Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 22–37

RIEMANNIAN HIGH-ORDER SPLINES

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ABSTRACT: This paper is an overview of the work of the authors about generalized polynomial curves and splines on Riemannian manifolds. The emphasis will be put on the variational approach that gives rise to such curves, and on the Hamiltonian formulation for the cubic case.

KEYWORDS: Riemannian polynomial splines, variational problems, Euler-Lagrange equations, generalized Jacobi fiels and conjugate points, m-exponential, optimal control, Hamiltonian equations.

1. Introduction

In this introduction we review the state of the art concerning variational methods that give rise to Riemannian splines, and briefly comment on other approaches to solve interpolation problems on manifolds. We also mention related problems and potential applications in control, geometric mechanics and robotics that have driven an increasing interest in this area.

It is well known that in Euclidean spaces polynomial splines of arbitrary odd degree $k = 2m-1$ are piecewise polynomials that enjoy remarkable optimum properties. These curves, that satisfy some boundary and interpolation conditions, arise as solutions of an optimization problem in which one minimizes an energy functional involving the norm of the m-th derivative. Cubic splines have a wide range of applications due to the fact that they minimize the acceleration cost.

A natural generalization to Riemannian manifolds is based on the simple idea of replacing the Euclidean metric by the Riemannian metric and usual derivatives by covariant derivatives. The definition of Riemannian k-polynomials was first proposed in Camarinha, Silva Leite and Crouch [18], and arose in continuity of the study of cubic splines, an interpolation method based on these curves developed in Gabriel and Kajiya [32], Noakes et al. [53], and Crouch and Silva Leite [28]. Although the extension of this definition has

Received November 10, 2022.
The final version of this paper is published in Proc. Steklov Inst. Math. 321 (2023).
http://dx.doi.org/10.1134/S0081543823020128.

arisen from a natural adaptation of the classical case, it opened up many theoretical challenges. The Riemannian cubic polynomials, which minimize the norm of the intrinsic acceleration, are challenging enough and garnered more attention, even though some results have also been obtained for arbitrary odd degree.

The Riemannian k-polynomials can be seen as a special case of polyharmonic maps. The study of polyharmonic curves is relatively recent ([49], [15], [52], [17]), although we could consider its beginning in 1964, when J. Eells and J. H. Sampson noted that harmonic maps between Riemannian manifolds were the first case of a more general theory [30].

The study of this m-order variational problem on Riemannian manifolds has been taken in many research directions and led to a wide range of applications. The literature on the subject is vast and divided into fields of different research interests, so it is not surprising that k-polynomials were also being called polyharmonic curves or m-geodesics (see for instance [63]).

From a geometrical viewpoint, the study of the problem was first carried out in [18] and the Riemannian k-polynomials were defined as solutions of the Euler-Lagrange equation. A theory showing the analogies of Riemannian k-polynomials with the classical theory of Riemannian geodesics has been developed in [20] for degree $k = 3$, and in [18] and [19] for arbitrary odd degree k. This study includes the generalization of Jacobi fields, conjugate points and the exponential map.

In [36], the m-order variational problem was studied from an analytic point of view and lower estimates on the number of Riemannian k-polynomials satisfying fixed arbitrary conditions were obtained. For $m = 2$, local existence and uniqueness of cubic polynomials were recently explored in [21] and [22]. Turning to the problem of proving the existence of cubic splines, the subject was recently addressed in [38].

With regard to these problems on Lie groups, Noakes and collaborators have an extensive work dedicated to the study of cubic polynomials (see, for instance, [57], [54], [58], and references therein). In particular, they derived the Lie reduction of the Euler-Lagrange equation and introduced the notion of null cubic polynomials. Riemannian cubics, or simply cubics, was the abbreviation used by Noakes for cubic polynomials, and the curves on the Lie algebra obtained by Lie reduction of cubics were designated by Lie quadratics. This terminology for cubic polynomials stood out due to the extension of their work.

For higher-order polynomials on Lie groups, the Lie reduction of these curves to the corresponding Lie algebra was addressed in [45] and [63], as an extension of the result obtained by Noakes for cubics. Zefran et al. [67] analysed the problem of generating smooth rigid body motions, studying left-invariant metrics in $SE(3)$. In [4] the problem is considered in the more general setting of semidirect product of Lie groups.

The second order variational problem was formulated as an optimal control problem in Crouch et al. [26] and analyzed through two different perspectives, a variational approach and a Hamiltonian methodology. The authors also showed that, for Lie groups, the variational approach leads to the Lie reduction of the Euler-Lagrange equation obtained by Noakes. More recently, Balseiro et al. [6] studied the problem from the point of view of the Pontryagin maximum principle in optimal control theory.

Bearing in mind applications to computational anatomy, Gay-Balmaz et al. [34], [33] developed the higher-order framework for Lagrangian and Hamiltonian reduction by symmetry in geometric mechanics and studied higher-order variational problems invariant under Lie group transformations. In this context, they presented the Euler-Poincaré equations for k-polynomials on Lie groups. The Hamiltonian and Hamilton-Ostrogradsky Lie-Poisson formulations of the higher-order Euler-Poincaré theory were also developed.

Apart from the studies on cubics parametrized by arclength, developed by Arroyo, Garay and Mencía [5] on space forms of constant sectional curvature, cubics have rarely been obtained. Noakes made great efforts to determine explicit solutions of the Euler-Lagrange equation in $SO(3)$ and $SO(1,2)$ and, almost invariably, approximated solutions [55].

While the theory of Riemannian splines is already fairly well established, one major problem with applications of this theory is related with difficulties of obtaining explicit solutions.

Numerical integration schemes are also potential powerful tools to obtain approximate solutions. In [16], a geometric integrator was developed for a class of higher-order mechanical systems. The design of geometric integrators proposed in [50] was extended in [24] to higher-order variational systems with boundary conditions. These theories offer different methods of obtaining approximations of k-splines and were illustrated with the Riemannian cubic example.

Alternative methods to solve interpolation problems on Riemannian manifolds that overcome the difficulties of the variational and Hamiltonian approaches have been developed. For instance, in [60] and [29], the classical De Casteljau algorithm was generalized to Riemannian manifolds in general and to some symmetric spaces in particular. More on this subject can also be found in [64]. A numerical algorithm for $\mathcal{C}^2$-smooth splines on symmetric spaces, based on generalized Bézier curves, has also been presented in [14]. These methods don't guarantee that the splines solve the Euler-Lagrange equation of the variational approach, but more recently there has been attempts to incorporate the optimality criteria in the De Casteljau algorithm [37].

Another alternative that produces explicit formulas for interpolating curves on manifolds involves rolling maps, an efficient technique that consists in rolling the data to a flat space so that classical methods can be used to solve the problem, and then roll the solution back to the original manifold. Details may be found, for instance, in [39] and [40]. But in spite of its simplicity, the optimization criteria of such interpolating curves is still in question.

The growing interest in the study of k-splines on manifolds, and in particular cubic splines, was essentially due to their optimality properties. The link with optimal control problems is quite natural.

In relation to optimal control problems, during the last few decades other developments have been achieved having in mind applications in robotics, aeronautics, quantum mechanics, computational anatomy, and many other areas. Unable to give credit to so many contributions, we emphasize some of the most relevant in the study of optimal control problems for mechanical systems evolving on Riemannian manifolds, which are more closely related to the subject of this paper. Without any particular order of importance, we mention [11], [12], [41], [7], [23], and references therein.

Extensions of the problems studied here are already quite vast, but we limit our references to the geometric splines in tension and to least squares problems on manifolds, since their approach is closely related to what is presented here.

Riemannian cubics in tension, also called elastic curves in some literature, are solutions of optimization problems whose the cost functional depends on both velocity and acceleration. They have been studied, for instance, in [66], [59], and [68].

The relation of Riemannian cubics to the Bernoulli elastica appears in [5], with focus on cubics for space forms of constant sectional curvature. This link was also explored later in [31] for Lie groups, and extended to homogeneous spaces in [68], where the authors made the difference between elastica and cubics in tension. The study of elastica on Lie groups was also incorporated into optimal control theory in [44]. Results were obtained through the Pontryagin's Maximum Principle on manifolds, using the Hamiltonian formalism.

In many applications involving smoothing data on non-Euclidean spaces, it is not crucial to find an interpolating curve, but rather a curve that passes reasonably close to the known data. This is the case when the collected data results from experimental tasks is corrupted by noise, or when a small deviation from the data can result in a significant decrease of the cost.

