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SOME STOCHASTIC ORDERING RESULTS FOR SUMS OF RANDOM VARIABLES

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Abstract: Given two random variables X and Y, we derive stochastic ordering results between X and the sum $X+Y$, using nonparametric assumptions on the shape of the distribution of X. Such results are extended to the comparisons of general sums of random variables.

Keywords: convolutions, likelihood ratio order, reversed hazard rate, log-concavit, sign variation.
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1. Introduction

Sums of independent random variables play an important role in different areas of probability and statistics, as they correspond to any model where accumulation is meaningful, such as the total amount of claims to an insurance company, total amount of input in a storage facility, the lifetime of a system obtained by replacing failed components with standby ones, to name a few common examples. Thus, it becomes of interest to address the stochastic behaviour of such sums of random quantities. We will seek for characterisations relying on stochastic ordering properties, as these may express comparisons with respect to size or variability behaviours. We refer the reader to the books by Shaked and Shantikumar [13] or Marshall and Olkin [10] for an account on general properties and characterisations of stochastic ordering notions. In this paper, we focus on the problem of comparing partial sums of random variables, that is, sums with a different number of addends, which, in fact, boils down to comparing random variables X and $X+Y$.

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Intuitively, when summing nonnegative random variables, it is reasonable to expect that the sum is stochastically increasing with respect to the number of addends (Shaked and Shantikumar [13], Theorem 1.A.4). On the other hand, for real valued random variables, one may argue that the sum gets "riskier", that is, more variable, or dispersed, when the number of terms increases. Finally, these two aspects may be combined: sums of random variables may be compared with respect to size and dispersion, simultaneously, according to the distributions of the addends. Referring to well known ordering notions, size aspects may be measured using the usual stochastic, the likelihood ratio, hazard rate and reversed hazard rate orders. On the other hand, variability aspects may be measured using the convex and dispersive orders, whereas the combination of size–variability aspects may be measured using the increasing concave or increasing convex orders. As a basic principle, stronger results may be found by focusing on special parametric families of distributions, or by adding nonparametric shape restrictions, such as monotone behaviours of the hazard rate or reversed hazard rate, unimodality or log-concavity, also known as strong unimodality (Marshall and Olkin [10]). For instance, Block and Savits [3] derive conditions for hazard rate ordering of partial sums of random variables with increasing hazard rate (IHR), while Droste and Wefelmeyer [5] derive conditions for the dispersive order for partial sums of random variables with log-concave densities.

In section 3, we prove that $X+Y$ dominates X in the reversed hazard rate order, provided that Y is nonnegative and X belongs to the decreasing reversed hazard rate (DRHR) family, extending the approach of Block and Savits [3] to models in which X does not necessarily have all moments. Moreover, if X has a log-concave density, which is a more stringent condition, we obtain a stronger result, namely that $X+Y$ dominates X in the likelihood ratio order. With regard to variability aspects, it is known that, if the random variable Y has zero-mean, then $X+Y$ is more variable than X in the convex order (Shaked and Shantikumar [13]). In section 4, we establish a stronger property, showing that, in the symmetric case, the cumulative distribution functions (CDFs) of X and Y are single-crossing, provided that X is unimodal. Eventually, we note that scale-invariant variability orders, such as the star and the Lorenz orders, have a different behaviour with regard to summation. Nevertheless, sufficient conditions for such orders may be derived for products of random variables, instead of sums.

2. Preliminaries

Throughout this paper, we will consider X to be a random variable with CDF F_X, probability density function f_X, and survival function $\overline{F}_X = 1 - F_X$. We define the hazard rate and the reversed hazard rate of X as $h_X = \frac{f_x}{\overline{F}_X}$ and $r_X = \frac{f_X}{F_X}$, respectively. We shall use similar notations when referring to any other random variables. Note that throughout this paper, "increasing" and "decreasing" are interpreted as "nondecreasing" and "nonincreasing", respectively. Moreover, a function g is said to be *star-shaped* if $\frac{g(x)}{x}$ is increasing.

We start by recalling some nonparametric classes of distributions that will allow us to establish ordering of partial sums, with respect to the previously mentioned stochastic orders.

