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HIGHER POLYNOMIAL IDENTITIES FOR MUTATIONS OF ASSOCIATIVE ALGEBRAS

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Abstract: We study polynomial identities satisfied by the mutation product $xpy-yqx$ on the underlying vector space of an associative algebra A, where p, q are fixed elements of A. We simplify known results for identities in degree 4, proving that only two identities are necessary and sufficient to generate them all; in degree 5, we show that adding one new identity suffices; in degree 6, we demonstrate the existence of a number of new identities.

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1. Introduction

Let A be an associative algebra over a field $\mathbb{F}$ of characteristic 0. Fix two elements $p, q \in A$ and define a new bilinear operation on the underlying vector space:

$$x *_{pq} y = xpy - yqx.$$

The resulting nonassociative algebra A_{pq} is called the *pq-mutation* of A.

Mutation algebras were introduced by theoretical physicists around 1980; see [8, equation (1.6b)] and [17, equation (66)]. For a survey of early work by mathematicians on this topic, see [15]. For a detailed exposition of the structure theory of mutation algebras, see [6]. For mutations of nonassociative algebras, see [2].

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To motivate the investigation of polynomial identities for mutation algebras, we paraphrase some comments from [6, Preface]. Mutation algebras are both Lie- and Jordan-admissible, but they also satisfy other more complex identities of higher degree; see [6, Chapter 5]. It is an open problem to determine a complete set of independent identities satisfied by all mutation algebras for arbitrary p, q. In fact, mutation algebras do not form a variety defined by polynomial identities. We note that Lie- and Jordan-admissible algebras were introduced by Albert [1, §IV.1-2].

Polynomial identities for mutation algebras were first studied systematically by Montaner [14] using the classical techniques of nonassociative algebra [16, 18]. That work did not consider the original operation $*_{pq}$ but decomposed it as the sum of commutative and anticommutative operations: $x *_{pq} y = \{x, y\} + [x, y]$ where

$$\{x, y\} = \tfrac{1}{2}(x *_{pq} y + y *_{pq} x), \qquad [x, y] = \tfrac{1}{2}(x *_{pq} y - y *_{pq} x).$$

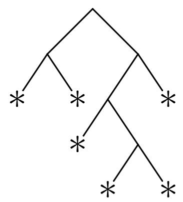
For further information about the notion of polarization of a binary operation, see [12]. Furthermore, the work of Montaner considered identities which are not necessarily multilinear, but hand calculation restricted the degree of the identities to $n \leq 4$.

We use a different approach which allows us to simplify the known results in degree 4, to determine a complete set of identities in degree 5, and to demonstrate the existence of a number of new identities in degree 6:

- We use elementary concepts from the theory of algebraic operads [3, 10, 11, 13].
- Our main tool is computer algebra, in particular:
  - Linear algebra over the rational numbers and finite prime fields: the row canonical form of a matrix using Gaussian elimination.
  - Linear algebra over the integers: the Hermite normal form of a matrix and the Lenstra-Lenstra-Lovász algorithm (LLL, see [5, 9]) for lattice basis reduction.
- We consider only multilinear identities for the original operation $x *_{pq} y$: this allows us to use the representation theory of the symmetric group [4] to decompose the computations into small pieces corresponding to irreducible representations.

2. Algebraic Operads

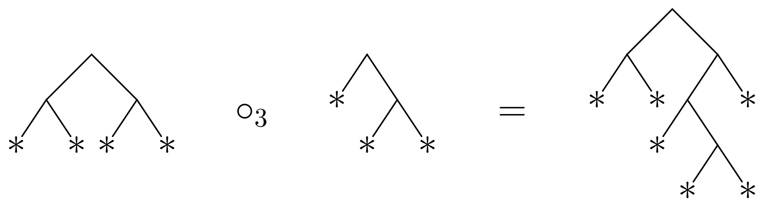
2.1. The free nonsymmetric operad. We write T_n for the set of all complete rooted plane binary trees with n leaves denoted by asterisks; for $n = 1$ there is only the exceptional tree with one leaf and no root, $T_1 = \{*\}$. Each tree in T_n contains $n-1$ internal nodes (including the root); hence the size of T_n is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$. We write U_n for the set of all association types in degree n: balanced placements of parentheses in a sequence of n asterisks. There is a bijection $\mu_n \colon T_n \to U_n$ defined recursively: $\mu_1(*) = *$; for every internal node v with left and right subtrees $t_1 \in T_{n_1}$ and $t_2 \in T_{n_2}$ we replace the subtree with root v by $(\, \mu_{n_1}(t_1)\, \mu_{n_2}(t_2)\,)$. For example,



$$\xrightarrow{\ \mu_6\ } \quad (**)((*(**))*)$$

(We omit the outermost pair of parentheses corresponding to the root of the tree.)