In this context, smoothing geometric splines were defined as solutions of a natural generalization to Riemannian manifolds of the classical least squares problems and studied in [48]. Such generalization is based in the formulation of a high order variational problem, depending on a smoothing parameter, whose solutions are smoothing curves minimizing the average norm of the covariant derivative of order $m \geq 1$ and, in addition, fitting a given set of points on a manifold at prescribed instants of time. This work followed the ideas behind the construction of smoothing splines for the S^2 sphere encountered in [42] and generalized the work in [47] for the case $m = 2$.

The organization of this survey paper is the following. In Section 2 the variational problem that gives rise to high-order splines on Riemannian manifolds (geometric splines) is formulated and the corresponding Euler-Lagrange equations are derived. In order to obtain sufficient optimality conditions, in Section 3 we extend several concepts from the theory of geodesics to geometric polynomials, in particular, we obtain the second variation formula and define generalized Jacobi fields and conjugate points. In Section 4, a generalization of the exponential map is introduced in order to address questions of existence and uniqueness of geometric polynomials. Finally, we formulate in Section 5 the variational problem giving rise to cubic polynomials as an optimal control problem and derive the corresponding Hamiltonian equations.

2. High-order splines on Riemannian manifolds

High-order polynomial splines on $\mathbb{R}^n$, equipped with the Euclidean metric $\langle ., . \rangle$, have been studied intensively in the past. It is well known that, for $m \geq$

2, there exists a unique $2m-2$ times continuously differentiable polynomial spline of degree $2m-1$, satisfying $N+1$ interpolation conditions $x(T_i) = x_i$, $0 \leq i \leq N$, and $2m-2$ prescribed derivatives at the end points, $(d^j/dt^j)x(0)$ and $(d^j/dt^j)x(T)$, $j = 1, \cdots, m-1$. It turns out that this polynomial spline also arises as the only solution of the optimization problem in which one minimizes the functional

$$J(x) = \frac{1}{2}\int_0^T \langle \frac{d^m x}{dt^m}, \frac{d^m x}{dt^m} \rangle \, dt, \tag{1}$$

among all $\mathcal{C}^m$ functions satisfying the same prescribed data. In this case, the Euler-Lagrange equation is

$$\frac{d^{2m}x(t)}{dt^{2m}} = 0, \quad \forall t \in [T_{i-1}, T_i] \text{ and } 1 \leq i \leq N. \tag{2}$$

We refer, for instance, to Prenter [65] for details.

The generalization to non-Euclidian spaces of high-order polynomial splines appeared first in Camarinha, Silva Leite and Crouch [18]. Instead of the Euclidean space, we now consider a Riemannian manifold M of dimension n, with Riemannian metric $\langle \cdot, \cdot \rangle$. Denote by ∇ the Levi-Civita connection on M, which is the symmetric connection compatible with that metric. If $t \mapsto x(t)$ is a curve in M with velocity vector field $V(x(t)) = dx(t)/dt = V(t) \in T_{x(t)}$ and W is a vector field defined on a neighborhood of the curve x, then the covariant derivative along x of W is defined by $DW(t)/dt = (\nabla_V W)(x(t))$. We can similarly define the covariant derivative of a covector field η along x by considering $D\eta(t)/dt = (\nabla_V \eta)(x(t))$, bearing in mind that now ∇ is the connection induced on covector fields by the Levi-Civita connection. In what follows we also denote by $D^i x(t)/dt^i$ the covariant derivative of order $i-1$ of the velocity vector field V, which is given by $D^i x(t)/dt^i = D(D^{i-1}x(t)/dt^{i-1})/dt$. We refer to Milnor [51] and Lee [46] for details concerning Riemannian geometry and, in particular, for properties of covariant derivatives. We now consider the following optimization problem, where the energy functional is the counterpart of (1).

Problem ($\mathcal{P}_1$):

Find the critical points of the functional

$$J(x) = \frac{1}{2}\int_0^T \langle \frac{D^m x}{dt^m}, \frac{D^m x}{dt^m} \rangle \, dt, \tag{3}$$

over the class Γ of $\mathcal{C}^{2m-3}$-smooth paths x on M satisfying $x|_{[T_{i-1},T_i]}$ is smooth,

$$x(T_i) = x_i, \;\; 0 \leq i \leq N, \tag{4}$$

for a distinct set of points $x_i \in M$ and fixed times T_i, $0 \leq i \leq N$, where $0 = T_0 < T_1 < \cdots < T_{N-1} < T_N = T$, and, in addition,

$$\frac{D^j x}{dt^j}(0) = v_{0j}, \quad \frac{D^j x}{dt^j}(T) = v_{Tj}, \quad 1 \leq j \leq m-1, \tag{5}$$

where v_{ij}, with $i = 0, T$ and $1 \leq j \leq m-1$, are fixed n-vectors.

Before presenting the main result of this section, we need to introduce a few more concepts.

For each curve $x \in \Gamma$ we define the tangent space $T_x\Gamma$ to Γ at x as the vector space of $\mathcal{C}^{2m-3}$-vector fields $t \mapsto W(t)$ along x such that W is smooth on each interval $[T_{i-1}, T_i], 1 \leq i \leq N$, and satisfies the interpolation conditions

$$W(T_i) = 0, \;\; 1 \leq i \leq N-1, \tag{6}$$

and the boundary conditions

$$W(0) = 0, \qquad \frac{D^j W}{dt^j}(0) = 0, \; j = 1, \ldots, m-1, \tag{7}$$

$$W(T) = 0, \qquad \frac{D^j W}{dt^j}(T) = 0, \; j = 1, \ldots, m-1. \tag{8}$$

Now let $\alpha : [0,T] \times (-\epsilon, +\epsilon) \mapsto M$, for some $\epsilon > 0$, be a one-parameter variation of x, defined through the exponential map on M and a vector field $W \in T_x\Gamma$ by $\alpha(t,u) = exp_{x(t)}(uW(t))$. This variation satisfies the following conditions:

$$
\begin{aligned}
&\circ\ \alpha \text{ is smooth on each interval } [T_{i-1},T_i]\times(-\epsilon,+\epsilon), 1\le i\le N;\\
&\circ\ \alpha(t,0)=x(t), \quad 0\le t\le T;\\
&\circ\ \frac{\partial\alpha}{\partial u}(t,0)=W(t),\ \ 0\le t\le T;\\
&\circ\ \frac{\partial\alpha}{\partial u}(T_i,0)=W(T_i)=0,\ \ 0\le i\le N;\\
&\circ\ \frac{D}{dt}\frac{\partial\alpha}{\partial u}(t,0)=\frac{DW}{dt}(t)\ \text{ is }\ \mathcal{C}^{2m-4}\ \text{ on }\ [0,T];\\
&\circ\ \frac{D^j}{dt^j}\frac{\partial\alpha}{\partial u}(0,0)=\frac{D^j}{dt^j}\frac{\partial\alpha}{\partial u}(T,0)=0,\ \ j=1,\cdots m-1.
\end{aligned}
\tag{9}
$$

To simplify notations, write $\alpha_u(t)$ for $\alpha(t,u)$. The following properties of the curvature tensor can be found in Milnor [51] and Nomizu [56] and will play an important role here. (We note that Milnor defines R with the opposite sign.) For X,Y,Z and W any vector fields in M one has:

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0; \tag{10}$$

$$\langle R(X,Y)Z,W\rangle=\langle R(W,Z)Y,X\rangle; \tag{11}$$

$$
\begin{aligned}
\nabla_W(R(X,Y)Z) &= (\nabla_W R)(X,Y)Z+R(\nabla_W X,Y)Z\\
&\qquad +R(X,\nabla_W Y)Z+R(X,Y)\nabla_W Z.
\end{aligned}
\tag{12}
$$

We also need the following result which can easily be proved by induction, using the fact that, in general, the covariant differentiation operators $D/\partial u$ and $D/\partial t$ do not commute (see, for instance, Milnor [51]).

Lemma 2.1. *Let α be a one-parameter variation of x, satisfying conditions* (9)*. Then, $\forall m\ge 2$, we have*

$$\frac{D}{\partial u}\frac{D^m\alpha_u}{\partial t^m} = \frac{D^m}{\partial t^m}\frac{\partial\alpha_u}{\partial u}+\sum_{j=2}^{m}\frac{D^{m-j}}{\partial t^{m-j}}\left(R(\frac{\partial\alpha_u}{\partial u},\frac{\partial\alpha_u}{\partial t})\frac{D^{j-1}\alpha_u}{\partial t^{j-1}}\right) \tag{13}$$

Theorem 2.2. *A necessary condition for x to be a minimizer of the functional* (3) *over the class of $\mathcal{C}^{2m-3}$ paths x on M such that $x|_{[T_{i-1},T_i]}$ is smooth and also satisfies* (4) *and* (5) *is that, x is $\mathcal{C}^{2m-2}$, and, $\forall t\in[T_{i-1},T_i]$ and $1\le i\le N$, the following holds*

$$\frac{D^{2m-1}V}{dt^{2m-1}}(t) + \sum_{j=2}^{m}(-1)^j R\left(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}(t), \frac{D^{j-2}V}{dt^{j-2}}(t)\right) V(t) = 0, \tag{14}$$

where $V(t) = dx(t)/dt$ and R is the curvature tensor of the Levi-Civita connection ∇ on M.

