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SOME PROPERTIES OF CONJUNCTIVITY (SUBFITNESS) IN GENERALIZED SETTINGS

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Abstract: The property of *subfitness* used in point-free topology (roughly speaking) to replace the slightly stronger T_1-separation, appeared (as *disjunctivity*) already in the pioneering Wallman's [16], then practically disappeared to reappear again (*conjunctivity, subfitness*), until it was in the recent decades recognized as an utmost important condition playing a very special role. Recently, it was also observed that this property (or its dual) appeared independently in general poset setting (e.g. as *separativity* in connection with forcing). In a recent paper [2], Delzell, Ighedo and Madden discussed it in the context of semilattices. In this article we discuss it on the background of the systems of meet-sets (subsets closed under existing infima) in posets of various generality (semilattices, lattices, distributive lattices, complete lattices) and present parallels of some localic (frame) facts, including a generalized variant of fitness.

Keywords: Separative poset, conjunctive poset, meet-set, semilattice, frame, locale, subfitness, fitness.

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Introduction

In his pioneering article [16] Wallman, using lattice theoretic techniques to prove topological results, needed a point-free replacement for the T_1-separation axiom. His disjunctivity axiom did the job excellently. In its dual form, as *conjunctivity*[1], it can be formulated as follows

if $a \not\leq b$ *then there is a* c *such that* $a \vee c = 1 \neq b \vee c$.

During the early development of point-free topology it was sort of remembered ([14, 15, 8]), but not really paid much attention. Then, in another pioneering article, Isbell's [4], it reappeared as *subfitness*: in this, equivalent,

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[1] Wallman worked with the lattice of *closed* sets; later it turned out that it is more natural to represent spaces modelling the behavior of the lattice of *open* sets.

condition one required that each open subobject was a join of closed subobjects; this property came together with *fitness* (where each closed subobject was an intersection of open ones). Since its categorical properties were not satisfactory (while those of fitness were) it was neglected again, until it turned out in the recent decades that it was in fact a very important property with lots of consequences (see, e.g., [10, 12, 1, 3, 13]).

From poset perspective, frames, in the context of which all this has happened, are very special order structures. But conjunctivity appeared in quite general partial orders (slightly hidden, and in a dual form again) in connection with forcing as *separativity* ([7]): a poset$(X, \leq)$ is *separative* if $x \not\leq y$ implies there is some $z \leq x$ that is incompatible with y; equivalently,

> *if $a \leq b$ then there is a $c \leq a$ such that $\{b, c\}$ has no lower bound.*

So one can translate conjunctivity in frames L (or more generally in bounded join-semilattices) to the condition that $L^{\mathrm{op}} \smallsetminus \{1\}$ is separative.

Thus, conjunctivity (or, subfitness) is a property the importance of which reaches far beyond the realm of frame (locale) theory. In the recent article [2], Delzell, Ighedo and Madden present a thorough study of this concept in the context of join semilattices. The article also indicates the importance of the study of this and related conditions relaxing the frame structure. Besides treating quite general questions, it points out the interest of discussing the particular situation in the Joyal-Tierney category of $\bigvee$-lattices and the role of meet-closed subsets as subobjects (see [5]): here, the generalization does not seem to be very radical, but the absence of the Heyting structure is in fact quite essential.

The background of our study is the system $\mathcal{M}(L)$ of meet-sets ($\equiv$ subsets closed under existing infima) in a poset L which in various contexts is treated in various generality (semilattice, lattice or complete lattice, distributive lattice). $\mathcal{M}(L)$ is easily seen to be a complete lattice. Particularly important meet-sets are the $\uparrow a = \{x \mid x \geq a\}$ extending the role of the closed sublocales in frames. Already in the quite general context they associate subfitness (conjunctivity) with (co)density: a nice subobject that meets each non-trivial $\uparrow a$ is the whole of L.

The closed meet-set $\uparrow a$ has in $\mathcal{M}(L)$ a complement that is also a pseudocomplement. This complement is not very interesting, but if our poset is a (semi)lattice, one has a natural extension of the concepts of open and semiopen sublocales, the open resp. semiopen meet-sets $\mathfrak{o}(a)$ resp $\mathfrak{so}(a)$, and the question naturally arises what their relation to the $\uparrow a$ is. For a complete lattice one has that L is subfit iff $\mathfrak{so}(a)$ are complements of $\uparrow a$ and, moreover, they are under this condition also supplements of $\uparrow a$. $\mathcal{M}(L)$ is not distributive and complements are not unique; here we have specified the interval of complements between the already mentioned pseudocomplement and this $\mathfrak{so}(a)$. The last is of a particular interest, also because, if we add the condition of (plain) distributivity we obtain the $\mathfrak{so}(a)$ coinciding with the open $\mathfrak{o}(a)$.

