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BMO–REGULARITY FOR A DEGENERATE TRANSMISSION PROBLEM

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ABSTRACT: We examine a transmission problem driven by a degenerate nonlinear operator with a natural interface condition. Two aspects of the problem entail genuine difficulties in the analysis: the absence of representation formulas for the operator and the degenerate nature of the diffusion process. Our arguments circumvent these difficulties and lead to new regularity estimates. We prove the local boundedness of weak solutions and establish an estimate for their gradient in BMO–spaces. The latter implies solutions are of class $C^{0,\mathrm{Log-Lip}}$ across the interface.

KEYWORDS: Transmission problems; p-Laplace operator; local boundedness; BMO gradient estimates; Log-Lipschitz regularity.
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1. Introduction

Transmission problems describe diffusive processes within heterogeneous media that change abruptly across certain interfaces. They find application, for example, in the study of electromagnetic conductivity and composite materials, and their mathematical formulation involves a domain split into sub-regions, where partial differential equations (PDEs) are prescribed. Since the PDEs vary from region to region, the problem may have discontinuities across the interfaces. Consequently, the geometry of these interfaces (which, in contrast to free boundary problems, are fixed and given a priori) and the structure of the underlying equations play a crucial role in analysing transmission problems.

This class of problems first appeared circa 1950, in the work of Mauro Picone [16], as an attempt to address heterogeneous materials in elasticity theory. Several subsequent works developed the basics of the theory and generalised it in various directions [4, 8, 9, 10, 12, 17, 19, 21]. We refer the interested reader to [5] for a comprehensive account of this literature.

Developments concerning the regularity of the solutions to transmission problems are much more recent. In [14], the authors study a class of elliptic equations in divergence form, with discontinuous coefficients, modelling

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composite materials with closely spaced interfacial boundaries, such as fibre-reinforced structures. The main result in that paper is the local Hölder continuity for the gradient of the solutions, with estimates. The findings in [14] are relevant from the applied perspective since the gradient of a solution accounts for the stress of the material, and estimating it shows the stresses remain uniformly bounded, even when fibres are arbitrarily close to each other. The vectorial counterpart of the results in [14] appeared in [13], where regularity estimates for higher-order derivatives of the solutions are obtained. See also the developments reported in [3].

A further layer of analysis concerns the proximity of sub-regions in limiting scenarios. In [1], the authors examine a domain containing two subregions, which are ε-apart, for some $\varepsilon > 0$. Within each sub-region, the diffusion process is given by a divergence-form equation with a diffusivity coefficient $A \neq 1$. In the remainder of the domain, the diffusivity is also constant but equal to 1. By setting $A = +\infty$, the authors examine the case of perfect conductivity. The remarkable fact about this model is that estimates on the gradient of the solutions deteriorate as the two regions approach each other. In [1], the authors obtain blow-up rates for the gradient norm in terms of $\varepsilon \to 0$. We also notice the findings reported in [2] extend those results to the context of multiple inclusions and also treat the case of perfect insulation $A = 0$. We also refer the reader to [6].

More recently, the analysis of transmission problems focused on the geometry of the interface. The minimum requirements on the transmission interface yielding regularity properties for the solutions are particularly interesting. In [7], the authors consider a domain split into two sub-regions. Inside each sub-region, the solution of the problem is required to be a harmonic function, and a flux condition is prescribed along the interface separating the sub-regions. By resorting to a representation formula for harmonic functions, the authors establish the existence of solutions to the problem and prove that solutions are of class $C^{0,\mathrm{Log-Lip}}$ across the interface. In addition, under the assumption that the interface is locally of class $C^{1,\alpha}$, they prove the solutions are of class $C^{1,\alpha}$ within each sub-region, *up to the transmission interface.* This fact follows from a new stability result allowing the argument to import information from the case of flat interfaces. In [20], the authors extend the analysis in [7] to the context of fully nonlinear elliptic operators. Under the assumption that the interface is of class $C^{1,\alpha}$, they prove that solutions are of class $C^{0,\alpha}$ across, and $C^{1,\alpha}$ up to the interface. Furthermore, if the interface is of class $C^{2,\alpha}$, then solutions became $C^{2,\alpha}$-regular, also up to the interface. The findings in [20] rely on a new Aleksandrov-Bakelman-Pucci estimate and variants of the maximum principle and the Harnack inequality.