Proof: A curve x is a critical path for the functional (3) if $\frac{\partial}{\partial u}J(\alpha_u)\big|_{u=0} = 0$, for all variations $\alpha : [0,T] \times (-\epsilon, +\epsilon) \to M$, satisfying conditions (9). Using the result of Lemma 2.1, one has

$$\begin{aligned} \frac{\partial}{\partial u}J(\alpha_u) &= \int_0^T \langle \frac{D}{\partial u}\frac{D^m\alpha_u}{\partial t^m}, \frac{D^m\alpha_u}{\partial t^m}\rangle dt = \int_0^T \Big\{ \langle \frac{D^m}{\partial t^m}\frac{\partial\alpha_u}{\partial u}, \frac{D^m\alpha_u}{\partial t^m}\rangle \\ &+ \sum_{j=2}^{m}\langle \frac{D^{m-j}}{\partial t^{m-j}}\left(R(\frac{\partial\alpha_u}{\partial u}, \frac{\partial\alpha_u}{\partial t})\frac{D^{j-1}\alpha_u}{\partial t^{j-1}}\right), \frac{D^m\alpha_u}{\partial t^m}\rangle \Big\}\, dt. \end{aligned} \tag{15}$$

Now, integrating m times by parts on each interval $[T_{i-1}, T_i]$, $\ 1 \le i \le N$, one gets:

$$\begin{aligned} \int_0^T & \langle \frac{D^m}{\partial t^m}\frac{\partial\alpha_u}{\partial u}, \frac{D^m\alpha_u}{\partial t^m}\rangle dt = \\ &= \sum_{i=1}^{N}\sum_{l=1}^{m}(-1)^{l+1}\left[\langle \frac{D^{m-l}}{\partial t^{m-l}}\frac{\partial\alpha_u}{\partial u}, \frac{D^{m+l-1}\alpha_u}{\partial t^{m+l-1}}\rangle\right]_{T_{i-1}^+}^{T_i^-} + \\ &+(-1)^m\int_0^T \langle \frac{\partial\alpha_u}{\partial u}, \frac{D^{2m}\alpha_u}{\partial t^{2m}}\rangle\, dt, \end{aligned} \tag{16}$$

and

$$\int_0^T \langle \frac{D^{m-j}}{\partial t^{m-j}} \left(R(\frac{\partial \alpha_u}{\partial u}, \frac{\partial \alpha_u}{\partial t}) \frac{D^{j-1}\alpha_u}{\partial t^{j-1}} \right), \frac{D^m \alpha_u}{\partial t^m} \rangle \, dt$$

$$\begin{aligned} &= \sum_{i=1}^{N} \Big[\langle \frac{D^{m-j-1}}{\partial t^{m-j-1}} \left(R(\frac{\partial \alpha_u}{\partial u}, \frac{\partial \alpha_u}{\partial t}) \frac{D^{j-1}\alpha_u}{\partial t^{j-1}} \right), \frac{D^m \alpha_u}{\partial t^m} \rangle \\ &\quad - \langle \frac{D^{m-j-2}}{\partial t^{m-j-2}} \left(R(\frac{\partial \alpha_u}{\partial u}, \frac{\partial \alpha_u}{\partial t}) \frac{D^{j-1}\alpha_u}{\partial t^{j-1}} \right), \frac{D^{m+1} \alpha_u}{\partial t^{m+1}} \rangle + \\ &\quad + \cdots + (-1)^{m-j-1} \langle R(\frac{\partial \alpha_u}{\partial u}, \frac{\partial \alpha_u}{\partial t}) \frac{D^{j-1}\alpha_u}{\partial t^{j-1}}, \frac{D^{2m-j-1}\alpha_u}{\partial t^{2m-j-1}} \rangle \Big]_{T_{i-1}^+}^{T_i^-} + \\ &\quad + (-1)^{m-j} \int_0^T \langle R\left(\frac{\partial \alpha_u}{\partial u}, \frac{\partial \alpha_u}{\partial t}\right) \frac{D^{j-1}\alpha_u}{\partial t^{j-1}}, \frac{D^{2m-j}\alpha_u}{\partial t^{2m-j}} \rangle \, dt. \end{aligned} \tag{17}$$

Replacing (16) and (17) in (15) and using properties (11) and (12) for the curvature tensor and also identities (9), we obtain, after several calculations, the following:

$$\begin{aligned} \frac{\partial}{\partial u} J(\alpha_u) \Big|_{u=0} \;=\; & \sum_{i=1}^{N-1} \langle \frac{DW}{dt}(T_i), \frac{D^{2m-3}V}{dt^{2m-3}}(T_i^+) - \frac{D^{2m-3}V}{dt^{2m-3}}(T_i^-) \rangle \\ & + (-1)^m \int_0^T \langle W, \frac{D^{2m-1}V}{dt^{2m-1}} + \\ & + \sum_{j=2}^{m} (-1)^j R\left(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}, \frac{D^{j-2}V}{dt^{j-2}} \right) V \rangle \, dt. \end{aligned} \tag{18}$$

Now choose a variation α of x with variation vector field W given by

$$W(t) = F(t) \left(\frac{D^{2m-1}V}{dt^{2m-1}}(t) + \sum_{j=2}^{m} (-1)^j R\left(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}(t), \frac{D^{j-2}V}{dt^{j-2}}(t) \right) V(t) \right),$$

where $F(t) > 0, t \in (T_{i-1}, T_i),\ 1 \le i \le N$, and $F(T_i) = (D/dt)F(T_i) = 0,\ 1 \le i \le N$. Then, it follows from (18) that, if $(\partial/\partial u) J(\alpha_u)|_{u=0} = 0$,

$$\frac{D^{2m-1}V}{dt^{2m-1}}(t) + \sum_{j=2}^{m} (-1)^j R\left(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}(t), \frac{D^{j-2}V}{dt^{j-2}}(t) \right) V(t) = 0, \tag{19}$$

$\forall t \in [T_{i-1}, T_i]$ and $1 \leq i \leq N$, and this shows that (19) is a necessary condition for x to be a minimizer.

To show that x is $\mathcal{C}^{2m-2}$, suppose that x is a minimizer. Then, under this condition, the second term in the right-hand side of (18) is zero. If α is a variation of x with variation vector field W satisfying

$$\frac{DW}{dt}(T_i) = \frac{D^{2m-3}V}{dt^{2m-3}}(T_i^+) - \frac{D^{2m-3}V}{dt^{2m-3}}(T_i^-), \quad 1 \leq i \leq N-1,$$

it follows from (18) that $(D^{2m-3}V/dt^{2m-3})(T_i^+) = (D^{2m-3}V/dt^{2m-3})(T_i^-), 1 \leq i \leq N-1$, so x is of class $\mathcal{C}^{2m-2}$. This completes the proof of Theorem 2.2

■

Remark 2.3. *If, in the statement of Theorem 2.2, we relax the smoothness condition for the curve x on each subinterval $[T_{i-1}, T_i]$ by considering that $x_{|[T_{i-1},T_i]}$ is piecewise smooth, one can still show that these segments are smooth.*

Definition 2.4. *We say that the curve $x \in \Gamma$ is a $\mathcal{C}^{2m-2}$-*geometric spline *on M if the velocity vector field V along x satisfies the equation (14), or equivalently, if*

$$\frac{D^{2m}x(t)}{dt^{2m}} + \sum_{j=2}^{m}(-1)^j R\left(\frac{D^{2m-j}x(t)}{dt^{2m-j}}, \frac{D^{j-1}x(t)}{dt^{j-1}}\right)\frac{Dx(t)}{dt} = 0, \tag{20}$$

$\forall t \in [T_{i-1}, T_i]$ and $1 \leq i \leq N$.
Equation (20) is the Euler-Lagrange equation *for the variational problem $(\mathcal{P}_1)$.*
In the absence of interpolating points, the solutions of the Euler-Lagrange equation are geometric polynomials *on M.*

Remark 2.5.