In the last section we briefly discuss the stronger condition of fitness. The extension of this property makes sense in any lattice. In distributive lattices it always implies subfitness, and is characterized by every $\uparrow a$ being an intersection of open meet-sets. In the complete case, furthermore, hereditary subfitness implies fitness.

1. Preliminaries

1.1. In a poset (partially ordered set) $(X, \leq)$ we write

$$\uparrow a \text{ for } \{x \mid x \geq a\} \quad \text{and} \quad \uparrow A = \{x \mid \exists\, a \in A,\ x \geq a\}.$$

The subsets $A \subseteq (X, \leq)$ such that $\uparrow A = A$ will be referred to as *up-sets*.

We will mostly deal with complete lattices, lattices or semilattices and denote the joins (suprema) resp. meets (infima) by $\bigvee A$ or $\bigvee_{a\in A} a$ resp. $\bigwedge A$ or $\bigwedge_{a\in A} a$ ($a \vee b$ resp. $a \wedge b$ in the finite case); this notation will be used also in the general case, where we will always mention that we speak on *existing* meets resp. joins.

1.1.1. The least resp. largest element (*bottom* or *top*, if it exists) will be denoted by 0 resp. 1.

A *pseudocomplement* of a is the largest element x such that $x \wedge a = 0$. It may not exist, but if it does then it is uniquely determined; it is then denoted by a^*. Similarly, the smallest element such that $x \vee a = 1$, if it exists, is called the *supplement* of a and denoted by $a^{\#}$.

A *complement* of a in a lattice is a b such that $a \wedge b = 0$ and $a \vee b = 1$. In general it is not uniquely determined. In distributive lattices it is (of course it does not have to exist), but our context will be typically non-distributive.

1.2. As usual, subsets $S, T \subseteq (X, \leq)$ are *cofinal* if for each $s \in S$ there is a $t \in T$ such that $s \leq t$, and for each $t \in T$ there is an $s \in S$ such that $t \leq s$.

1.3. Separativeness and conjunctivity. In connection with forcing ([7]) one uses the concept of a *separative* poset as such that $a \not\leq b$ implies that there is a $c \leq a$ such that c and b have no lower bound. This formulation makes sense only in posets without bottom. It is perhaps more transparent if we formally add 0. Then the formulation transforms to

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ 0 \neq c \leq a \text{ and } \inf\{b, c\} = 0. \tag{sep}$$

In the special case of (complete) lattices it appeared already in 1938 in Wallman's pioneering article [16] as *disjunctivity*

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ 0 \neq c \leq a \text{ and } b \wedge c = 0. \tag{disj}$$

It was used there as a (slightly weaker) replacement of the T_1 axiom in the point-free treatment of topology.

Thus, separativeness is just disjunctivity generalized for posets without special properties. We will use the term (or, rather, its dual variant, *co-separativity*) just to emphasize we have in mind the general context.

1.3.1. Conjunctivity. Disjunctivity was introduced in view of the lattice of closed sets. During the development of point-free topology it turned out that the perspective of the lattice of *open sets*, dual to the previous one, is more natural, and the concept was replaced by the dual

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ b \leq c \neq 1 \text{ and } a \vee c = 1, \tag{conj}$$

called *conjunctivity.*[2] Because of another pioneering article [4], where it was introduced independently (in a different form, and as a weaker variant of another property, *fitness*), another term, namely *subfitness* took hold. We will use "conjunctivity" and "subfitness" interchangeably.

The natural extension in general posets,

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ 1 \neq c \geq b \text{ and } \sup\{a, c\} = 1 \tag{co-sep}$$

will be referred to as *co-separability* (in (conj) we have a formally added top).

[2] It is usually presented in the equivalent form

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ b \vee c \neq 1 \text{ and } a \vee c = 1$$

(replace c by $c \vee b$). The less suggestive form we use here has, however, some technical advantages, and the relation with the general concept is more straightforward.

1.4. Adjunction. Monotone maps $f\colon X \to Y$ and $g\colon Y \to X$ between posets are (*Galois*) *adjoint, f to the left and g to the right*, if

$$f(x) \le y \iff x \le g(y),$$

equivalently, if $fg \le \mathrm{id}$ and $gf \ge \mathrm{id}$. It is standard that

(1) left adjoints preserve all existing suprema and right adjoints preserve all existing infima,
(2) and if X, Y are complete lattices then each $f\colon X \to Y$ preserving all suprema is a left adjoint (has a right adjoint), and each $g\colon Y \to X$ preserving all infima is a right adjoint.

1.5. Frames. A frame is a complete lattice satisfying

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$$

for all $A \subseteq L$ and $b \in L$ (which – recall 1.4 – induces a Heyting structure $a \wedge b \le c$ iff $a \le b \to c$) and frame homomorphisms preserve all joins and finite meets.