Our gist in this paper is to extend the results of [7] to the nonlinear degenerate case of p-harmonic functions in each sub-region. We first prove that weak solutions, properly defined and whose existence follows from well-known methods, are locally bounded. The proof combines delicate inequalities with the careful choice of auxiliary test functions and a cut-off argument to produce a variant of the weak Harnack inequality. Working under a C^1 interface geometry, our main contribution is an integral estimate for the gradient, leading to regularity in BMO–spaces. As a corollary, we infer that solutions are of class $C^{0,\mathrm{Log-Lip}}$ across the fixed interface. This transmission problem driven by the degenerate p-Laplace operator presents genuine difficulties compared to the Laplacian's linear case. Firstly, the operator lacks representation formulas, and the strategy developed in [7] is no longer available. Secondly, the degenerate nature of the problem rules out the approach put forward in [20]. Consequently, one must develop new machinery to examine the regularity of the solutions.

Another fundamental question concerns the optimal regularity *up to the interface*. As mentioned before, results of this type appear in the recent works [7] and [20]; see also [11]. In the context of the p-Laplacian operator, the problem remains open. We believe the analysis of the boundary behaviour of p-harmonic functions may yield helpful information in this direction.

The remainder of this article is organised as follows. Section 2 contains the precise formulation of the problem, details the existence of a unique solution and gathers basic material used in the paper. In Section 3, we put forward the proof of the local boundedness. The proof of the BMO–regularity and its consequences is the object of Section 4.

2. Setting of the problem and auxiliary results

In this section, we precisely state our transmission problem, introduce the notion of a weak solution and prove its existence and uniqueness. We then collect several auxiliary results.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and fix $\Omega_1 \Subset \Omega$. Define $\Omega_2 := \Omega \setminus \overline{\Omega_1}$ and consider the interface $\Gamma := \partial\Omega_1$, which we assume is a $(d-1)$-surface of class C^1. For a function $u : \overline{\Omega} \to \mathbb{R}$, we set

$$u_1 := u\big|_{\overline{\Omega_1}} \qquad \text{and} \qquad u_2 := u\big|_{\overline{\Omega_2}}.$$

Note that we necessarily have $u_1 = u_2$ on Γ. Denoting with ν the unit normal vector to Γ pointing inwards to Ω_1, we write

$$(u_i)_\nu = \frac{\partial u_i}{\partial \nu} = Du_i \cdot \nu, \quad i = 1, 2.$$

For $p > 2$ and $g \in L^\infty(\Gamma)$, we consider the problem of finding a function $u : \overline{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} \operatorname{div}(|Du_1|^{p-2} Du_1) = 0 & \text{in} \quad \Omega_1 \\ \operatorname{div}(|Du_2|^{p-2} Du_2) = 0 & \text{in} \quad \Omega_2, \end{cases} \tag{1}$$

and the additional conditions

$$\begin{cases} u = 0 & \text{on} \quad \partial\Omega \\ |Du_2|^{p-2}(u_2)_\nu - |Du_1|^{p-2}(u_1)_\nu = g & \text{on} \quad \Gamma. \end{cases} \tag{2}$$

The precise definition of solution we have in mind is the object of the following definition.

Definition 1. *A function $u \in W_0^{1,p}(\Omega)$ is a weak solution of* (1)-(2) *if*

$$\int_\Omega |Du|^{p-2} Du \cdot Dv \,\mathrm{d}x = \int_\Gamma gv \,\mathrm{d}\mathcal{H}^{d-1}, \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{3}$$

We use the Hausdorff measure $\mathcal{H}^{d-1}$ in the surface integral to emphasise that $\Delta_p u$ is a measure supported along the interface, and we write

$$-\Delta_p u = g \,\mathrm{d}\mathcal{H}^{d-1}\big|_\Gamma.$$

To justify the former definition, we multiply both equations in (1) by a test function $\varphi \in C_c^\infty(\Omega)$, and formally integrate by parts to get

$$\int_{\Omega_1} |Du_1|^{p-2} Du_1 \cdot D\varphi \,\mathrm{d}x = -\int_\Gamma \left(|Du_1|^{p-2} Du_1 \cdot \nu\right) \varphi \,\mathrm{d}\mathcal{H}^{d-1}$$

and

$$\int_{\Omega_2} |Du_2|^{p-2} Du_2 \cdot D\varphi \,\mathrm{d}x = \int_\Gamma \left(|Du_2|^{p-2} Du_2 \cdot \nu\right) \varphi \,\mathrm{d}\mathcal{H}^{d-1}.$$

