to a pseudovariety of semigroups V, extending what was done in [18] for the pseudovariety of finite groups. We give simple conditions on V guaranteeing that the free product of pro-V semigroups embeds naturally in their V-coproduct.

We introduce a restricted form of projectivity. The profinite semigroups with this property are called KR-covers and turn out to be equidivisible semigroups. We show that the class of pro-V KR-covers is closed under V-coproduct when V is closed under two-sided Karnofsky–Rhodes expansion. This expansion is a two-sided analog of the so-called Karnofsky–Rhodes expansion [8, 14], which has recently found new applications beyond semigroup theory [12, 13].

One of the motivations for searching for new examples of equidivisible profinite semigroups comes from the fact that several results in [5] were stated for equidivisible profinite semigroups, frequently with the additional requirement that they satisfy a certain cancellation property. Semigroups satisfying this cancellation property are called letter super-cancellative in [4] and finitely cancellable in [5]. They include the finitely generated relatively free profinite semigroups that are equidivisible but not completely simple. In this paper we provide a characterization of the class of all finitely generated equidivisible letter super-cancellative profinite semigroups, involving the notion of strong KR-cover, which we introduce. We show that the subclass consisting of pro-V semigroups is also closed under taking finite V-coproducts when V is closed under two-sided Karnofsky–Rhodes expansion. We also exhibit an element of the class that is not relatively free (the existence of such an example was left open in [5]).

2. Preliminaries

The reader is referred to standard references for general background on profinite semigroups and pseudovarieties [1, 2, 15]. For the remainder of the section, we introduce briefly specific notions and terminology needed in the sequel.

For a semigroup S, let S^I be the semigroup which is obtained by adjoining a new identity element, denoted I, even if S is itself a monoid. The semigroup

S is *equidivisible* when, for every $u, v, x, y \in S$, the equality $uv = xy$ implies the existence of $t \in S^I$ such that $x = ut$ and $ty = v$, or $xt = u$ and $y = tv$. Equivalently, any two factorizations of the same element of S have a common refinement.

A *pseudovariety (of semigroups)* is a class of finite semigroups closed under taking homomorphic images, subsemigroups and finite direct products. For the remainder of the paper, V denotes an arbitrary pseudovariety.

A *topological semigroup* is a semigroup S endowed with a topology such that the semigroup multiplication $S \times S \to S$ is continuous. We then say that a mapping $\varphi : X \to S$, with X a nonempty set, is a *generating mapping* if $\varphi(X)$ generates a dense subsemigroup of S. When the generating mapping is understood, we may simply say that S is X*-generated*. When no topology is mentioned, we consider semigroups endowed with the discrete topology.

Throughout the paper, when we consider compact spaces, we assume that they are Hausdorff. A *pro-*V *semigroup* is a compact semigroup which is residually V. A *profinite semigroup* is a pro-S semigroup for the pseudovariety S of all finite semigroups. Since in a finite semigroup, for every element s the sequence $(s^{n!})_n$ converges, and the limit is idempotent, the same holds in an arbitrary profinite semigroup; the limit of the sequence is denoted s^ω. More generally, the sequence $(s^{n!+k})_{n \geqslant |k|}$ converges for every integer k and the limit is denoted $s^{\omega+k}$.

Pro-V semigroups may be alternatively described as the inverse limits of inverse systems of finite semigroups. Here, by an *inverse system*, we mean a family $(S_i)_{i\in I}$ of compact semigroups, where I is an upper directed set, together with continuous homomorphisms $\varphi_{i,j} : S_i \to S_j$ whenever $i \geqslant j$ such that $\varphi_{i,i}$ is the identity mapping and $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$ whenever $i \geqslant j \geqslant k$. We say that the inverse system is an *inverse quotient system* if every mapping φ_i is onto. The inverse limit $\varprojlim_{i\in I} S_i$ of the inverse system $(S_i)_{i\in I}$ is the subsemigroup of the direct product $\prod_{i\in I} S_i$ consisting of the elements $(s_i)_{i\in I}$ of such that $\varphi_{i,j}(s_i) = s_j$ whenever $i \geqslant j$. The restriction to $\varprojlim_{i\in I} S_i$ of the j-component projection is a continuous homomorphism $\varphi_j : \varprojlim_{i\in I} S_i \to S_j$. In case the S_i are pro-V semigroups, so is $\varprojlim_{i\in I} S_i$. In the case of an inverse quotient system, we also say that $\varprojlim_{i\in I} S_i$ is an *inverse quotient limit*; in this case, the mappings φ_i are onto.

Given a nonempty set X, there is a *free pro-*V *semigroup on* X given by a mapping $\iota : X \to \overline{\Omega}_X\mathsf{V}$ with the following universal property: for every mapping $\varphi : X \to S$ into a pro-V semigroup S, there is a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_X\mathsf{V} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. Such semigroups are well known to exist and may be constructed as inverse limits of X-generated semigroups from V or, in the case X is finite, as completions of the free semigroup X^+ on X with respect to a natural pseudometric. From the universal property of $\overline{\Omega}_X\mathsf{V}$ it follows immediately that ι is a generating mapping and, unless V is the trivial pseudovariety, consisting only of singleton semigroups, the mapping ι is injective and we then identify each element $x \in X$ with its image $\iota(x)$.

A homomorphism $\psi : S \to T$ of finite semigroups is said to be a V-*morphism* if $\varphi^{-1}(e) \in \mathsf{V}$ for every idempotent e of T. For pseudovarieties V and W, their *Mal'cev product* is the pseudovariety $\mathsf{V} \textcircled{m} \mathsf{W}$ generated by the class of all finite semigroups S for which there is a V-morphism $S \to T$ with $T \in \mathsf{W}$. For example, it is well known that $\mathsf{J} = \mathsf{N} \textcircled{m} \mathsf{Sl}$, where J, N and Sl are, respectively, the pseudovarieties of all finite $\mathcal{J}$-trivial semigroups nilpotent semigroups and semilattices.

3. Coproduct of profinite semigroups

Let V be a pseudovariety of finite semigroups. Given a nonempty family $(S_i)_{i\in I}$ of pro-V semigroups, their V-*coproduct* is a pro-V semigroup S together with a collection of continuous homomorphisms $\varphi_i : S_i \to S$ such that the following universal property holds: for every pro-V semigroup T and every collection of continuous homomorphisms $\psi_i : S_i \to T$, there is a unique continuous homomorphism $\psi : S \to T$ such that Diagram 3.1 commutes for every $i \in I$.

$$\begin{tikzcd} & S_i \arrow[dl, "\varphi_i"'] \arrow[dr, "\psi_i"] & \\ S \arrow[rr, dashed, "\psi"] & & T. \end{tikzcd} \tag{3.1}$$

Note that, by the usual "abstract nonsense", if such a pro-V semigroup S exists, then it is unique up to isomorphism. It is then denoted $\coprod_{i\in I}^{\mathsf{V}} S_i$. In the case of a finite family $(S_i)_{i=1,\dots,n}$ we also write $S_1 \amalg^{\mathsf{V}} \cdots \amalg^{\mathsf{V}} S_n$ to denote $\coprod_{i=1}^{\mathsf{V}} S_i$.

3.1. Construction and basic properties of V-coproducts.

The following is the extension to arbitrary profinite semigroups of the special case of profinite groups considered in [18, Proposition 9.1.2].

Proposition 3.1. *Let $(S_i)_{i\in I}$ be a nonempty family of pro-V semigroups. Then the V-coproduct $\coprod_{i\in I}^{\mathsf{V}} S_i$ exists.*

In the proof of Proposition 3.1 we use the following alternative characterization of the V-coproduct, consisting in replacing "pro-V semigroup T" by "semigroup T from V" in the above definition of V-coproduct.

Proposition 3.2. *Consider a nonempty family $(S_i)_{i\in I}$ of pro-V semigroups. The pro-V semigroup S, together with the collection of continuous homomorphisms $\varphi_i : S_i \to S$, is the V-coproduct of the family $(S_i)_{i\in I}$ if and only if for every semigroup T from V and every collection of continuous homomorphisms $\psi_i : S_i \to T$, there is a unique continuous homomorphism $\psi : S \to T$ such that Diagram (3.1) commutes for every $i \in I$.*

Proof: The "only if" part of the proposition is trivial.

Conversely, let us assume that the restricted version of the universal property holds for S whenever T belongs to V. Take now an inverse limit $T = \varprojlim_{\lambda\in\Lambda} T_\lambda$ of semigroups T_λ from V. For each $\lambda \in \Lambda$, denote by π_λ the associated projection $T \to T_\lambda$, and for $\mu, \lambda \in \Lambda$ such that $\lambda \leqslant \mu$, let $\pi_{\mu,\lambda}$ be the connecting homomorphism $T_\mu \to T_\lambda$. Consider a collection of continuous homomorphisms $\psi_i : S_i \to T$. By hypothesis, for each $\lambda \in \Lambda$, there is a continuous homomorphism $\psi_\lambda : S \to T_\lambda$ such that Diagram 3.2 commutes for every $i \in I$.

$$\begin{tikzcd} & S_i \arrow[dl, "\varphi_i"'] \arrow[dr, "\pi_\lambda\circ\psi_i"] & \\ S \arrow[rr, dashed, "\psi_\lambda"] & & T_\lambda. \end{tikzcd} \tag{3.2}$$

Given $\mu, \lambda \in \Lambda$ such that $\lambda \leqslant \mu$, we have

$$\pi_{\mu,\lambda} \circ \psi_\mu \circ \varphi_i = \pi_{\mu,\lambda} \circ \pi_\mu \circ \psi_i = \pi_\lambda \circ \psi_i$$

for every $i \in I$. Since ψ_λ is the unique continuous homomorphism for which Diagram (3.2) commutes for all $i \in I$, we deduce that $\pi_{\mu,\lambda} \circ \psi_\mu = \psi_\lambda$. By the definition of inverse limit, we conclude that there is a continuous homomorphism $\psi : S \to T$ for which Diagram (3.3) commutes whenever $\mu, \lambda \in \Lambda$ satisfy $\mu \leqslant \lambda$.

$$\begin{tikzcd} & & T_\mu \arrow[dd, "\pi_{\mu,\lambda}"] \\ S \arrow[r, dashed, "\psi"] \arrow[urr, bend left, "\psi_\mu"] \arrow[drr, bend right, "\psi_\lambda"'] & T \arrow[ur, "\pi_\mu"] \arrow[dr, "\pi_\lambda"'] & \\ & & T_\lambda \end{tikzcd} \tag{3.3}$$

Let $i \in I$. Since for every $\lambda \in \Lambda$ we have $\pi_\lambda \circ (\psi \circ \varphi_i) = \psi_\lambda \circ \varphi_i = \pi_\lambda \circ \psi_i$, we conclude that $\psi \circ \varphi_i = \psi_i$, thus establishing the proposition. ∎

We may now proceed with the proof of Proposition 3.1. It is well known that coproducts exist in the category of semigroups (see [11, Theorem I.13.5]). We denote the coproduct, also known as free product, of a nonempty family $(S_i)_{i\in I}$ of semigroups by $S = \ast_{i\in I} S_i$.

Proof of Proposition 3.1: We start by taking the free product $S = \ast_{i\in I} S_i$ in the category of semigroups and the corresponding natural homomorphisms $\tilde{\varphi}_i : S_i \to S$. We consider the family $\mathcal{C}$ of all congruences θ on S such that $S/\theta \in \mathsf{V}$ and, for each $i \in I$, the congruence $(\tilde{\varphi}_i \times \tilde{\varphi}_i)^{-1}(\theta)$ is a clopen subset of $S_i \times S_i$. The latter condition expresses that the composite of $\tilde{\varphi}_i$ followed by the natural quotient mapping $q_\theta : S \to S/\theta$ is continuous. Given $\theta, \rho \in \mathcal{C}$, note that $\theta \cap \rho \in \mathcal{C}$ since V is closed under taking subsemigroups and finite direct products. Hence, the quotients S/θ with $\theta \in \mathcal{C}$ form an inverse system, for which we may consider the inverse limit, which we denote S_{V}. Thus, S_{V} is a pro-V semigroup. Let $\iota : S \to S_{\mathsf{V}}$ be the natural homomorphism and let $\varphi_i = \iota \circ \tilde{\varphi}_i$ $(i \in I)$.

For every $\theta \in \mathcal{C}$, the natural projection $\pi_\theta : S_{\mathsf{V}} \to S/\theta$ satisfies $\pi_\theta \circ \iota = q_\theta$, and so we have the equalities

$$\pi_\theta \circ \varphi_i = \pi_\theta \circ \iota \circ \tilde{\varphi}_i = q_\theta \circ \tilde{\varphi}_i,$$

for which the reader may wish to refer to Diagram (3.4) below. Since, by the definition of $\mathcal{C}$, the mapping $q_\theta \circ \tilde{\varphi}_i$ is continuous for every $\theta \in \mathcal{C}$, it follows that each mapping φ_i is continuous.