Frames will be in the background of our geometric intuition in investigating more general posets,

In particular the treatment of special subobjects is motivated by the behavior of sublocales, generalized subspaces of frames viewed as spaces. See e.g. [6, 11].

2. Topology of posets

2.1. Our study is motivated by the general properties of conjunctivity (dual separativity, subfitness) and related properties. It plays a fundamental geometric role in the context of frames; it is only natural to consider the behavior of subsets extending the generalized subspaces, in particular the topologically most important ones: closed and open sublocales. The context will vary in generality: it will concern

- general posets, or semilattices,
- general lattices,
- distributive lattices,
- complete lattices, and
- distributive complete lattices.

For technical reasons it will be of advantage to assume the existence of maximal element (denoted by 1).

2.2. Subobjects. Motivated by sublocales in frames and the natural generalization of the category of frames in the category of sup-lattices ([5]) we will be interested in subsets closed under meets. That is, we will speak about

(1) *meet-sets* (or $\bigwedge$*-sets*), subsets $S \subseteq (X, \leq)$ closed under all *existing* meets, and about

(2) *adjoint-meet-sets* $S \subseteq (X, \leq)$ such that the embedding $j : S \subseteq X$ has a left adjoint.

In case of complete lattices, meet-sets are automatically adjoint-meet-sets.

2.2.1. The complete lattice $\mathcal{M}(L)$. Meet-sets of a poset L constitute a complete lattice

$$\mathcal{M}(L)$$

with meets resp. joins

$$\bigwedge_i S_i = \bigcap_i S_i \quad \text{resp.} \quad \bigvee_i S_i = \{\bigwedge M, \text{ if it exists} \mid M \subseteq \bigcup_i S_i\}.$$

In particular we have the void join (bottom element)

$$\mathsf{O} = \{1\}.$$

2.3. Closed subobjects. Extending the concept of closed sublocales of frames we introduce the *closed* meet-sets $\uparrow a$. Note that whenever the infimum $\bigwedge_i a_i$ in L makes sense we have

$$\bigcap_i \uparrow a_i = \uparrow \bigwedge_i a_i.$$

Thus, in case of a complete lattice L the meets of closed subobjects in $\mathcal{M}(L)$ are always closed, in analogy with the behavior of closed subsets in topological spaces.

The lattice $\mathcal{M}(L)$ is not distributive, but we have

2.3.1. Proposition. *For any system of meet-sets S_i in a complete lattice we have*

$$\uparrow a \cap \bigvee_i S_i = \bigvee_i (\uparrow a \cap S_i).$$

Proof: $\supseteq$ is trivial. Now let $x \in \uparrow a \cap \bigvee_i S_i$ then $x \geq a$ and $x = \bigwedge x_i$ with $x_i \in S_i$. Then for all i, $x_i \geq x \geq a$, hence $x_i \in \uparrow a \cap S_i$, and $x \in \bigvee_i (\uparrow a \cap S_i)$. ∎

2.4. Open subobjects. The definition of an open sublocale of á frame can be extended to a general lattice L. Namely, we have the *open* subobject associated with $a \in L$,

$$\mathfrak{o}(a) = \{u \mid x \wedge a \leq u \;\Rightarrow\; x \leq u\}. \tag{open}$$

(It is a meet-set: if $u_i \in \mathfrak{o}(a)$ and $u = \bigwedge_i u_i$ exists and if $x \wedge a \leq u$ then $x \wedge a \leq u_i$ for all i, hence $x \leq u_i$ for all i, and finally $x \leq u$.)

2.4.1. Lemma. 1. *$\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$.*
2. *$\uparrow a \cap \mathfrak{o}(a) = \mathsf{O}$.*
3. *In a distributive lattice,*

$$a \vee c = 1 \quad \textit{iff} \quad \uparrow a \subseteq \mathfrak{o}(c).$$

Proof: 1: If $a \leq b$ and $u \in \mathfrak{o}(a)$ then obviously $u \in \mathfrak{o}(b)$. Hence $\mathfrak{o}(-)$ is monotone and we have the inclusion $\subseteq$. Now let $u \in \mathfrak{o}(a) \cap \mathfrak{o}(b)$. If $x \wedge a \wedge b \leq u$ then $x \wedge a \leq u$ because $u \in \mathfrak{o}(b)$ and then $x \leq u$ because it is also in $\mathfrak{o}(a)$.

2: If $u \geq a$ and $x \wedge a \leq u \;\Rightarrow\; x \leq u$ consider $x = 1$. Since $1 \wedge a \leq u$, we have $1 \leq u$.

