Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 22–49

LAX COMMA CATEGORIES OF ORDERED SETS

MARIA MANUEL CLEMENTINO AND FERNANDO LUCATELLI NUNES

In memory of Horst Herrlich

ABSTRACT: Let Ord be the category of (pre)ordered sets. Unlike Ord/X, whose behaviour is well-known, not much can be found in the literature about the lax comma 2-category $\mathsf{Ord}//X$. In this paper we show that, when X is complete, the forgetful functor $\mathsf{Ord}//X \to \mathsf{Ord}$ is topological. Moreover, $\mathsf{Ord}//X$ is complete and cartesian closed if and only if X is. We end by analysing descent in this category. Namely, when X is complete and cartesian closed, we show that, for a morphism in $\mathsf{Ord}//X$, being pointwise effective for descent in Ord is sufficient, while being effective for descent in Ord is necessary, to be effective for descent in $\mathsf{Ord}//X$.

KEYWORDS: effective descent morphisms, lax comma 2-categories, comma categories, exponentiability, cartesian closed categories, topological functors, enriched categories, Ord-enriched categories.
MATH. SUBJECT CLASSIFICATION (2000): 06A07, 18A25, 18A30, 18N10, 18D20, 18E50.

Introduction

Janelidze-Galois theory [7, 8] neatly gives a common ground for many Galois-type theories, prominently including Magid's Galois theory of commutative rings, Grothendieck's theory of étale covering of schemes, and central extension of groups. There is a deep connection between Janelidze-Galois theory and factorization systems [9, 2].

Motivated by this connection and the theory of *lax orthogonal factorization systems* [3, 4], we have started a project whose aim is to investigate two-dimensional extensions of the basic ideas and results of Janelidze-Galois theory.

It has been noticeable that the so-called *lax comma 2-categories* play an important role in our work (*c.f.* [5]). Although they are quite natural (appearing, for instance, in [14] and [15, 2.2]), it seems that the literature still

Received December 27, 2022.
The research was supported through the programme "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2022, and partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

lacks a systematic study of their fundamental properties; namely, topologicity, exponentiability, and descent.

Since these properties are essential to our endeavour, we give herein an exposition on the lax comma 2-categories of Ord, the 2-category of ordered sets (also called preordered sets). We prove that the forgetful functor $\mathsf{Ord}/\!/X \to \mathsf{Ord}$ is topological. Moreover, we show that $\mathsf{Ord}/\!/X$ is complete and cartesian closed if and only if X is. We end by analysing descent in this category. Namely, when X is complete and cartesian closed, we show that, for a morphism in $\mathsf{Ord}/\!/X$, being pointwise effective for descent in Ord is sufficient, while being effective for descent in Ord is necessary, to be effective for descent in $\mathsf{Ord}/\!/X$.

Although further enriched and 2-dimensional aspects of lax comma objects are essential to our project (see, for instance, [5] for an overall view of our ongoing work's setting), they are not relevant to the present note and, hence, will not be dealt herein.

The main intent of this paper is threefold: (1) give an exposition of lax comma 2-categories of Ord, showing some of its nice properties; (2) provide background to our future work in descent and Galois theory regarding Ord-enriched categories; (3) give a guiding template for our most general systematic study of lax comma 2-categories. Finally, we also want to pick the community's attention to the problem of studying lax comma 2-categories, showing that, even in the case of Ord, there are still facets to be better explored.

1. The forgetful functor $U\colon \mathsf{Ord}/\!/X \to X$

Let X be an ordered set. Here by *order* it is meant a reflexive and transitive binary relation, not necessarily antisymmetric (also called preorder).

The category $\mathsf{Ord}/\!/X$ has as objects monotone maps $a\colon Y \to X$, where Y is an ordered set, and as morphisms $f\colon (Y,a) \to (Z,b)$ monotone maps $f\colon Y \to Z$ such that $a \leqslant bf$:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "a"'] & \leqslant & Z \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

with the usual composition. Given two morphisms $f, g\colon (Y,a) \to (Z,b)$, we say that $f \leqslant g$ if $f(y) \leqslant g(y)$ for all $y \in Y$, that is to say, if $f \leqslant g$ in Ord. This makes $\mathsf{Ord}/\!/X$ an Ord-enriched category.

The category Ord/X is a non-full subcategory of $\mathsf{Ord}//X$, having the same objects, and morphisms $f\colon (Y,a) \to (Z,b)$ those morphisms in $\mathsf{Ord}//X$ such that $a = bf$. The Ord-enrichment in Ord/X is the same as $\mathsf{Ord}//X$; that is, the inclusion $\mathsf{Ord}/X \to \mathsf{Ord}//X$ is locally full.