Definition 1. *X is said to be*

(1) *IHR if h_X is increasing,*
(2) *DRHR if r_X is decreasing,*
(3) *log-concave if $\ln f_X$ is concave.*

A random variable X with unimodal distribution is said to be *strongly* unimodal if the sum of X with any other unimodal random variable is still unimodal. It is important to remark that log-concavity is equivalent to strong unimodality (Ibragimov [6]). Moreover, it is well known that log-concavity implies the IHR and DRHR conditions (Barlow and Proschan [1]).

The following orders are typically used to compare random variables in terms of size, or magnitude, of their values (Shaked and Shantikumar [13]).

Definition 2. *X is said to be smaller than Y in the*

(1) *usual stochastic order, denoted by $X \le_{st} Y$, if $F_X(x) \ge F_Y(x)$,*
(2) *hazard rate order, denoted by $X \le_{hr} Y$, if $h_X(x) \ge h_Y(x)$,*
(3) *reversed hazard rate order, denoted by $X \le_{rhr} Y$, if $r_X(x) \le r_Y(x)$,*
(4) *likelihood ratio order, denoted by $X \le_{lr} Y$, if the ratio $\frac{f_Y}{f_X}$ is increasing.*

Define, for $x \in (0,1)$, $R(x) = F_Y(F_X^{-1}(x))$, known as the *ordinal dominance curve*, and the complementary curve $\widetilde{R}(x) = 1 - R^{-1}(1-x)$. The following useful characterisations hold.

Proposition 3 (Lehmann and Rojo [8])**.**

(1) *$X \le_{st} Y$ if and only if $R(x) \le x$, for every $x \in (0,1)$,*

(2) $X \leq_{hr} Y$ *if and only if* $\widetilde{R}(x)$ *is star-shaped,*
(3) $X \leq_{rhr} Y$ *if and only if* $R(x)$ *is star-shaped,*
(4) $X \leq_{lr} Y$ *if and only if* $R(x)$ *is convex.*

Note that the star-shapedness of $R(x)$, equivalent to $X \leq_{rhr} Y$, means that $R(0) = 0$, hence the left endpoint of the support of X must be smaller or equal than that of Y. On the other hand, the star-shapedness also implies that the same inequalities holds for the right endpoints of the supports. A similar behaviour occurs when $X \leq_{hr} Y$, by considering $\widetilde{R}(x)$. Since convexity of R or $\tilde{R}(x)$ implies its star-shapedness, which, in turn, implies that R and $\widetilde{R}$ are dominated by the identity function, it is straightforward to verify that $X \leq_{lr} Y \implies X \leq_{hr} Y \implies X \leq_{st} Y$, and similarly $X \leq_{lr} Y \implies X \leq_{rhr} Y \implies X \leq_{st} Y$.

We will also consider the following stochastic orders, that address variability properties.

Definition 4. *X is smaller than Y in the*

(1) *convex (concave) order, denoted by $X \leq_{cx} (\leq_{cv})Y$, if $E\phi(X) \leq E\phi(Y)$, for every convex (concave) function $\phi : \mathbb{R} \to \mathbb{R}$, provided that the expectations exist.*
(2) *increasing convex (concave) order, denoted by $X \leq_{icx} (\leq_{icv})Y$, if $E\phi(X) \leq E\phi(Y)$, for every increasing convex (concave) function $\phi : \mathbb{R} \to \mathbb{R}$, provided that the expectations exist.*
(3) *Lorenz order, denoted by $X \leq_L Y$, if*
$$\frac{1}{EX}\int_0^p F_X^{-1}(u)\,du \geq \frac{1}{EY}\int_0^p F_Y^{-1}(u)\,du,$$
for $p \in [0,1]$.
(4) *dispersive order, denote by $X \leq_{disp} Y$, if $F_X^{-1}(\alpha) - F_X^{-1}(\beta) \leq F_Y^{-1}(\alpha) - F_Y^{-1}(\beta)$, for every $0 \leq \alpha < \beta \leq 1$.*
(5) *star order, denoted by $X \leq_* Y$, if $F_Y^{-1}(F_X(x))$ is star-shaped, and $F_X(0) = F_Y(0) = 0$.*