3: If $\uparrow a \subseteq \mathfrak{o}(c)$ then $\uparrow(a \vee c) = \uparrow a \cap \uparrow c \subseteq \mathfrak{o}(c) \cap \uparrow c = \{1\}$. Hence $a \vee c = 1$.
If $a \vee c = 1$ and $u \geq a$, if $x \wedge c \leq u$ then $x = x \wedge (a \vee c) \leq a \vee (x \vee c) \leq u$. ■

2.5. (Technical) "supplement" sets, and semiopen subobjects. Set

$$\mathsf{s}(a) = \{x \mid x \vee a = 1\}$$

and its meet-extension, the *semiopen* subobject (cf. [10])

$$\mathfrak{so}(a) = \{\inf M \text{ if it exists } \mid M \subseteq \mathsf{s}(a)\}.$$

2.5.1. Lemma. 1. *$\uparrow a \cap \mathfrak{so}(a) = \mathsf{O}$.*
2. *$a \vee c = 1$ iff $\uparrow a \subseteq \mathfrak{so}(c)$ iff $\uparrow a \subseteq \mathsf{s}(c)$.*
3. *In a distributive lattice, $\mathfrak{so}(a) \subseteq \mathfrak{o}(a)$.*

Proof: 1: Let $\inf M \geq a$ and $M \subseteq \mathsf{s}(a)$. Then for every $m \in M$, $m \geq a$ and $m \vee a = 1$, hence $m = 1$.

2: Compare with 2.4.1.3; here, distributivity is not needed. If $\uparrow a \subseteq \mathfrak{so}(c)$ then

$$\uparrow(a \vee c) = \uparrow a \cap \uparrow c \subseteq \mathfrak{so}(c) \cap \uparrow c = \mathsf{O}$$

and hence $a \vee c = 1$; if $\uparrow a \subseteq \mathsf{s}(c)$ then, of course, $\uparrow a \subseteq \mathfrak{so}(c)$.
If $a \vee c = 1$ then $a \in \mathsf{s}(c)$ and since $\mathsf{s}(c)$ is an up-set, $\uparrow a \subseteq \mathsf{s}(c) \subseteq \mathfrak{so}(c)$.

3: Obviously $\mathfrak{o}(a)$ is closed under meets, hence it suffices to prove that $\mathsf{s}(a) \subseteq \mathfrak{o}(a)$. If $x \vee a = 1$ and $y \wedge a \leq x$ then $y = y \wedge (x \vee a) \leq (y \wedge x) \vee x = x$. ■

2.6. Note. The lattice $\mathcal{M}(L)$ is not distributive and hence complements need not be uniquely defined. In particular, $\uparrow a$ has the trivial pseudocomplement $(L \smallsetminus \uparrow a) \cup \{1\}$ which is by far not unique (and also not very interesting). In the conjunctive case there is another, more interesting one, namely $\mathfrak{so}(a)$ (which happens to be a supplement: quite a bizzare situation, distinct pseudocomplement and supplement, both of them complements). In the distributive conjunctive case, one has a satisfactory open complement.

3. Some simple facts about separativeness

3.1. The proposition below will be formulated in the original setting, that is, without the formally added 1,

$$a \not\leq b \quad \Rightarrow \quad \exists c \geq b \;\text{ such that } \{a, c\} \text{ has no upper bound} \qquad \text{(co-sep)}$$

(the cofinality statements will be simpler).

Note the obvious

3.1.1. Observation. *The subset $\uparrow a$ of any co-separative poset resp. conjunctive join-semilattice is co-separative resp. conjunctive.*

3.2. The left adjoint of the embedding $j\colon S \subseteq X$ of an adjoint-meet-set $S \subseteq L$ will be denoted by

$$\nu_S\colon L \to S.$$

Note that for a meet-set in a complete lattice we have

$$\nu_S(a) = \bigwedge\{s \in S \mid a \leq s\}.$$

3.3. Proposition. *In the following statements about $(X, \leq)$ we have* (1)⇒(2)⇒(3)⇒(4)

(1) *$(X, \leq)$ is co-separative.*
(2) *An adjoint-meet-set S is cofinal in $(X, \leq)$ only if $S = X$.*
(3) *If $S \subsetneq X$ for an adjoint-meet-set $S \subseteq (X, \leq)$ then there is a c such that $\uparrow c \cap S = \emptyset$.*
(4) *If $\uparrow a \neq X$ then there is a b such that $\uparrow b \cap \uparrow a = \emptyset$.*
(5) *If $\{a, x\}$ has an upper bound for every x then a is the bottom of $(X, \leq)$.*

Proof: (1)⇒(2): Let $x \in X \smallsetminus S$. Then $x < \nu_S(x)$ and hence there is a c, $c \geq x$ such that there is no upper bound of$\{\nu(x), c\}$. By cofinality take a

$t \in S$ such that $t \geq c$. Then $t \leq x$ and hence $t \geq \nu(x)$. Consequently, $t \geq \nu(x), c$, a contradiction.

(2)⇒(3): Since $S \neq X$, S is not cofinal in X and hence there is a c such that $s \ngeq c$ for all s with $s \in S$. Thus, ${\uparrow}c \cap S = \emptyset$.