Adding and using (2), we obtain, by density,

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \,\mathrm{d}x = \int_\Gamma g\varphi \,\mathrm{d}\mathcal{H}^{d-1}, \quad \forall\, \varphi \in W_0^{1,p}(\Omega).$$

2.1. Existence and uniqueness of a weak solution. To prove the existence of a unique weak solution to (1)-(2), we resort to standard methods in the literature. Indeed, the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$\langle Au, v\rangle := \int_\Omega |Du|^{p-2} Du \cdot Dv \,\mathrm{d}x$$

is bounded, hemicontinuous, strictly monotone and coercive, and hence it is bijective. Since, due to the trace theorem and Poincaré's inequality, the right-hand side in (3) defines an element in $W^{-1,p'}(\Omega)$, we obtain the result.

Additionally, we remark that the weak solution is the global minimiser of the functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega |Du|^p \,\mathrm{d}x - \int_\Gamma g u \,\mathrm{d}\mathcal{H}^{d-1}, \tag{4}$$

whose Euler-Lagrange equation, in its weak formulation, is precisely (3).

2.2. Auxiliary results. We now collect some auxiliary material which will be instrumental in the proofs of the main results. We start with a technical inequality (c.f. [18, Lemma 2]).

Lemma 1. *Let $p > 0$, and $N \in \mathbb{N}$. Let also $a_1, \ldots, a_N, q_1, \ldots, q_N$ be real numbers such that $0 < a_i < \infty$ and $0 \le q_i < p$, for every $i = 1, \ldots, N$. Suppose that $z, z_1, \ldots, z_N$ are positive real numbers satisfying*

$$z^p \le \sum_{i=1}^N a_i z_i^{q_i}.$$

Then there exists $C > 0$ such that

$$z \le C \sum_{i=1}^N a_i^{\gamma_i}$$

where $\gamma_i = (p - q_i)^{-1}$, for $i = 1, \ldots, N$. Finally, $C = C(N, p, q_1, \ldots, q_N)$.

Although standard in the field, the following result lacks detailed proof in the literature. We include it here for completeness and future reference.

Lemma 2. *Fix $R_0 > 0$ and let $\phi : [0, R_0] \to [0, \infty)$ be a non-decreasing function. Suppose there exist constants $C_1, \alpha, \beta > 0$, and $C_2, \mu \ge 0$, with $\beta < \alpha$, satisfying*

$$\phi(r) \le C_1\left[\left(\frac{r}{R}\right)^\alpha + \mu\right]\phi(R) + C_2 R^\beta,$$

for every $0 < r \leq R \leq R_0$*. Then, for every* $\sigma \leq \beta$*, there exists* $\mu_0 = \mu_0(C_1, \alpha, \beta, \sigma)$ *such that, if* $\mu < \mu_0$*, for every* $0 < r \leq R \leq R_0$*, we have*

$$\phi(r) \leq C_3\Big(\frac{r}{R}\Big)^{\sigma}\big(\phi(R) + C_2 R^{\sigma}\big),$$

where $C_3 = C_3(C_1, \alpha, \beta, \sigma) > 0$*. Moreover,*

$$\phi(r) \leq C_4 r^{\sigma},$$

where $C_4 = C_4(C_2, C_3, R_0, \phi(R_0), \sigma)$*.*

Proof: For clarity, we split the proof into two steps. First, an induction argument leads to an inequality at discrete scales; then, we pass to the continuous case and conclude the argument.

Step 1 - We want to verify that

$$\phi(\theta^{n+1}R) \leq \theta^{(n+1)\delta}\phi(R) + C_2\theta^{n\beta}R^{\beta}\sum_{j=0}^{n}\theta^{j(\delta-\beta)}, \tag{5}$$

for every $n \in \mathbb{N}$. We notice it suffices to prove the estimate for $\sigma = \beta$ and work in this setting. For $0 < \theta < 1$ and $0 < R \leq R_0$ the assumption of the lemma yields

$$\phi(\theta R) \leq C_1\left[\left(\frac{\theta R}{R}\right)^{\alpha} + \mu\right]\phi(R) + C_2R^{\beta} = \theta^{\alpha}C_1(1 + \mu\theta^{-\alpha})\phi(R) + C_2R^{\beta}.$$