These two categories have very different behaviour, as we will see throughout this text. We start by comparing the two locally full forgetful functors of the diagram

$$\begin{tikzcd} \mathsf{Ord}/X \arrow[rr] \arrow[dr, "\overline{U}"'] & & \mathsf{Ord}//X \arrow[dl, "U"] \\ & \mathsf{Ord} & \end{tikzcd}$$

It is well-known that *the forgetful functor* $\overline{U}\colon \mathsf{Ord}/X \to \mathsf{Ord}$ *is* Ord*-comonadic, and therefore it reflects isomorphisms, and creates (*Ord*-weighted) colimits and absolute equalizers.* Moreover, the category Ord/X is complete but $\overline{U}$ does not preserve limits in general; indeed, it preserves equalizers and pullbacks but not products: in Ord/X the terminal object is $1_X\colon X \to X$, and products are formed via pullbacks. On the contrary, *the forgetful functor* $U\colon \mathsf{Ord}//X \to \mathsf{Ord}$ *does not reflect isomorphisms, but in turn it is topological* [6]:

Theorem 1.1. *If X is a complete ordered set, then the forgetful functor $U\colon \mathsf{Ord}//X \to \mathsf{Ord}$ is topological.*

Proof: Given a family $(f_i\colon Y \to (Z_i,b_i))_{i\in I}$ of monotone maps (where I may be a proper class), we define $a\colon Y \to X$ by $a(y) = \bigwedge_{i\in I} b_i(f_i(y))$. Then, by construction, a is monotone and $a \leqslant bf$; that is, for each $i \in I$, $f_i\colon (Y,a) \to (Z_i,b_i)$ is a morphism in $\mathsf{Ord}//X$. Moreover, given any family of morphisms $(g_i\colon (W,c) \to (Z_i,b_i))_{i\in I}$ and a monotone map $h\colon W \to Y$ such that $f_ih = g_i$ for every i, then it is easily checked that $c \leqslant ah$, i.e. $h\colon (W,c) \to (Y,a)$ is a morphism in $\mathsf{Ord}//X$ (and clearly the unique whose image under U is $h\colon W \to Y$). ■

Corollary 1.2. *The category $\mathsf{Ord}//X$ is (co)complete if, and only if, X is (co)complete.*

Proof: It remains to check that X is complete provided that $\mathsf{Ord}//X$ is. Let $(x_i)_{i\in I}$ be a family of elements of X and consider the identities $(1 \to (1,x_i))_{i\in I}$ (here we will use the same notation for the element x of X and the constant map $Y \to X$ which assigns x to every element of Y). The initial structure x on 1, with respect to this family, is clearly $\bigwedge_{i\in I} x_i$ in X: so that $(1,x) \to (1,x_i)$

is a morphism, $x \leqslant x_i$ for every i; the universal property of the lifting gives that, if $y \in X$ is such that $y \leqslant x_i$ for every $i \in I$, then $y \leqslant x$.

For the corresponding result for cocompleteness the proof is analogous. ∎

From now on, X is a complete ordered set. We find it worth to describe how limits and colimits are built in $\mathsf{Ord}/\!/X$. Given a family $(X_i, a_i)_{i\in I}$ of objects of $\mathsf{Ord}/\!/X$, the structure $a\colon \prod X_i \to X$ in the product $\prod X_i$ is defined by $a\left((x_i)_i\right) = \bigwedge_i a_i(x_i)$, while the structure in its coproduct $\coprod X_i$ is given by $b\colon \coprod X_i \to X$, with $b(y) = a_i(y)$ when $y \in X_i$. Equalisers are built as expected: given morphisms $f, g\colon (Y, a) \to (Z, b)$, its equaliser is $m\colon (M = \{y \in Y\,;\, f(y) = g(y)\}, \widehat{a}) \to (Y, a)$, where $\widehat{a}$ is the restriction of a to M. Coequalisers are given by Kan extensions, as we show next.

Lemma 1.3. *Given morphisms f, g, h in $\mathsf{Ord}/\!/X$ as in the diagram*

$$\begin{tikzcd} Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "a"'] & Z \arrow[r, "h"] \arrow[d, "b"] \arrow[dl, phantom, "\leqslant"] \arrow[dr, phantom, "\leqslant"] & W \arrow[dl, "c"] \\ & X & \end{tikzcd}$$

h is the coequaliser of f, g in $\mathsf{Ord}/\!/X$ if, and only if:

(1) *h is the coequaliser of f, g in Ord;*
(2) *c is the right Kan extension of b along h.*

Proof: Assume that $h\colon (Z, b) \to (W, c)$ is the coequaliser of f, g in $\mathsf{Ord}/\!/X$. Since the forgetful functor into Ord is topological, h is the coequaliser of f, g in Ord. To check that c is the right Kan extension of b along h, let $c'\colon W \to X$ be such that $b \leqslant c'h$. Then, by the universal property of the coequaliser, there exists $t\colon (W, c) \to (W, c')$ such that $th = h$; surely $t = 1_W$ since h is an epimorphism, and therefore $c \leqslant c'$ as claimed.

Conversely, assume that $h\colon (Z, b) \to (W, c)$ satisfies conditions (1), (2). Then trivially $hf = hg$, and, for any $h'\colon (Z, b) \to (W', c')$ with $h'f = h'g$ there is a (unique) monotone map $t\colon W \to W'$ such that $th = h'$. Since, by assumption, $b \leqslant c'h' = c'th$, by (2) we conclude that $c \leqslant c't$, that is, $t\colon (W, c) \to (W', c')$ is a morphism in $\mathsf{Ord}/\!/X$ and our conclusion follows. ∎

Remark 1.4 (Weighted (co)limits). We refer to [1, pages 7 & 8] for Ord-enriched weighted (co)limits. Recall that an Ord-enriched category is Ord-(co)complete whenever it has conical (co)limits and the so-called Ord-(co)tensors, called herein Ord-(co)powers (see, for instance, [12, Theorem 3.73] for the general enriched setting).