(3)⇒(4): Apply (3) for $S = {\uparrow}a$.

(4)⇒(5): (5) is a translation of (4). ■

3.4. Notes. 1. In frames, (1) is subfitness and (5) is weak subfitness ([10]), hence we may also speak of *weak conjunctivity.* Thus the corresponding statement on meet-semilattices can be formulated as follows.

We have the implications (1)⇒(2)⇒(3)⇒(4)⇒(5) in

(1) *L is conjunctive.*
(2) *For every adjoint-meet-set $S \subseteq L$, $S \smallsetminus \{0\}$ is cofinal in $L \smallsetminus \{1\}$ only if $S = L$.*
(3) *If $S \subsetneq L$ for an adjoint-meet-set $S \subseteq L$ then there is a $c \neq 1$ such that ${\uparrow}c \cap S = \{0\}$.*
(4) *If ${\uparrow}a \neq L$ then there is a $c \neq 1$ such that ${\uparrow}c \cap {\uparrow}a = \{0\}$.*
(5) *L is weakly conjunctive.*

2. Again, conjunctivity is the same as weak conjunctivity of all the closed meet-sets.

3. (2) corresponds to the characterization of subfitness as the property that a congruence trivial on 1 is equality.

4. For frames one has (3)⇒(1), hence the equivalence of the first three statements. In the next section we will see that this (and more) holds already for meet-sets in complete lattices (extending the facts about sublocales in frames).

3.5. A stronger variant of separativity. Let us add a short note on a stronger variant of separativity which shows that the study of properties akin to conjunctivity (subfitness) in a general setting is profitable. This property resp. its dual does not make sense in (co)frames (not even in distributive lattices – see 3.5.1 below) but applied in the general order context it has a very useful application (see [9]).
The separativity from 1.3 in the form

$$a \notin {\downarrow}b \quad \Rightarrow \quad \exists c,\ 0 \neq c \leq a \text{ such that } {\downarrow}b \cap {\downarrow}c = \{0\} \tag{sep}$$

was in [9] naturally strengthened, replacing the one-point set $\{b\}$ by a finite one, to

$$\begin{aligned}&\text{for every finite } B \subseteq X,\\ &a \notin {\downarrow}B \quad\Rightarrow\quad \exists c,\ 0 \neq c \leq a \text{ such that } {\downarrow}B \cap {\downarrow}c = \{0\}\end{aligned} \tag{Sep}$$

obtaining a characterization of the posets $(X, \leq)$ making certain spaces based on $(X, \leq)$ subfit (which in turn was used to construct spaces with desired properties). For meet-semilattices L with a bottom it makes

$$\begin{aligned}&\text{for every finite } B \subseteq L,\\ &(\forall b \in B,\ a \nleq b) \quad\Rightarrow\quad \exists c,\ 0 \neq c \leq a \text{ s. t. } (\forall b \in B,\ b \wedge c = 0).\end{aligned} \tag{Sep}$$

3.5.1. Observation. *No distributive lattice L satisfies* (*Sep*)*, not even with just two-element sets B.*

Proof: First, L cannot be non-trivial linear, because then it would not be even separative. Thus, there have to be $b_1, b_2 \in L$ such that $b_i \neq b = b_1 \vee b_2$, that is, $b \nleq b_i$ and we have a $0 < c \leq b$ such that $b_i \wedge c = 0$. Then, however, $c = b \wedge c = (b_1 \vee b_2) \wedge c = (b_1 \wedge c) \vee (b_2 \vee c) = 0$, a contradiction. ■

4. Conjunctivity in the category $\bigvee$Lat

4.1. Category $\bigvee$Lat and meet-sets. Recall from [5] the *category of sup-lattices*

$$\textstyle\bigvee\mathbf{Lat}$$

with objects complete lattices and morphisms join-preserving maps.

4.2. The background of the geometric intuition: frames. We can view the category $\bigvee$**Lat** as an extension of the category **Frm** of frames.
The natural subobjects of frames are the *sublocales*, that are the subsets closed under meets satisfying, moreover, the requirement that $s \in S$ implies that $x \to s \in S$ for any x.[3] Among them, a particular role is played by the closed and open ones,

$$\mathfrak{c}(a) = {\uparrow}a \quad\text{and}\quad \mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\}$$

(which precisely correspond to closed resp. open subspaces in the spatial context) and which we imitate in our more general reasoning.

[3]The sublocales of a frame L constitute a co-frame $\mathsf{S}(L)$ (hence in particular, a distributive lattice), with meets and joins defined by the formulas $(*)$ as above again; thus $\mathsf{S}(L)$ is a complete sublattice of $\mathcal{M}(L)$.