- *If $M = \mathbb{R}^n$, the curvature tensor is zero and the covariant derivative is the usual derivative. In this case, the equation (20) reduces to the equation (2) of a Euclidean polynomial spline of degree $2m-1$.*
- *If $m = 1$ and there are no interpolating points, the equation (14) reduces to the well known equation of a geodesic in M.*
- *If $m = 2$, the equation (20) reduces on each subinterval $[T_{i-1}, T_i]$ to*

$$\frac{D^4x}{dt^4} + R(\frac{D^2x}{dt^2}, \frac{Dx}{dt})\frac{Dx}{dt} = 0, \tag{21}$$

giving the geometric cubic splines in Crouch and Silva Leite [27] *and Noakes, Heinzinger and Paden* [53].

The equation (20) is highly nonlinear and there is no hope to solve it explicitly even for $m = 2$.

From the Euler-Lagrange equation (14), it is easy to obtain the following invariant along a geometric polynomial.

$$\sum_{j=2}^{m}(-1)^j\langle\frac{D^{2m-j}V}{dt^{2m-j}},\frac{D^{j-2}V}{dt^{j-2}}\rangle+\frac{1}{2}(-1)^{m+1}\langle\frac{D^{m-1}V}{dt^{m-1}},\frac{D^{m-1}V}{dt^{m-1}}\rangle=k, \tag{22}$$

for a real constant k. When $m = 2$, this invariant reduces to

$$\langle\frac{D^2V}{dt^2},V\rangle-\frac{1}{2}\langle\frac{DV}{dt},\frac{DV}{dt}\rangle=k. \tag{23}$$

This first integral turns out to be the Hamiltonian function associated with the Hamiltonian formulation of the variational problem, as will be clear later on.

3. Jacobi fields along geometric polynomials

The Euler-Lagrange equation gives a necessary condition for x to be a solution of problem $(\mathcal{P}_1)$. Contrary to what happens in Euclidean spaces, for general Riemannian manifolds it is much harder to show whether or not being a solution of the Euler-Lagrange equation is also a sufficient condition. In order to find a possible answer to this question, we must find the second variation of the functional J given by (3) and develop a variational theory based on the notions of Jacobi fields and conjugate points. This is the objective of the present section.

In the absence of interpolating points, the extended class of curves mentioned in Remark 2.3 reduces to the class Ω of $\mathcal{C}^{2m-3}$ piecewise smooth curves x on M satisfying the boundary conditions

$$x(T_0)=x_0,\;\; x(T_N)=x_N, \tag{24}$$

in addition with (5). For each curve $x \in \Omega$ we define the tangent space $T_x\Omega$ at x as the vector space of $\mathcal{C}^{2m-3}$ piecewise smooth vector fields X along x verifying the boundary conditions (7)–(8), and consider a two-parameter admissible variation of $x(t)$ $\alpha : [0,T]\times(-\epsilon,\epsilon)\times(-\delta,\delta)\mapsto M$, defined for some $\epsilon > 0$ and $\delta > 0$ by the exponential mapping on M through the equality $\alpha(t,r,s) = exp_{x(t)}(rX(t) + sY(t))$, where $X, Y \in T_x\Omega$. To simplify notations write $\alpha_{r,s}(t)$ for $\alpha(t,r,s)$.

3.1. Second variation of the functional J. In order to obtain sufficient optimality conditions, first we need to obtain the second variation of J. From now on, x denotes a geometric polynomial of degree $k = 2m - 1$.

Theorem 3.1. *If α is the two-parameter admissible variation of x associated with the vector fields X and Y belonging to $T_x\Omega$, then*

$$\frac{\partial^2}{\partial s \partial r} J(\alpha_{r,s})_{|r=s=0} = \int_0^T \left(\langle \frac{D^m X}{dt^m}, \frac{D^m Y}{dt^m} \rangle + (-1)^m \langle X, F(Y,V) \rangle \right) dt, \quad (25)$$

where F is defined by

$$\begin{aligned} F(Y,V) = & \sum_{j=2}^{2m} \frac{D^{2m-j}}{dt^{2m-j}} [R(Y,V) \frac{D^{j-2}V}{dt^{j-2}}] \\ & + \sum_{j=2}^{m} (-1)^j \Bigg[(\nabla_Y R)(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}, \frac{D^{j-2}V}{dt^{j-2}})V \\ & + R(\frac{D^{2m-j}Y}{dt^{2m-j}}, \frac{D^{j-2}V}{dt^{j-2}})V + R(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}, \frac{D^{j-1}Y}{dt^{j-1}})V \\ & + R(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}, \frac{D^{j-2}V}{dt^{j-2}}) \frac{DY}{dt} \\ & + \sum_{i=2}^{2m-j} R(\frac{D^{2m-j-i}}{dt^{2m-j-i}} [R(Y,V) \frac{D^{i-2}V}{dt^{i-2}}], \frac{D^{j-2}V}{dt^{j-2}})V \\ & + \sum_{i=2}^{j-1} R(\frac{D^{2m-j-1}V}{dt^{2m-j-1}}, \frac{D^{j-1-i}}{dt^{j-1-i}} [R(Y,V) \frac{D^{i-2}V}{dt^{i-2}}])V \Bigg]. \end{aligned} \quad (26)$$

The proof of this result involves calculations but is otherwise straightforward.

According to the last theorem, the expression (25) defines a bilinear form on $T_x\Omega$. It is called the *second variation* of J at x and will be denoted by I_x. The second variation of J may also be defined by the alternative formula:

$$
\begin{aligned}
I_x(X,Y) = \sum_{i=1}^{l-1}\sum_{k=1}^{m}(-1)^k\langle\frac{D^{m-k}X}{dt^{m-k}}(t_i), \Delta_i\frac{D^{m+k-1}Y}{dt^{m+k-1}}\rangle \\
+(-1)^m\int_0^T \langle X, \frac{D^{2m}Y}{dt^{2m}} + F(Y,V)\rangle\, dt,
\end{aligned}
\tag{27}
$$

where, $\Delta_i Z = Z(t_i{}^+) - Z(t_i{}^-)$ for a vector field Z along x and a partition of $[0,T]$, $0 = t_0 < t_1 < \cdots < t_l = T$, such that $\alpha_{r,s|[t_{i-1},t_i]}$, $i = 1,\cdots,l$, is smooth.

3.2. Generalized Jacobi fields and conjugate points.

Definition 3.2. *A smooth vector field W along a geometric polynomial x is said to be a Jacobi field if W solves the equation*

$$
\frac{D^{2m}W}{dt^{2m}} + F(W,V) = 0, \tag{28}
$$

called the Jacobi equation.

The notion of Jacobi fields is strictly related with the following definition, which plays an important role in the discussion of the problem.

Definition 3.3. The points $t = t_1$ and $t = t_2$, $t_1, t_2 \in [0,T], t_1 \neq t_2$, are said *conjugate* along the geometric polynomial x, if there exists a non-zero Jacobi field W along x satisfying

$$
W(t_1) = 0, \qquad W(t_2) = 0,
$$

$$
\frac{D^jW}{dt^j}(t_1) = 0, \ \frac{D^jW}{dt^j}(t_2) = 0, \ \ j = 1,\ldots,m-1.
$$

The *multiplicity* of $t = t_1$ and $t = t_2$ as conjugate points is the dimension of the vector space consisting of all such Jacobi fields.

We may also say that the points $(x(t_1), \frac{dx}{dt}(t_1), \frac{D^2x}{dt^2}(t_1), \cdots, \frac{D^{m-1}x}{dt^{m-1}}(t_1))$ and $(x(t_2), \frac{dx}{dt}(t_2), \frac{D^2x}{dt^2}(t_2), \cdots, \frac{D^{m-1}x}{dt^{m-1}}(t_2))$ are *conjugate* along x, if there is no ambiguity, that is, if the points do not coincide.

The following theorem establishes the relationship between Jacobi fields and the nullspace of the second variation. This general result has much in common with analogous theorems in the theory of geodesics and the theory of generalized cubic polynomials developed in Crouch and Silva Leite [28] and Camarinha et al. [20].

Theorem 3.4. *The vector field $W \in T_x\Omega$ belongs to the nullspace of I_x if and only if W is a Jacobi field along x. Hence, I_x is degenerate if and only if $t = 0$ and $t = T$ are conjugate along x.*

4. A generalization of the exponential map

Consider the k-tangent bundle $T^kM = \bigcup_{q\in M}(T_qM)^k$, with $k = 2m-1$. A geometric polynomial x of order k is uniquely defined by the initial conditions

$$x(0) = x_0, \quad \frac{D^jx}{dt^j}(0) = v_{0j}, \quad 1 \leq j \leq k, \tag{29}$$

where $x_0 \in M$ and $v_{i,j} \in T_{x_0}M$, with $i = 0, T$ and $1 \leq j \leq m-1$. Such a polynomial will be denoted by $x_{v_0^{(k)}}$, with $v_0^{(k)} = (x_0, v_{0,1}, v_{0,2}, \cdots, v_{0,k}) \in T^kM$.