We establish herein that, when X is (co)complete, $\mathsf{Ord}/\!/X$ is Ord-complete and cocomplete. More precisely, assuming that X is (co)complete, the conical (co)limits described in Section 1 are Ord-enriched. Furthermore, for a pair $(W,(Y,a)) \in \mathsf{Ord} \times \mathsf{Ord}/\!/X$:

- the Ord-copower $W \otimes (Y,a)$ is given by $(W \times Y, W \otimes a)$ where $W \otimes a(w,y) = a(y)$.
- the Ord-power $W \pitchfork (Y,a)$ is given by $\left(Y^W, a^\top\right)$ where

$$a^\top(f) = \bigwedge_{w\in W} a(f(w)),$$

in which Y^W is the exponential in Ord.

2. Exponentiability

In order to investigate under which conditions $\mathsf{Ord}/\!/X$ is a cartesian closed category, we first recall two well-known results.

First of all, the (complete) ordered set X, as a thin category, is cartesian closed if, and only if, it is an Heyting algebra; that is, it has a binary operation $X \times X \to X$, assigning to each pair (x,y) an element y^x such that $z \leqslant y^x$ if and only if $z \wedge x \leqslant y$, for every $x,y,z \in Z$. This is in fact equivalent to X being a frame, i.e. in X arbitrary joins distribute over finite meets.

Secondly, Ord is a cartesian closed category. For each ordered set Y, the right adjoint functor $(\ )^Y$ to $(\ ) \times Y : \mathsf{Ord} \to \mathsf{Ord}$ assigns to each ordered set Z the set $Z^Y = \{f : Y \to Z\,;\ f \text{ is a monotone map}\}$, equipped with the pointwise order; that is, for $f,g \in Z^Y$, $f \leqslant g$ if, for all $y \in Y$, $f(y) \leqslant g(y)$.

On the contrary, Ord/X is not cartesian closed in general. The following result can be found in [16].

Theorem 2.1. *Given a monotone map $a\colon Y \to X$, the functor*

$$(\ ) \times (Y,a)\colon \mathsf{Ord}/X \to \mathsf{Ord}/X$$

has a right adjoint if, and only if, for all $y_0 \leqslant y_1$ in Y, $x \in X$, if $a(y_0) \leqslant x \leqslant a(y_1)$, then there exists $y \in Y$ with $y_0 \leqslant y \leqslant y_1$ and $a(y) = x$.

Again, $\mathsf{Ord}/\!/X$ behaves differently:

Theorem 2.2. *For a complete ordered set X, the following assertions are equivalent:*

(i) *X is cartesian closed;*
(ii) *$\mathsf{Ord}/\!/X$ is cartesian closed.*

Proof: (i)⇒(ii): Given two objects $a\colon Y \to X$ and $b\colon Z \to X$ in $\mathsf{Ord}//X$, in order to define $(Z,b)^{(Y,a)}$ first we consider the ordered set Z^Y as defined above, and then the map $b^a\colon Z^Y \to X$ defined by

$$b^a(f) = \bigwedge_{y\in Y} b(f(y))^{a(y)}.$$

The map b^a is monotone: if $f,g\colon Y \to Z$ are monotone maps, with $f \leqslant g$, then, for every $y \in Y$,

$$b(f(y))^{a(y)} \wedge a(y) \leqslant b(f(y)) \leqslant b(g(y)) \;\Rightarrow\; b(f(y))^{a(y)} \leqslant b(g(y))^{a(y)}.$$

The monotone map ev is a morphism in $\mathsf{Ord}//X$

$$\begin{tikzcd} Z^Y \times Y \arrow[rr, "\mathrm{ev}"] \arrow[dr, "b^a\wedge a"'] & \arrow[d, phantom, "\leqslant"] & Z \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

since, by definition of b^a, for all $f \in Z^Y$ and $y \in Y$,

$$b^a(f) \wedge a(y) = \bigwedge_{y'\in Y} b(f(y'))^{a(y')} \wedge a(y) \leqslant b(f(y))^{a(y)} \wedge a(y) \leqslant b(f(y)).$$

To check its universality let $c\colon W \to X$ be an object and $h\colon (W,c) \times (Y,a) \to (Z,b)$ a morphism in $\mathsf{Ord}//X$. Then $\overline{h}\colon W \to Z^Y$, with $\overline{h}(w)\colon Y \to Z$ defined by $\overline{h}(w)(y) = h(w,y)$ for every $w \in W$ and $y \in Y$, is a morphism in $\mathsf{Ord}//X$:

$$c(w)\wedge a(y) \leqslant b(h(w,y)) \Rightarrow c(w) \leqslant b(\overline{h}(w)(y))^{a(y)} \Rightarrow c(w) \leqslant \bigwedge_{y\in Y} b(\overline{h}(w)(y))^{a(y)}.$$

Therefore $(\ ) \times (Y,a)$ has a right adjoint $(\ )^{(Y,a)}$ assigning (Z^Y, b^a) to each (Z,b) in $\mathsf{Ord}//X$.