4.2.1. Thus we have in particular the closed and open meet-sets $\uparrow a$, $\mathfrak{o}(a) = \{u \mid x \wedge a \leq u \;\Rightarrow\; x \leq u\}$ and $\mathfrak{so}(a)$ as in Section 2, and we have (see 2.2.1, 2.4.1, 2.4.2)

$$\bigcap_i \uparrow a_i = \uparrow \bigvee_i a_i, \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b), \quad \uparrow a \cap \mathfrak{o}(a) = \mathsf{O}$$

and in the distributive case also

$$\uparrow a \vee \uparrow b = \uparrow (a \wedge b) \text{ and } \mathfrak{so}(a) \subseteq \mathfrak{o}(a).$$

4.3. Proposition. $\mathfrak{so}(a) = \bigvee\{\uparrow b \mid \uparrow b \cap \uparrow a = \mathsf{O}\} = \bigvee\{\uparrow b \mid b \vee a = 1\}.$

Proof: Of course $\uparrow b \cap \uparrow a = \mathsf{O}$ iff $b \vee a = 1$. Since $\mathsf{s}(a)$ is an up-set, it is the union $\bigcup\{\uparrow b \mid b \vee a = 1\}$. Use the definition of join in $\mathcal{M}(L)$. ■

4.4. Proposition. *Let $S \in \mathcal{M}(L)$ such that $S \vee \uparrow a = L$. Then $\mathfrak{so}(a) \subseteq S$.*

Proof: If for $S \in \mathcal{M}(L)$, $S \vee \uparrow a = L$ then for each $x \in L$ there are $s_x \in S$ and $b_x \geq a$ such that $x = b_x \wedge s_x$. Take a $\bigwedge M$ with $M \subseteq \mathsf{s}(a)$. Then for each $m \in M$,

$$1 = a \vee (s_m \wedge b_m) \quad \text{and hence} \quad 1 = a \vee b_m = b_m$$

so that $m = s_m \in S$, and finally $\bigwedge M \in S$. ■

4.5. Proposition. *The following are equivalent for any complete lattice L.*

(1) *L is conjunctive.*
(2) *For every meet-set $S \subseteq L$, $S \smallsetminus \{1\}$ is cofinal in $L \smallsetminus \{1\}$ only if $S = L$.*
(3) *If for a meet-set $S \subseteq L$, $S \subsetneq L$ then there is a $c \neq 1$ such that $\uparrow c \cap S = \{0\}$.*
(4) *For every $a \in L$, $\uparrow a \vee \mathfrak{so}(a) = L$.*

Proof: (1)⇒(2)⇒(3) are in 3.3.

(3)⇒(4): Suppose $\uparrow c \cap (\uparrow a \vee \mathfrak{so}(a)) = \mathsf{O}$. Then

$$\uparrow c \cap \uparrow a = \uparrow (a \vee c) = \mathsf{O}$$

and hence $c \vee a = 0$. Considering $M = \{c\}$ we observe that $c \in \uparrow c \cap \mathfrak{so}(a)$; since also $\uparrow c \cap \mathfrak{so}(a) = \mathsf{O}$, $c = 1$ and hence $\uparrow a \vee \mathfrak{so}(a)$ cannot be smaller than L.

(4)⇒(1): Let $a \nleq b$. Then $\mathfrak{so}(a) \nsubseteq \mathfrak{so}(b)$: indeed, if $\mathfrak{so}(a) \subseteq \mathfrak{so}(b)$ we have by (4) and 3.2,

$$\uparrow b = \uparrow b \cap (\uparrow a \vee \mathfrak{so}(a)) = (\uparrow b \cap \uparrow a) \vee (\uparrow b \cap \mathfrak{so}(a)) = \uparrow b \cap \uparrow a$$

and hence $a \le b$. Hence $\{x \mid x \vee a\} \nsubseteq \{x \mid x \vee b\}$ and there is an x with $x \vee a = 1$ and $x \vee b \neq 1$. Set $c = x \vee b$. ■

4.5.1. From 2.4.1. and 2.5.1.3 we now immediately obtain

Corollary. *In a distributive complete lattice, $\mathfrak{o}(a)$ is a complement of $\uparrow a$.*

4.6. Lemma. *Let L be a distributive complete lattice and let $c \in L$. If the only $s \geq c$ that is in $\mathfrak{o}(u)$ is the top then $c \geq u$.*

Proof: If the only $s \geq c$ that is in $\mathfrak{o}(u)$ is the top then $\uparrow c \cap \mathfrak{o}(u) = \mathsf{O}$. By 4.5.1 in the current text, $\uparrow u \vee \mathfrak{o}(u) = L$ and hence

$$\uparrow c = \uparrow c \cap (\uparrow u \vee \mathfrak{o}(u)) = \uparrow c \cap \uparrow u = \uparrow (c \vee u),$$

hence $c = c \vee u$. ■

4.6.1. Proposition. *Let a distributive complete lattice L be subfit and let $u \in L$. Then each $\mathfrak{o}(u) \subseteq L$ is subfit.*