Note that $X \leq_{cx} (\leq_{cv})Y$ means that X exhibits less (more) variability compared to Y. Differently, $X \leq_L Y$ generally has a different interpretation, that is, X exhibits less "inequality" than Y (Marshall et al. [11]). In fact, it is also important to remark that the Lorenz order is a scale-free version of the convex order, in that, if $EX = EY$, $X \leq_L Y$ if and only if $X \leq_{cx} Y$, or, put otherwise, $X \leq_L Y$ if and only if $\frac{X}{EX} \leq_{icx} \frac{Y}{EY}$. The star order is often used as a stronger condition to obtain the Lorenz order, since $X \leq_* Y \implies X \leq_L Y$. Finally, it is easily seen that $X \leq_{st} Y \implies X \leq_{icx} (\leq_{icv}) Y$, consequently, the latter orders may be used to measure size and variability aspects at the same time.

3. Size results

Given two independent variables X and Y, it is easy to see that $X \leq_{st} X+Y$, if Y is nonnegative. We will see that under additional conditions on the distribution of X, one may obtain stronger order relations.

Block and Savits [3] proved that if X and Y are nonnegative independent random variables, and X is IHR, then $X \leq_{hr} X+Y$. By an analogous approach, it is easily seen that this still holds if we consider that only Y is nonnegative.

Theorem 5. *Let X and Y be independent random variables. If X is IHR and Y is nonnegative, then $X \leq_{hr} X+Y$.*

Similarly, we may establish that if X is DRHR, X is smaller than $X+Y$ in the reversed hazard rate order.

Theorem 6. *Let X and Y be independent random variables. If X is DRHR and Y is nonnegative, then $X \leq_{rhr} X+Y$.*

Proof: The reversed hazard rate of $X+Y$ is

$$r_{X+Y}(t) = \frac{f_{X+Y}(t)}{F_{X+Y}(t)} = \frac{\int_0^{+\infty} f_X(t-y) f_Y(y)\, dy}{\int_0^{+\infty} F_X(t-y) f_Y(y)\, dy}.$$

The result follows by proving that, for every t, $r_X(t) \leq r_{X+Y}(t)$, which is equivalent to $r_X(t)F_{X+Y}(t) - f_{X+Y}(t) \leq 0$. Since $f_X(t) = r_X(t)F_X(t)$, we have that,

$$r_X(t)F_{X+Y}(t) - f_{X+Y}(t) = \int_0^{+\infty} F_X(t-y) f_Y(y) [r_X(t) - r_X(t-y)]\, dy,$$

which is not positive, since X is DRHR. ■

Remark 7. *The proof of Theorem 6 may suggest that, taking X to have an increasing reversed hazard rate, one would obtain $X+Y \leq_{rhr} X$. But note that this is not possible, because the right-endpoint of the support of*

$X+Y$ is strictly larger than that of X, which is sufficient to conclude that $X+Y \not\leq_{rhr} X$ (in fact, in this case we have $X \leq_{st} X+Y$).

Note that Theorem 5 is based on the assumption that X is IHR, which implies that X must have all moments. Differently, the DRHR condition does not require existence of moments, and it is compatible with heavy tailed distributions. From this point of view, our Theorem 6 extends the scope of applicability of Theorem 5. Differently from Theorem 6, the following result represents a refinement of that of Theorem 5. In fact, by imposing a stronger condition on X, namely, log-concavity, we obtain the stronger likelihood ratio order. The proof is based on a "total positivity" property, recalled below, which is known to be equivalent to log-concavity (Marshall and Olkin [10]).