Suppose that $T \in \mathsf{V}$ and $(\psi_i)_{i\in I}$ is a family of continuous homomorphisms $\psi_i : S_i \to T$. By the universal property of the free product S, there exists a unique homomorphism $\gamma : S \to T$ such that $\gamma \circ \tilde{\varphi}_i = \psi_i$ $(i \in I)$. Let θ be the kernel of γ. Then $\theta \in \mathcal{C}$ and there is a unique homomorphism $\beta : S/\theta \to T$ such that $\beta \circ q_\theta = \gamma$, and so the non-dashed part of Diagram (3.4) is commutative.

$$\begin{tikzcd}
& S_i \arrow[dl, "\tilde{\varphi}_i"'] \arrow[dr, "\psi_i"] \arrow[ddl, bend right=60, "\varphi_i"'] & \\
S \arrow[rr, "\gamma"] \arrow[d, "\iota"'] \arrow[drr, "q_\theta"] & & T \\
S_{\mathsf{V}} \arrow[urr, dashed, "\psi"] \arrow[rr, "\pi_\theta"] & & S/\theta \arrow[u, "\beta"']
\end{tikzcd} \tag{3.4}$$

Therefore, for the continuous homomorphism $\psi = \beta \circ \pi_\theta$, we have $\psi \circ \varphi_i = \psi_i$, for every $i \in I$.

Since the restriction to $\iota(S)$ of each π_θ is onto, as so is q_θ, and the preimages of points under the π_θ form a base of the topology of S_{V}, we see that $\iota(S)$ is dense in S_{V}. On the other hand, S is generated by $\bigcup_{i\in I} \tilde{\varphi}_i(S_i)$. Thus, ψ is completely determined by its restriction to the union $\bigcup_{i\in I} \varphi_i(S_i)$, and so ψ is the unique continuous homomorphism from S_{V} to T such that, for every $i \in I$, the following diagram commutes:

$$\begin{tikzcd} & S_i \arrow[dl, "\varphi_i"'] \arrow[dr, "\psi_i"] & \\ S_{\mathsf{V}} \arrow[rr, dashed, "\psi"] & & T. \end{tikzcd}$$

Since T is an arbitrary semigroup from V, it follows from Proposition 3.2 that the profinite semigroup S_{V}, together with the family of continuous homomorphisms $(\varphi_i)_{i\in I}$, provides a V-coproduct of the family $(S_i)_{i\in I}$. ■

Let $k \in I$. The continuous semigroup homomorphism $\varphi_k : S_k \to \coprod_{i\in I}^{\mathsf{V}} S_i$ introduced in the definition of V-coproduct is said to be the *natural* mapping from S_k into $\coprod_{i\in I}^{\mathsf{V}} S_i$.

Proposition 3.3. *If $(S_i)_{i\in I}$ is a nonempty family of pro-V semigroups then, for every $k \in I$, the natural mapping $\varphi_k : S_k \to \coprod_{i\in I}^{\mathsf{V}} S_i$ is injective, whence an embedding of topological semigroups.*

Proof: Let $k \in I$. Since S_k is pro-V, in Diagram 3.1 we may take $T = S_k$, choose ψ_k as the identity Id_{S_k} on S_k, and if $i \in I \setminus \{k\}$, we choose ψ_i as any constant mapping from S_i onto an idempotent of S_k. We may then consider the continuous homomorphism ψ as in Diagram 3.1, for which we have in particular $\psi \circ \varphi_k = \mathrm{Id}_{S_k}$, and so the proposition holds. ■

In view of Proposition 3.3, we may henceforth see each S_i as a closed subsemigroup of $\coprod_{i\in I}^{\mathsf{V}} S_i$, with φ_i being the inclusion.

Let Sl denote the pseudovariety of all finite semilattices. The following technical observation will be convenient later on.

Lemma 3.4. *Suppose that V contains Sl. Let $(S_i)_{i\in I}$ be a nonempty family of nontrivial pro-V semigroups. Let A be a generating subset of the V-coproduct $\coprod_{i\in I}^{\mathsf{V}} S_i$ as a topological semigroup. Then $A \cap S_i$ generates S_i as a topological semigroup for every $i \in I$.*

Proof: Let $S = *_{i\in I} S_i$ and let $S_{\mathsf{V}} = \coprod^{\mathsf{V}}_{i\in I} S_i$. Note that $\bigcup_{i\in I} S_i$ generates S_{V}. We should show that every element of S_i is the limit of a net of products of elements of $A \cap S_i$. With this aim, we first claim that there is a continuous homomorphism ψ_{V} from S_{V} to the two-element semilattice $\{0, 1\}$ such that $\psi_{\mathsf{V}}^{-1}(1) = S_i$. It follows that $S_{\mathsf{V}} \setminus S_i = \psi_{\mathsf{V}}^{-1}(0)$ is an ideal of S_{V} containing $A \setminus S_i$. Hence, in a net of products of elements of A converging to an element of S_i all products from some point on can only involve factors from $A \cap S_i$.

To establish the claim, we use the universal properties of S and S_{V} to define a homomorphism ψ and a continuous homomorphism ψ_{V} by considering the constant homomorphisms φ_k on the S_k, with value 1 for $k = i$ and value 0 for $k \in I \setminus \{i\}$. More precisely, we have the following commutative diagram for each $k \in I$:

$$\begin{tikzcd} S \arrow[d, "\iota"'] \arrow[dr, "\psi"] & S_k \arrow[l, hook'] \arrow[d, "\varphi_k"] \\ S_{\mathsf{V}} \arrow[r, "\psi_{\mathsf{V}}"] & \{0,1\}. \end{tikzcd}$$

Since ψ_{V} is continuous and $\iota(S)$ is dense in S_{V}, we obtain the following formula:

$$\psi_{\mathsf{V}}^{-1}(1) = \overline{\iota(\psi^{-1}(1))} = \iota(S_i) = S_i$$

which establishes the claim and completes the proof of the lemma. ∎

Note that the hypothesis that V contains Sl may not be omitted in Lemma 3.4. For instance, if G is a cyclic group of order 2 with generator a and H is a cyclic group of order 3 with generator b, then ab^2 is a generator of $G \amalg^{\mathsf{Ab}} H$ (since $(ab^2)^2 = b$ and $(ab^2)^3 = a$) but $ab^2 \notin G \cup H$.

The next couple of facts (that one should bear in mind, albeit not needed for the sequel) may also be easily proved with the universal property of the V-coproduct:

1. The V-coproduct is associative: if the nonempty set I has a partition $I = \biguplus_{j\in J} I_j$, and $(S_i)_{i\in I}$ is a family of pro-V semigroups, then we have an isomorphism
$$\textstyle\coprod^{\mathsf{V}}_{i\in I} S_i \cong \coprod^{\mathsf{V}}_{j\in J}(\coprod^{\mathsf{V}}_{i\in I_j} S_i)$$
of profinite semigroups.
2. For a subpseudovariety V, the V-coproduct of free pro-V semigroups is also free pro-V.

3.2. On the injectivity of the mapping ι. It is natural to ask under what conditions the natural mapping

$$\iota : \mathop{\ast}_{i\in I} S_i \to \coprod_{i\in I}^{\mathsf{V}} S_i$$

is injective for pro-V semigroups S_i $(i \in I)$. A partial solution to the analogous question for pseudovarieties of groups in the category of groups can be found in [18, Proposition 9.1.8]. Note, that in, that category, the free product of trivial groups is trivial, whereas the free product of more than one trivial semigroup (in every category of semigroups containing semigroups with more than one idempotent) is an infinite idempotent-generated semigroup. Our main result of this section gives a sufficient condition for injectivity of the function ι. Our sufficient condition holds for a large class of pseudovarieties considered in the remainder the paper, namely equidivisible pseudovarieties containing Sl. We leave as an open problem the complete characterization of the pseudovarieties for which the function ι is injective.

We start by some simple observations.

Remark 3.5. Let $\varphi_i : S_i \to T_i$ $(i \in I)$ be continuous homomorphisms between pro-V semigroups. By the universal property of V-coproducts, there is a unique continuous homomorphism φ_{V} such that the following diagram commutes:

$$\begin{tikzcd} S_i \arrow[r, "\varphi_i"] \arrow[d, hook] & T_i \arrow[d, hook] \\ \coprod_{i\in I}^{\mathsf{V}} S_i \arrow[r, "\varphi_{\mathsf{V}}"] & \coprod_{i\in I}^{\mathsf{V}} T_i. \end{tikzcd}$$

Remark 3.6. Let X be a set and suppose that x and y are distinct elements of X such that x^ω is a factor y^ω in the semigroup $\overline{\Omega}_X\mathsf{V}$. Then, V is contained in LG. Indeed, it follows that, in every semigroup from V, all idempotents are factors of each other, a property that characterizes membership in LG.

Lemma 3.7. *Let* V *be a pseudovariety of semigroups containing* J. *Then the natural mapping* $\iota : \ast_{i\in I} S_i \to \coprod_{i\in I}^{\mathsf{V}} S_i$ *is injective whenever the* S_i *are trivial semigroups.*

Proof: Let $S_i = \{e_i\}$ $(i \in I)$ and let $I \to X$ be a bijection given by $i \mapsto x_i$. By the universal property of V-coproducts, there is a unique continuous homomorphism $\varphi : \coprod_{i\in I}^{\mathsf{V}}\{e_i\} \to \overline{\Omega}_X\mathsf{J}$ such that $\varphi(e_i) = x_i^\omega$.

Given elements $e_{i_1}e_{i_2}\cdots e_{i_m}$ and $e_{j_1}e_{j_2}\cdots e_{j_n}$ in $\ast_{i\in I}\{e_i\}$ (with adjacent indices distinct), suppose that their images under ι coincide. Then, their images under $\varphi\circ\iota$ also coincide, giving the equality

$$x_{i_1}^{\omega}x_{i_2}^{\omega}\cdots x_{i_m}^{\omega}=x_{j_1}^{\omega}x_{j_2}^{\omega}\cdots x_{j_n}^{\omega}. \tag{3.5}$$

Hence, it suffices to show that, for every equality in $\overline{\Omega}_X\mathsf{J}$ of the form (3.5), with $x_k\in X$ for every index k, if all adjacent indices are distinct, then we must have $n=m$ and $i_k=j_k$ for every $k\in\{1,\dots,n\}$. This is a special case of [1, Theorem 8.2.8]. ∎

Lemma 3.8. *Let* V *be a pseudovariety of semigroups containing* Sl *such that* $\mathsf{N}\ ⓜ\ \mathsf{V}=\mathsf{V}$ *and let* S *be a member of* V. *Then, the natural mapping* $\iota: S\ast\{e\}\to S\amalg^{\mathsf{V}}\{e\}$ *is injective.*

Proof: Let $u=s_1es_2e\cdots es_m$ and $v=t_1et_2e\cdots et_n$ be distinct elements of $S\ast\{e\}$, where $s_2,\dots,s_{m-1},t_2,\dots,t_{n-1}\in S$ and $s_1,s_m,t_1,t_n\in S^I$. Consider the ideal $J=\bigcup_{k>\max\{m,n\}}S^Ie(Se)^kS^I$ of $S\ast\{e\}$. Since the elements u and v are not in J, they are distinguished in the Rees quotient $T=S\ast\{e\}/J$. Note that, since S is finite, so is T. To complete the proof, it suffices to establish that T belongs to V, since then the natural homomorphism $S\ast\{e\}\to T$ factors through ι.

Let K be the complement of $S\cup\{e\}$ in T. Then K is an ideal and a nilpotent subsemigroup of T. Thus, if we show that $T/K\in\mathsf{V}$, then it follows that $T\in\mathsf{N}\ ⓜ\ \mathsf{V}=\mathsf{V}$. But, T/K is the zero sum of the semigroups S and $\{e\}$, which is a quotient of the direct product $S\times U$ where U is the three-element semilattice which is the zero sum of two trivial semigroups. This shows that indeed $T\in\mathsf{V}$. ∎

Theorem 3.9. *Let* V *be a pseudovariety of semigroups containing* Sl *such that* $\mathsf{N}\ ⓜ\ \mathsf{V}=\mathsf{V}$. *Then the natural mapping* $\iota:\ast_{i\in I}S_i\to\coprod_{i\in I}^{\mathsf{V}}S_i$ *is injective whenever the* S_i *are pro-*V *semigroups.*

Proof: Consider constant homomorphisms $\varphi_i: S_i\to\{e_i\}$, the induced continuous homomorphism $\varphi_{\mathsf{V}}:\coprod_{i\in I}^{\mathsf{V}}S_i\to\coprod_{i\in I}^{\mathsf{V}}\{e_i\}$ given by Remark 3.5, and an analogous abstract homomorphism $\varphi:\ast_{i\in I}S_i\to\ast_{i\in I}\{e_i\}$. We get the following commutative diagram, where ι' is the natural mapping:

$$\begin{tikzcd} \ast_{i\in I} S_i \arrow[r, "\varphi"] \arrow[d, "\iota"] & \ast_{i\in I}\{e_i\} \arrow[d, "\iota'"] \\ \coprod^{\mathsf{V}}_{i\in I} S_i \arrow[r, "\varphi_{\mathsf{V}}"] & \coprod^{\mathsf{V}}_{i\in I}\{e_i\}. \end{tikzcd}$$

Given elements $u = s_{i_1}s_{i_2}\cdots s_{i_m}$ and $v = t_{j_1}t_{j_2}\cdots t_{j_n}$ of $\ast_{i\in I} S_i$, with $s_i, t_i \in S_i$ and no two adjacent indices equal, suppose that their images under ι coincide. As $\mathsf{J} = \mathsf{N} ⓜ \mathsf{SI} \subseteq \mathsf{N} ⓜ \mathsf{V} = \mathsf{V}$, we see that ι' is injective by Lemma 3.7. Hence, the images of u and v under φ must be equal, too. We conclude that $m = n$ and $i_1 = j_1, \ldots, i_m = j_m$.