Proof: Let $a, b \in \mathfrak{o}(u)$ and $a \nleq b$. Then $a \wedge u \nleq b$ and hence there is a $c \in L$ such that $(a \wedge u) \vee c = 1$ and $b \leq c \neq 1$. Then

$$(a \vee c) \wedge (u \vee c) = 1 \quad \text{and hence in particular} \quad u \vee c = 1.$$

There has to be a $c' \in \mathfrak{o}(u)$ with $c \leq c' \neq 1$ since otherwise we would have by Lemma 4.6 that $c = c \vee u = 1$. ■

4.6.2. Corollary. *Let a distributive complete lattice L be subfit and let $u \in L$. Then each $\mathfrak{o}(u) \subseteq L$ is a join of closed subobjects (that is, $\mathfrak{o}(u) = \mathfrak{so}(u)$).*

(Else $\mathfrak{so}(u) = \bigvee\{\uparrow b \mid b \vee a = 1\} \subsetneq \mathfrak{o}(u)$. Use 4.5(3).)

4.6.3. Remark. It follows from 4.6.2, 4.5.1 and 4.5 that a distributive complete L is conjunctive iff $\mathfrak{so}(u) = \mathfrak{o}(u)$ for every $u \in L$.

4.7. Dense and codense maps. Let $h\colon L \to M$ be a sup-lattice homomorphism.

4.7.1. Observations. *1. In the following statements about h and its right adjoint h_* we have* (1)⇔(2)⇔(3) *and* (4)⇔(5)⇔(6).

(1) $h(1) = 1$.
(2) $h_*(y) = 1 \;\Rightarrow\; y = 1$.
(3) $h_*[M \smallsetminus \{1\}] = h_*[M] \smallsetminus \{1\}$.
(4) $h_*(0) = 0$.

(5) $h(x) = 0 \;\Rightarrow\; x = 0$.
(6) $h[L \smallsetminus \{0\}] = h[L] \smallsetminus \{0\}$.

2. If (1) *holds then* $h_*[M \smallsetminus \{1\}]$ *is cofinal in* $L \smallsetminus \{1\}$ *iff* $h(x) = 1 \;\Rightarrow\; x = 1$.

3. If (4) *holds then* $h[L \smallsetminus \{0\}]$ *is down cofinal in* $M \smallsetminus \{0\}$ *iff* $h_*(y) = 0 \;\Rightarrow\; y = 0$.

($\Rightarrow$: Let $y > 0$. By the assumption, $y \geq h(x) > 0$ for some $x \in L$. Hence $h_*(y) \geq h_*h(x) \geq x > 0$.

$\Leftarrow$: Let $y > 0$ in M. Then $h_*(y) > 0$ and $hh_*(y) > 0$, hence $y \geq hh_*(y) \in h[L \smallsetminus \{0\}] = h[L] \smallsetminus \{0\}$.)

4.7.2. h is a *dense map* if satisfies property (4) above and $h[L \smallsetminus \{0\}]$ is down cofinal in $M \smallsetminus \{0\}$. Dually, h is *codense* if (1) holds and $h_*[M \smallsetminus \{1\}]$ is cofinal in $L \smallsetminus \{1\}$.

4.7.3. Proposition. *A complete lattice L is subfit if and only if each codense sup-lattice morphism $h\colon L \to M$ is injective.*

Proof: $\Rightarrow$: Let $h\colon L \to M$ be a codense map with L subfit. Then $h_*[M]$ is a meet-set of L and by 4.5 h_* is onto, that is, h is injective.
$\Leftarrow$: Let $S \subseteq L$ be a meet-set such that $S \smallsetminus \{1\}$ is cofinal in $L \smallsetminus \{1\}$. By 4.5, it suffices to show that $S = L$. But $\nu_S\colon L \to S$, the left adjoint to the inclusion $j\colon S \subseteq L$, is a codense sup-lattice homomorphism. Hence it is one-one, that is, $S = j[S] = L$. ∎

5. Fitness

5.1. Let us rewrite the standard first order formula for frame fitness avoiding the explicit use of the Heyting operation.