(ii)⇒(i): Assuming that $\mathsf{Ord}//X$ is cartesian closed, for each $x \in X$, let $(W,c) = (X,1_X)^{(1,x)}$ be the exponential in $\mathsf{Ord}//X$. It is easily checked that $W \cong \mathsf{Ord}(1,X) \cong X$. We will show that $y^x = c(y)$, where $y\colon 1 \to X$ is the map assigning y to the only element of 1, for $y \in X$. Using

$$\begin{tikzcd} W \times 1 \arrow[rr, "\mathrm{ev}"] \arrow[dr, "c\wedge x"'] & \arrow[d, phantom, "\leqslant"] & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

one concludes that $c(y) \wedge x \leqslant y$; moreover, from the universality of ev it follows that, if $z \wedge x \leqslant y$, then the morphism $y\colon (1,z) \times (1,x) \to (X, 1_X)$

$$\begin{tikzcd} 1 \times 1 \arrow[rr, "y"] \arrow[dr, "z \wedge x"'] & \arrow[d, phantom, "\leqslant"] & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

induces, by universality of ev, a morphism $\overline{u}\colon (1,z) \to (W,c)$ such that $\mathrm{ev}(\overline{u} \times 1) = y$, and thus $z \leqslant c(y)$ as required. ∎

A careful analysis of this proof allows us to conclude the following

Corollary 2.3. *Let X be a complete ordered set and (Y,a) an object of $\mathsf{Ord}//X$. The following conditions are equivalent:*

(i) *(Y,a) is exponentiable in $\mathsf{Ord}//X$;*
(ii) *For all $y \in Y$, $a(y)$ is exponentiable in X.*

Proof: (i) $\Rightarrow$ (ii) is shown exactly as in the Theorem above, observing that to define the exponentials with exponent (Y,a) we only need exponentials in X with exponent $a(y)$, for $y \in Y$.

(ii) $\Rightarrow$ (i): Assuming that (Y,a) is exponentiable, let $(W,c) = (X, 1_X)^{(Y,a)}$. Then it is easy to check that $W = \{f\colon Y \to X\,;\ f \text{ is monotone}\}$. Let $x \in X$, and $g\colon Y \to X$ be defined by $g(y) = x$ for all $y \in Y$. Then, on one hand, $c(g) \wedge a(y) \leqslant x$ because $\mathrm{ev}\colon (W,c) \times (Y,a) \to (X,1_X)$ is a morphism in $\mathsf{Ord}//X$, and, on the other hand, if $z \in X$ is such that $z \wedge a(y) \leqslant x$ then the map $h\colon (1,z) \times (Y,a) \to (X,1_X)$ constantly equal to x is a morphism in $\mathsf{Ord}//X$ and so there is $\overline{h}\colon (1,z) \to (W,c)$ such that $\mathrm{ev}(\overline{h} \times 1) = h$. Necessarily $\overline{h}(*) = g$ and therefore $z \leqslant c(g)$. ∎

3. Descent

We could not find a reference on the literature on the behaviour of the change-of-base functors for morphisms of Ord/X. Indeed their study does not provide extra information since they reduce to change of base functors for morphisms in Ord: if $f\colon (Y,a) \to (Z,b)$ is a morphism in Ord/X, then pulling back along f in Ord/X is the same as pulling back along f in Ord:

$$((\mathsf{Ord}/X)/(Z,b) \xrightarrow{f^*} (\mathsf{Ord}/X)/(Y,a)) \;\cong\; (\mathsf{Ord}/Z \xrightarrow{f^*} \mathsf{Ord}/Y)$$

since, for any (Y,a), $(\mathsf{Ord}/X)/(Y,a)$ is isomorphic to Ord/Y and pullbacks are formed exactly in the same way.

This is not the case for $\mathsf{Ord}/\!/X$ as we show next. We will investigate effective descent morphisms in $\mathsf{Ord}/\!/X$, showing in particular that being effective for descent in Ord is necessary but not sufficient for a morphism to be effective for descent in $\mathsf{Ord}/\!/X$.

Throughout we assume that X, as a category, is complete and cartesian closed (i.e., X is a complete Heyting algebra).

We start by characterizing (stable) regular epimorphisms in $\mathsf{Ord}/\!/X$.

Lemma 3.1. *For a morphism $f\colon (Y,a)\to(Z,b)$ in $\mathsf{Ord}/\!/X$, the following conditions are equivalent:*

(i) *f is a regular epimorphism in $\mathsf{Ord}/\!/X$;*
(ii) *f is a regular epimorphism in Ord and*

$$(\forall z\in Z)\;\; b(z)=\bigvee_{f(y)\leqslant z} a(y). \tag{3.i}$$

Proof: What remains to show is that (3.i) is equivalent to $b=\mathrm{ran}_f a$. Given a regular epimorphism $f\colon Y\to Z$ in Ord and a monotone map $a\colon Y\to X$, (3.i) defines a monotone map $b\colon Z\to X$ such that $a\leqslant bf$. Moreover, if $a\leqslant b'f$ for some monotone map $b'\colon Z\to X$, then, for every $z\in Z$ and $y\in Y$ with $f(y)\leqslant z$, $a(y)\leqslant c'(f(y))\leqslant c'(z)$, and so $c\leqslant c'$. The converse is shown analogously. ∎