It remains to show that $s_{i_k} = t_{i_k}$ for $k = 1, \ldots, m$. Suppose, on the contrary, that $s_j \neq t_j$ for some $j = i_k$ with $k \in \{1, \ldots, m\}$. Since S_j is a pro-V semigroup, there is a continuous homomorphism $\psi : S_j \to F$ into some $F \in \mathsf{V}$ such that $\psi(s_j) \neq \psi(t_j)$. We also consider the unique homomorphisms $S_i \to \{e\}$ for each $i \in I \setminus \{j\}$. Together, these mappings induce the horizontal mappings in the following commutative diagram:

$$\begin{tikzcd} \ast_{i\in I} S_i \arrow[r, "\bar{\psi}"] \arrow[d, "\iota"] & F \ast \{e\} \arrow[d, "\iota'"] \\ \coprod^{\mathsf{V}}_{i\in I} S_i \arrow[r, "\bar{\psi}_{\mathsf{V}}"] & F \amalg^{\mathsf{V}} \{e\} \end{tikzcd}$$

where ι' is the natural mapping. As $\iota(u) = \iota(v)$, we get $\iota'(\bar{\psi}(u)) = \iota'(\bar{\psi}(v))$. Since ι' is injective by Lemma 3.8, we conclude that $\bar{\psi}(u) = \bar{\psi}(v)$, which implies that $\psi(s_j) = \psi(t_j)$, contradicting the choice of ψ. ■

4. The two-sided Karnofsky–Rhodes expansion

The two-sided Karnofsky–Rhodes expansion plays a central role in [4] when studying equidivisible relatively free profinite semigroups. We recall here this expansion.

We adopt the following notational conventions. Let S be a semigroup. Given a mapping $\varphi : A \to S$, we may denote the unique homomorphism $A^+ \to S$ extending φ also by φ. Similarly, if S is a pro-V semigroup, then the unique continuous homomorphism $\overline{\Omega}_A\mathsf{V} \to S$ extending $\varphi : A \to S$ may also be denoted by φ. Conversely, given a homomorphism $\varphi : A^+ \to S$, or a continuous homomorphism $\varphi : \overline{\Omega}_A \mathsf{V} \to S$, its restriction to A may also be denoted by φ, if no confusion arises.

Let φ be a homomorphism from A^+ onto a semigroup S. We denote by Γ_φ the *two-sided Cayley graph*, whose set of vertices is $S^I \times S^I$, and where an edge from (s_1, t_1) to (s_2, t_2) is a triple $((s_1, t_1), a, (s_2, t_2))$, with $a \in A$, such that $s_1\varphi(a) = s_2$ and $t_1 = \varphi(a)t_2$. We see Γ_φ as a labeled directed graph, by labeling each edge $((s_1, t_1), a, (s_2, t_2))$ with the letter a. By the *label* of a directed path in Γ_φ we mean the word obtained by concatenating the successive labels of its edges.

A *transition edge* of a directed graph is an edge $x \to y$ such that there is no directed path from y to x. For each path p in the two-sided Cayley graph Γ_φ, we denote by $T(p)$ the set of transition edges of Γ_φ that occur in p. For each $u \in A^+$, let p_u be the unique path of Γ_φ from $(I, \varphi(u))$ to $(\varphi(u), I)$ that is labeled by the word u. Consider the binary relation $\equiv_\varphi$ on A^+ defined by $u \equiv_\varphi v$ if $\varphi(u) = \varphi(v)$ and $T(p_u) = T(p_v)$. Then one can easily check that $\equiv_\varphi$ is a congruence, which is of finite index if both S and A are finite. Consider the quotient semigroup $S_\varphi^{\mathrm{KR}} = A^+/{\equiv_\varphi}$ and the corresponding quotient homomorphism $\varphi^{\mathrm{KR}} : A^+ \to S_\varphi^{\mathrm{KR}}$.

We also consider the unique homomorphism $\pi_\varphi : S_\varphi^{\mathrm{KR}} \to S$ such that the following diagram commutes:

$$\begin{tikzcd} & A^+ \arrow[dl, "\varphi^{\mathrm{KR}}"'] \arrow[d, "\varphi"] \\ S_\varphi^{\mathrm{KR}} \arrow[r, "\pi_\varphi"'] & S. \end{tikzcd} \tag{4.1}$$

Remark 4.1. Suppose that the homomorphism $\varphi : A^+ \to S$ is such that A and S are finite, so that so is S_φ^{KR}. It is folklore that π_φ is an LI-morphism; in fact, for $\mathsf{W} = [\![xyz = xz]\!]$, it follows easily from the definition of S_φ^{KR} that π_φ is a W-morphism, and one clearly has $\mathsf{W} \subseteq \mathsf{LI}$.

The homomorphism φ^{KR} is the *two-sided Karnofsky–Rhodes expansion* of φ, and S_φ^{KR} is a *two-sided Karnofsky–Rhodes expansion* of S. The correspondence $(S, \varphi) \mapsto (S^{\mathrm{KR}}, \varphi^{\mathrm{KR}})$ is an expansion cut to generators [14]. In fact, a more general property holds, which we state in the next proposition.

Proposition 4.2. *Let $\varphi : A^+ \to S$ and $\psi : B^+ \to T$ be two homomorphisms from finitely generated free semigroups onto finite semigroups. Suppose that* *the homomorphisms* $\lambda : S \to T$ *and* $\alpha : A^+ \to B^+$ *are such that* $\lambda \circ \varphi = \psi \circ \alpha$. *Then, there is a unique homomorphism* $\Lambda : S_\varphi^{\mathrm{KR}} \to T_\psi^{\mathrm{KR}}$ *such that the diagram*

$$
\begin{tikzcd}
S_\varphi^{\mathrm{KR}} \arrow[rrr, dashed, "\Lambda"] \arrow[ddd, "\pi_\varphi"'] & & & T_\psi^{\mathrm{KR}} \arrow[ddd, "\pi_\psi"] \\
& & & \\
& A^+ \arrow[r, "\alpha"] \arrow[uul, "\varphi^{\mathrm{KR}}"'] \arrow[dl, "\varphi"'] & B^+ \arrow[uur, "\psi^{\mathrm{KR}}"] \arrow[dr, "\psi"] & \\
S \arrow[rrr, "\lambda"] & & & T
\end{tikzcd}
\tag{4.2}
$$

is commutative.

The analog of Proposition 4.2 in the category of monoids appears as part of [7, Proposition 4.4] (see also [7, Proposition 4.10]). Since the original proof is somewhat indirect, for the sake of completeness we present here a direct proof for the category of semigroups.

Proof of Proposition 4.2: Let $u, v \in A^+$. Suppose that $\varphi^{\mathrm{KR}}(u) = \varphi^{\mathrm{KR}}(v)$. We want to show that $\psi^{\mathrm{KR}}(\alpha(u)) = \psi^{\mathrm{KR}}(\alpha(v))$. By the commutativity of the left triangle and of the lower trapezoid in Diagram (4.2), we know that $\psi(\alpha(u)) = \psi(\alpha(v))$. We need to show that the coterminal paths $p_{\alpha(u)}$ and $p_{\alpha(v)}$ of Γ_ψ contain the same transition edges of Γ_ψ.

Suppose that τ is a transition edge of Γ_ψ that occurs in the path $p_{\alpha(u)}$. Then, there are words $w_1, w_2 \in B^*$ and $b \in B$ such that $\alpha(u) = w_1 b w_2$ and τ is the edge

$$\tau : (\psi(w_1), \psi(bw_2)) \xrightarrow{b} (\psi(w_1 b), \psi(w_2)).$$

Moreover, there are $u_1, u_2 \in A^*$, $a \in A$, and $w_1', w_2' \in B^*$ such that $u = u_1 a u_2$, $w_1 = \alpha(u_1) w_1'$, $\alpha(a) = w_1' b w_2'$ and $w_2 = w_2' \alpha(u_2)$. The reader may wish to refer to Figure 1.

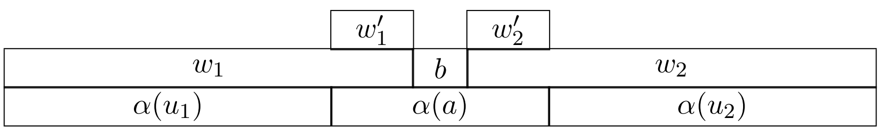

FIGURE 1. Factorizations of $\alpha(u)$.

Note that the edge τ' of Γ_φ given by

$$\tau' : (\varphi(u_1), \varphi(au_2)) \xrightarrow{a} (\varphi(u_1 a), \varphi(u_2))$$

belongs to the path p_u of Γ_φ. We claim that τ' is a transition edge of Γ_φ. Suppose it is not. Then, there is some $z \in A^+$ such that

$$\varphi(u_1az) = \varphi(u_1) \quad \text{and} \quad \varphi(u_2) = \varphi(zau_2).$$

By the commutativity of the lower trapezoid of Diagram (4.2), we get

$$\psi(\alpha(u_1az)) = \psi(\alpha(u_1)) \quad \text{and} \quad \psi(\alpha(u_2)) = \psi(\alpha(zau_2)).$$

Hence, we have

$$\begin{aligned}\psi(w_1b \cdot w_2'\alpha(z)w_1') &= \psi(\alpha(u_1)w_1'bw_2'\alpha(z)w_1')\\ &= \psi(\alpha(u_1az)w_1')\\ &= \psi(\alpha(u_1)w_1')\\ &= \psi(w_1),\end{aligned}$$

and similarly

$$\psi(w_2) = \psi(w_2'\alpha(z)w_1' \cdot bw_2).$$

Therefore, the graph Γ_ψ contains the path

$$(\psi(w_1b), \psi(w_2)) \xrightarrow{w_2'\alpha(z)w_1'} (\psi(w_1), \psi(bw_2)).$$

with contradicts τ being a transition edge of Γ_ψ. To avoid the contradiction, the edge τ' must be a transition edge of Γ_φ. Since the paths p_u and p_v of Γ_φ contain the same transition edges of Γ_φ, and in view of the commutativity of the lower trapezoid in Diagram (4.2), we conclude that

$$(\psi(\alpha(u_1)), \psi(\alpha(au_2))) \xrightarrow{\alpha(a)} (\psi(\alpha(u_1a)), \psi(\alpha(u_2)))$$

is a path of Γ_ψ contained in the path $p_{\alpha(v)}$. But this path factorizes as

$$(\psi(\alpha(u_1)), \psi(\alpha(au_2))) \xrightarrow{w_1'} (\psi(w_1), \psi(bw_2)) \xrightarrow{b} (\psi(w_1b), \psi(w_2)) \xrightarrow{w_2'} (\psi(\alpha(u_1a)), \psi(\alpha(u_2))),$$

thus having τ as one of its edges. This shows that every transition edge of Γ_ψ that belongs to $p_{\alpha(u)}$ also belongs to $p_{\alpha(v)}$. By symmetry, we conclude that $\psi^{\mathrm{KR}}(\alpha(u)) = \psi^{\mathrm{KR}}(\alpha(v))$.

Since, for every $u, v \in A^+$, the equality $\varphi^{\mathrm{KR}}(u) = \varphi^{\mathrm{KR}}(v)$ implies the equality $\psi^{\mathrm{KR}}(\alpha(u)) = \psi^{\mathrm{KR}}(\alpha(v))$, and since φ^{KR} is onto, we conclude that there exists a unique homomorphism Λ such that Diagram (4.2) commutes. ■

A pseudovariety of semigroups V is said to be closed under two-sided Karnofsky–Rhodes expansion when $S \in \mathsf{V}$ implies $S_\varphi^{\mathrm{KR}} \in \mathsf{V}$, for every onto homomorphism

$A^+ \to S$ and every finite alphabet A. A proof of the following characterization of such pseudovarieties may be found in [4], where one sees it as a direct consequence of a deep result of Rhodes et al. [16, 17]

Theorem 4.3. *A pseudovariety of semigroups* V *is closed under two-sided Karnofsky–Rhodes expansion if and only if* $\mathsf{V} = \mathsf{LI} \ⓜ \mathsf{V}$.

5. KR-covers

We are now ready to introduce the following new definition, playing a central role in this paper.

Definition 5.1 (KR-cover of a finite semigroup)**.** Let S be a profinite semigroup, and let T be a finite semigroup. We say that S is a *KR-cover of* T when T is a continuous homomorphic image of S and for every continuous onto homomorphism $\varphi : S \to T$ there is a generating mapping $\psi : A \to T$, for some finite alphabet A depending on φ, and a continuous homomorphism $\varphi_\psi : S \to T_\psi^{\mathrm{KR}}$ such that the following diagram commutes:

$$\begin{tikzcd} & S \arrow[dl, dashed, "\varphi_\psi"'] \arrow[d, "\varphi"] \\ T_\psi^{\mathrm{KR}} \arrow[r, "\pi_\psi"'] & T. \end{tikzcd} \tag{5.1}$$

The generating mapping ψ that appears in Definition 5.1 may in fact be any generating mapping of T, as shown next. Thus, S is in a sense more general than all two-sided Karnofsky–Rhodes expansions of T.