Choose $\theta \in (0, 1)$ such that $2C_1\theta^{\alpha} = \theta^{\delta}$ with $\beta < \delta < \alpha$. Notice that θ depends only on C_1, α, δ. Take $\mu_0 > 0$ such that $\mu_0\theta^{-\alpha} < 1$. For every $R \leq R_0$ we then have

$$\phi(\theta R) \leq \theta^{\delta}\phi(R) + C_2R^{\beta} \tag{6}$$

and the base case follows. Suppose the statement has already been verified for some $k \in \mathbb{N}$, $k \geq 2$; then

$$\phi(\theta^k R) \leq \theta^{k\delta}\phi(R) + C_2\theta^{(k-1)\beta}R^{\beta}\sum_{j=0}^{k-1}\theta^{j(\delta-\beta)}.$$

Thanks to (6), we have

$$\begin{aligned}\phi(\theta^{k+1}R) = \phi\big(\theta^k(\theta R)\big) &\le \theta^{k\delta}\phi(\theta R) + C_2\theta^{(k-1)\beta}(\theta R)^\beta \sum_{j=0}^{k-1}\theta^{j(\delta-\beta)} \\ &\le \theta^{k\delta}\big[\theta^\delta\phi(R) + C_2R^\beta\big] + C_2\theta^{k\beta}R^\beta\sum_{j=0}^{k-1}\theta^{j(\delta-\beta)} \\ &= \theta^{(k+1)\delta}\phi(R) + C_2\theta^{k\delta}R^\beta + C_2\theta^{k\beta}R^\beta\sum_{j=0}^{k-1}\theta^{j(\delta-\beta)} \\ &= \theta^{(k+1)\delta}\phi(R) + C_2\theta^{k\beta}R^\beta\sum_{j=0}^{k}\theta^{j(\delta-\beta)}.\end{aligned}$$

Hence, (5) holds for every $k \in \mathbb{N}$, and the induction argument is complete.

Step 2 - Next, we pass from the discrete to the continuous case. In particular, we claim that

$$\phi(r) \le C_3\Big(\frac{r}{R}\Big)^\beta\big(\phi(R) + C_2R^\beta\big),$$

for every $0 < r \le R \le R_0$.

Indeed,

$$\begin{aligned}\phi(\theta^{k+1}R) \le& \theta^{(k+1)\delta}\phi(R) + C_2\theta^{k\beta}R^\beta\frac{1}{1-\theta^{\delta-\beta}} \\ =& \theta^{(k+1)\delta}\phi(R) + C_2R^\beta\frac{\theta^{(k+1)\beta}}{\theta^\beta - \theta^\delta} \\ \le& C_3\theta^{(k+1)\beta}\big(\phi(R) + C_2R^\beta\big),\end{aligned}$$

for every $k \in \mathbb{N}$. Taking $k \in \mathbb{N}$ such that $\theta^{k+2}R \le r < \theta^{k+1}R$, up to relabeling the constant C_3, we get

$$\begin{aligned}\phi(r) \le& \phi(\theta^{k+1}R) \le C_3\theta^{(k+1)\beta}\big(\phi(R) + C_2R^\beta\big) \\ =& C_3\theta^{(k+2)\beta}\theta^{-\beta}\big(\phi(R) + C_2R^\beta\big) \\ \le& C_3\Big(\frac{r}{R}\Big)^\beta\big(\phi(R) + C_2R^\beta\big).\end{aligned}$$

Finally, one notices

$$\phi(r) \leq C_3 \frac{1}{R_0^\beta}\big(\phi(R_0) + C_2 R_0^\beta\big) r^\beta =: C_4 r^\beta,$$

and the proof is complete. ■

3. Local boundedness

In this section, we prove the local boundedness for the weak solutions to the problem. Our argument is inspired by the one put forward in [18].

Theorem 1 (Local Boundedness). *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to (1)-(2). Then for any $B_R := B_R(x_0) \Subset \Omega$, there exists a constant $C = C\big(d, p, R, \|g\|_{L^\infty(\Gamma)}\big) > 0$ such that*

$$\|u\|_{L^\infty(B_{R/2})} \leq C R^{-\frac{d}{p}}\big(\|u\|_{L^p(B_R)} + R^{\frac{d}{p}+1}\|g\|_{L^\infty(\Gamma)}\big)$$

and

$$\|Du\|_{L^p(B_{R/2})} \leq C R^{-1}\big(\|u\|_{L^p(B_R)} + R^{\frac{d}{p}+1}\|g\|_{L^\infty(\Gamma)}\big).$$

Proof: Fix $R > 0$ such that $B_R \Subset \Omega$ and set $k := R\|g\|_{L^\infty(\Gamma)}$. Define $\overline{u} : \Omega \to \mathbb{R}$ as

$$\overline{u}(x) := |u(x)| + k$$

for all $x \in \Omega$. Fix $q \geq 1$ and $\ell > k$. For $t \in \mathbb{R}$, denote $\bar{t} := |t| + k$. To ease the presentation, we split the remainder of the proof into four steps.