If $t_1 \in T_{n_1}$ and $t_2 \in T_{n_2}$ then the partial composition $t_1 \circ_i t_2 \in T_{n_1+n_2-1}$ for $1 \le i \le n_1$ is obtained by grafting the right tree into the left at position i: that is, identifying leaf i of t_1 (from left to right) with the root of t_2. For example,



In terms of association types, $\mu_{n_1}(t_1) \circ_i \mu_{n_2}(t_2)$ corresponds to substitution of $(\, \mu_{n_2}(t_2)\,)$ for argument i of $\mu_{n_1}(t_1)$; we omit the parentheses if $n_2 = 1$. For example, $(**)(**) \circ_3 *(**) = (**)((*(**))*)$.

Partial composition is nonassociative but satisfies sequential and parallel axioms [3, Definition 3.2.2.3] (see also [10, 13]). We state these axioms following [11, Definition 11, Figure 1]. If $f \in T_m$, $g \in T_n$, $h \in T_p$ then

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+p-1} g & 1 \le i \le j-1 \\ f \circ_j (g \circ_{i-j+1} h) & j \le i \le n+j-1 \\ (f \circ_{i-n+1} h) \circ_j g & n+j \le i \le m+n-1 \end{cases}.$$

Let T denote the disjoint union of the T_n for $n \geq 1$:

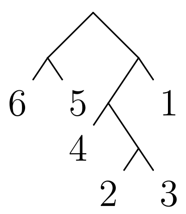
$$T = \bigsqcup_{n\geq 1} T_n.$$

The set T together with all partial compositions is isomorphic to the free nonsymmetric set operad generated by one binary operation ω corresponding to the tree with root and two leaves. (Nonsymmetric means that we have not yet introduced the action of the symmetric group on the arguments.) Let $T(n)$ denote the vector space with basis T_n. On the direct sum

$$\mathcal{T} = \bigoplus_{n\geq 1} T(n),$$

we extend partial compositions so that they are linear in each factor. The vector space $\mathcal{T}$ together with the extended partial compositions is isomorphic to the free nonsymmetric vector operad generated by ω.

2.2. The free symmetric operad. Consider an integer $n \geq 1$ and the set of indeterminates $\{x_1, \dots, x_n\}$. We write S_n for the symmetric group of all $n!$ permutations of $\{1, \dots, n\}$. For each $\alpha \in S_n$ and $t \in T_n$ we obtain the labelled tree αt consisting of t with leaves labelled $\alpha(1), \dots, \alpha(n)$ from left to right. We write LT_n for the set of all such labelled trees. Similarly, we apply the association type $\mu_n(t)$ for $t \in T_n$ to the multilinear associative monomial $x_{\alpha(1)} \cdots x_{\alpha(n)}$ and obtain the nonassociative monomial $\alpha\mu_n(t)$. We write LU_n for the set of all such nonassociative monomials. The bijection $\mu_n \colon T_n \to U_n$ extends in the obvious way to the bijection $\lambda\mu_n \colon LT_n \to LU_n$. For example,



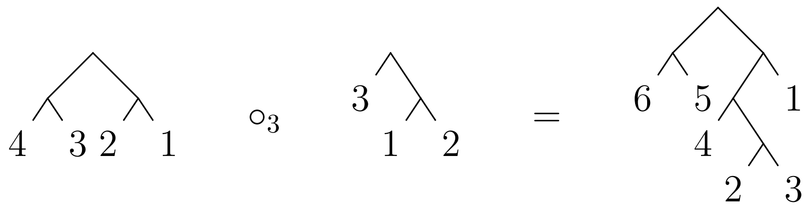
$$\xrightarrow{\lambda\mu_6} \quad (x_6x_5)((x_4(x_2x_3))x_1)$$

We extend partial compositions to labelled trees. Consider two labelled trees $\alpha t \in LT_m$ and $\beta u \in LT_n$. If $1 \leq i \leq m$ then the partial composition $\alpha t \circ_i \beta u \in LT_{m+n-1}$ must be a tree with $m+n-1$ leaves labelled by a permutation in S_{m+n-1}. (Simple grafting of one labelled tree onto the other does not produce a permutation.) This must be done in a manner which is equivariant with respect to the action of the symmetric group. Following [13, Definion 1.37] with minor changes, we have:

- A leaf of αt with label j for $1 \leq j \leq \alpha(i) - 1$ retains its label.

- A leaf of βu with label j for $1 \leq j \leq n$ is relabelled $j + \alpha(i) - 1$.
- A leaf of αt with label j for $\alpha(i) + 1 \leq j \leq m$ is relabelled $j + n - 1$.