Lemma 5.2. *Suppose that the profinite semigroup S is a KR-cover of the finite semigroup T. Let $\varphi : S \to T$ be a continuous homomorphism from S onto T. For every finite alphabet A and generating mapping $\psi : A \to T$, there is a continuous homomorphism $\varphi_\psi : S \to T_\psi^{\mathrm{KR}}$ such that Diagram* (5.1) *commutes.*

Proof: Let $\psi : A^+ \to T$ be any homomorphism from A^+ onto T, defined on a finite alphabet A. Because S is a KR-cover of T, there is an onto homomorphism $\zeta : B^+ \to T$, for some finite alphabet B, inducing a homomorphism $\varphi_\zeta : S \to T_\zeta^{\mathrm{KR}}$ such that the leftmost triangle of the following diagram is commutative:

$$\begin{tikzcd}
 & T_\zeta^{\mathrm{KR}} \arrow[rrr, dotted, "\Lambda"] \arrow[dd, "\pi_\zeta"'] & & & T_\psi^{\mathrm{KR}} \arrow[dd, "\pi_\psi"] \\
S \arrow[ur, "\varphi_\zeta"] \arrow[dr, "\varphi"'] & & B^+ \arrow[ul, "\zeta^{\mathrm{KR}}"'] \arrow[dl, "\zeta"] \arrow[r, dashed, "\alpha"] & A^+ \arrow[ur, "\psi^{\mathrm{KR}}"] \arrow[dr, "\psi"'] & \\
 & T \arrow[rrr, equal, "\mathrm{Id}_T"] & & & T.
\end{tikzcd} \tag{5.2}$$

Since ψ is onto, there is a homomorphism $\alpha : B^+ \to A^+$ such that $\zeta = \psi \circ \alpha$, that is, such the lower trapezoid in Diagram (5.2) commutes. By Proposition 4.2, there is a homomorphism $\Lambda : T_\zeta^{\mathrm{KR}} \to T_\psi^{\mathrm{KR}}$ such that Diagram (5.2) commutes. Therefore, if φ_ψ is the homomorphism $\Lambda \circ \varphi_\zeta$, then Diagram (5.1) commutes. ∎

Letting T vary in Definition 5.1, we are led to the following stronger property.

Definition 5.3 (KR-cover)**.** A profinite semigroup S is a *KR-cover* if it is a KR-cover of each of its finite continuous homomorphic images.

The notion of KR-cover is reminiscent of that of profinite projective semigroup, which we recall here. Consider a pseudovariety V of semigroups. A pro-V semigroup S is said to be V-*projective* if, whenever T and R are pro-V semigroups and $f : S \to T$ and $g : R \to T$ are continuous homomorphisms with g onto, there is some continuous homomorphism $f' : S \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
 & S \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\
R \arrow[r, "g"'] & T.
\end{tikzcd}$$

A profinite projective semigroup is just an S-projective semigroup, where S is the pseudovariety of all finite semigroups (cf. [15, Remark 4.1.34]). Every free pro-V semigroup is an example of a V-projective semigroup.

The next simple observation gives our first examples of KR-covers and provides a more precise connection with projectivity.

Proposition 5.4. *If* V *is a semigroup pseudovariety closed under two-sided Karnofsky–Rhodes expansion, then every* V*-projective semigroup is a KR-cover.*

Proof: Let S be a V-projective semigroup, and let φ be a continuous homomorphism from S onto a finite semigroup T. Note that $T \in \mathsf{V}$ (see, for instance, [2, Proposition 3.7]). Consider any two-sided Karnofsky–Rhodes expansion T_{ψ}^{KR} of T, with $\psi : A \to T$ a generating mapping with finite domain. Since $T_{\psi}^{\mathrm{KR}} \in \mathsf{V}$ and S is V-projective, there is a continuous homomorphism $\varphi' : S \to T_{\psi}^{\mathrm{KR}}$ such that $\pi_{\psi} \circ \varphi' = \varphi$. ■

Proposition 5.4 is complemented by the next proposition.

Proposition 5.5. *Every profinite completely simple semigroup is a KR-cover.*

Proposition 5.5 follows easily (explicit details are given below) from a simple property, expressed in the next lemma. Let A be the pseudovariety of all finite aperiodic semigroups.

Lemma 5.6. *If $\pi : S \to T$ is an onto A-morphism of finite semigroups, and K is a $\mathcal{J}$-class of T which is a subsemigroup of T, then there is a subsemigroup K' of S such that the restriction $\pi : K' \to K$ is an isomorphism.*

Proof: As the $\mathcal{J}$-class K is regular, there is a regular $\mathcal{J}$-class J of S such that $\pi(J) = K$ and such that every element of $\pi^{-1}(K)$ is a factor of the elements of J (cf. [15, Lemma 4.6.10]). Moreover, since K is a union of groups, J must be a union of groups, and each maximal subgroup of S contained in J is mapped by π onto a maximal subgroup of T contained in K.

Fix an idempotent e of K. Let X be the set of idempotents $\mathcal{R}$-equivalent to e, and let Y be the set of idempotents $\mathcal{L}$-equivalent to e. Choose an idempotent γ_e in J such that $\pi(\gamma_e) = e$. If $f \in X \setminus \{e\}$ then, since $f = ef$, we may choose an idempotent $\gamma_f \in J$ such that $\pi(\gamma_f) = f$ and $\gamma_f = \gamma_e\gamma_f$. Similarly, if $f \in Y \setminus \{e\}$ then we may choose an idempotent $\gamma_f \in J$ such that $\pi(\gamma_f) = f$ and $\gamma_f = \gamma_f\gamma_e$. Note that $X' = \{\gamma_f \mid f \in X\}$ is contained in the $\mathcal{R}$-class of γ_e, and that $Y' = \{\gamma_f \mid f \in Y\}$ is contained in the $\mathcal{L}$-class of γ_e.

For each idempotent $g \in X' \cup Y'$, consider the $\mathcal{H}$-class H_g of g. Let K' be the subsemigroup of T generated by $\bigcup_{g \in X' \cup Y'} H_g$. Then K' is a completely semigroup contained in J and such that $\pi(K') \subseteq K$. Moreover, each $\mathcal{H}$-class H of K contains an element of the form yx, with $y \in Y$ and $x \in X$, and so if H' is the $\mathcal{H}$-class of $\gamma_y\gamma_x \in K'$, then we have $\pi(H') = H$. Hence, we actually have $\pi(K') = K$.

Since K' has $|X| = |X'|$ $\mathcal{R}$-classes and $|Y| = |Y'|$ $\mathcal{L}$-classes, we know that K' has the same number of $\mathcal{H}$-classes as K. On the other hand, the restriction of an A-morphism to a regular $\mathcal{H}$-class is injective (cf. [15, Lemma 4.4.4]). We then conclude that π restricts to an isomorphism $K' \to K$. ■

Proof of Proposition 5.5: Let S be a profinite completely simple semigroup, and let $\varphi : S \to T$ is a continuous homomorphism onto a finite semigroup T. Then T is a completely simple semigroup, and, for every generating mapping $\psi : A \to T$ such that A is a finite alphabet, the homomorphism $\pi_\psi : T_\psi^{\mathrm{KR}} \to T$ is an A-morphism (cf. Remark 4.1). Applying Lemma 5.6, we deduce that there is a subsemigroup T' of T_ψ^{KR} such that the restriction $\pi_\psi|_{T'} : T' \to T$ is an isomorphism, whose inverse we denote ρ. Then the continuous homomorphism $\varphi_\psi : S \to T_\psi^{\mathrm{KR}}$ such that $\varphi_\psi = \rho \circ \varphi$ satisfies $\pi_\psi \circ \varphi_\psi = \varphi$. This establishes that S is a KR-cover. ∎

By Proposition 5.4, the class of finite projective semigroups includes examples of finite KR-covers that are not among those provided by Proposition 5.5: see [15, Lemma 4.1.39 and Exercise 4.1.43] for simple examples of such kind. For a complete characterization of the finite projective semigroups, see [21]. All finite projective semigroups are bands whose $\mathcal{J}$-classes form a chain (for a proof of this fact, alternative to [21] and implicitly using the equidivisibility of the free profinite semigroup, see [19]).

The following gives a necessary and sufficient condition for a finite semigroup to be a KR-cover.

Proposition 5.7. *A finite semigroup S is a KR-cover if and only if there are a finite set A, a generating mapping $\psi : A \to S$, and a homomorphism θ such that the following diagram commutes:*

$$\begin{tikzcd} & S \arrow[dl, dashed, "\theta"'] \arrow[d, "\mathrm{Id}"] \\ S_\psi^{\mathrm{KR}} \arrow[r, "\pi_\psi"'] & S. \end{tikzcd} \tag{5.3}$$

Moreover, then the same property holds for every generating mapping $\psi : A \to S$ with A finite.

Proof: The possibility of completing the diagram for every generating mapping $\psi : A \to S$ with A finite follows directly from the assumption that S is a finite semigroup and a KR-cover. Conversely, if the diagram can be completed for a generating mapping $\psi : A \to S$ with A finite then, for every onto homomorphism $\varphi : S \to T$, we may consider the following commutative diagram, where

θ is given by hypothesis and φ^{KR} by Proposition 4.2:

$$\begin{tikzcd}
 & S \arrow[dl, dashed, "\theta"'] \arrow[d, "\mathrm{Id}"] \\
S_{\psi}^{\mathrm{KR}} \arrow[r, "\pi_{\psi}"'] \arrow[d, "\varphi^{\mathrm{KR}}"'] & S \arrow[d, "\varphi"] \\
T_{\psi\circ\varphi}^{\mathrm{KR}} \arrow[r, "\pi_{\varphi\circ\psi}"] & T.
\end{tikzcd}$$

Hence, S is a KR-cover. ■

An immediate consequence of Proposition 5.7 is that it is decidable whether a finite semigroup is a KR-cover.

If S is a profinite semigroup, then the monoid S^I is also a profinite semigroup, where we endow S^I with the sum topology of S with $\{I\}$.

Proposition 5.8. *Let S be a profinite semigroup. If S is a KR-cover, then S^I is a KR-cover.*

Proof: Given a profinite semigroup S which is a KR-cover, let $\Phi : S^I \to R$ be a continuous homomorphism onto a finite semigroup R. Set $T = \Phi(S)$. Denote φ the restriction of Φ to S, which is an onto continuous homomorphism $S \to T$. We may take a generating mapping $\psi : A \to T$ such that A is finite. Since S is a KR-cover, there is a continuous homomorphism $\rho : S \to T_{\psi}^{\mathrm{KR}}$ such that $\varphi = \pi_{\psi} \circ \rho$.

Consider the alphabet $B = A \cup \{b\}$, for some letter b not in A. Denote Ψ the extension $B \to R$ of ψ such that

$$\Psi(b) = \Phi(I).$$

Then Ψ generates R. Since R_{Ψ}^{KR} is finite, we may take a positive integer n such that $x^n = x^{\omega}$ for every $x \in R_{\Psi}^{\mathrm{KR}}$. Consider the homomorphism $\alpha : A^+ \to B^+$ such that $\alpha(a) = b^n a b^n$ for every $a \in A$. By Proposition 4.2, there is a homomorphism $\Lambda : T_{\psi}^{\mathrm{KR}} \to R_{\Psi}^{\mathrm{KR}}$ such that Diagram (5.4) commutes.

$$\begin{tikzcd}
T_{\psi}^{\mathrm{KR}} \arrow[rrr, dashed, "\Lambda"] \arrow[d, "\pi_{\psi}"'] & & & R_{\Psi}^{\mathrm{KR}} \arrow[d, "\pi_{\Psi}"] \\
 & A^+ \arrow[ul, "\psi^{\mathrm{KR}}"] \arrow[r, "\alpha"] \arrow[dl, "\psi"] & B^+ \arrow[ur, "\Psi^{\mathrm{KR}}"'] \arrow[dr, "\Psi"] & \\
T \arrow[rrr, hook] & & & R
\end{tikzcd} \tag{5.4}$$

For every $a \in A^+$, we have $\Psi^{\mathrm{KR}} \circ \alpha(a) = \Psi^{\mathrm{KR}}(b)^\omega \cdot \Psi^{\mathrm{KR}}(a) \cdot \Psi^{\mathrm{KR}}(b)^\omega$ and so the image of $\Psi^{\mathrm{KR}} \circ \alpha$ is contained in the subsemigroup M of R_Ψ^{KR} defined by

$$M = \Psi^{\mathrm{KR}}(b)^\omega \cdot R_\Psi^{\mathrm{KR}} \cdot \Psi^{\mathrm{KR}}(b)^\omega.$$

Since $\Lambda \circ \psi^{\mathrm{KR}} = \Psi^{\mathrm{KR}} \circ \alpha$ and ψ^{KR} is onto, it follows that the image of Λ is contained in M. Let $\theta = \Lambda \circ \rho$. Note that M is a monoid, whose neutral element is $\Psi^{\mathrm{KR}}(b)^n$. Therefore, the homomorphism $\theta : S \to M$ extends to a monoid homomorphism $\widetilde{\theta} : S^I \to M$. The reader may wish to refer to Diagram (5.5).

$$\begin{tikzcd}
& S \arrow[r, hook] \arrow[dl, "\rho"'] \arrow[d, "\theta"] & S^I \arrow[dl, dashed, "\widetilde{\theta}"'] \arrow[d, "\Phi"] \\
T_\psi^{\mathrm{KR}} \arrow[r, "\Lambda"] & R_\Psi^{\mathrm{KR}} \arrow[r, "\pi_\Psi"] & R
\end{tikzcd} \tag{5.5}$$

If $s \in S$, then in view of the commutativity of Diagram (5.4), we have

$$\pi_\Psi \circ \widetilde{\theta}(s) = \pi_\Psi \circ \Lambda \circ \rho(s) = \pi_\psi \circ \rho(s) = \varphi(s) = \Phi(s).$$

On the other hand, we also have

$$\pi_\Psi \circ \widetilde{\theta}(I) = \pi_\Psi(\Psi^{\mathrm{KR}}(b)^n) = \Psi(b)^n = \Phi(I)^n = \Phi(I).$$

Therefore, the equality $\pi_\Psi \circ \widetilde{\theta} = \Phi$ holds. This shows that Diagram (5.5) is commutative and that S^I is a KR-cover. ∎

6. KR-covers are equidivisible

We highlight the following property of KR-covers, to be shown below.