Step 1 - Let $F : [k, \infty) \to \mathbb{R}$ be defined as

$$F(s) := \begin{cases} s^q & \text{if} \quad k \leq s \leq \ell \\ q\ell^{q-1}s - (q-1)\ell^q & \text{if} \quad \ell < s. \end{cases}$$

Then $F \in C^1\big([k,\infty)\big)$ and $F \in C^\infty\big([k,\infty) \setminus \{\ell\}\big)$. Let $G : \mathbb{R} \to \mathbb{R}$ be defined as

$$G(t) := \operatorname{sgn}(t)\big(F(\bar{t})F'(\bar{t})^{p-1} - q^{p-1}k^\beta\big), \quad \forall t \in \mathbb{R},$$

where $\beta = p(q-1) + 1 > 1$. A simple computation yields

$$G'(t) = \begin{cases} q^{-1}\beta F'(\bar{t})^p & \text{if} \quad |t| < \ell - k \\ F'(\bar{t})^p & \text{if} \quad |t| > \ell - k. \end{cases}$$

Notice that

$$|G(u)| \leq F(\overline{u})F'(\overline{u})^{p-1}$$

and

$$\overline{u}F'(\overline{u}) \leq qF(\overline{u}).$$

Step 2 - In this step, we introduce auxiliary test functions, which build upon the former inequalities. Fix $0 < r < R$. Let $\eta \in C_c^\infty(B_R)$, $0 \leq \eta \leq 1$, $\eta = 1$ in B_r, $|D\eta| \leq (R-r)^{-1}$. Let $v = \eta^p G(u)$. Since $G \in C^1\big(\mathbb{R} \setminus \{\pm(\ell - k)\}\big)$ is continuous, with bounded derivative, it follows that $G(u) \in W^{1,p}(\Omega)$. Hence v is an admissible test function. We have

$$Dv = \begin{cases} p\eta^{p-1}G(u)D\eta + \eta^p G'(u)Du & \text{if} \quad u \neq \pm(\ell - k) \\ p\eta^{p-1}G(u)D\eta & \text{if} \quad u = \pm(\ell - k). \end{cases}$$

Set $w(x) = F\big(\overline{u}(x)\big)$. Notice that $q^{-1}\beta \geq 1$; hence $G'(u) \leq q^{-1}\beta F'(\overline{u})^p$. Notice also that $|Du| = |D\overline{u}|$.

Using the trace theorem and the Poincaré inequality, we get

$$\int_\Omega |Du|^{p-2}Du \cdot Dv \,\mathrm{d}x \leq \|g\|_{L^\infty(\Gamma)} \int_\Gamma |v| \,\mathrm{d}\mathcal{H}^{d-1} \leq C \int_\Omega |Dv| \,\mathrm{d}x. \tag{7}$$

Now we estimate the left-hand side of (7) from below. We get

$$\begin{aligned} \int_{B_1} |Du|^{p-2}Du \cdot Dv \,\mathrm{d}x = & \int_{B_1} |Du|^{p-2}Du \cdot \big(p\eta^{p-1}G(u)D\eta + \eta^p G'(u)Du\big) \,\mathrm{d}x \\ = & p \int_{B_1} \eta^{p-1}G(u)|Du|^{p-2}Du \cdot D\eta \,\mathrm{d}x \\ & + \int_{B_1} \eta^p G'(u)|Du|^p \,\mathrm{d}x \\ \geq & - p \int_{B_1} \eta^{p-1}F(\overline{u})F'(\overline{u})^{p-1}|D\overline{u}|^{p-1}|D\eta| \,\mathrm{d}x \\ & + \int_{B_1} \eta^p F'(\overline{u})^p|D\overline{u}|^p \,\mathrm{d}x \\ = & - p \int_{B_1} \eta^{p-1} w|Dw|^{p-1}|D\eta| \,\mathrm{d}x \\ & + \int_{B_1} \eta^p |Dw|^p \,\mathrm{d}x \\ \geq & - p\|wD\eta\|_{L^p(B_1)}\|\eta Dw\|^{p-1}_{L^p(B_1)} + \|\eta Dw\|^p_{L^p(B_1)}. \end{aligned} \tag{8}$$