For example,



Let $LT(n)$ denote the S_n-module with linear basis LT_n; we use the natural left action on labels (not on positions). The direct sum of these S_n-modules,

$$\mathcal{LT} = \bigoplus_{n \geq 1} LT(n),$$

together with the bilinear extension of the partial compositions, is isomorphic to the free symmetric vector operad generated by ω. (This binary operation has no symmetry: it is neither commutative nor anticommutative.)

An ideal $\mathcal{I}$ in the free symmetric operad $\mathcal{LT}$ is a graded subspace (that is, $\mathcal{I}(n) \subseteq \mathcal{LT}(n)$ for $n \geq 1$) such that

- $S_n \cdot \mathcal{I}(n) = \mathcal{I}(n)$: each homogeneous space $\mathcal{I}(n)$ is an S_n-module (that is, closed under the action of the symmetric group), and
- if $f \in \mathcal{I}(m)$ and $g \in \mathcal{LT}(n)$ then $f \circ_i g$ $(1 \leq i \leq m)$ and $g \circ_j f$ $(1 \leq j \leq n)$ belong to $\mathcal{I}(m{+}n{-}1)$ (that is, $\mathcal{I}$ is closed under partial compositions).

The ideal $\langle f_1, f_2, \dots \rangle \subseteq \mathcal{LT}$ generated by homogeneous elements $f_1, f_2, \dots$ is the smallest ideal of $\mathcal{LT}$ containing $f_1, f_2, \dots$ If $\mathcal{I} = \langle f_1, f_2, \dots \rangle$ then we say that $\mathcal{G} = \{f_1, f_2, \dots\}$ is a minimal set of generators for $\mathcal{I}$ if no proper subset of $\mathcal{G}$ generates $\mathcal{I}$; this condition does not uniquely determine $\mathcal{G}$.

2.3. Associativity, nullary operations, and the expansion map. In general, an n-ary operation ($n \geq 0$) on a vector space V is a multilinear map $f \colon V^n \to V$. For $n = 1$ we have $V^1 = V$ so a unary operation is simply a linear operator on V; for $n = 0$ we have $V^0 = \mathbb{F}$ so a nullary operation is equivalent to the choice of a constant vector $f(1) \in V$. If we write $\mathrm{End}_n(V)$ for the vector space of all n-ary operations on V, then the direct sum $\mathrm{End}(V) = \bigoplus_{n \geq 0} \mathrm{End}_n(V)$, together with partial compositions (substitution of the output of one operation for an input of another operation), is the *endomorphism operad* of V.

Let p, q be symbols denoting nullary operations on some underlying vector space. For $n \geq 1$, consider monomials $v_1 v_2 \cdots v_{2n-1}$ with an odd number of factors such that the n odd-indexed factors v_{2i-1} $(1 \leq i \leq n)$ form a multilinear associative monomial $x_{\alpha(1)} \cdots x_{\alpha(n)}$ for some $\alpha \in S_n$, and each of the $n-1$ even-indexed factors is either p or q. We write W_n for the set of all such monomials; S_n acts by permuting the odd-indexed factors. For $v \in W_m$ and $w \in W_n$, we define $v \circ_i w \in W_{m+n-1}$ for $1 \leq i \leq m$ by substituting w for v_{2i-1} (with the appropriate change of labels). We write $W(n)$ for the vector space whose basis consists of all such monomials. The direct sum $\mathcal{W} = \bigoplus_{n \geq 1} W(n)$ is a suboperad of the symmetric associative operad with two nullary operations.

Definition 2.1. The *expansion map* $X_n \colon LT(n) \to W(n)$ on monomials $\alpha t \in LT_n$ is defined recursively. For a leaf with label i, we set $X_n(i) = x_i$. If tu denotes an internal node with left and right subtrees $t \in LT_r$ and $u \in LT_s$ with $r + s = n$ then

$$X_n(tu) = X_r(t) p X_s(u) - X_s(u) q X_r(t).$$

That is, we replace each internal node by the operation $*_{pq}$.

If we represent trees by nonassociative monomials and leaf labels by letters then $X_2(ab) = apb - bqa$ and

$$\begin{aligned} X_3((ab)c) &= (apb{-}bqa)pc - cq(apb{-}bqa) = apbpc - bqapc - cqapb + cqbqa, \\ X_3(a(bc)) &= ap(bpc{-}cqb) - (bpc{-}cqb)qa = apbpc - apcqb - bpcqa + cqbqa. \end{aligned} \tag{1}$$

For the expansions in degree 4, see Figure 1.

Definition 2.2. For each $n \geq 1$, the expansion map $X_n \colon \mathcal{LT}(n) \to \mathcal{W}(n)$ is a morphism of S_n-modules; we write $\mathcal{K}(n) = \ker(X_n)$. Combining all the expansion maps we obtain the morphism of operads $X \colon \mathcal{LT} \to \mathcal{W}$ with kernel $\mathcal{K} = \bigoplus_{n \geq 1} \mathcal{K}(n)$. The polynomial identities satisfied by $*_{pq}$ for all associative algebras A and all $p, q \in A$ coincide with $\mathcal{K}$, which is an operad ideal in $\mathcal{LT}$. These identities are the linear dependence relations among the expansions of the nonassociative monomials. We refer to $\mathcal{K}(n)$ as the S_n-module of *all identities* in degree n.