Definition 8 (Karlin [7])**.** *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be totally positive of order 2 (TP2) if for every real numbers $x_1 < x_2$ and $y_1 < y_2$, $f(x_1, y_2)f(x_2, y_1) \leq f(x_1, y_1)f(x_2, y_2)$.*

Theorem 9. *Let X and Y be independent random variables. If X is log-concave and Y is nonnegative, then $X \leq_{lr} X+Y$.*

Proof: The result follows by proving that $F_{X+Y}(F_X^{-1}(x))$ is convex, which is equivalent to showing that, for every $x \leq y$, $f_X(x)f_{X+Y}(y) \geq f_X(y)f_{X+Y}(x)$. Since $f_{X+Y}(x) = \int_0^{+\infty} f_X(x-t)f_Y(t)\,dt$, this is still equivalent to

$$\int_0^{+\infty} f_Y(t)f_X(x)f_X(y)\left[\frac{f_X(y-t)}{f_X(y)} - \frac{f_X(x-t)}{f_X(x)}\right] dt \geq 0.$$

Taking into account 21.B.8 in Marshall and Olkin [10], f_X is log-concave if and only if $h(x,t) = f(x-t)$ is TP2. Therefore, for $x \leq y$ and $0 \leq t$, we have that $\frac{f_X(y-t)}{f_X(y)} \geq \frac{f_X(x-t)}{f_X(x)}$. Hence, the conclusion follows. ■

One may wonder if Theorem 9 holds under the weaker IHR and DRHR assumptions, as in Theorem 5 and Theorem 6. In the case of the DRHR assumption, this is generally not true, as it can be seen by the following counterexample. Take for instance $X =_d 2T + W$, where $T \sim LL(1,1)$, $W \sim LL(1/2, 1)$, and $LL(a,b)$ denotes a log-logistic distribution with shape parameter a and scale parameter b. Take also $Y \sim LL(1,1)$, which is DRHR but not log-concave. The density of the sum $X+Y$ has a closed form expression (although quite complicated), and it can be seen that the likelihood ratio $\frac{f_{X+Y}}{f_X}$ is non-monotone (see Figure 1), so that $X \not\leq_{lr} X+Y$. One may

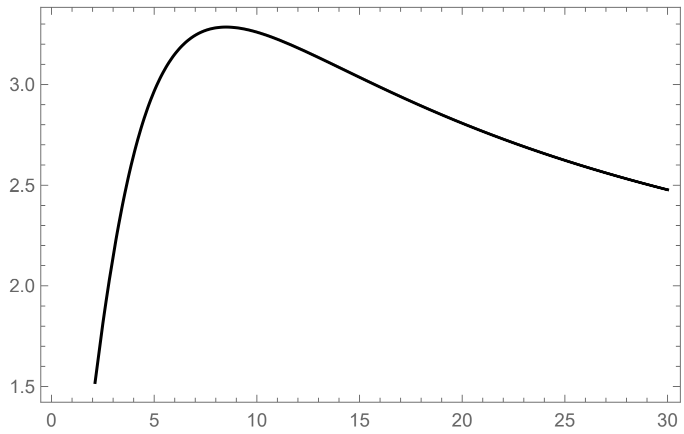

FIGURE 1. Likelihood ratio $\frac{f_{X+Y}}{f_X}$.

expect a similar behaviour in the case when X is IHR but not log-concave, although we did not find examples of this kind.

As a consequence of the previous theorems, it is easy to derive the hazard rate, reversed hazard rate and the likelihood ratio orderings between partial sums of random variables, taking into account that the log-concavity, IHR and DRHR properties are closed under convolutions (see Proposition 4.B.3 and Theorem 4.C.3 in Marshal and Olkin [10], and Corollary 3.3 in Barlow et al. [2], respectively).

Corollary 10. *Let $X_1, X_2, \ldots, X_n$ be nonnegative independent random variables. The following hold, for every $k \leq m \leq n$, $k, m \in \mathbb{N}$.*

(1) *If X_i is IHR, for every $i = 1, \ldots, n$, then $\sum_{i=1}^k X_i \leq_{hr} \sum_{i=1}^m X_i$.*
(2) *If X_i is DRHR, for every $i = 1, \ldots, n$, then $\sum_{i=1}^k X_i \leq_{rhr} \sum_{i=1}^m X_i$.*
(3) *If X_i is log-concave, for every $i = 1, \ldots, n$, then $\sum_{i=1}^k X_i \leq_{lr} \sum_{i=1}^m X_i$.*

4. Variability results

When considering real valued random variables X and Y, it is generally expected that, by summing Y to X, the variability increases. The following result shows that this is indeed the case.