Proposition 3.2. *For a morphism $f\colon (Y,a)\to(Z,b)$ in $\mathsf{Ord}/\!/X$, the following conditions are equivalent:*

(i) *f is a stable regular epimorphism in $\mathsf{Ord}/\!/X$;*
(ii) *f is a stable regular epimorphism in Ord, that is, for each $z_0\leqslant z_1$ in Z there exist $y_0\leqslant y_1$ in Y with $f(y_i)=z_i$ ($i=0,1$), and*

$$(\forall z\in Z)\;\; b(z)=\bigvee_{f(y)=z} a(y). \tag{3.ii}$$

Proof: (i) $\Rightarrow$ (ii): The forgetful functor $U\colon \mathsf{Ord}/\!/X\to\mathsf{Ord}$ preserves regular epimorphisms and pullbacks, hence every stable regular epimorphism in $\mathsf{Ord}/\!/X$ is also stably a regular epimorphism in Ord. If (3.ii) does not hold, that is, if there exists $z\in Z$ with $b(z)>\bigvee_{f(y)=z}a(y)$, then we consider the pullback of f along $g\colon (1,b(z))\to(Z,b)$ with $g(*)=z$. It is easy to check that in the pullback diagram

$$
\begin{tikzcd}
f^{-1}(z) \arrow[rr, "\pi_2"] \arrow[d, "\pi_1"] \arrow[ddr, bend right=60, "a\pi_1"'] & & 1 \arrow[d, "g"] \arrow[ddl, bend left=60, "b(z)"] \\
Y \arrow[rr, "f"] \arrow[dr, "a"'] \arrow[rr, phantom, "\leqslant"', shift right=4ex] & & Z \arrow[dl, "b"] \\
& X &
\end{tikzcd}
$$

π_2 is not a regular epimorphism in $\mathsf{Ord}//X$ since it does not satisfy (3.i).

(ii) $\Rightarrow$ (i): If $f\colon (Y,a) \to (Z,b)$ satisfies (ii), for any pullback diagram in $\mathsf{Ord}//X$

$$
\begin{tikzcd}
Y \times_Z W \arrow[rr, "\pi_2"] \arrow[d, "\pi_1"] \arrow[ddr, bend right=60, "a\pi_1 \wedge c\pi_2"'] & & W \arrow[d, "g"] \arrow[ddl, bend left=60, "c"] \\
Y \arrow[rr, "f"] \arrow[dr, "a"'] \arrow[rr, phantom, "\leqslant"', shift right=4ex] \arrow[d, phantom, "\leqslant"] & & Z \arrow[dl, "b"] \arrow[d, phantom, "\geqslant"] \\
{} & X & {}
\end{tikzcd}
$$

by assumption π_2 is a regular epimorphism in Ord, so it remains to be shown that π_2 satisfies (3.i): for any $w \in W$, $c(w) \leqslant b(g(w)) = \bigvee_{f(y)=g(w)} a(y)$; hence

$$
\begin{aligned}
c(w) \leqslant b(g(w)) \wedge c(w) &= \Big(\bigvee_{f(y)=g(w)} a(y)\Big) \wedge c(w) \\
&= \bigvee_{f(y)=g(w)} (a(y) \wedge c(w)) \quad (X \text{ is cartesian closed}) \\
&\leqslant \bigvee_{f(y')=g(w'),\, w' \leqslant w} (a\pi_1 \wedge c\pi_2)(y', w').
\end{aligned}
$$

■

Next we investigate effective descent morphisms in $\mathsf{Ord}/\!/X$. We will show that, for a given morphism $f\colon (Y,a)\to(Z,b)$ in $\mathsf{Ord}/\!/X$,

$$(\forall z_0\leqslant z_1\leqslant z_2 \text{ in } Z)\ (\exists y_0\leqslant y_1\leqslant y_2 \text{ in } Y)\ :\ \begin{array}{l} f(y_i)=z_i\ (i=0,1,2)\\ \text{and } a(y_0)=b(z_0)\end{array} \tag{3.iii}$$

$$\Downarrow$$

$$f \text{ is effective for descent in } \mathsf{Ord}/\!/X$$

$$\Downarrow$$

$$(\forall z_0\leqslant z_1\leqslant z_2 \text{ in } Z)\ (\exists y_0\leqslant y_1\leqslant y_2 \text{ in } Y)\ :\ f(y_i)=z_i\ (i=0,1,2). \tag{3.iv}$$

We start by showing the latter implication.