Theorem 6.1. *Let S be a profinite semigroup. If S is a KR-cover, then it is equidivisible.*

Before the proof of Theorem 6.1, we formulate an improvement of the main theorem of [4]. Let CS be the pseudovariety of all finite completely simple semigroups, that is, of all finite semigroups with only one (nonempty) ideal.

Theorem 6.2. *The following conditions are equivalent for a pseudovariety V not contained in CS:*

(1) for every finite alphabet A, $\overline{\Omega}_A\mathsf{V}$ is equidivisible;
(2) for every alphabet A, $\overline{\Omega}_A\mathsf{V}$ is equidivisible;
(3) the equality $\mathsf{LI} ⓜ \mathsf{V} = \mathsf{V}$ holds;
(4) the pseudovariety V is closed under two-sided Karnofsky–Rhodes expansion.

Proof: The equivalence of (1) and (3) is the main result of [4], while the equivalence of (3) and (4) is given by Theorem 4.3. To finish the proof, just note that (4) implies (2) by Theorem 6.1 in view of Proposition 5.4, since free pro-V semigroups are V-projective. ■

As in [4], we say that a pseudovariety V is *equidivisible* if it satisfies Property (1) of Theorem 6.2.

The proof of Theorem 6.1 relies on the following lemma.

Lemma 6.3. *Let T be a finite semigroup, and let $\psi : A^+ \to T$ be an onto homomorphism, where A is a finite alphabet. Suppose that $u, v, x, y \in T_\psi^{\mathrm{KR}}$ are such that $uv = xy$. Then, there is $t \in T^I$ such that at least one of the following situations occurs:*

(1) $\pi_\psi(u)t = \pi_\psi(x)$ and $\pi_\psi(v) = t\pi_\psi(y)$;
(2) $\pi_\psi(u) = \pi_\psi(x)t$ and $t\pi_\psi(v) = \pi_\psi(y)$.

Proof: Along the proof, the reader may wish to refer to the following commutative diagram:

$$\begin{tikzcd} & A^+ \arrow[dl, dashed, "\psi^{\mathrm{KR}}"'] \arrow[d, "\psi"] \\ T_\psi^{\mathrm{KR}} \arrow[r, "\pi_\psi"'] & T. \end{tikzcd}$$

Let $\bar{u}, \bar{v}, \bar{x}, \bar{y} \in A^+$ be such that $\psi^{\mathrm{KR}}(\bar{u}) = u$, $\psi^{\mathrm{KR}}(\bar{v}) = v$, $\psi^{\mathrm{KR}}(\bar{x}) = x$ and $\psi^{\mathrm{KR}}(\bar{y}) = y$. The equality

$$\psi^{\mathrm{KR}}(\bar{u}\bar{v}) = \psi^{\mathrm{KR}}(\bar{x}\bar{y})$$

means that $\psi(\bar{u}\bar{v}) = \psi(\bar{x}\bar{y})$ and that, in the graph Γ_ψ, the coterminal paths $p_{\bar{u}\bar{v}}$ and $p_{\bar{x}\bar{y}}$ have the same transition edges. Since the pair $(\psi(\bar{u}), \psi(\bar{v}))$ is a vertex in the path $p_{\bar{u}\bar{v}}$ and $(\psi(\bar{x}), \psi(\bar{y}))$ is a vertex in the path $p_{\bar{x}\bar{y}}$, we know that at least one of the following situations occurs in the graph Γ_ψ:

- there is a (possibly empty) path from vertex $(\psi(\bar{u}), \psi(\bar{v}))$ to vertex $(\psi(\bar{x}), \psi(\bar{y}))$;
- there is a (possibly empty) path from vertex $(\psi(\bar{x}), \psi(\bar{y}))$ to vertex $(\psi(\bar{u}), \psi(\bar{v}))$.

In the first case, we have $\psi(\bar{u})t = \psi(\bar{x})$ and $\psi(\bar{v}) = t\psi(\bar{y})$, for some $t \in T^I$. It then suffices to note that, since $\pi_\psi \circ \psi^{\mathrm{KR}} = \psi$, we get $\pi_\psi(u)t = \pi_\psi(x)$ and $\pi_\psi(v) = t\pi_\psi(y)$. The second case is analogous. ■

We are now ready to establish Theorem 6.1.

Proof of Theorem 6.1: Let $\{\varphi_{j,i} : S_j \to S_i \mid i, j \in I,\, i \leqslant j\}$ be an inverse system of homomorphisms between finite semigroups such that S is its inverse limit. For each $i \in I$, the canonical projection $S \to S_i$ is denoted φ_i. Let $u, v, x, y \in S$ be such that $uv = xy$. Take $i \in I$. Since S is a KR-cover, there is a generating mapping $\theta : A \to S_i$, for some finite alphabet A, and a homomorphism $(\varphi_i)_\theta : S \to (S_i)_\theta^{\mathrm{KR}}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & S \arrow[dl, dashed, "(\varphi_i)_\theta"'] \arrow[d, "\varphi_i"] \\
(S_i)_\theta^{\mathrm{KR}} \arrow[r, "\pi_\theta"'] & S_i.
\end{tikzcd}$$

Then, we have $(\varphi_i)_\theta(uv) = (\varphi_i)_\theta(xy)$. By Lemma 6.3, there is $t_i \in S_i^I$ such that at least one of the following situations occurs:

(1) $\varphi_i(u)t_i = \varphi_i(x)$ and $\varphi_i(v) = t_i\varphi_i(y)$;
(2) $\varphi_i(u) = \varphi_i(x)t_i$ and $t_i\varphi_i(v) = \varphi_i(y)$.

Let I_1 (respectively, I_2) be the subset of elements i of I for which the first (respectively, second) situation occurs. Since $I = I_1 \cup I_2$, at least one of the sets I_1 or I_2 is cofinal. Without loss of generality, suppose that I_1 is cofinal (note that, since the conjuction of $i \in I_1$ and $k \leqslant i$ implies $k \in I_1$, we then actually have $I_1 = I$). By a standard compactness argument, we conclude that there is $t \in S^I$ such that $ut = x$ and $v = ty$. ∎

The following result shows that the converse of Theorem 6.1 fails.

Proposition 6.4. *Let $G^0 = G \uplus \{0\}$ be the semigroup obtained by adjoining a zero to a finite group G. Then G^0 is equidivisible, while G^0 is a KR-cover if and only if G is trivial.*

Proof: It is easy to check that G^0 satisfies the definition of equidivisible semigroup. If G is trivial, then G^0 is a two-element semilattice, which is well known to be projective, whence a KR-cover (cf. [15, Lemma 4.1.39]).

The remainder of the proof consists in showing that $S = G^0$ is not a KR-cover when G is a finite nontrivial group. For that purpose we let $\varphi : A \cup \{b\} \to S$ be a generating mapping, where $\varphi(A) \subseteq G$ (so that $\varphi(b) = 0$). In view of Remark 4.1, we know that $\pi_\varphi^{-1}(0)$ is a subsemigroup of S_φ^{KR} satisfying the identity $xyz = xz$. It follows that every element of $\pi_\varphi^{-1}(0)$ is of one of the forms

$$ubv/{\equiv_\varphi} \text{ or } ubvbw/{\equiv_\varphi}, \text{ where } u,v,w \in A^*. \tag{6.1}$$

Moreover, in both cases, the occurrences of b label transition edges in the corresponding paths from $(I,0)$ to $(0,I)$: for instance for the first occurrence of b, the corresponding edge is of the form $(g,0) \to (0,s)$ with $g \in G$ and, therefore, it must be a transition edge; the argument is similar for the last occurrence of b taking into account instead the second component of the vertices. Hence, the idempotents in $\pi_\varphi^{-1}(0)$ are the elements of the second form in (6.1).

Next, we show that $ubvbwt \equiv_\varphi ubvbw$ with $u,v,w \in A^*$ and $t \in A^+$ implies $\varphi(t)=1$. Indeed, as the paths p_{ubvbwt} and p_{ubvbw} use the same transition edges and the b's label such edges, comparing the second components of the end vertex of the edge corresponding to the second b, we conclude that $\varphi(wt)=\varphi(w)$. Hence, $\varphi(t)$ is equal to the identity element of the group G.

Suppose that there is a homomorphism $\theta : S \to S_\varphi^{\mathrm{KR}}$ completing Diagram (5.3). Let $g \in G$. As $\pi_\varphi(\theta(g)) = g \neq 0$, there is some $t \in A^+$ such that $\theta(g) = t/{\equiv_\varphi}$. Moreover, we have $\theta(0)\theta(g)=\theta(0)$. Since $\theta(0)$ is an idempotent in $\pi_\varphi^{-1}(0)$, we already know that it is of the form $ubvbw/{\equiv_\varphi}$ for some $u,v,w \in A^*$. It follows from the previous paragraph that $g = \pi_\varphi(\theta(g)) = \pi_\varphi(t/{\equiv_\varphi}) = \varphi(t)$ is the identity of G. This shows that G is trivial. ∎

It is routine to check that an inverse quotient limit of equidivisible compact semigroups is equidivisible. In the context of this paper, it is worthy to record the following similar fact.

Proposition 6.5. *An inverse quotient limit of KR-covers is a KR-cover.*

Proof: Let $S = \varprojlim_{i\in I} S_i$ be an inverse quotient limit of the KR-covers S_i. For each $i \in I$, let p_i be the canonical projection $S \to S_i$. Suppose that $\varphi : S \to T$ is a continuous homomorphism onto a finite semigroup T. Then there is $k \in I$ for which there is a factorization $\varphi = \varphi_k \circ p_k$ such that $\varphi_k : S_k \to T$ is a continuous onto homomorphism (see, for instance, [15, Lemma 3.1.37]). As S_k is a KR-cover, there is a finite alphabet A and a generating mapping $\psi : A \to T$ for which there is a continuous homomorphism $\rho : S_k \to T_\psi^{\mathrm{KR}}$ satisfying $\varphi_k = \pi_\psi \circ \rho$. Since the continuous homomorphism $\varphi_\psi = \rho \circ p_k$ satisfies $\pi_\psi \circ \varphi_\psi = \varphi$, we conclude that S is a KR-cover. The diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "p_k"'] \arrow[dr, "\varphi_\psi"' near end] & T \\ S_k \arrow[r, "\rho"] \arrow[ur, "\varphi_k" near start] & T_\psi^{\mathrm{KR}} \arrow[u, "\pi_\psi"'] \end{tikzcd}.$$

may help visualizing the various homomorphisms involved in this proof. ∎

To finish this section we observe that a profinite KR-cover may not be, up to isomorphism, an inverse limit of finite KR-covers. Indeed, KR-covers are equidivisible by Theorem 6.1. Now, by [10, Theorem 1.9] (which is attributed to Rees), elements of finite order of an equidivisible semigroup lie in groups; in particular, finite KR-covers are unions of groups and, therefore, so are their inverse limits. On the other hand, the semigroups $\overline{\Omega}_A\mathsf{V}$ of Theorem 6.2 (that is, with V closed under two-sided Karnofsky–Rhodes expansion) are KR-covers by Proposition 5.4 but they are never unions of groups since V contains LI.

7. Profinite coproducts of KR-covers

In combination with Theorem 6.1, the following property provides a way of producing new examples of profinite equidivisible semigroups.

Theorem 7.1. *For every pseudovariety of semigroups V closed under two-sided Karnofsky–Rhodes expansion, the class of all pro-V KR-covers is closed under V-coproducts.*

Proof: Let $(S_i)_{i\in I}$ be a family of pro-V KR-covers. Let S be their V-coproduct, with associated continuous homomorphisms $\varphi_i : S_i \to S$.