Theorem 11. *Let X and Y be independent random variables with finite means. If $EY = 0$, then $X \leq_{cx} X + Y$. If $EY < 0$, then $X + Y \leq_{icv} X$. If $EY > 0$, then $X \leq_{icx} X + Y$.*

Proof: The case $EY = 0$ coincides with Theorem 3.A.34 in Shaked and Shantikumar [13]. Consider that $EY < 0$. Since $E(Y - EY) = 0$, it follows from the previous case that $X \le_{cx} X+Y-EY$, which is equivalent to $X+Y-EY \le_{cv} X$. Hence, for every increasing concave function ϕ, $E\phi(X) \ge E\phi(X+Y-EY) \ge E\phi(X+Y)$. Since $X \le_{icx} Y$ if and only if $-Y \le_{icv} -X$ (see Theorem 4.A.1 in Shaked ans Shantikumar [13]), the case $EY > 0$ follows immediately. ■

A similar result to Corollary 10 can be established for the increasing convex and concave orders.

Corollary 12. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with finite means. The following hold, for every $k \le m \le n$, $k, m \in \mathbb{N}$.*

(1) *If $EX_i = 0$, for every $i = 1, \ldots, n$, then $\sum_{i=1}^k X_i \le_{cx} \sum_{i=1}^m X_i$,.*
(2) *If $EX_i < 0$, for every $i = 1, \ldots, n$, then $\sum_{i=1}^m X_i \le_{icv} \sum_{i=1}^k X_i$.*
(3) *If $EX_i > 0$, for every $i = 1, \ldots, n$, then $\sum_{i=1}^k X_i \le_{icx} \sum_{i=1}^m X_i$.*

The comparison of variability of two distributions can also be achieved through the study of the number of crossings of the underlying distribution functions (see for example, Theorems 3.A.44 or 4.A.22 in Shaked and Shantikumar [13]). Single crossing results are particularly meaningful, as they imply the variability orders discussed, making it possible to control the behaviour of the distribution functions in a more precise way. To this end, we introduce some notation. Given a function V, we denote by $S^-(V) = n(\sigma)$, $n \in \mathbb{N}$ represents the number of sign variations of V, where $\sigma \in \{+,-\}$ indicates the starting sign of the function (Karlin [7]). We will be omitting σ in the cases when we are only interested in the number of sign variations of the function. The following theorem provides sufficient conditions for establishing that the CDFs of X and $X+Y$ cross once.

Theorem 13. *Let X and Y be independent random variables with symmetric density functions. If $EY = 0$ and f_X is unimodal, then $S^-(F_{X+Y} - F_X) = 1(+)$.*

Proof: Without loss of generality, we may consider that X has a density symmetric at 0. Denote by $V(x) = F_{X+Y}(x) - F_X(x)$. First, observe that $F_{X+Y}(x) = A(x) + B(x)$, with $A(x) = \int_{-\infty}^0 f_X(x-t)F_Y(t)\,dt$ and $B(x) = \int_0^{+\infty} f_X(x-t)F_Y(t)\,dt$. Given that Y has a symmetric density at 0, we may write

$$A(x) = \int_{-\infty}^0 f_X(x-t)\overline{F}_Y(-t)\,dt = \int_0^{+\infty} f_X(x+u)\overline{F}_Y(u)\,du,$$

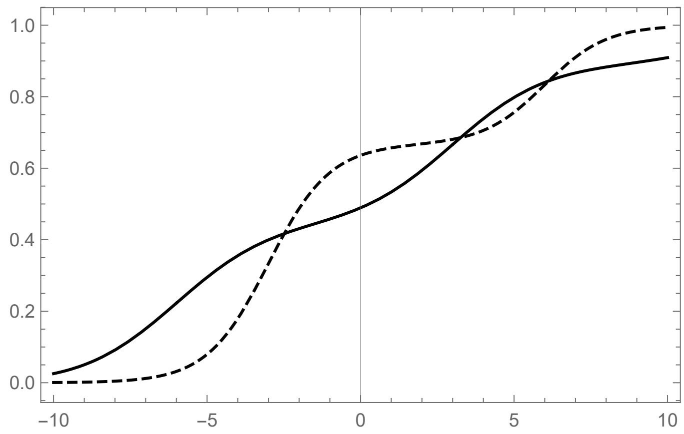

FIGURE 2. CDFs of F_{X+Y} (solid) and F_X (dashed).