Let B be a neighborhood of $v_0^{(k)}$ and δ a positive real number such that, for each $v_1^{(k)} \in B$, there exists a unique k-polynomial $x_{v_1^{(k)}}$ defined in the interval $(-\delta, \delta)$. Consider a neighborhood D of $v_0^{(m,k)} = (v_{0,m}, v_{0,m+1}, \cdots, v_{0,k})$ in $(T_{x_0}M)^m$ such that $\{v_0^{(m-1)}\} \times D \subset B$. Also assume $0 < t < \delta$.

The m-exponential is the map $m\text{-}\exp^t_{v_0^{(m-1)}} : D \to T^mM$ that assigns, to each point $v_1^{(m,k)}$, the point $(x(t), \frac{dx}{dt}(t), \frac{D^2x}{dt^2}(t), \cdots, \frac{D^{m-1}x}{dt^{m-1}}(t))$, where $x = x_{v_1^{(k)}}$. This map is well defined and smooth on D.

Now, we recall that it is possible to interpret m-conjugate points in terms of the m-exponential map if we assume that the map $m\text{-}\exp^t_{v_0^{(m-1)}}$ is defined in a neighborhood of $v_0^{(m,k)}$, for each $t \in [0, T]$.

Proposition 4.1. *The points $t = 0$ and $t = t_0$, $t_0 \in (0, T]$, are m-conjugate along the k-polynomial $x_{v_1^{(k)}}$ if and only if the map $m\text{-}exp^{t_0}_{v_0^{(m-1)}}$ is critical at $v_0^{(m,k)}$.*

The m-exponential map establishes the connection between the initial conditions (29) and the boundary conditions given in (4)–(5) and, as happens in the theory of geodesics, it is a key ingredient to address local existence and uniqueness of geometric polynomials satisfying boundary conditions. Actually, this map can be seen as a generalization of the exponential map exp_{x_0}, though, in general, there is no guarantee the geometric polynomials can be defined at $t = 1$.

A first attempt to study this map has been made in [21] and [22], for cubic polynomials ($m = 2$). The m-exponential map is, in this case, called *biexponential map.*

Proposition 4.2 ([21]). *Let* $v_0^{(3)} = (x_0, v_{0,1}, v_{0,2}, v_{0,3}) \in T^3M$, $2\text{-}exp^T_{(x_0,v_{0,1})}$ *the biexponential map defined in a neighborhood* D *of* $(v_{0,2}, v_{0,3})$ *in* $(T_pM)^2$ *and* $(x_1, v_{1,1}) = 2\text{-}exp^T_{(x_0,v_{0,1})}(v_{0,2}, v_{0,3})$, *with* $T > 0$. *If the biexponential map* $2\text{-}exp^T_{(x_0,v_{0,1})}$ *is not critical at* $(v_{0,2}, v_{0,3})$, *then there exist two neighborhoods* W_1 *and* W_2 *of* $(x_0, v_{0,1})$ *and* $(x_1, v_{1,1})$, *respectively, and a neighborhood* U *of* $(x_0, v_{0,1}, v_{0,2}, v_{0,3})$ *such that:*

(1) For each $(\tilde{x}_0, \tilde{v}_{0,1}) \in W_1$ *and* $(\tilde{x}_1, \tilde{v}_{1,1}) \in W_2$, *there exists a unique cubic* y *satisfying*

$$y(0) = \tilde{x}_0, \;\; y(T) = \tilde{x}_1, \;\; \frac{dy}{dt}(0) = \tilde{v}_{0,1}, \;\; \frac{dy}{dt}(T) = \tilde{v}_{1,1}, \tag{30}$$

$$(y(0), \frac{dy}{dt}(0), \frac{D^2y}{dt^2}(0), \frac{D^3y}{dt^3}(0)) \in U. \tag{31}$$

(2) This cubic y *depends smoothly on the points* $(\tilde{x}_0, \tilde{v}_{0,1})$ *and* $(\tilde{x}_1, \tilde{v}_{1,1})$, *in the sense that the following map is smooth.*

$$\begin{array}{ccc} [0,T] \times W_1 \times W_2 & \to & M \\ (t, (\tilde{x}_0, \tilde{v}_{0,1}), (\tilde{x}_1, \tilde{v}_{1,1})) & \mapsto & y(t) \end{array}.$$

(3) $2\text{-}exp^T_{(x_0,v_{0,1})}$ *maps an open set* $C \subset D$ *in* $(T_{x_0}M)^2$ *diffeomorphically onto an open set* $Z \supset W_2$ *in* TM.

The simplest case when $(v_{0,2}, v_{0,3})$ is a regular point of $2\text{-}\exp^T_{(x_0,v_{0,1})}$ is obtained by considering the zero vector at x_0 for each $v_{0,j}$, $j = 1, 2, 3$. In this case, Proposition 4.2 reduces to the following result.

Proposition 4.3 ([21]). *Let* $x_0 \in M$ *and* $T > 0$. *There exists a neighborhood* W *of* $(x_0, 0)$ *in* TM *and a positive real number* ϵ *so that:*

(1) For each points $(\tilde{x}_0, \tilde{v}_{0,1})$ *and* $(\tilde{x}_1, \tilde{v}_{1,1}) \in W$, *there exists a unique cubic* y *such that*

$$y(0) = \tilde{x}_0, \; y(T) = \tilde{x}_1, \; \frac{dy}{dt}(0) = \tilde{v}_{0,1}, \; \frac{dy}{dt}(T) = \tilde{v}_{1,1},$$

$$\|\frac{D^2y}{dt^2}(0)\| < \epsilon, \;\; \|\frac{D^3y}{dt^3}(0)\| < \epsilon.$$

(2) This cubic y *depends smoothly on the points* $(\tilde{x}_0, \tilde{v}_{0,1})$ *and* $(\tilde{x}_1, \tilde{v}_{1,1})$.

(3) 2-$exp^T_{(x_0,0)}$ maps the open set

$$B_\epsilon(0,0) = \{(\tilde{v}_{0,2}, \tilde{v}_{0,3}) : \|\tilde{v}_{0,2}\| < \epsilon, \|\tilde{v}_{0,3}\| < \epsilon\}$$

in $(T_{x_0}M)^2$ diffeomorphically onto an open set $Z \supset W$ in TM.

Proposition 4.3 establishes the existence and uniqueness of cubics for boundary data with sufficiently short length. The extension of this result to a more general boundary data follows naturally for nonzero tangent vectors $v_{0,1}$, although a compromise between the length of the tangent vector $v_{0,1}$ and the time t is needed.

Proposition 4.4 ([22])**.** *Let $p \in M$. There exists a positive number ϵ such that, for $v \in T_pM$ and $T > 0$ verifying $T\|v\| < \epsilon$, the biexponential map 2-$exp^T_{(x_0,v_{0,1})}$ is not critical at $(0,0)$.*

From the previous result, the hypothesis of Proposition 4.2 are satisfied for boundary data $(x_0, v_{0,1})$ and $(x_1, v_{1,1})$ of sufficiently small arcs of geodesic and we can guarantee the existence and uniqueness of cubics with boundary data close to the boundary data $(x_0, v_{0,1})$ and $(x_1, v_{1,1})$.

Proposition 4.5 ([22])**.** *Consider $(x_0, v_{0,1}) \in TM$, $T > 0$ and $\epsilon > 0$ in the conditions of Proposition 4.4. Then the biexponential map 2-$exp^T_{(x_0,v_{0,1})}$ is not critical at $(0,0)$ and, therefore, if x is the geodesic defined on $[0,T]$ by the initial data $(x_0, v_{0,1})$, with $(x_1, v_{1,1}) = (x(T), \dfrac{dx}{dt}(T))$, then there exist two neighborhoods W_1 and W_2 of $(x_0, v_{0,1})$ and $(x_1, v_{1,1})$, respectively, and a neighborhood U of $(x_0, v_{0,1}, 0, 0)$ in T^3M verifying the conditions 1,2 and 3 of Proposition 4.2.*

In [22] the authors address the problem of extending these results to a more general boundary data and explore methods to study the existence and uniqueness of cubics close to a reference cubic $x = x_{v_0^{(3)}}$ defined in $[0,T]$ by the initial data $v_0^{(3)} \in T^3M$. If the vector fields $V = \frac{dx}{dt}$, $Y = \frac{D^2x}{dt^2}$ and $Z = \frac{D^3x}{dt^3}$ have sufficiently small length along x, then the reference cubic is close to a geodesic in a certain sense, this is, the cubic x can be interpreted as a nearly geodesic cubic. This analysis can, in particular, be applied to a geodesic x with velocity of sufficiently small length, and reduces to Proposition 4.5.