Our ultimate goal is to determine a set of generators for $\mathcal{K}$.

$$
\boxed{
\begin{array}{rcl}
((ab)c)d & \longmapsto & apbpcpd - dqapbpc - cqapbpd + dqcqapb \\
 & & - bqapcpd + dqbqapc + cqbqapd - dqcqbqa, \\
(a(bc))d & \longmapsto & apbpcpd - dqapbpc - bpcqapd + dqbpcqa \\
 & & - apcqbpd + dqapcqb + cqbqapd - dqcqbqa, \\
(ab)(cd) & \longmapsto & apbpcpd - cpdqapb - apbpdqc + dqcqapb \\
 & & - bqapcpd + cpdqbqa + bqapdqc - dqcqbqa, \\
a((bc)d) & \longmapsto & apbpcpd - bpcpdqa - apdqbpc + dqbpcqa \\
 & & - apcqbpd + cqbpdqa + apdqcqb - dqcqbqa, \\
a(b(cd)) & \longmapsto & apbpcpd - bpcpdqa - apcpdqb + cpdqbqa \\
 & & - apbpdqc + bpdqcqa + apdqcqb - dqcqbqa.
\end{array}
}
$$

FIGURE 1. Expansions of basis monomials in degree 4

3. Polynomial Identities in Degree $n \leq 3$

Definition 3.1. In a nonassociative algebra, the *Lie-admissible identity* is

$$L(a, b, c) = \sum_{\sigma \in S_3} \varepsilon(\sigma)\big((a^\sigma b^\sigma)c^\sigma - a^\sigma(b^\sigma c^\sigma)\big),$$

where $\varepsilon \colon S_3 \to \{\pm 1\}$ is the sign homomorphism. If $L(a, b, c) \equiv 0$ then the commutator $xy - yx$ satisfies the Jacobi identity.

We provide a different proof of the next result using elementary linear algebra.

Theorem 3.2. [14]. *Over a field of characteristic 0, every multilinear polynomial identity in degree $n \leq 3$ satisfied by every mutation of every associative algebra is a consequence of the Lie-admissible identity.*

Proof: It is straightforward to verify that $\ker X_n = \{0\}$ for $1 \leq n \leq 2$. The monomial basis of $\mathcal{LT}(3)$ consists of 12 elements ordered first by association type and then by permutation of the variables:

$$
\begin{array}{cccccc}
(ab)c, & (ac)b, & (ba)c, & (bc)a, & (ca)b, & (cb)a, \\
a(bc), & a(cb), & b(ac), & b(ca), & c(ab), & c(ba).
\end{array}
\tag{2}
$$

The monomial basis of $\mathcal{W}(3)$ consists of 24 elements ordered first by lex order of the pair of nullary operations (pp, pq, qp, qq) and then by permutation of the variables:

$$\begin{array}{cccccc}
apbpc, & apcpb, & bpapc, & bpcpa, & cpapb, & cpbpa, \\
apbqc, & apcqb, & bpaqc, & bpcqa, & cpaqb, & cpbqa, \\
aqbpc, & aqcpb, & bqapc, & bqcpa, & cqapb, & cqbpa, \\
aqbqc, & aqcqb, & bqaqc, & bqcqa, & cqaqb, & cqbqa.
\end{array} \tag{3}$$

The expansion map $X_3 \colon \mathcal{LT}(3) \to \mathcal{W}(3)$ is determined by its values on the nonassociative monomials with the identity permutation of the arguments; see (1). We apply all permutations in S_3 to the arguments a, b, c and store the coefficients of the monomials in the 24×12 matrix E_3 representing X_3 with respect to the ordered bases; see Figure 2. That is, the (i, j) entry of E_3 is the coefficient of the ith associative monomial (3) in the expansion of the jth nonassociative monomial (2). It is easy to check that this matrix has rank 11 and hence nullity 1, and that a basis for its nullspace is the coefficient vector of the Lie-admissible identity. ■

$$\begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & -1 & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
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\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}$$

FIGURE 2. The matrix E_3 representing the expansion map X_3 (here a dot represents a zero entry)

4. Polynomial Identities in Degree $n = 4$

Montaner [14] (see also [6, Chapter 5]) showed that every identity in degree $n \leq 4$ satisfied by every mutation algebra is a consequence of the Lie-admissible identity, the Jordan-admissible identity, and two further identities; furthermore, none of these identities is a consequence of the other three. In this section we use computer algebra to simplify this result: we discover two new multilinear identities in degree 4, which are not consequences of the Lie-admissible identity, and which generate all identities in degree 4 (including the Jordan-admissible identity).