Consider a continuous homomorphism $\psi : S \to T$ onto a finite semigroup T and let $\psi_i = \psi \circ \varphi_i$. Let T_i be the image of ψ_i. Since S_i is a KR-cover, there are a finite set A_i, a generating mapping $\delta_i : A_i \to T_i$, depending on T_i only (not on i), and a continuous homomorphism β_i such that the following diagram commutes:

$$\begin{tikzcd} & S_i \arrow[r, "\varphi_i"] \arrow[dl, "\beta_i"'] \arrow[d, "\psi_i"] & S \arrow[d, "\psi"] \\ (T_i)_{\delta_i}^{\mathrm{KR}} \arrow[r, "\pi_{\delta_i}"] & T_i \arrow[r, hook] & T. \end{tikzcd} \tag{7.1}$$

We may assume that $T_i \neq T_j$ implies $A_i \cap A_j = \emptyset$, and we let $A = \bigcup_{i\in I} A_i$. Since T is finite, the set $\{T_i \mid i \in I\}$ is finite; moreover, its union is T since the union of the images of the φ_i generates a dense subsemigroup of S. The union $\delta = \bigcup_{i\in I} \delta_i$ is then a generating mapping $A \to T$ with finite domain.

By Proposition 4.2, there are homomorphisms $\eta_i : (T_i)^{\mathrm{KR}}_{\delta_i} \to T^{\mathrm{KR}}_{\delta}$, with η_i depending only on T_i, such that the lower rectangle of the following diagram commutes:

$$\begin{tikzcd}
 & S \arrow[d, dashed, "\beta"'] \arrow[dr, "\psi"] & \\
S_i \arrow[ur, "\varphi_i"] \arrow[dr, "\beta_i"'] & T^{\mathrm{KR}}_{\delta} \arrow[r, "\pi_\delta"] & T \\
 & (T_i)^{\mathrm{KR}}_{\delta_i} \arrow[u, "\eta_i"'] \arrow[r, "\pi_{\delta_i}"] & T_i. \arrow[u, hook]
\end{tikzcd} \tag{7.2}$$

Since T is a finite continuous homomorphic image of the pro-V semigroup S, we know that T belongs to V (again, see, for instance, [2, Proposition 3.7]). And since V is closed under two-sided Karnofsky–Rhodes expansion, T^{KR}_{δ} also belongs to V. Therefore, by the definition of V-coproduct, the homomorphisms $\eta_i \circ \beta_i$ $(i \in I)$ induce a unique continuous homomorphism $\beta : S \to T^{\mathrm{KR}}_{\delta}$ such that the left triangle in Diagram (7.2) commutes for each $i \in I$. Then, taking also into account the commutativity of Diagram (7.1), we deduce that for every $i \in I$ and every $s \in S_i$, the following chain of equalities holds:

$$\psi \circ \varphi_i(s) = \psi_i(s) = \pi_{\delta_i} \circ \beta_i(s) = \pi_\delta \circ \eta_i \circ \beta_i(s) = \pi_\delta \circ \beta \circ \varphi_i(s).$$

Since $\bigcup_{i\in I} \varphi_i(S_i)$ generates a dense subsemigroup of S, we conclude that $\psi = \pi_\delta \circ \beta$ (that is, Diagram (7.2) commutes). This completes the proof that S is a KR-cover. ■

For a set A, one may consider the A-indexed V-coproduct $\coprod^{\mathsf{V}}_{a\in A}\{1\}$ of trivial semigroups. Note that it is precisely the free object on A in the category of idempotent-generated pro-V semigroups. By Theorem 7.1, such semigroups are KR-covers whenever V is closed under two-sided Karnofsky–Rhodes expansion, whence they are equidivisible by Theorem 6.1.

8. Letter super-cancellative equidivisible profinite semigroups

In this section we completely characterize a class of equidivisible profinite semigroups, defined by a cancellation property (Definition 8.1), that was considered in [4, 5].

8.1. Letter super-cancellative semigroups. In the following definition, we adopt the terminology of [4].

Definition 8.1 (Letter super-cancellative semigroup)**.** Let S be a compact semigroup and suppose that S is generated, as a topological semigroup, by a finite subset A. Say that S is *letter super-cancellative* (with respect to A) when, for every $a, b \in A$ and $u, v \in S^I$, the following holds: if we have $ua = vb$ or $au = bv$, then we have $a = b$ and $u = v$.

Remark 8.2. As observed in [5, Lemma 6.1], if S is letter super-cancellative with respect to A and also with respect to B, then $A = B$. In [5], a letter super-cancellative semigroup is called *finitely cancellable*.

Example 8.3. By [4, Proposition 6.3], for an equidivisible pseudovariety V not contained in CS and a finite set A, $\overline{\Omega}_A\mathsf{V}$ is letter super-cancellative. In view of Theorem 6.2, this holds precisely when V is closed under two-sided Karnofsky–Rhodes expansion.

An *epigroup* is a semigroup S such that every element x of S has some power x^n lying in a subgroup of S, with n a positive integer. For example, finite semigroups and completely simple semigroups are epigroups. It is easy to see that no profinite epigroup is letter super-cancellative. The argument extends to the following proposition.

Proposition 8.4. *Let* V *be a pseudovariety containing* Sl*. If a nonempty family of nontrivial pro-*V *semigroups includes some epigroup and the* V*-coproduct of the family is finitely generated as a topological semigroup, then that* V*-coproduct is not letter super-cancellative.*

Proof: Consider a nonempty family $(S_i)_{i\in I}$ of nontrivial semigroups. Let $i_0 \in I$ be such that S_{i_0} is an epigroup. Let A be a finite generating subset of the profinite semigroup $S = \coprod_{i\in I}^{\mathsf{V}} S_i$. By Lemma 3.4 there is $a \in A \cap S_{i_0}$. Since S_{i_0} is an epigroup, there is a positive integer k such that $a^k = a^{\omega+k}$. Since $a^k \cdot I = a^k \cdot a^{\omega-k}$ but $I \neq a^{\omega-k}$, we conclude that S is not letter super-cancellative with respect to A. ∎

8.2. Strong KR-covers. The following somewhat subtly strengthened version of KR-cover is crucial in our main result of this section.

Definition 8.5 (Strong KR-cover)**.** Consider a profinite semigroup with a generating mapping $\kappa : A \to S$ such that A is finite. Let T be a continuous finite homomorphic image of S. We say that S is a *strong KR-cover of* T *with respect to* κ if, for every continuous onto homomorphism $\varphi : S \to T$, there is a continuous homomorphism $\varphi_\kappa : S \to T^{\mathrm{KR}}_{\varphi\circ\kappa}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[d, "(\varphi\circ\kappa)^{\mathrm{KR}}"'] & S \arrow[d, "\varphi"] \arrow[dl, dashed, "\varphi_\kappa"'] \\ T^{\mathrm{KR}}_{\varphi\circ\kappa} \arrow[r, "\pi_{\varphi\circ\kappa}"'] & T. \end{tikzcd} \tag{8.1}$$

The profinite semigroup S is a *strong KR-cover of* T if it is a strong KR-cover of T with respect to some such κ. Finally, S is a *strong KR-cover* if it is a strong KR-cover of each of its finite continuous homomorphic images.

Remark 8.6. Every strong KR-cover (of a finite semigroup T) is a KR-cover (of T).

The definition of strong KR-cover is motivated by the following link with the property of being letter super-cancellative.

Proposition 8.7. *Let S be a profinite semigroup with a generating mapping $\kappa : A \to S$, where A is a finite alphabet. If S is a strong KR-cover of the trivial semigroup with respect to κ, then S is letter super-cancellative with respect to $\kappa(A)$.*

Proof: Let $x, y \in S$ and $a, b \in A$ be such that $x \cdot \kappa(a) = y \cdot \kappa(b)$. We want to show that $x = y$ and that $\kappa(a) = \kappa(b)$. Since, by Theorem 6.1, the semigroup S is equidivisible, we know that there is $t \in S^I$ such that $xt = y$ and $\kappa(a) = t\kappa(b)$, or such that $x = ty$ and $\kappa(a)t = \kappa(b)$. Without loss generality, we assume that $xt = y$ and $\kappa(a) = t\kappa(b)$.

Arguing by contradiction, suppose $t \neq I$. Let $\varphi : S \to T$ be the continuous homomorphism from S onto the trivial semigroup T. As S is a strong KR-cover of T, there is a continuous onto homomorphism $\varphi_\kappa : S \to T^{\mathrm{KR}}_{\varphi\circ\kappa}$ such that Diagram 8.1 commutes. Because $t \neq I$, we know that $\varphi_\kappa(\kappa(a))$ belongs to $T^{\mathrm{KR}}_{\varphi\circ\kappa} \cdot \varphi_\kappa(\kappa(b))$. Therefore, and since $\varphi_\kappa \circ \kappa = (\varphi \circ \kappa)^{\mathrm{KR}}$, there are $c \in A$ and $u \in A^*$ such that $(\varphi \circ \kappa)^{\mathrm{KR}}(a) = (\varphi \circ \kappa)^{\mathrm{KR}}(cub)$. The latter equality means that $\varphi(\kappa(a)) = \varphi(\kappa(cub))$ and that the coterminal paths p_a and p_{cub} of the two-sided Cayley graph $\Gamma_{\varphi\circ\kappa}$ have the same transition edges. But p_a has length one, while p_{cub} has at least two distinct transition edges of $\Gamma_{\varphi\circ\kappa}$, namely the edges

$$(I, \varphi(\kappa(cub))) \xrightarrow{c} (\varphi(\kappa(c)), \varphi(\kappa(ub)))$$

and

$$(\varphi(\kappa(cu)), \varphi(\kappa(b))) \xrightarrow{b} (\varphi(\kappa(cub)), I).$$

We reached a contradiction, resulting from assuming that $t \neq I$. This shows that indeed we have $x = y$ and $\kappa(a) = \kappa(b)$.

Symmetrically, if $\kappa(a) \cdot x = \kappa(b) \cdot y$ holds, then $x = y$ and $\kappa(a) = \kappa(b)$. ∎

Remark 8.8. In view of Proposition 8.7 and Remark 8.2, up to the name of generators, there can be only one injective generating mapping $\kappa : A \to S$ with respect to which the profinite semigroup S is a strong KR-cover.

Note that no finite semigroup is a strong KR-cover: indeed, finite semigroups are epigroups and we already observed that epigroups are not letter super-cancellative, while strong KR-covers of the trivial semigroup are letter super-cancellative by Proposition 8.7. On the other hand, there are several examples of finite KR-covers (see Section 5).

More generally, for every pseudovariety of semigroups V containing SI and closed under two-sided Karnofsky–Rhodes expansion, if S is the V-coproduct of a nonempty finite family of finitely generated pro-V semigroups which are KR-covers, with at least one being an epigroup, then S is a KR-cover which is not a strong KR-cover, thanks to Theorem 7.1 and also Propositions 8.4 and 8.7.

Next is a complete characterization of the strong KR-covers.

Theorem 8.9. *Let S be a finitely generated profinite semigroup. The following conditions are equivalent:*

(1) S is equidivisible and letter super-cancellative;
(2) S is a strong KR-cover;
(3) S is a KR-cover, and S is a strong KR-cover of the trivial semigroup.

The proof of Theorem 8.9, given in this section, is inspired by the proof of [4, Theorem 8.3].

Note that, for an equidivisible pseudovariety V not contained in CS, Property (1) of Theorem 8.9 holds for $\overline{\Omega}_A\mathsf{V}$ whenever A is finite (cf. Example 8.3). Thus, strong KR-covers may be seen as a generalization of finitely generated free profinite semigroups over pseudovarieties closed under two-sided Karnofsky–Rhodes expansion.

We recall the following definition used in [4].

Definition 8.10 (Transition edge for a pseudoword)**.** Let A be a finite alphabet and $u \in \overline{\Omega}_A\mathsf{S}$. Consider a continuous homomorphism φ from $\overline{\Omega}_A\mathsf{S}$ onto a finite semigroup. Suppose that $(u_n)_n$ is a sequence of elements of A^+ converging to u. A *transition edge* for u in Γ_φ is an edge of Γ_φ which is a transition edge for u_n in Γ_φ for all sufficiently large n.

Moreover, a sequence of edges of Γ_φ is said to be a *sequence of transition edges for* u in Γ_φ if it is the sequence of all transition edges of u_n for all sufficiently large n, where $(u_n)_n$ is a sequence of elements of A^+ such that $u_n \to u$.

Since $\varphi^{\mathrm{KR}}(u_n) \to \varphi^{\mathrm{KR}}(u)$, the property of being a transition edge (or of being a sequence of transition edges) for u in Γ_φ does not depend on the choice of the sequence u_n, it only depends on φ and u.

For proving Theorem 8.9, we need the following property, contained in [4, Lemma 8.1].

Lemma 8.11. *Let φ be a continuous homomorphism from $\overline{\Omega}_A\mathsf{S}$ onto a finite semigroup, where A is a finite alphabet. Let $u \in \overline{\Omega}_A\mathsf{S}$. If $((s_1,t_1),a,(s_2,t_2))$ is a transition edge for u in Γ_φ, then there is a factorization $u = u_1au_2$ of u, with $u_1,u_2 \in (\overline{\Omega}_A\mathsf{S})^I$, such that $\varphi(u_1) = s_1$ and $\varphi(u_2) = t_2$.*

Proof of Theorem 8.9: The implication (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (1) follows from Theorem 6.1 and Proposition 8.7. It remains to show (1)$\Rightarrow$ (2). Suppose that S is letter super-cancellative with respect to the finite set A. Denote by κ the continuous onto homomorphism $\overline{\Omega}_A\mathsf{S} \to S$ extending the inclusion of A in S.