Finally, we remark the role bi-Jacobi fields play in these studies. In fact, we should mention that Noakes [54] and later on Noakes and Ratiu [55] have also used bi-Jacobi fields to compute approximations of nearly geodesic cubics for studies on $SO(3)$ with bi-invariant and left-invariant Riemannian metrics.

5. Geometric polynomials and optimal control

The Pontryagin Maximum Principle (PMP) is the key result in optimal control theory. R. Gamkrelidze made fundamental contributions to this theory, having been the first to prove the PMP while setting the grounds of the Mathematical Theory of Optimal Processes, together with L. Pontryagin, V. Boltyanskii and E. Mishchenko, in the late 1950's ([61], [62]). New development and generalizations of the Pontryagin Maximum Principle, to include broader classes of optimal control problems, have evolved since then. We refer to [2] for a modern geometric point of view of the mathematical theory of optimal control for manifolds, and to [43] for the case of Lie groups.

Many problems on geometric mechanics and geometry can be incorporated into optimal control. The Hamiltonian formalism at the bottom of the PMP opens new perspectives of research that enrich the subject in hand.

Back to the present case, Riemannian cubics arise in optimal control for mechanical systems. In this situation, one may encounter high order bundles that make the Hamiltonian approach quite cumbersome. One situation that illustrates this, but is still manageable, is related to the optimal control problem associated to the cubic geometric polynomials which are solutions of the unconstrained variational problem $(\mathcal{P}_1)$, with $m = 2$ and $N = 1$.

Consider the optimal control problem $(\mathcal{P}_2)$:

$$\min_u \int_0^T \frac{1}{2}\langle u, u\rangle dt \tag{32}$$

subject to:

$$\dot{x} = V, \quad \frac{DV}{dt} = u, \tag{33}$$

$$\begin{aligned} x(0) &= x_0, & \dot{x}(0) &= v_0, \\ x(T) &= x_T, & \dot{x}(T) &= v_T, \end{aligned} \tag{34}$$

where x_0 and x_T are given points in M, v_0 and v_T are tangent vectors to M, at x_0 and x_T respectively.

The system of equations (33) may already be viewed as the reduction to TM, of a system which is viewed in the Hamiltonian setting as one in TT^*M. To solve the optimal control problem, however, the maximum principle instructs us that extremal solutions are projections of a Hamiltonian flow in TT^*TM. This situation is already complicated, using the canonical symplectic form on T^*TM. Here we explore the possibility of writing down the extremal solutions of the problem (32) - (33) - (34) as a flow on the space

$E = \cup_{q\in M} T_qM \oplus T_q^*M \oplus T_q^*M$. We exhibit the extremal equations in Hamiltonian form and identify the correct symplectic form, but our result is dependent upon a choice of frame for TM. Thus we obtain global results, only in the case M is parallelizable. For the particular case when $M = G$, a Lie group, our results are global and the flow reduces to a flow on $G\times\mathcal{G}\times\mathcal{G}^*\times\mathcal{G}^*$ where $\mathcal{G}$ is the Lie algebra of G. In this section, we follow closely [26].

Before proceeding, let $\{X_1, \cdots, X_n\}$ be a frame of vector fields on M and $\{\omega_1, \cdots, \omega_n\}$ a co-frame of covector fields such that $\omega_k(X_j) = \delta_{kl}$. This selection must be local, unless M is parallelizable. In terms of these frames we may write any vector field Y and covector field η along a curve $t \to x(t)$ as:

$$Y(x(t)) = Y(t) = \sum_{i=1}^{n} y_i(t)X_i(x(t)) \in T_{x(t)}M,$$
$$\eta(x(t)) = \eta(t) = \sum_{i=1}^{n} \eta_i(t)w_i(x(t)) \in T^*_{x(t)}M.$$

Thus, although the X_i's and the w_i's are defined on some open set in M, $Y(t)$ and $\eta(t)$ are only defined along the curve $x(t)$. Setting

$$\dot{x}(t) = \sum_{i=1}^{n} v_i(t)X_i(x(t)) = V(t) \in T_{x(t)}M,$$

it follows that the covariant derivatives of Y and η, along the curve $t \to x(t)$ with velocity vector field V, are given by:

$$\tfrac{DY}{dt}(t) = \sum_{i=1}^{n} \dot{y}_i(t)X_i(x(t)) + \sum_{i=1}^{n} y_i(t)(\nabla_V X_i)(x(t)),$$

$$\tfrac{D\eta}{dt}(t) = \sum_{i=1}^{n} \dot{\eta}_i(t)w_i(x(t)) + \sum_{i=1}^{n} \eta_i(t)(\nabla_V w_i)(x(t)).$$

We denote these expressions by the contracted forms

$$\frac{DY}{dt} = \dot{Y} + \nabla_V Y, \quad \frac{D\eta}{dt} = \dot{\eta} + \nabla_V \eta. \tag{35}$$

5.1. A variational approach. We first look at the optimal control problem (32) - (33) - (34) as a constrained variational problem, and seek for a solution using the Lagrange multipliers as co-states. So, we briefly consider solving the optimal control problem through the following variational problem, where $p_1(t),\ p_2(t)$ belong to $T^*_{x(t)}M$, and $u(t),\ V(t)$ belong to $T_{x(t)}M$:

$$\min_u J(x,V,p_1,p_2,u) = \int_0^T (p_1(\dot{x}-V) + p_2(\tfrac{DV}{dt}-u) + \tfrac{1}{2}\langle u,u\rangle)dt, \qquad (36)$$

subject to the dynamics (33) *and to the boundary conditions* (34).

Using the fundamental theorem of the calculus of variations, one may derive the corresponding Euler-Lagrange equations.

In what follows, $\Sigma : TM \to T^*M$ is the fiber-linear map associated to the Riemannian metric $\langle .,.\rangle$ and defined by

$$(\Sigma X)(Y) = \langle X,Y\rangle, \quad X,Y \in \Gamma(TM), \qquad (37)$$

where $\Gamma(TM)$ denotes the set of smooth vector fields on M.

Theorem 5.1 ([25]). *The extremals of the optimal control problem* (32) - (33) - (34) *may be expressed as solutions of the following system of equations, relative to the local choice of frame and co-frame for TM and T^*M:*

$$\left\{\begin{array}{lcl} \dot{x} & = & V \\ \dot{V} & = & \Sigma^{-1}p_2 - \nabla_V V \\ \dot{p}_1 & = & -dp_1(V,.) + p_2(\nabla(\Sigma^{-1}p_2)) \\ \dot{p}_2 & = & -p_1 + p_2(\nabla V) - \nabla_V p_2 \end{array}\right. . \qquad (38)$$

Notice that the optimal control u^* is given by $u^* = \Sigma^{-1}p_2$ and that, since $V \in T_xM,\ \ p_1,p_2 \in T^*_xM$, this system evolves on $\cup_{x\in M} T_xM \oplus T^*_xM \oplus T^*_xM$.

Theorem 5.2 ([26]). *If V is a solution of* (38), *then V also solves*

$$\frac{D^3V}{dt^3} + R(\frac{DV}{dt},V)V = 0, \qquad (39)$$

which is the Euler-Lagrange equation (21) for the unconstrained variational problem.

5.2. A Hamiltonian approach. The Hamiltonian function associated with the optimal control problem (32) - (33) - (34) is given by

$$H(q,V,p_1,p_2) = -\tfrac{1}{2}\langle\Sigma^{-1}p_2,\Sigma^{-1}p_2\rangle + p_1(V) + p_2(\dot{V}) \qquad (40)$$

or, equivalently, by

$$H(q,V,p_1,p_2) = -\tfrac{1}{2}\langle\Sigma^{-1}p_2,\Sigma^{-1}p_2\rangle + p_1(V) - p_2(\nabla_V V). \qquad (41)$$

Our next objective is to exhibit the extremal equations in Hamiltonian form, for this Hamiltonian, and identify the correct symplectic form. Since for any vector fields X, Y and covector field p along a curve $t \to q(t)$, one has

$$p(\nabla_X Y) = -(\nabla_X p)(Y), \tag{42}$$

we also have the following alternative formula for the Hamiltonian:

$$H(q, V, p_1, p_2) = \tfrac{1}{2} p_2(\Sigma^{-1} p_2) + (\nabla_V p_2)(V) + p_1(V). \tag{43}$$

Theorem 5.3. *Let* $E = \cup_{q \in M} T_q M \oplus T_q^* M \oplus T_q^* M$. *The extremals of the optimal control problem* (32) - (33) - (34) *satisfy:*

$$\begin{cases} \dot{q} & = & D_{p_1} H \\ \dot{V} & = & D_{p_2} H \\ \dot{p}_1 + dp_1(\dot{q}, .) & = & -D_q H \\ \dot{p}_2 & = & -D_V H \end{cases}, \tag{44}$$

where D_{p_1}, D_{p_2}, *and* D_V *are fiber derivatives in the bundle* E.