Definition 4.1. In a nonassociative algebra, the *linearized Jordan identity* is

$$((bc)a)d + ((bd)a)c + ((cd)a)b - (ab)(cd) - (ac)(bd) - (ad)(bc).$$

If we expand each nonassociative product xy as the anticommutator $xy + yx$ then we obtain the *Jordan-admissible identity*:

$$\begin{aligned} J(a,b,c,d) = {} & ((bc)a)d + ((bd)a)c + ((cb)a)d + ((cd)a)b + ((db)a)c + ((dc)a)b \\ & + (a(bc))d + (a(bd))c + (a(cb))d + (a(cd))b + (a(db))c + (a(dc))b - (ab)(cd) \\ & - (ab)(dc) - (ac)(bd) - (ac)(db) - (ad)(bc) - (ad)(cb) - (ba)(cd) - (ba)(dc) \\ & - (bc)(ad) - (bc)(da) - (bd)(ac) - (bd)(ca) - (ca)(bd) - (ca)(db) - (cb)(ad) \\ & - (cb)(da) - (cd)(ab) - (cd)(ba) - (da)(bc) - (da)(cb) - (db)(ac) - (db)(ca) \\ & - (dc)(ab) - (dc)(ba) + b((cd)a) + b((dc)a) + c((bd)a) + c((db)a) + d((bc)a) \\ & + d((cb)a) + b(a(cd)) + b(a(dc)) + c(a(bd)) + c(a(db)) + d(a(bc)) + d(a(cb)). \end{aligned}$$

If $J(a,b,c,d) \equiv 0$ then the anticommutator $xy + yx$ satisfies the Jordan identity.

Definition 4.2. In a nonassociative algebra, we consider the following identities where $(x,y,z) = (xy)z - x(yz)$ and $x \circ y = xy + yx$:

$$H(a,b,c,d) = \big((a,c,b) + (b,a,c) + (c,b,a)\big)d - \sum_{\sigma \in S_3} \big((a^\sigma b^\sigma)(c^\sigma d) - a^\sigma((b^\sigma c^\sigma)d)\big),$$

$$\begin{aligned} I(a,b,c,d) = {} & (bc,a,d) - (a,bc,d) + (a,d,bc) + (b,a \circ d,c) - (b,d,c) \circ a \\ & - (b,a,c) \circ d. \end{aligned}$$

In identity H, the first three terms include a cyclic sum of associators [7, Equation (5)], and each term in the summation can be written as the difference of two associators.

The next result improves [14, Theorem 2.3]: we have only two new identities in degree 4, not three. In addition, even though our new identities H and I are multilinear, each contains only 18 nonassociative monomials (after expanding the associators and anticommutators), whereas the new identities of [14] have 48 monomials (the Jordan-admissible identity), 20 monomials (identity E), and 52 monomials (identity K). Furthermore, our new identities have only coefficients ± 1, whereas identity K has coefficients 1, 2, 4, 6. Finally, we show that the new identities also generate the consequences of the Lie-admissible identity, and thus the latter do not need to be considered.

Theorem 4.3. *Every identity in degree 4 satisfied by every mutation algebra follows from the identities H and I from Definition 4.2.*

Proof: We first consider the expansion matrix. The monomial basis of $\mathcal{LT}(4)$ consists of 120 elements ordered first by association type and then by lex order of permutations $\sigma \in S_4$ (indicated by superscripts):

$$((a^\sigma b^\sigma)c^\sigma)d^\sigma,\ \ (a^\sigma(b^\sigma c^\sigma))d^\sigma,\ \ (a^\sigma b^\sigma)(c^\sigma d^\sigma),\ \ a^\sigma((b^\sigma c^\sigma)d^\sigma),\ \ a^\sigma(b^\sigma(c^\sigma d^\sigma)). \tag{4}$$

The monomial basis of $\mathcal{W}(4)$ consists of 192 elements ordered first by lex order of the triple of nullary operations and then by lex order of permutations $\sigma \in S_4$:

$$\begin{array}{llll} a^\sigma p\, b^\sigma p\, c^\sigma p\, d^\sigma, & a^\sigma p\, b^\sigma p\, c^\sigma q\, d^\sigma, & a^\sigma p\, b^\sigma q\, c^\sigma p\, d^\sigma, & a^\sigma p\, b^\sigma q\, c^\sigma q\, d^\sigma, \\ a^\sigma q\, b^\sigma p\, c^\sigma p\, d^\sigma, & a^\sigma q\, b^\sigma p\, c^\sigma q\, d^\sigma, & a^\sigma q\, b^\sigma q\, c^\sigma p\, d^\sigma, & a^\sigma q\, b^\sigma q\, c^\sigma q\, d^\sigma. \end{array} \tag{5}$$

The expansion map $X_4 \colon \mathcal{LT}(4) \to \mathcal{W}(4)$ is determined by its values on the nonassociative monomials with the identity permutation of the arguments (Figure 1). We apply all permutations in S_4 to the arguments a, b, c, d in the expansions and store the coefficients in the 192×120 matrix E_4 representing X_4 with respect to the ordered bases (4) and (5). The (i, j) entry of E_4 is the coefficient of the ith associative monomial in the expansion of the jth nonassociative monomial. Thus each column of E_4 contains 1 and -1 each four times.