Let $\varphi : S \to T$ be a continuous homomorphism onto a finite semigroup. Let $\Phi = \varphi \circ \kappa$. We then have the following diagram, where the outer square commutes.

$$\begin{tikzcd}
\overline{\Omega}_A\mathsf{S} \arrow[r, "\kappa"] \arrow[d, "\Phi^{\mathrm{KR}}"'] & S \arrow[d, "\varphi"] \arrow[dl, dashed, "\varphi_\kappa"'] \\
T_\Phi^{\mathrm{KR}} \arrow[r, "\pi_\Phi"'] & T
\end{tikzcd} \tag{8.2}$$

Our aim is to show that there exists a continuous homomorphism φ_κ such that the whole diagram commutes. For that purpose, take $u, v \in \overline{\Omega}_A\mathsf{S}$ such that $\kappa(u) = \kappa(v)$. We claim that $\Phi^{\mathrm{KR}}(u) = \Phi^{\mathrm{KR}}(v)$. Let $(\varepsilon_i)_{i\in\{1,\dots,n\}}$ and $(\delta_i)_{i\in\{1,\dots,m\}}$ be the sequences of transition edges in Γ_Φ respectively for u and for v. Without loss of generality, we may assume that $n \leqslant m$. For an edge

ε of the graph Γ_Φ, $\alpha(\varepsilon)$ and $\omega(\varepsilon)$ denote the beginning and end vertices of ε, respectively.

Suppose that the set

$$\{i \in \{1,\dots,n\} \mid \varepsilon_i \neq \delta_i\} \tag{8.3}$$

is nonempty, and let j be its minimum. By Lemma 8.11, there are factorizations

$$u = u_1 a u_2 \quad \text{and} \quad v = v_1 b v_2$$

with $a, b \in A$ and $u_1, u_2, v_1, v_2 \in (\overline{\Omega}_A \mathsf{S})^I$, such that

$$\varepsilon_j = ((\Phi(u_1), \Phi(au_2)), a, (\Phi(u_1 a), \Phi(u_2))$$

and

$$\delta_j = ((\Phi(v_1), \Phi(bv_2)), b, (\Phi(v_1 b), \Phi(v_2)).$$

Note that $\alpha(\varepsilon_j)$ and $\alpha(\delta_j)$ belong to the same strongly connected component of Γ_Φ, by the minimality of the index j.

If $u_1 = I$, then $j = 1$, which in turn implies that $v_1 = I$. We then have $\kappa(au_2) = \kappa(u) = \kappa(v) = \kappa(bv_2)$. Because S is finitely cancellable with respect to A, we deduce that $a = b$ and $\kappa(u_2) = \kappa(v_2)$, and so we get that $\varepsilon_j = \delta_j$, a contradiction. Hence we have $u_1 \neq I$, and, analogously, $v_1 \neq I$.

Similarly, if $u_2 = v_2 = I$, then $j = n = m$ and $\varepsilon_j = \delta_j$, a contradiction.

Suppose that $u_2 = I$ and $v_2 \neq I$. We then have $\kappa(u_1 a) = \kappa(v_1 b v_2)$. Since S is letter super-cancellative with respect to A, it follows that there is a pseudoword $v_2' \in (\overline{\Omega}_A \mathsf{S})^I$ such that $v_2 = v_2' a$ and $\kappa(u_1) = \kappa(v_1 b v_2')$. This implies the existence of a path in Γ_Φ from $\omega(\delta_j) = (\Phi(v_1 b), \Phi(v_2))$ to $\alpha(\varepsilon_j) = (\Phi(v_1 b v_2'), \Phi(a))$ labeled by a word v_2'' of A^* such that $\Phi(v_2'') = \Phi(v_2')$. As $\alpha(\varepsilon_j)$ and $\alpha(\delta_j)$ belong to the same strongly connected component of Γ_Φ, we deduce that there is in Γ_Φ a path from $\omega(\delta_j)$ to $\alpha(\delta_j)$, contradicting the fact that δ_j is a transition edge of Γ_Φ. Therefore, we must have $u_2 \neq I$.

Since S is equidivisible, and we have $\kappa(u_1 a \cdot u_2) = \kappa(v_1 \cdot b v_2)$ with none of the elements $\kappa(u_1 a)$, $\kappa(u_2)$, $\kappa(v_1)$, $\kappa(b v_2)$ being equal to I, we know that there is $t \in (\overline{\Omega}_A \mathsf{S})^I$ such that

$$\kappa(u_1 a t) = \kappa(v_1) \quad \text{and} \quad \kappa(u_2) = \kappa(t b v_2), \tag{8.4}$$

or

$$\kappa(v_1 t) = \kappa(u_1 a) \quad \text{and} \quad \kappa(b v_2) = \kappa(t u_2). \tag{8.5}$$

If Case (8.4) holds, then there is in Γ_Φ a (possibly empty) path from $\omega(\varepsilon_j)$ to $\alpha(\delta_j)$, labeled by a word $t_0 \in A^*$ such that $\Phi(t_0) = \Phi(t)$. But, since $\alpha(\varepsilon_j)$ and

$\alpha(\delta_j)$ are in the same strongly connected component, we reach a contradiction with the hypothesis that ε_j is a transition edge of Γ_Φ.

Therefore, Case (8.5) holds with $t \neq I$. Since S is letter super-cancellative with respect to A, there is $t' \in (\overline{\Omega}_A\mathsf{S})^I$ with $t = t'a$ and

$$\kappa(v_1t') = \kappa(u_1) \quad \text{and} \quad \kappa(bv_2) = \kappa(t'au_2). \tag{8.6}$$

Suppose that $t' \neq I$. Again because S is letter super-cancellative with respect to A, it follows from (8.6) that there is $t'' \in (\overline{\Omega}_A\mathsf{S})^I$ with $t' = bt''$ and

$$\kappa(v_1b \cdot t'') = \kappa(u_1) \quad \text{and} \quad \kappa(v_2) = \kappa(t'' \cdot au_2). \tag{8.7}$$

This implies the existence of a path in Γ_Φ from $\omega(\delta_j)$ to $\alpha(\varepsilon_j)$, which once more contradicts the definition of a transition edge.

Therefore, we have $t' = I$, and so, once again because S is letter super-cancellative with respect to A, from (8.6) we get $\kappa(v_1) = \kappa(u_1)$, $a = b$ and $\kappa(v_2) = \kappa(u_2)$. This yields $\varepsilon_j = \delta_j$, which contradicts the initial assumption. Therefore, the set (8.3) is empty. In particular, $\varepsilon_n = \delta_n$ holds. Since ε_n is the last transition edge for u, we have $\omega(\delta_n) = (\Phi(u), I)$, which means that δ_n is the last transition edge for v, whence $m = n$ and $\varepsilon_i = \delta_i$ for every $i \in \{1, \dots, n\}$.

We have therefore established the claim that $\Phi^{\mathrm{KR}}(u) = \Phi^{\mathrm{KR}}(v)$ holds whenever $\kappa(u) = \kappa(v)$, and so there is a unique continuous homomorphism $\varphi_\kappa : S \to T_\Phi^{\mathrm{KR}}$ such that Diagram 8.2 commutes, thus showing that S is a strong KR-cover. ■

8.3. A strong KR-cover which is not relatively free. Let S be a finite semigroup and let $\varphi : A \to S$ be a generating mapping, where A is a finite alphabet. We define the onto homomorphism $\varphi^{\mathrm{KR}^n} : A^+ \to S_\varphi^{\mathrm{KR}^n}$ recursively by

$$\varphi^{\mathrm{KR}^0} = \varphi \quad \text{and} \quad \varphi^{\mathrm{KR}^{n+1}} = (\varphi^{\mathrm{KR}^n})^{\mathrm{KR}} \quad (n \geqslant 0).$$

For $m \geqslant n$, let $\varrho_{m,n}$ be the unique (onto) homomorphism $S_\varphi^{\mathrm{KR}^m} \to S_\varphi^{\mathrm{KR}^n}$ such that the following diagram commutes:

$$\begin{tikzcd} & A^+ \arrow[dl, "\varphi^{\mathrm{KR}^n}"'] \arrow[dr, "\varphi^{\mathrm{KR}^m}"] & \\ S_\varphi^{\mathrm{KR}^n} & & S_\varphi^{\mathrm{KR}^m}. \arrow[ll, "\varrho_{m,n}"] \end{tikzcd}$$

The family of homomorphisms $\{\varrho_{m,n} \mid m, n \in \mathbb{N}, m \geqslant n\}$ defines an inverse system of A-generated semigroups. Consider its inverse limit, the profinite semigroup $S_\varphi^{\mathrm{KR}^\omega} = \varprojlim S_\varphi^{\mathrm{KR}^n}$, with generating mapping $\varphi^{\mathrm{KR}^\omega} : A \to S_\varphi^{\mathrm{KR}^\omega}$. The associated projection $S_\varphi^{\mathrm{KR}^\omega} \to S_\varphi^{\mathrm{KR}^n}$, denoted ϱ_n, is an onto continuous homomorphism, since the connecting homomorphisms defining the inverse limit are onto (see, for instance, [15, Lemma 3.1.26]).

Using results that can be found in [15] (namely, Corollary 5.3.22 and Theorem 3.6.4), one may show that for an arbitrary pseudovariety V, the following equality holds, where two-sided Karnofsky–Rhodes expansion needs to be extended to profinite semigroups (as in [14]) when V is not locally finite:

$$(\overline{\Omega}_A \mathsf{V})^{\mathrm{KR}^\omega} = \overline{\Omega}_A(\mathsf{LI} ⓜ \mathsf{V}).$$

Proposition 8.12. *Let S be a finite semigroup generated by $\varphi : A \to S$, where A is a finite alphabet. The profinite semigroup $S_\varphi^{\mathrm{KR}^\omega}$ is a strong KR-cover with respect to the generating mapping $\varphi^{\mathrm{KR}^\omega} : A \to S_\varphi^{\mathrm{KR}^\omega}$.*

Proof: Let $\kappa = \varphi^{\mathrm{KR}^\omega}$ and ψ be a continuous homomorphism from $S_\varphi^{\mathrm{KR}^\omega}$ onto a finite semigroup T. Choose an integer $n \geqslant 1$ for which ψ has a factorization $\psi = \psi_n \circ \varrho_n$ for some homomorphism $\psi_n : S_\varphi^{\mathrm{KR}^n} \to T$ (its existence being guaranteed, for instance, by [15, Lemma 3.1.37]). Observe that the non-dashed part of Diagram (8.8) is commutative.

$$\begin{tikzcd}
S_\varphi^{\mathrm{KR}^n} \arrow[dd, "\psi_n"'] & & & S_\varphi^{\mathrm{KR}^{n+1}} \arrow[lll, "\varrho_{n+1,n}"'] \arrow[dd, dashed, "(\psi_n)^{\mathrm{KR}}"] \\
& S_\varphi^{\mathrm{KR}^\omega} \arrow[ul, "\varrho_n"'] \arrow[urr, "\varrho_{n+1}"] \arrow[dl, "\psi"] & A^+ \arrow[l, "\kappa"] \arrow[ur, "\varphi^{\mathrm{KR}^{n+1}}"] \arrow[dr, "(\psi\circ\kappa)^{\mathrm{KR}}"'] & \\
T & & & T_{\psi\circ\kappa}^{\mathrm{KR}} \arrow[lll, "\pi_{\psi\circ\kappa}"]
\end{tikzcd} \tag{8.8}$$

Applying Proposition 4.2, we see that there is a unique homomorphism $(\psi_n)^{\mathrm{KR}} : S_\varphi^{\mathrm{KR}^{n+1}} \to T_{\psi\circ\kappa}^{\mathrm{KR}}$ for which Diagram (8.9) commutes. Since the topmost triangle in Diagram (8.9) is the rightmost triangle in Diagram (8.8), we deduce that the latter is commutative.

$$
\begin{tikzcd}
S_{\varphi}^{\mathrm{KR}^{n+1}} \arrow[rr, dashed, "(\psi_n)^{\mathrm{KR}}"] \arrow[dd, "\varrho_{n+1,n}"'] & & T_{\psi\circ\kappa}^{\mathrm{KR}} \arrow[dd, "\pi_{\psi\circ\kappa}"] \\
& A^{+} \arrow[ul, "\varphi^{\mathrm{KR}^{n+1}}"] \arrow[ur, "(\psi\circ\kappa)^{\mathrm{KR}}"'] \arrow[dl, "\varphi^{\mathrm{KR}^{n}}"'] \arrow[dr, "\psi\circ\kappa"] & \\
S_{\varphi}^{\mathrm{KR}^{n}} \arrow[rr, "\psi_n"] & & T
\end{tikzcd}
\tag{8.9}
$$

Note that $(\psi_n)^{\mathrm{KR}}$ is onto, as $(\psi \circ \kappa)^{\mathrm{KR}}$ is onto. Let ψ_κ be the onto continuous homomorphism $(\psi_n)^{\mathrm{KR}} \circ \varrho_{n+1}$. The commutativity of Diagram (8.8) entails in particular that

$$(\psi \circ \kappa)^{\mathrm{KR}} = (\psi_n)^{\mathrm{KR}} \circ \varrho_{n+1} \circ \kappa = \psi_\kappa \circ \kappa$$

and so Diagram (8.10) commutes.

$$
\begin{tikzcd}
A^{+} \arrow[r, "\kappa"] \arrow[d, "(\psi\circ\kappa)^{\mathrm{KR}}"'] & S_{\varphi}^{\mathrm{KR}^{\omega}} \arrow[d, "\psi"] \arrow[dl, dashed, "\psi_\kappa"'] \\
T_{\psi\circ\kappa}^{\mathrm{KR}} \arrow[r, "\pi_{\psi\circ\kappa}"] & T
\end{tikzcd}
\tag{8.10}
$$

This establishes that $S_{\varphi}^{\mathrm{KR}^{\omega}}$ is a strong KR-cover. ■

Example 8.13. Let $S = \{a, b\}$ be the 2-element semilattice, where b is the minimum element. For the alphabet $A = \{a, b\}$, the description of S determines an onto homomorphism $\varphi : A^+ \to S$. By Proposition 8.12, the profinite semigroup $T = S_{\varphi}^{\mathrm{KR}^{\omega}}$ is a strong KR-cover with respect to the natural generating mapping $\varphi^{\mathrm{KR}^{\omega}} : \{a, b\} \to T$. Since $\varphi^{\mathrm{KR}^{\omega}}(a) \neq \varphi^{\mathrm{KR}^{\omega}}(b)$, we see $\{a, b\}$ as a subset of T.