L is fit iff

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \text{ and } c \to b \not\leq b,$$

or, equivalently,

$$(a \vee c = 1 \;\Rightarrow\; c \to b \leq b) \quad \Rightarrow \quad a \leq b.$$

The expression $c \to b \leq b$, that is,

$$x \leq c \to b \;\Rightarrow\; x \leq b,$$

can be rewritten as

$$x \wedge c \leq b \;\Rightarrow\; x \leq b.$$

Thus, fitness for (bounded) lattices can be defined by the formula

$$(a \vee c = 1 \;\Rightarrow\; (x \wedge c \leq b \;\Rightarrow\; x \leq b)) \quad\Rightarrow\quad a \leq b, \tag{5.1.1}$$

or, equivalently,

$$a \nleq b \quad\Rightarrow\quad \exists\, c, x, \;\; a \vee c = 1 \;\&\; x \wedge c \leq b \;\&\; x \nleq c. \tag{5.1.2}$$

Using the definition of an open meet-set we finally obtain

$$(a \vee c = 1 \;\Rightarrow\; b \in \mathfrak{o}(c)) \quad\Rightarrow\quad a \leq b. \tag{fit}$$

5.1.1. Note. Recall 2.4.3. In a distributive complete lattice the last can be rewritten as

$$(\uparrow\! a \in \mathfrak{o}(c) \;\Rightarrow\; b \in \mathfrak{o}(c)) \quad\Rightarrow\quad a \leq b. \tag{fit}$$

Compare it with the formula for subfitness in the distributive case (2.3.4 used again):

$$(\uparrow\! a \subseteq \mathfrak{o}(c) \;\Rightarrow\; \uparrow\! b \subseteq \mathfrak{o}(c)) \quad\Rightarrow\quad a \leq b. \tag{sfit}$$

Look at the subtle difference between $\{b\}$ and $\uparrow\! b$ in the first implication of the two definitions.

Also note that the formula (fit) throws some light on the connection between the first-order and second order formula for fitness: it is just the first order formula rewritten, but it somehow includes "closeds as intersections of opens" one in a straightforward way.

5.2. Proposition. *In a (bounded) distributive lattice, fitness implies subfitness.*

Proof: Let L be subfit and let $a \leq b$. Then there is a d and x such that $a \vee d = 1$, $x \wedge d \leq b$ and $x \nleq b$. Set $c = b \vee d$. Then $c \neq 1$ since otherwise $x = x \wedge c = (x \wedge b) \vee (x \wedge d) \leq b$. ■

5.3. Fitness and subfitness of meet-sets. .

Proposition. *Let L be a distributive complete lattice. If each of its meet-sets $S \subseteq L$ is subfit then L is fit.*

Proof: Let L not be fit. Then there are $a \nleq b$ such that

$$a \vee c = 1 \;\Rightarrow\; (x \wedge c \leq b \;\Rightarrow\; x \leq b).$$

Set

$$S = \{y \mid a \vee c = 1 \;\Rightarrow\; (x \wedge c \leq y \;\Rightarrow\; x \leq y)\}.$$

Obviously S is a meet-set, and $b \in S$.

Further, we have

$$\uparrow a \subseteq S. \tag{$*$}$$

Indeed, if $y \geq a$ and $a \vee c = 1$ then $y \vee c = 1$ and if $x \wedge c \leq y$ then $x = x \wedge (y \vee c) = (x \wedge y) \vee (x \wedge c) \leq y$ and hence $y \in S$.

We will prove that S is not subfit. Suppose it is. Obviously, $b \in S$ and by $(*)$ also $a \in S$. There has to be a $1 \neq c$ such that the join of a and c in S is 1; by $(*)$ again, however, this join coincides with $a \vee c = 1$ in L. Now since $c \in S$ and $a \vee c = 1$, we have

$$x \wedge c \leq c \;\Rightarrow\; x \leq c.$$

Choosing $x = 1$ we obtain a contradiction $c = 1$. ■

5.4. Theorem. *A distributive lattice L is fit iff each $\uparrow a$ is a meet of open objects.*

Proof: I. Let L be fit. Then trivially

$$\uparrow a \subseteq \bigcap\{\mathfrak{o}(c) \mid \uparrow a \subseteq \mathfrak{o}(c)\} = \bigcap\{\mathfrak{o}(c) \mid a \vee c = 1\}$$

and if $u \in \bigcap\{\mathfrak{o}(c) \mid a \vee c = 1\}$ then

$$a \vee c = 1 \;\Rightarrow\; u \in \mathfrak{o}(c)$$

and by 3, $u \in \uparrow a$.

II. Let $\uparrow a = \bigcup_i \mathfrak{o}(c_i)$. Then $\bigcap\{\mathfrak{o}(c) \mid a \vee c = 1\} \subseteq \uparrow a$ and if $a \vee c = 1$ implies $b \in \mathfrak{o}(c)$ then $b \in \bigcap\{\mathfrak{o}(c) \mid a \vee c = 1\}$ and hence $b \geq a$. ■

5.5. Corollary. *In a fit distributive complete lattice,*

$$\uparrow a = \bigcap\{\mathfrak{so}(c) \mid \uparrow a \subseteq \mathfrak{so}(c)\} = \bigcap\{\mathfrak{so}(c) \mid a \vee c = 1\}.$$

(By 2.4.1, if $a \vee c = 1$, $\uparrow a \subseteq \mathfrak{so}(c) \subseteq \mathfrak{o}(c)$.)

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