Next, we consider the consequences of the Lie-admissible identity. The identity $L(a, b, c) \in \mathcal{LT}(3)$ is skew-symmetric:

$$L(a^\sigma, b^\sigma, c^\sigma) = \epsilon(\sigma) L(a, b, c).$$

We write $\mathcal{L} \subset \mathcal{LT}$ for the operad ideal generated by L; clearly $\mathcal{L} \subseteq \mathcal{K}$. The homogeneous component $\mathcal{L}(4)$ is generated as an S_4-module by the partial compositions

$$
\begin{aligned}
L \circ_1 \omega &= L(\omega(a,b),c,d) = L(ab,c,d), \\
\omega \circ_1 L &= \omega(L(a,b,c),d) = L(a,b,c)d, \\
\omega \circ_2 L &= \omega(a,L(b,c,d)) = aL(b,c,d).
\end{aligned} \tag{6}
$$

We refer to the elements of $\mathcal{L}(4)$ as the *old identities* in degree 4. Applying all permutations $\sigma \in S_4$ to the generators (6) allows us to represent $\mathcal{L}(4)$ as the row space of the 72×120 matrix C_4 whose columns are labelled by the monomials (4). The row space of C_4 is a subspace (in fact an S_4-submodule) of the nullspace of the matrix E_4. We set $o_4 = \mathrm{rank}(C_4)$ and write $\overline{C}_4$ for the $o_4 \times 120$ matrix in RCF (row canonical form) whose row space equals that of C_4.

Finally, we consider the new identities. The elements of the nullspace of E_4 are the coefficient vectors of $\mathcal{K}(4)$. We set $a_4 = \mathrm{nullity}(E_4)$ and write N_4 for the $a_4 \times 120$ matrix in RCF whose row space is the nullspace of E_4. The rows of N_4 span the S_4-module of all identities in degree 4. Clearly the row space of $\overline{C}_4$ is a subspace of the row space of N_4, and hence $o_4 \leq a_4$. The quotient $\mathcal{K}(4)/\mathcal{L}(4)$ is the S_4-module of *new identities* in degree 4, and its dimension is $a_4 - o_4$.

Let $A, O \subseteq \{1, \dots, 120\}$ be the column indices of the leading 1s in N_4 and $\overline{C}_4$ respectively. A linear basis of $\mathcal{K}(4)/\mathcal{L}(4)$ corresponds to (the cosets of) the rows of N_4 whose leading 1s have column indices in $A \setminus O$. It is straightforward using the module generators algorithm [4] to compute a subset of this linear basis which represents a set of S_4-module generators for the quotient module. Computations with the computer algebra system SageMath show that $a_4 = 88$ and hence N_4 has rank $n_4 = 32$; the nonzero entries of N_4 are $\pm\frac{1}{2}, \pm 1, -\frac{3}{2}$. For each row of N_4, we multiply the coefficients by the LCM of their denominators to obtain integers and then divide by the GCD of these integers to obtain vectors with relatively prime integer coefficients. The squared Euclidean lengths of the resulting vectors with multiplicities in parentheses are

$$
12\,(4),\ 18\,(4),\ 42\,(8),\ 48\,(2),\ 56\,(1),\ 60\,(4),\ 64\,(1),\ 72\,(2),\ 74\,(3),\ 82\,(1),\ 100\,(2).
$$

We sort the rows of the new integer matrix, also called N_4, by increasing length. Further SageMath computations show that $o_4 = 19$, which implies that the quotient module $\mathcal{K}(4)/\mathcal{L}(4)$ has dimension 13.

We next use the module generators algorithm again to determine the smallest subset of the shortest rows of N_4 which generates the quotient module

$\mathcal{K}(4)/\mathcal{L}(4)$. We obtain two identities and verify that neither is a consequence of the other. The first has 18 terms and coefficients ± 1 (squared length 18); the second has 33 terms and coefficients $\pm 1, \pm 2$ (squared length 42).

We can obtain better results using linear algebra over the integers; this requires replacing the RCF by the HNF (Hermite normal form), and applying the LLL algorithm [5] to determine shorter integer vectors.