In [5], some properties of the equidivisible profinite semigroups which are letter supper-cancellative were studied. But the question of whether such semigroups may not be relatively free was left open (cf. [5, Section 9]). Example 8.13 provides an answer to that question, as shown next.

Proposition 8.14. *The semigroup T from Example* (8.13) *is an equidivisible profinite semigroup which is letter super-cancellative but not a relatively free profinite semigroup.*

Proof: We already observed that T is a strong KR-cover. It then follows from Theorem 8.9 that T is equidivisible and letter super-cancellative. We proceed to show that T is not a relatively free profinite semigroup.

We claim that the idempotent $\varphi^{\mathrm{KR}^\omega}(b^\omega) \in T$ belongs to the minimum ideal of T. To avoid overloaded notation, denote $\varphi^{\mathrm{KR}^n}(v)$ by $[v]_n$. We establish the claim by proving that the idempotent $[b^\omega]_n$ belongs to the minimum ideal of $T_n = S_\varphi^{\mathrm{KR}^n}$, for every $n \geqslant 0$. We do this by showing inductively on n that

$$[b^\omega w b^\omega]_n = [b^\omega]_n \quad \text{for every } w \in A^*. \tag{8.11}$$

The base case $n = 0$ is immediate, as $\varphi^{\mathrm{KR}^0} = \varphi$. Assume that (8.11) holds for a certain value of $n \geqslant 0$. Let k be an integer such that $[b^\omega]_{n+1} = [b^k]_{n+1}$. Note that, since $\varphi^{\mathrm{KR}^n} = \varrho_{n+1,n} \circ \varphi^{\mathrm{KR}^{n+1}}$, we also have $[b^\omega]_n = [b^k]_n$. In the graph $\Gamma = \Gamma_{\varphi^{\mathrm{KR}^n}}$, the path $p_{b^{2k}}$ is the concatenation of the path q from $(I, [b^k]_n)$ to $([b^k]_n, [b^k]_n)$ labeled by b^k, and the path q' from the latter vertex to $([b^k]_n, I)$, which is also labeled by b^k. On the other hand, by the induction hypothesis, we know that, for every $w \in A^*$, the path $p_{b^{2k}wb^{2k}}$ of Γ decomposes as qrq' where r is the circuit at the vertex $([b^k]_n, [b^k]_n)$ labeled by $b^k w b^k$ (see Figure 2). In particular, the two coterminal paths $p_{b^{2k}}$ and $p_{b^{2k}wb^{2k}}$ use the same transition edges of the graph Γ, and so the equality $[b^{2k}]_{n+1} = [b^{2k} w b^{2k}]_{n+1}$ holds. That is, we have (8.11) for $n+1$ in the place of n. This concludes the inductive step of the proof, and shows that b^ω belongs to the minimum ideal of T.

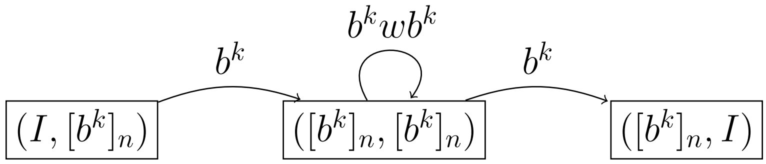

FIGURE 2. The path $p_{b^{2k}wb^{2k}}$ of the graph $\Gamma_{\varphi^{\mathrm{KR}^n}}$.

Arguing by contradiction, suppose that T is a relatively free profinite semigroup $\overline{\Omega}_X\mathsf{V}$, for some semigroup pseudovariety V and alphabet X. As the semilattice $S = \{a, b\}$ is a continuous homomorphic image of T, the set X has at least two elements and V contains the pseudovariety Sl (of all finite semilattices). Let π be the unique continuous homomorphism $\overline{\Omega}_X\mathsf{V} \to \overline{\Omega}_X\mathsf{Sl}$ mapping each generator to itself. Then $\pi(b) = \pi(b^\omega)$ belongs to the minimum ideal K of $\overline{\Omega}_X\mathsf{Sl}$ by the claim, and since $\{a, b\}$ generates $T = \overline{\Omega}_X\mathsf{V}$, we conclude that

$\overline{\Omega}_X\mathsf{SI} \setminus K \subseteq \pi(a^*) = \{\pi(a)\}$. But since X has at least two elements, which belong to the set $\overline{\Omega}_X\mathsf{SI} \setminus K$, we reach a contradiction. To avoid the contradiction, the only possibility is that T is not a relatively free profinite semigroup. ∎

9. Profinite coproducts of letter super-cancellative equidivisible profinite semigroups

The proof of the following proposition is an adaptation of the proof of Theorem 7.1.

Proposition 9.1. *If* V *is a semigroup pseudovariety containing* LI*, then the class of all pro-*V *strong KR-covers of the trivial semigroup is closed under finite* V*-coproducts.*

Proof: We start by observing that, by Remark 4.1, V contains all two-sided Karnofsky–Rhodes expansions of the trivial semigroup.

Let $(S_i)_{i\in I}$ be a finite family of pro-V strong KR-covers of the trivial semigroup T. Let S be their V-coproduct, with associated continuous homomorphisms $\varphi_i : S_i \to S$. Consider the unique mapping $\psi : S \to T$ and let $\psi_i = \psi \circ \varphi_i$. Since S_i is a strong KR-cover of T, we know that there are a finite set A_i, a generating mapping $\kappa_i : A_i \to S_i$, and a continuous homomorphism β_i such that the diagram

$$\begin{tikzcd}
A_i^+ \arrow[r, "\kappa_i"] \arrow[d, "(\psi_i\circ\kappa_i)^{\mathrm{KR}}"'] & S_i \arrow[d, "\psi_i"] \arrow[dl, dashed, "\beta_i"'] \\
T^{\mathrm{KR}}_{\psi_i\circ\kappa_i} \arrow[r, "\pi_{\psi_i\circ\kappa_i}"'] & T
\end{tikzcd}$$

commutes. We may assume that the sets A_i are pairwise disjoint. Let $A = \bigcup_{i\in I} A_i$. The union $\kappa = \bigcup_{i\in I} \kappa_i$ is then a generating mapping $A \to S$ with finite domain.

Applying Proposition 4.2, we see that for each $i \in I$ there is a homomorphism $\eta_i : T^{\mathrm{KR}}_{\psi_i \circ \kappa_i} \to T^{\mathrm{KR}}_{\psi \circ \kappa}$ such that Diagram (9.1) commutes.

$$
\begin{tikzcd}
T^{\mathrm{KR}}_{\psi_i \circ \kappa_i} \arrow[rrr, dashed, "\eta_i"] \arrow[dd, "\pi_{\psi_i \circ \kappa_i}"'] & & & T^{\mathrm{KR}}_{\psi \circ \kappa} \arrow[dd, "\pi_{\psi \circ \kappa}"] \\
& A_i^+ \arrow[ul, "{\psi_i \circ \kappa_i}^{\mathrm{KR}}"'] \arrow[dl, "\psi_i \circ \kappa_i"] \arrow[r, hook] & A^+ \arrow[ur, "(\psi \circ \kappa)^{\mathrm{KR}}"] \arrow[dr, "\psi \circ \kappa"'] & \\
T \arrow[rrr, equal, "\mathrm{Id}_T"] & & & T
\end{tikzcd}
\tag{9.1}
$$

Therefore, the non-dashed part of Diagram (9.2) commutes.

$$
\begin{tikzcd}
A_i^+ \arrow[rrr, hook] \arrow[dr, "\kappa_i"] \arrow[ddd, "(\psi_i \circ \kappa_i)^{\mathrm{KR}}"'] & & & A^+ \arrow[dl, "\kappa"'] \arrow[ddd, "(\psi \circ \kappa)^{\mathrm{KR}}"] \\
& S_i \arrow[r, "\varphi_i"] \arrow[d, "\psi_i"] \arrow[ddl, "\beta_i"'] & S \arrow[dl, "\psi"] \arrow[ddr, dashed, "\beta"] & \\
& T & & \\
T^{\mathrm{KR}}_{\psi_i \circ \kappa_i} \arrow[ur, "\pi_{\psi_i \circ \kappa_i}"'] \arrow[rrr, "\eta_i"] & & & T^{\mathrm{KR}}_{\psi \circ \kappa} \arrow[ull, "\pi_{\psi \circ \kappa}"']
\end{tikzcd}
\tag{9.2}
$$

From the observation at the beginning of the proof, we know that $T^{\mathrm{KR}}_{\psi \circ \kappa}$ belongs to V. Therefore, by the definition of V-coproduct, there is a unique continuous homomorphism $\beta : S \to T^{\mathrm{KR}}_{\psi \circ \kappa}$ such that $\beta \circ \varphi_i = \eta_i \circ \beta_i$ for each $i \in I$, thus completing Diagram (9.2), which we next show to be commutative. Indeed, using the already established commutativity of the non-dashed part of Diagram (9.2), we see that for every $i \in I$ and every $a \in A_i$, the following chain of equalities holds:

$$\beta \circ \kappa(a) = \beta \circ \varphi_i \circ \kappa_i(a) = \eta_i \circ \beta_i \circ \kappa_i(a) = \eta_i \circ (\psi_i \circ \kappa_i)^{\mathrm{KR}}(a) = (\psi \circ \kappa)^{\mathrm{KR}}(a).$$

This shows that $\beta \circ \kappa = (\psi \circ \kappa)^{\mathrm{KR}}$. In particular, we conclude that Diagram (9.3)

$$
\begin{tikzcd}
A^+ \arrow[r, "\kappa"] \arrow[d, "(\psi \circ \kappa)^{\mathrm{KR}}"'] & S \arrow[d, "\psi"] \arrow[dl, dashed, "\beta"'] \\
T^{\mathrm{KR}}_{\psi \circ \kappa} \arrow[r, "\pi_{\psi \circ \kappa}"'] & T
\end{tikzcd}
\tag{9.3}
$$

commutes, thereby completing the proof that S is a strong KR-cover of the trivial semigroup T. ■

We are ready to deduce the following theorem.

Theorem 9.2. *For every pseudovariety of semigroups* V *closed under two-sided Karnofsky–Rhodes expansion, the class of letter super-cancellative equidivisible finitely generated pro-*V *semigroups is closed under finite* V*-coproducts.*

Proof: Let $\mathcal{C}$ be the class of all pro-V KR-covers, and let $\mathcal{D}$ be the class of all pro-V strong KR-covers of the trivial semigroup. By Theorem 4.3, V contains LI.* Both $\mathcal{C}$ and $\mathcal{D}$ are closed under taking finite V-coproducts, by Theorem 7.1 and Proposition 9.1, respectively. By Theorem 8.9, the class of letter super-cancellative equidivisible pro-V semigroups is the intersection $\mathcal{C} \cap \mathcal{D}$. Combining these observations, we immediately obtain the theorem. ■

A natural question arising from Theorem 9.2 is the following problem, which we leave open.

Problem 9.3. *Suppose that* V *is a pseudovariety of semigroups closed under two-sided Karnofsky–Rhodes expansion. Is it true that the class of all equidivisible pro-*V *semigroups is closed under* V*-coproducts? A perhaps simpler question is the following: is the class of all finitely generated equidivisible pro-*V *semigroups closed under finite* V*-coproducts?*

It would also be interesting to have a complete characterization of the KR-covers, in the spirit of the characterization of the strong KR-covers, established in Theorem 8.9.

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*In fact, we only need that V contains LI to invoke Proposition 9.1, where the proof starts by observing that it follows that V contains all two-sided Karnofsky–Rhodes expansions of the trivial semigroup. So, one could avoid applying Theorem 4.3 here. We preferred to keep the statement of Proposition 9.1 the simplest possible.

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