Theorem 3.3. *If $f\colon (Y,a)\to(Z,b)$ is effective for descent in $\mathsf{Ord}/\!/X$, then $Uf\colon Y\to Z$ is effective for descent in Ord.*

Proof: With $f\colon (Y,a)\to(Z,b)$ also its pullback $f_\perp$ along $(Z,\perp)\to(Z,b)$

$$\begin{tikzcd} (Y,\perp) \arrow[r, "f_\perp"] \arrow[d, "1"'] & (Z,\perp) \arrow[d, "1"] \\ (Y,a) \arrow[r, "f"'] & (Z,b) \end{tikzcd}$$

is effective for descent. Observing that the change of base functors of $f_\perp$ in $\mathsf{Ord}/\!/X$ and of $Uf_\perp = Uf$ in Ord are isomorphic:

$$(\ (\mathsf{Ord}/\!/X)/(Z,\perp) \xrightarrow{\ f_\perp^*\ } (\mathsf{Ord}/\!/X)/(Y,\perp)\ ) \cong (\ \mathsf{Ord}/Z \xrightarrow{\ (Uf)^*\ } \mathsf{Ord}/Y\ )$$

we conclude that Uf is effective for descent in Ord, which, thanks to [10, Proposition 3.4], is equivalent to (3.iv). ∎

To show that (3.iii) is sufficient for f to be effective for descent, we will make use of the chain of pullback preserving (faithful) inclusions

$$\mathsf{Ord}/\!/X \xrightarrow{\ \Pi\ } [X^{\mathrm{op}},\mathsf{Ord}] \longrightarrow [X^{\mathrm{op}},\mathsf{Rel}] \longrightarrow [X^{\mathrm{op}},\mathsf{Gph}], \tag{3.v}$$

where $\Pi(Y,a)\colon X^{\mathrm{op}}\to\mathsf{Ord}$ is defined by $\Pi(Y,a)(x)=Y_x=\{y\in Y\,;\,x\leqslant a(y)\}$, $\Pi(Y,a)(x\geqslant x')$ is the inclusion of Y_x in $Y_{x'}$ and $\Pi(f\colon (Y,a)\to(Z,b))_x$

is the (co)restriction $Y_x \to Z_x$ of f, and the following Theorem, that can be found, for instance, in [11, pag. 260] or [13, Theorem 1.4].

Theorem 3.4. *Let $\mathbb{A}$ and $\mathbb{B}$ be categories with pullbacks. If $F\colon \mathbb{A} \to \mathbb{B}$ is a fully faithful pullback preserving functor and $F(f)$ is of effective descent in $\mathbb{B}$, then f is of effective descent if, and only if, it satisfies the following property: whenever the diagram below is a pullback in $\mathbb{B}$, there is an object A in $\mathbb{A}$ such that $F(A) \cong B$*

$$\begin{tikzcd} F(P) \arrow[r] \arrow[d] & B \arrow[d] \\ F(Y) \arrow[r, "F(f)"'] & F(Z). \end{tikzcd}$$

Indeed, using (3.v) we will show that in $[X^{\mathrm{op}}, \mathsf{Ord}]$ a natural transformation is effective for descent if, and only if, it is pointwise effective for descent in Ord, and that f is effective for descent in $\mathsf{Ord}//X$ provided that Πf is effective for descent in $[X^{\mathrm{op}}, \mathsf{Ord}]$.

Proposition 3.5. *In $[X^{\mathrm{op}}, \mathsf{Gph}]$ a morphism $\alpha\colon F \to G$ is effective for descent if, and only if, it is an epimorphism.*

Proof: The category $[X^{\mathrm{op}}, \mathsf{Gph}]$ is a topos. ∎

Proposition 3.6. *For a morphism $\alpha\colon F \to G$ in $[X^{\mathrm{op}}, \mathsf{Rel}]$, the following conditions are equivalent:*

(i) *α is effective for descent;*
(ii) *α is a stable regular epimorphism;*
(iii) *α is a regular epimorphism;*
(iv) *$(\forall x \in X)$ α_x is a regular epimorphism in Rel;*
(v) *$(\forall x \in X)$ $(\forall (z_0, z_1) \in G(x))$ $(\exists (y_0, y_1) \in F(x))$: $\alpha_x(y_0, y_1) = (z_0, z_1)$;*
(vi) *$(\forall x \in X)$ α_x is effective for descent in Rel.*

Proof: Applying Theorem 3.4 for the inclusion $[X^{\mathrm{op}}, \mathsf{Rel}] \to [X^{\mathrm{op}}, \mathsf{Gph}]$, and knowing that pullbacks in $[X^{\mathrm{op}}, \mathsf{Rel}]$ are formed pointwise and regular epimorphisms are pullback stable, one concludes that (i)⇔(ii)⇔(iii)⇔(iv). The characterizations of regular epimorphisms and effective descent morphisms in Rel of [10, Propositions 2.1 and 3.3] give (iv)⇔(v)⇔(vi). ∎

Theorem 3.7. *In $[X^{\mathrm{op}}, \mathsf{Ord}]$ a morphism $\alpha\colon F \to G$ is effective for descent if, and only if,*

$$(\forall x \in X)\ \alpha_x \text{ is effective for descent in } \mathsf{Ord}. \tag{3.vi}$$