The entries of the matrix E_4 belong to $\{0, \pm 1\}$. We compute the HNF of the transpose E_4^t, denoted by V, and a square matrix U with $\det(U) = \pm 1$ such that $UE_4^t = V$. Since E_4^t has rank 88, the bottom 32 rows of V are zero, and hence the bottom 32 rows of U form a matrix N whose rows form a lattice basis of the left nullspace of E_4^t (the right nullspace of E_4). (By a lattice basis we mean a set of free generators for a submodule of a free $\mathbb{Z}$-module.)

After applying the LLL algorithm to the lattice generated by the rows of N, we obtain a matrix N_{LLL} whose nonzero entries are ± 1 and whose rows have the following squared Euclidean lengths with multiplicities given in parentheses:

$$12\,(13),\ 18\,(12),\ 24\,(1),\ 26\,(1),\ 28\,(1),\ 32\,(3),\ 34\,(1).$$

Further computations show that the quotient module $\mathcal{K}(4)/\mathcal{L}(4)$ is generated by two rows of N_{LLL} with squared length 18. These are the coefficient vectors of the identities $I(a,b,c,d)$ and $H(a,b,c,d)$. Moreover, the S_4-module generated by $I(a,b,c,d)$ and $H(a,b,c,d)$ has dimension 32, so it coincides with $\mathcal{K}(4)$. ∎

Remark 4.4. Since the dimension of $\mathcal{K}(4)$ is 32 and the permutation of variables of a multilinear identity of degree 4 can produce at most $4! = 24$ linearly independent identities, a lower bound on the number of generators of $\mathcal{K}(4)$ as an S_4-module is $\lceil 32/24 \rceil = 2$. Therefore the set of generators $\{H, I\}$ of $\mathcal{K}(4)$ has minimum cardinal.

Corollary 4.5. *Consider these three consequences of the Lie-admissible identity:*

$$P(a,b,c,d) = L(ab,c,d),\ Q(a,b,c,d) = L(a,b,c)d,\ R(a,b,c,d) = aL(b,c,d).$$

Then

$$P(a,b,c,d) = I(c,a,b,d) - I(d,a,b,c),$$
$$Q(a,b,c,d) = H(a,c,b,d) - H(a,b,c,d),$$

$$\begin{aligned} 2R(a,b,c,d) &= \sum_{\sigma\in S_3} \varepsilon(\sigma)\big(H(a,b^\sigma,c^\sigma,d^\sigma) + I(a,b^\sigma,c^\sigma,d^\sigma) + I(c^\sigma,a,b^\sigma,d^\sigma)\big) \\ &\quad + \sum_{\sigma\in S_2} \varepsilon(\sigma) H(b,c^\sigma,d^\sigma,a), \\ 2J(a,b,c,d) &= \sum_{\sigma\in S_3} \big(H(a,b^\sigma,c^\sigma,d^\sigma) + I(a,b^\sigma,c^\sigma,d^\sigma) + I(c^\sigma,a,b^\sigma,d^\sigma)\big) \\ &\quad + \sum_{\sigma\in S_2} H(b,c^\sigma,d^\sigma,a), \end{aligned}$$

where $J(a,b,c,d)$ stands for the Jordan-admissible identity.

Proof: Straightforward computation. ■

5. Polynomial Identities in Degree 5

In degree 5 there are 14 association types and hence $14 \cdot 5! = 1680$ multilinear nonassociative monomials; there are $5! \cdot 2^4 = 1920$ associative *pq*-monomials.

Recall that in degree 4, identities H and I from Definition 4.2 generate the kernel $\mathcal{K}(4)$ of the expansion map as an S_4-module. Each identity U in degree 4 produces six consequences in degree 5:

$U(ab,c,d,e)$, $U(a,bc,d,e)$, $U(a,b,cd,e)$, $U(a,b,c,de)$, $U(a,b,c,d)e$, $aU(b,c,d,e)$.

Theorem 5.1. *Every identity in degree 5 satisfied by every mutation algebra follows from the consequences of H and I in degree 4, and the new identity G in degree 5 displayed in Figure 3.*

Proof: The proof is similar to that of degree 4. We order the monomial bases of $\mathcal{LT}(5)$ and $\mathcal{W}(5)$ as in Theorem 4.3. We need to perform computations on the 1920×1680 matrix E_5 representing the expansion map $X_5 : \mathcal{LT}(5) \to \mathcal{W}(5)$ (with respect to the monomial bases above). To this end, we use the class of rational sparse matrices in SageMath.