by a change of variable. Since $B(x) = F_X(x) - \int_0^{+\infty} f_X(x-t)\overline{F}_Y(t)\,dt$, it follows that $V(x) = \int_0^{+\infty} \overline{F}_Y(t)\,(f_X(x+t) - f_X(x-t))\,dt$. Assume that $x \leq 0$. In this case f_X is increasing, since X is unimodal and symmetric. Thus, if $x+t \leq 0$, it follows that $x+t \geq x-t$, for every x and $t \geq 0$. implying that $f_X(x+t) \geq f_X(x-t)$. On the other and, if $x+t \geq 0$, then $-x-t \leq 0$, and $f_X(x+t)-f_X(x-t) = f_X(-x-t)-f_X(x-t) \geq 0$. Therefore, $V(x) \geq 0$, for $x \leq 0$. Consider now $x \geq 0$. In this case f_X is decreasing. If $x-t \geq 0$, it follows that $f_X(x+t) \leq f_X(x-t)$, since $x-t \leq x+t$, while if $x-t \leq 0$, then $-x+t \geq 0$. Hence, by the symmetry of f_X at 0, we have that $f_X(x+t) - f_X(x-t) = f_X(x+t) - f_X(-x+t) \leq 0$. Thus, $V(x) \leq 0$, for $x \geq 0$. ■

The conclusion of Theorem 13 may be also stated as follows: $X^+ \leq_{st} (X+Y)^+$ and $X^- \leq_{st} (X+Y)^-$, where $X^+ = \max(X,0)$ and $X^- = \max(-X,0)$ represent the positive and negative parts of X. This means that both negative and positive parts of the sum $X+Y$ are stochastically larger when compared to X.

Note that the symmetry conditions of Theorem 13 are important, since we cannot ensure the same behaviour without them. Take for instance $X =_d Y =_d 2T + W$, where $T \sim L(-3,1)$, $W \sim L(6,1)$, and $L(a,b)$ denotes a logistic distribution with location and scale parameters a and b, respectively. Clearly, $EY = 0$, but f_X and f_Y are not symmetric. The CDF of $X+Y$ has a closed form expression and it can be seen, in Figure 2, that it crosses that of Y 3 times.

Chung [4] gave an example that shows that unimodality is not closed under convolutions. However, long-concavity, which is also known as strong unimodality, makes it possible to preserve unimodality under convolution, allowing us to extend Theorem 13 to partial sums of random variables.

Corollary 14. *Let $X_1, X_2, \ldots, X_n$ be independent random variables, such that, for every $i = 1, \ldots, n$, $n \in \mathbb{N}$, X_i has a log-concave density, symmetric at* 0. *Let $T_j = \sum_{i=1}^{j} X_i$. Then $S^-(F_{T_m} - F_{T_k}) = 1(+)$ for every $k \leq m \leq n$, $k, m \in \mathbb{N}$.*

It is interesting to note that log-concavity often plays an important role in determining sign change properties. In particular, the variation diminishing property (see Theorem 3.1 in Karlin [7]), implies the folowing result.

Proposition 15. *Let Z be a log-concave random variable, independent of X and Y. If $S^-(F_Y - F_X) \leq 1$, then $S^-(F_{Y+Z} - F_{X+Z}) \leq 1$.*

This property basically establishes that, if the CDFs of X and Y are at most single crossing, the same behaviour is preserved after summation with an independend random "noise" Z, provided that Z is log-concave. This means that, somewhat surprisingly, one might even have $Y \not\leq_{st} X$ but $Y + Z \leq_{st} X + Z$. In this regard, it is interesting to note that Pomatto et al. [12] recently proved that, if $EX < EY$, there exists some Z independent of X and Y such that $X + Z \leq_{st} Y + Z$, that is, $S^-(F_{Y+Z} - F_{X+Z}) = 0(+)$.