Proof: Now we apply Theorem 3.4 to the full inclusion $[X^{\mathrm{op}}, \mathsf{Ord}] \to [X^{\mathrm{op}}, \mathsf{Rel}]$. Since it preserves pullbacks, to prove that $\alpha\colon F \to G$ satisfying (3.vi) is effective for descent in $[X^{\mathrm{op}}, \mathsf{Ord}]$ it is sufficient to show that in the pullback diagram

$$\begin{tikzcd} F \times_G H \arrow[r, "\rho"] \arrow[d, "\pi"'] & H \arrow[d, "\beta"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

if $F \times_G H$ belongs to $[X^{\mathrm{op}}, \mathsf{Ord}]$, then also H does. For each $x \in X$, consider the pullback diagram

$$\begin{tikzcd} F(x) \times_{G(x)} H(x) \arrow[r, "\rho_x"] \arrow[d, "\pi_x"'] & H(x) \arrow[d, "\beta(x)"] \\ F(x) \arrow[r, "\alpha_x"'] & G(x). \end{tikzcd}$$

If α_x is effective for descent in Ord, then $H(x) \in \mathsf{Ord}$ since $F(x) \times_{G(x)} H(x)$ does by assumption.

Conversely, let us assume that α is effective for descent, and let $x \in X$ and $z_0 \leqslant z_1 \leqslant z_2$ in $G(x)$. Consider the functor $H\colon X^{\mathrm{op}} \to \mathsf{Rel}$ defined by

$$H(x') = \begin{cases} \{(z_0, z_1), (z_1, z_2)\} & \text{if } x' \cong x \\ \{(z_0, z_0)\} & \text{if } x' < x \\ \varnothing & \text{otherwise.} \end{cases}$$

with $H(x'' \geqslant x')\colon H(x'') \to H(x')$ given by $\varnothing \to H(x')$ if $x'' \nleqslant x$, the constant map $H(x'') \to H(x')$ if $x' < x$ and $x'' \leqslant x$, and the identity otherwise. Since by assumption α is effective for descent and H does not belong to $[X^{\mathrm{op}}, \mathsf{Ord}]$ (since $H(x)$ is not transitive), also $F \times_G H$ does not belong to $[X^{\mathrm{op}}, \mathsf{Ord}]$. If $x' \ncong x$, then $F(x') \times_{G(x')} H(x')$ is either $\varnothing$ or isomorphic to $F(x')$, hence an ordered set. Therefore there must exist $x' \cong x$ (and so we may consider $x' = x$ since images of isomorphic elements will be isomorphic too) so that $F(x) \times_{G(x)} H(x)$, that is,

$$\{((y, y'), (z_0, z_1))\,;\, \alpha_x(y, y') = (z_0, z_1)\} \cup \{((y, y'), (z_1, z_2))\,;\, \alpha_x(y, y') = (z_1, z_2)\}$$

is not an ordered set. Failure of transitivity at $F(x) \times_{G(x)} H(x)$ means that, necessarily, there exist $((y, y'), (z_0, z_1))$ and $((y', y''), (z_1, z_2))$ in $F(x) \times_{G(x)} H(x)$; then $\pi_x(y, y')$ and $\pi_x(y', y'')$ gives that $y \leqslant y' \leqslant y''$ in $F(x)$. Computing now α_x gives $\alpha_x(y) = z_0$, $\alpha_x(y') = z_1$ and $\alpha_x(y'') = z_2$. ■

Theorem 3.8. *If* $f\colon (Y,a)\to(Z,b)$ *is a morphism in* $\mathsf{Ord}//X$ *satisfying* (3.iii)*, that is,*

$$(\forall z_0\leqslant z_1\leqslant z_2 \text{ in } Z)\ (\exists y_0\leqslant y_1\leqslant y_2 \text{ in } Y)\ :\quad \begin{array}{l} f(y_i)=z_i\ (i=0,1,2)\\ \text{and } a(y_0)=b(z_0),\end{array}$$

then f *is effective for descent in* $\mathsf{Ord}//X$*.*

Proof: Let $f\colon (Y,a)\to(Z,b)$ satisfy the condition above. Applying Theorem 3.4, what we need to show is that, given a pullback diagram in $[X^{\mathrm{op}},\mathsf{Ord}]$

$$\begin{tikzcd} U(P,c) \arrow[r, "\rho"] \arrow[d, "U\pi"'] & G \arrow[d, "\beta"] \\ U(Y,a) \arrow[r, "Uf"'] & U(Z,b) \end{tikzcd}$$

$G\cong U(W,d)$ for some $d\colon W\to X$ in $\mathsf{Ord}//X$.

First we show that, for every $x\in X$, $G(x\geqslant\perp)\colon G(x)\to G(\perp)$ is an injective map:

$$\begin{tikzcd} P_\perp \arrow[rrr, "\rho_\perp"] \arrow[ddd, "\pi_\perp"'] & & & G(\perp) \arrow[ddd, "\beta_\perp"] \\ & P_x \arrow[r, "\rho_x"] \arrow[d, "\pi_x"'] \arrow[ul] & G(x) \arrow[d, "\beta_x"] \arrow[ur] & \\ & Y_x \arrow[r, "f_x"'] \arrow[dl] & Z_x \arrow[dr] & \\ Y_\perp \arrow[rrr, "f_\perp"'] & & & Z_\perp. \end{tikzcd}$$

Indeed, if $w_1,w_2\in G(x)$ are such that $G(x\geqslant\perp)(w_1)=G(x\geqslant\perp)(w_2)=w$, then $\beta_x(w_1)=\beta_x(w_2)$. Let $y\in Y_x$ be such that $f_x(y)=\beta_x(w_1)$. Then (y,w_1) and (y,w_2) belong to P_x, hence they also belong to $P_\perp=P$, with $\rho_\perp(y,w_1)=\rho_\perp(y,w_2)=w$, $\pi_\perp(y,w_1)=\pi_\perp(y,w_2)=y$; hence $w_1=w_2$. Therefore also the maps $G(x'\geqslant x)\colon G(x')\to G(x)$ are injective, and so we may assume they are inclusions.