The kernel $\mathcal{K}(5)$ of the expansion map is an S_5-module of dimension 778 (comprising all identities). The twelve consequences (in degree 5) of identities H and I generate the S_5-module $\mathcal{O}(5)$ of old identities, which has dimension 747. Hence the quotient module $\mathcal{K}(5)/\mathcal{O}(5)$ of new identities has dimension 31. We compute the HNF, denoted by V, of E_5^t and a square matrix U with $\det(U) = \pm 1$ such that $UE_5^t = V$. The bottom 778 rows of U produce a matrix N whose rows form a lattice basis of the right nullspace of E_5. Next, we apply the LLL algorithm to the lattice generated by the rows of N to obtain the matrix N_{LLL}; we find that the S_5-module $\mathcal{K}(5)/\mathcal{O}(5)$ is generated by one row of N_{LLL} having 48 nonzero ± 1 entries, which is the coefficient vector of identity $G(a,b,c,d,e)$. The computations required around 4GB of RAM, and had a runtime of 90 minutes, in an AMD Ryzen 5 5600X processor at 3.70GHz running SageMath 9.2 on Windows 10. ■

$$
\boxed{
\begin{aligned}
&G(a,b,c,d,e) = \sum_{\substack{\sigma\in S_2(\{a,c\})\\ \tau\in S_2(\{d,e\})}} \varepsilon(\sigma)\Big(\{e,d,a^\sigma(c^\sigma b)\} - (\{e,d,b\}a^\sigma)c^\sigma + (e,b,a^\sigma(c^\sigma d))\\
&- ((ea^\sigma)c^\sigma,b,d) + (e,d,(ba^\sigma)c^\sigma) - (e,(ba^\sigma)c^\sigma,d) + ((e,b,d)a^\sigma)c^\sigma - a^\sigma(c^\sigma(e,d,b))\\
&+ ((d^\tau a^\sigma)c^\sigma)(e^\tau b) - d^\tau(a^\sigma(c^\sigma(e^\tau b))) + d^\tau(((e^\tau b)a^\sigma)c^\sigma) - d^\tau(((e^\tau a^\sigma)c^\sigma)b)\Big),\\
&\text{where } \{x,y,z\} := (xy)z + y(xz), \text{ and } S_2(\{x,y\}) \text{ denotes } S_2 \text{ acting on the set } \{x,y\}.
\end{aligned}
}
$$

FIGURE 3. The new identity in degree 5

Remark 5.2. The dimension of $\mathcal{K}(5)$ is 778 and permuting the variables of a multilinear identity of degree 5 can produce at most $5! = 120$ linearly independent identities, so a lower bound on the number of generators of $\mathcal{K}(5)$ as an S_5-module is $\lceil 778/120 \rceil = 7$. In Theorem 5.1 we have obtained a set with 13 generators: the 12 consequences of identities H and I plus a new identity G. In fact, it can be checked that $\mathcal{K}(5)$ is already generated by identity G, the consequences of identity I, and consequences $H(ab,c,d,e)$, $H(a,b,c,d)e$ and $aH(b,c,d,e)$ of identity H. Therefore an upper bound on the minimum number of generators of $\mathcal{K}(5)$ as an S_5-module is 10.

6. Polynomial Identities in Degree 6

In degree 6 there are 42 association types and hence $42 \cdot 6! = 30240$ multilinear nonassociative monomials, and there are $6! \cdot 2^5 = 23040$ associative pq-monomials. So, to represent the expansion map X_6 as a whole we would need to use a matrix of size 30240×23040, which is too large to manipulate with our computer system. We use the representation theory of the symmetric group to reduce the problem to a set of matrices of smaller sizes and demonstrate the existence of a number of new identities in degree 6. We choose a set of conjugacy class representatives in S_6 and calculate the matrices representing these permutations on the modules of old and all identities.

Comparing the traces of these matrices with the character table of S_6, we obtain the multiplicities of the irreducible representations of the S_6-modules.

Theorem 6.1. *For each of the 11 partitions λ of 6, the following table contains the multiplicity of each irreducible representation in the S_6-modules of all identities (the kernel of the expansion map), the old identities (the consequences of the identities of lower degree), and the quotient module of new identities (the difference of the previous two multiplicities):*

λ	6	51	42	411	33	321	3111	222	2211	21111	111111
$\dim(\lambda)$	1	5	9	10	5	16	10	5	9	5	1
$\mathrm{all}(\lambda)$	41	205	369	410	205	656	410	205	369	205	41
$\mathrm{old}(\lambda)$	29	136	237	268	131	422	267	131	236	133	28
$\mathrm{new}(\lambda)$	12	69	132	142	74	234	143	74	133	72	13

Furthermore, the dimension of the quotient module of new identities is

$$\sum_{\lambda} \mathrm{new}(\lambda)\dim(\lambda) = 10449.$$

Proof: These methods have been described in detail in [4, §§2.4–2.7]. ∎

Conjecture 6.2. The kernel of the expansion map in all degrees, that is, the operad ideal $\mathcal{K} = \bigoplus_{n\geq 1}\mathcal{K}(n)$ (see Definition 2.2), is not finitely generated. In other words, no finite set of identities generates all the identities satisfied by all mutation algebras.

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