Proof: From (41) it is clear that $D_{p_1} H = V$ and so the first equation in (38) implies that $D_{p_1} H = \dot{q}$. From (41) and the second equation in (38) we have $D_{p_2} H = \Sigma^{-1} p_2 - \nabla_V V = \dot{V}$. We now proceed to get the fourth equation. From (38) and (35) respectively, one gets $\frac{Dp_2}{dt} = -p_1 + p_2(\nabla V)$ and $\frac{Dp_2}{dt} = \dot{p}_2 + \nabla_V p_2$. From these identities and (42) it follows that

$$\dot{p}_2 = -p_1 - (\nabla p_2)(V) - \nabla_V p_2. \tag{45}$$

On the other hand, it follows from (43) that $D_V H = p_1 + (\nabla p_2)(V) + \nabla_V p_2$, and so, according to (45), $D_V H = -\dot{p}_2$ as required. Finally, to obtain the equation for $\dot{p}_1$ we consider H in the form given in (40),

$$H = -\frac{1}{2} p_2(\Sigma^{-1} p_2) + p_2(\dot{V}) + p_1(V).$$

Thus, for any vector field X along q, $dH(X) = X(H) = -p_2\left(\nabla_X(\Sigma^{-1} p_2)\right)$, since $p_2(\dot{V}) + p_1(V)$ does not depend on q. But, from the third equation in (38), we may write $dH(X) = -\dot{p}_1 - dp_1(\dot{q}, .)$ as required. ■

Lemma 5.4 ([25]). *The first integral (23) associated to the flow (39) corresponds to the Hamiltonian function (41).*

Having derived equations (44), we now explain in what sense they are Hamiltonian. We first define a map

$$\begin{array}{cccc} \sigma: & E & \longrightarrow & \hat{E} \\ & (q, V, p_1, p_2) & \to & (q, v_i, p_1^i, p_2^i) \end{array}, \tag{46}$$

where $E = \cup_{q\in M} T_q M \oplus T_q^* M \oplus T_q^* M$, $\hat{E} = M \oplus \mathbb{R}^n \oplus \mathbb{R}^{n*} \oplus \mathbb{R}^{n*}$ and v_i, p_1^i, p_2^i are respectively the coordinates of the vector field V and covector fields p_1, p_2 in M, with respect to the global frames $\{X_1, \cdots, X_n\}$ and $\{w_1, \cdots, w_n\}$. That is,

$$V = \sum_i v_i X_i,\ p_1 = \sum_i p_1^i w_i,\ p_2 = \sum_i p_2^i w_i.$$

σ is a diffeomorphism between the differentiable manifolds E and $\hat{E}$.

Lemma 5.5. *$\hat{E}$ is a symplectic manifold with symplectic form*

$$\hat{\Omega} = \sum_i (w_i \wedge dp_1^i - p_1^i dw_i + dv_i \wedge dp_2^i). \tag{47}$$

Proof: Since $\sum_i dv_i \wedge dp_2^i$ is the natural symplectic structure on $\mathbb{R}^n \oplus \mathbb{R}^{n*}$, it is enough to show that $\sum_i (w_i \wedge dp_1^i - p_1^i dw_i) = -d(\sum_i p_1^i w_i)$ is a symplectic form on $T^*M \simeq M \oplus \mathbb{R}^{n*}$. Assume that $q_1, \cdots, q_n$ are local coordinates on M, so that $\{dq_1, \cdots, dq_n\}$ is a basis of T_q^*M. So, any 1-form on M, in particular the w_i's, may be expressed in terms of that basis and we may write, $\sum_i p_1^i w_i = \sum_i \hat{p}_1^i dq_i = \theta$, where $p_1^i = \sum_j \hat{p}_1^j dq_j(X_i)$. Since the matrix $[dq_j(X_i)]_{i,j}$ is nondegenerate everywhere, $\{\hat{p}_1^i\}_i$ is another set of local coordinates for $\mathbb{R}^{n*}$. It is therefore sufficient to show that $d\theta$ is nondegenerate, but this is a simple exercise in local coordinates. So, $\sum_i (w_i \wedge dp_1^i - p_1^i dw_i)$ is the symplectic form $-d\theta$ on T^*M. ■

Since $\sigma : E \to \hat{E}$ is a diffeomorphism and $\hat{\Omega}$ is a symplectic form on $\hat{E}$, we may define the pull back $\sigma^*\hat{\Omega}$, of $\hat{\Omega}$ by σ, by

$$(\sigma^*\hat{\Omega})_x (X_1, X_2) = \Omega_{\sigma(x)}(\sigma_*(X_1), \sigma_*(X_2)),$$

where $x \in E$, $X_1, X_2 \in T_x E$ and σ_* is the derivative of σ at x. Ω is clearly a symplectic form on E. However Ω is complicated to explicitly write down.

Now let $\hat{H} = (\sigma^{-1})^* H$ be the pull back of H by σ^{-1}. That is,

$$\hat{H}(q, v_i, p_1^i, p_2^i) = H \circ \sigma^{-1}(q, v_i, p_1^i, p_2^i) = H(q, V, p_1, p_2).$$

Theorem 5.6. *On the symplectic space $(\hat{E}, \hat{\Omega})$, the equations*

$$\begin{cases} \dot{q} & = \sum_i \frac{\partial \hat{H}}{\partial p_1^i} X_i \\ \dot{v}_i & = \frac{\partial \hat{H}}{\partial p_2^i} \\ \dot{p}_1^i + dp_1(\dot{q}, X_i) & = -d\hat{H}(X_i) \\ \dot{p}_2^i & = -\frac{\partial \hat{H}}{\partial v_i} \end{cases} \tag{48}$$

are Hamiltonian, with Hamiltonian function $\hat{H}$.

Proof: Expand the equation $\hat{\Omega}(X_{\hat{H}}, .) = d\hat{H}$, in the coordinates (q, v_i, p_1^i, p_2^i), with $X_{\hat{H}} = (q, v_i, p_1^i, p_2^i)$. ■

Theorem 5.7. *The system (44) is a Hamiltonian system with Hamiltonian H on the symplectic space (E, Ω).*

Proof: The map σ maps the dynamics (44) onto the dynamics (48) and by construction σ is a symplectic morphism of (E, Ω) onto $(\hat{E}, \hat{\Omega})$. ■

5.3. The Lie group case. We now specialize to the case where $M = G$, is a compact or semi-simple Lie group, with Lie algebra $\mathcal{G}$. In this case M is parallelizable and the equations (38) and (44), may be given a global interpretation. In this case we also have an explicit expression for the connection corresponding to the unique bi-invariant metric on G, $\nabla_X Y = \frac{1}{2}[X, Y]$, (see, for instance, [51]). This corresponds to the choice where $\Sigma : \mathcal{G} \to \mathcal{G}^*$ is defined by $(\Sigma X)(Y) = \langle Y, X \rangle$.

We may assume that $\{X_1, \ldots, X_n\}$ is a basis of left-invariant vector fields and $\{w_1, \ldots, w_n\}$ is a dual basis of left-invariant one-forms. It follows that the equations (38) are indeed globally defined and we may identify V, p_1, p_2, as elements of $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}^*$ respectively.

We recall here a few formulas when M is a Lie group G. If for $X \in \mathcal{G}$, ad_X denotes the adjoint map

$$\begin{array}{cccc} ad_X & : & \mathcal{G} & \to & \mathcal{G} \\ & & Y & \mapsto & [X, Y] \end{array},$$

the co-adjoint map of ad_X is defined by:

$$ad_X^* \eta(Y) = -\eta(ad_X Y) = -\eta([X, Y]), \quad \eta \in \mathcal{G}^*, \, Y \in \mathcal{G}. \tag{49}$$

Now, if X and Y are left-invariant vector fields on G and η is a left-invariant one-form on G, then $Y(\eta(X)) = 0, \ \forall X, Y,$ and consequently

$$\eta(\nabla_X Y) = -(\nabla_X \eta)Y, \quad d\eta(X,Y) = -\eta([X,Y]). \tag{50}$$

Also taking into account that $\nabla_Y X = \frac{1}{2}[Y,X]$, it follows from (50) that

$$d\eta(X,Y) = -2\eta(\nabla_X Y) = 2(\nabla_X \eta)Y,$$

and, also, using (49)

$$\nabla_X \eta = \frac{1}{2} ad_X^* \eta. \tag{51}$$

We now turn our attention to the problem of identifying the correct Hamiltonian and symplectic structure, which will be a generalization of that from T^*G to T^*TG.