Now we consider $W=G(\perp)$ and define $d\colon W\to X$ by

$$d(w)=\bigvee\{x\in X\,;\,w\in G(x)\}.$$

Then:

– $w \in G(d(w))$: if $z = \beta_\perp(w)$, then, for all $x \in X$, if $w \in G(x)$ then $z \in Z_x$, i.e. $x \leqslant b(z)$; hence $d(w) \leqslant b(z)$, and so $Z \in Z_{d(w)}$. Let $y \in Y_{d(w)}$ be such that $f(y) = z$. Then, for all $x \in X$, if $w \in G(x)$ then $(y, w) \in P_x$, or, equivalently, $x \leqslant c(y, w)$, which implies $d(w) \leqslant c(y, w)$. Hence $w \in G(c(y, w)) \subseteq G(d(w))$.
– d is monotone: it follows from the fact that, for each $x \in X$, $G(x)$ is upwards-closed; indeed, if $w \leqslant w'$ in W and $w \in G(x)$, then $\beta_\perp(w) \leqslant \beta_\perp(w')$ and both belong to Z_x. Let $y \leqslant y'$ in Y_x be such that $f(y) = \beta_\perp(w)$ and $f(y') = \beta_\perp(w')$. Then $(y, w) \leqslant (y', w')$ in P and $(y, w) \in P_x$ implies $(y', w') \in P_x$, since P_x is upwards-closed. This gives $w' \in G(x)$ as claimed.

■

Remark 3.9. As we pointed out at the beginning of this section, Uf effective for descent in Ord does not imply $f\colon (Y, a) \to (Z, b)$ effective for descent in $\mathsf{Ord}//X$, since it does not even imply that f is a regular epimorphism in $\mathsf{Ord}//X$. *It is an open problem to know whether every stable regular epimorphism f with Uf effective for descent in* Ord *is effective for descent in* $\mathsf{Ord}//X$.

References

[1] J. Adámek, L. Sousa, and J. Velebil. Kan injectivity in order-enriched categories, *Math. Structures Comput. Sci.* 25:6–45, 2015.

[2] A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré. On localization and stabilization for factorization systems. *Appl. Categ. Structures* 5(1):1–58, 1997.

[3] M.M. Clementino and I. López Franco. Lax orthogonal factorisation systems. *Adv. Math.*, 302:458–528, 2016.

[4] M.M. Clementino and I. López Franco. Lax orthogonal factorisations in ordered structures. *Theory Appl. Categ.*, 35:No. 36, 1379–1423, 2020.

[5] M.M. Clementino and F. Lucatelli Nunes. Lax comma 2-categories and admissible 2-functors. arXiv 2002.03132. 2020.

[6] H. Herrlich, Topological functors, *General Top. Appl.*, 4:125–142, 1974.

[7] G. Janelidze. Pure Galois theory in categories. *J. Algebra*, 132(2):270–286, 1990.

[8] F. Borceux and G. Janelidze. *Galois theories*, volume 72 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2001.

[9] G. Janelidze and G. M. Kelly. The reflectiveness of covering morphisms in algebra and geometry. *Theory Appl. Categ.*, 3:No. 6, 132–159, 1997.

[10] G. Janelidze and M. Sobral. Finite preorders and topological descent I. *J. Pure Appl. Algebra*, 175:187–205, 2002.

[11] G. Janelidze and W. Tholen. Facets of descent I, *Appl. Categ. Structures*, 2:245–281, 1994.

[12] G. M. Kelly. *Basic Concepts of Enriched Category Theory.* Number 10 in Reprints in Theory and Applications of Categories. Cambridge University Press, 2005.

[13] F. Lucatelli Nunes. Pseudo-Kan extensions and descent theory. *Theory Appl. Categ.*, 33:No. 15, 390–444, 2018.

[14] S. Mac Lane. *Categories for the working mathematician, second edition*, Graduate Texts in Mathematics, 5, Springer-Verlag, New York, 1998.
[15] R. Street, W. Tholen, M. Wischnewsky, and H. Wolff. Semi-topological functors III: Lifting of monads and adjoint functors, *J. Pure Appl. Algebra*, 16:299–314, 1980.
[16] W. Tholen. Injectives, exponentials, and model categories, in: Abstracts of the International Summer Conference in Category Theory, Como, Italy, pp.183–190, 2000.

Maria Manuel Clementino
University of Coimbra, CMUC, Department of Mathematics, Portugal
E-mail address: mmc@mat.uc.pt

Fernando Lucatelli Nunes
Departement Informatica, Universiteit Utrecht, Nederland & University of Coimbra, CMUC, Department of Mathematics, Portugal
E-mail address: f.lucatellinunes@uu.nl