Back to the case of nonnegative random variables, we know from Section 3 that, in general, $X + Y$ is "larger" than X in terms of size. However, given the dispersion results discussed earlier, it is also natural to wonder if sums of random variables may exhibit more or less variability, as measured by the Lorenz order. Contrary to what has been obtained previously, it is possible to conclude that $X \leq_L X + Y$ does not hold in general. In fact, the Lorenz order is a variability order of different type from those considered earlier, being scale-independent. As a counterexample, if X is an exponential random variable, and Y is a Weibull random variable with shape parameter larger than 1, it can be seen that X and $X + Y$ are not comparable with respect to the Lorenz order.

Nevertheless, ordering results may be obtained by replacing the sum $X+Y$ with the product XY. In particular, since $X \leq_* XY$ implies $X \leq_L XY$, the following results provide sufficient conditions to derive the Lorenz ordering, through the star order.

Proposition 16. *Let X and Y be nonnegative independent random variables. Then $X \leq_* XY$ if and only if* $\ln(X)$ *is log-concave.*

Proof: Taking into account Theorem 4.B.1 in Shaked and Shantikumar [13], $X \leq_* XY$ is equivalent to $\ln(X) \leq_{disp} \ln(X) + \ln(Y)$. The result follows by Proposition 2 in Droste and Welfemeyer [5]. ■

The following result is an immediate consequence of Proposition 16 and the closure of log-concavity under convolution.

Corollary 17. *Let $X_1, X_2, \ldots, X_n$ be nonnegative independent random variables, such that* $\ln(X_i)$ *is log-concave, for every $i = 1, \ldots, n$, $n \in \mathbb{N}$. Then $\prod_{i=1}^{k} X_i \leq_* \prod_{i=1}^{m} X_i$, for every $k \leq m \leq n$, $m, k \in \mathbb{N}$.*

Similarly, one can obtain the star order between products of random variables.

Proposition 18. *Let X and Y be nonnegative random variables, such that $X \leq_* Y$. Let Z be another nonnegative random variable independent of X and Y. Then $XZ \leq_* YZ$ if and only if the random variable* $\ln(Z)$ *is log-concave.*

Proof: Taking into account Theorem 4.B.1 in Shaked and Shantikumar [13], $X \leq_* Y$ if and only if $\ln(X) \leq_{disp} \ln(Y)$. Hence, the result follows from Theorem 7 in Lewis and Thompson [9]. ■

The following result may be established by repeated applications of Theorem 18.

Corollary 19. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be nonnegative independent random variables, such that $X_i \leq_* Y_i$, for $i = 1, \ldots, n$. Then $\Pi_{i=1}^{n} X_i \leq_* \Pi_{i=1}^{n} Y_i$, for every $n \in \mathbb{N}$, if* $\ln(X_i)$ *and* $\ln(Y_i)$ *are log-concave, for every $i = 1, \ldots, n$.*

Proof: We prove the result by induction. When $n = 2$, it follows immediately from Theorem 18 and the transitivity of the star order that $X_1 X_2 \leq_* Y_1 Y_2$. Assume now that $\Pi_{i=1}^{k} X_i \leq_* \Pi_{i=1}^{k} Y_i$. Since $\ln(X_{k+1})$ has log-concave density, the previous result implies that $\Pi_{i=1}^{k} X_i X_{k+1} \leq_* \Pi_{i=1}^{k} Y_i X_{k+1}$. Taking into account that a convolution of random variables with log-concave densities still has log-concave density, and given that $X_{k+1} \leq Y_{k+1}$, we may conclude by Theorem 18 that $\Pi_{i=1}^{k} Y_i X_{k+1} \leq_* \Pi_{i=1}^{k} Y_i Y_{k+1}$. Thus, it follows by transitivity that $\Pi_{i=1}^{k+1} X_i \leq_* \Pi_{i=1}^{k+1} Y_i$. ■

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