Lemma 5.8. *In the case of a compact or semisimple Lie group G with Lie algebra $\mathcal{G}$, the extremal equations (38) may be written in the form*

$$\begin{cases} \dot{q} = L_{q*}(V) \\ \dot{V} = \Sigma^{-1} p_2 \\ \dot{p}_1 = -ad_V^* p_1 \\ \dot{p}_2 = -p_1 \end{cases}, \tag{52}$$

where $V \in \mathcal{G}, p_1, p_2 \in \mathcal{G}^*$, L_q is left translation in the Lie group G and L_{q*} is the derivative of L_q.

Proof: As a consequence of (50) and(51) we may write the equations (38), with $V \in \mathcal{G}, p_1, p_2 \in \mathcal{G}^*$, in the form

$$\begin{cases} \dot{q} = L_{q*}(V) \\ \dot{V} = \Sigma^{-1} p_2 \\ \dot{p}_1 = -ad_V^* p_1 + \frac{1}{2} ad^*_{\Sigma^{-1} p_2} p_2 \\ \dot{p}_2 = -p_1 \end{cases}.$$

But from (50) and the definition of Σ,

$$ad^*_{\Sigma^{-1}p_2} p_2(X) = -p_2([\Sigma^{-1}p_2, X]) = -\langle \Sigma^{-1}p_2, [\Sigma^{-1}p_2, X]\rangle = 0.$$

■

Note that the system (52) is Hamiltonian on $G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^* = E$ with Hamiltonian function $H = \frac{1}{2} p_2(\Sigma^{-1}p_2) + p_1(V)$ and symplectic form

$$\begin{aligned} \Omega(\dot{q}, \dot{V}, \dot{p}_1, \dot{p}_2, \dot{\bar{q}}, \dot{\bar{V}}, \dot{\bar{p}}_1, \dot{\bar{p}}_2) = & \textstyle\sum_i \left(\omega_i(\dot{q})\, \dot{\bar{p}}_1^i - \omega_i(\dot{\bar{q}})\, \dot{p}_1^i\right) \\ & -(ad^*{}_{L_q{}^{-1}{}_*(\dot{q})} p_1)(L_q{}^{-1}{}_*(\dot{\bar{q}})) + \textstyle\sum_i \left(\dot{v}_i\, \dot{\bar{p}}_2^i - \dot{\bar{v}}_i\, \dot{p}_2^i\right). \end{aligned}$$

Corollary 5.9. *The system of equations (52) satisfies*

$$\dddot{V} + [V, \ddot{V}] = 0, \tag{53}$$

which is the Euler-Lagrange equation (39) specialized to Lie groups.

Equation (53) was first written down in this generality in [53].

We can also write the extremal equations in Lemma 5.8 using the natural identification of elements of $\mathcal{G}$ and $\mathcal{G}^*$. More precisely, if one defines $p_1 = \langle A_1, . \rangle$, $p_2 = \langle A_2, . \rangle$, $A_1, A_2 \in \mathcal{G}$, and $\hat{E} = G \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$, then,

Theorem 5.10. *The extremal equations on $\hat{E}$ have the form*

$$\begin{cases} \dot{q} = L_{q_*}(V) \\ \dot{V} = A_2 \\ \dot{A}_1 - [A_1, V] = 0 \\ \dot{A}_2 = -A_1 \end{cases}. \tag{54}$$

Proof: We only need to prove that the third equation is satisfied. It follows from identity (49) and $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$ that

$$\dot{p}_1 = -ad_V^* p_1 \Leftrightarrow \langle \dot{A}_1, . \rangle = \langle A_1, [V, .] \rangle \Leftrightarrow \langle \dot{A}_1, . \rangle = -\langle [V, A_1], . \rangle,$$

which completes the proof. ■

Theorem 5.11. *$\hat{E} = G \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is a symplectic manifold with symplectic form*

$$\hat{\Omega}(\dot{q}, \dot{V}, \dot{A}_1, \dot{A}_2, \dot{\overline{q}}, \dot{\overline{V}}, \dot{\overline{A}}_1, \dot{\overline{A}}_2) = \langle \dot{\overline{A}}_1, {L_q}^{-1}{}_*(\dot{q}) \rangle - \langle \dot{A}_1, {L_q}^{-1}{}_*(\dot{\overline{q}}) \rangle$$

$$+ \langle \dot{V}, \dot{\overline{A}}_2 \rangle - \langle \dot{\overline{V}}, \dot{A}_2 \rangle - \langle ad_{{L_q}^{-1}{}_*(\dot{q})} A_1, {L_q}^{-1}{}_*(\dot{\overline{q}}) \rangle.$$

Moreover, the equations (54) are Hamiltonian with Hamiltonian function

$$\hat{H} = \frac{1}{2} \langle A_2, A_2 \rangle + \langle A_1, V \rangle.$$

5.4. Example. A specific problem in optimal control of the form (32)–(33)–(34) was treated in [13] where an analysis was made between the Hamiltonian and the Lagrangian formulation of higher order optimal control problems. We treat the example again here in a slightly different setting.

Consider the problem:

$$\min_u \int_0^T \frac{1}{2}\langle u, u\rangle dt, \qquad \textit{subject to:}$$

$$\begin{cases} \dot{Q} = \Omega_1 Q \\ \dot{\Omega}_1 = u \end{cases} \qquad Q \in SO(n),\ u, \Omega_1 \in so(n),$$

and boundary conditions

$$\begin{array}{ll} Q(0) = Q_0, & Q(T) = Q_T, \\ \dot{Q}(0) = \dot{Q}_0, & \dot{Q}(T) = \dot{Q}_T. \end{array}$$

Here $\langle A, B\rangle = trace(A^T B)$. To solve the problem we construct the Hamiltonian

$$H(u, Q, \Omega_1, p_1, p_2) = \langle p_2, u\rangle + \langle p_1, \Omega_1 Q\rangle - 1/2\langle u, u\rangle. \tag{55}$$

Thus the optimal control is $u^* = p_2 \in so(n)$, from which we get

$$H = 1/2\langle p_2, p_2\rangle + \langle p_1, \Omega_1 Q\rangle.$$

Using properties of the trace of a matrix we obtain

$$\dot{p}_2 = -1/2(p_1 Q^T - Q p_1^T), \quad \dot{p}_1 = -\Omega_1^T p_1. \tag{56}$$

We hypothesize a solution where $p_1 = \Omega_2 Q$, with $\Omega_2 \in so(n)$. If we make this assumption it follows from (56) that $\dot{p}_2 = -\Omega_2, \quad \dot{\Omega}_2 = [\Omega_1, \Omega_2]$ and so, the full extremal equations may be written as

$$\begin{cases} \dot{Q} = \Omega_1 Q \\ \dot{\Omega}_1 = p_2 \\ \dot{p}_2 = -\Omega_2 \\ \dot{\Omega}_2 = [\Omega_1, \Omega_2] \end{cases} . \tag{57}$$

The equations (57) are precisely the equations of Theorem 5.10 and the corresponding Hamiltonian function (55) is

$$H = 1/2\langle p_2, p_2\rangle + \langle p_1, \Omega_1 Q\rangle = 1/2\langle p_2, p_2\rangle + \langle \Omega_2 Q, \Omega_1 Q\rangle = 1/2\langle p_2, p_2\rangle + \langle \Omega_2, \Omega_1\rangle.$$

Note that the symplectic structure on $SO(n) \times so(n) \times so(n) \times so(n)$, is that in Theorem 5.11. We also note that the equations

$$\begin{cases} \dot{Q} = \Omega_1 Q \\ \dot{\Omega}_1 = p_2 \\ \dot{p}_2 = -1/2(p_1 Q^T - Q p_1^T) \\ \dot{p}_1 = -\Omega_1^T p_1 \end{cases}$$

are Hamiltonian with respect to the Hamiltonian function

$$H = \frac{1}{2}\langle p_2, p_2\rangle + \frac{1}{2}\langle p_1 Q^T - Q p_1^T, \Omega_1\rangle.$$

Acknowledgments

The work of Margarida Camarinha was partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES. Fátima Silva Leite acknowledges Fundação para a Ciência e Tecnologia (FCT-Portugal) and COMPETE 2020 program for financial support to the project UIDB/00048/2020.

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