We also control the right-hand side of (7) by computing

$$
\begin{aligned}
C\int_{B_1}|Dv|\,\mathrm{d}x =& C\int_{B_1}\frac{\overline{u}^{p-1}}{\overline{u}^{p-1}}|p\eta^{p-1}G(u)D\eta+\eta^pG'(u)Du|\,\mathrm{d}x\\
\leq& Ck^{1-p}p\int_{B_1}\overline{u}^{p-1}\eta^{p-1}|G(u)D\eta|\,\mathrm{d}x\\
&+Ck^{1-p}\int_{B_1}\overline{u}^{p-1}\eta^pG'(u)|Du|\,\mathrm{d}x\\
\leq& C\int_{B_1}\overline{u}^{p-1}\eta^{p-1}F(\overline{u})F'(\overline{u})^{p-1}|D\eta|\,\mathrm{d}x\\
&+Cq^{-1}\beta\int_{B_1}\overline{u}^{p-1}\eta^pF'(\overline{u})^p|Du|\,\mathrm{d}x\\
\leq& C\int_{B_1}\eta^{p-1}q^{p-1}F(\overline{u})^{p-1}F(\overline{u})|D\eta|\,\mathrm{d}x\\
&+Cq^{-1}\beta\int_{B_1}q^{p-1}F(\overline{u})^{p-1}\eta^pF'(\overline{u})|Du|\,\mathrm{d}x\\
=& Cq^{p-1}\int_{B_1}(\eta w)^{p-1}w|D\eta|\,\mathrm{d}x\\
&+Cq^{p-2}\beta\int_{B_1}(\eta w)^{p-1}\eta|Dw|\,\mathrm{d}x\\
\leq& Cq^{p-1}\|\eta w\|^{p-1}_{L^p(B_1)}\|wD\eta\|_{L^p(B_1)}\\
&+Cq^{p-2}\beta\|\eta w\|^{p-1}_{L^p(B_1)}\|\eta Dw\|_{L^p(B_1)}.
\end{aligned}\tag{9}
$$

From (7), combining (9) with (8), we get

$$
\begin{aligned}
\|\eta Dw\|^p_{L^p(\Omega)} \leq& p\|wD\eta\|_{L^p(\Omega)}\|\eta Dw\|^{p-1}_{L^p(\Omega)}\\
&+Cq^{p-1}\|\eta w\|^{p-1}_{L^p(\Omega)}\|wD\eta\|_{L^p(\Omega)}\\
&+Cq^{p-1}\|\eta w\|^{p-1}_{L^p(\Omega)}\|\eta Dw\|_{L^p(\Omega)},
\end{aligned}\tag{10}
$$

where we have used

$$\beta = pq-p+1\leq pq-p+q\leq pq+q=(p+1)q.$$

Step 3 - Set

$$z=\frac{\|\eta Dw\|_{L^p(\Omega)}}{\|wD\eta\|_{L^p(\Omega)}},\quad \zeta=\frac{\|\eta w\|_{L^p(\Omega)}}{\|wD\eta\|_{L^p(\Omega)}}.$$

By dividing (10) for $\|wD\eta\|^p_{L^p(\Omega)}$, we have

$$\begin{aligned} z^p \leq & p z^{p-1} + Cq^{p-1}\frac{\|\eta w\|^{p-1}_{L^p(\Omega)}}{\|wD\eta\|^{p-1}_{L^p(\Omega)}} + Cq^{p-1}\frac{\|\eta w\|^{p-1}_{L^p(\Omega)}}{\|wD\eta\|^{p-1}_{L^p(\Omega)}}\frac{\|\eta Dw\|_{L^p(\Omega)}}{\|wD\eta\|_{L^p(\Omega)}} \\ = & p z^{p-1} + Cq^{p-1}\zeta^{p-1} + Cq^{p-1}\zeta^{p-1} z. \end{aligned}$$

An application of Lemma 1, implies

$$z \leq C\big(p + q^{\frac{p-1}{p}}\zeta^{\frac{p-1}{p}} + q\zeta\big) \leq Cq(1+\zeta),$$

giving

$$\|\eta Dw\|_{L^p(\Omega)} \leq Cq\big(\|\eta w\|_{L^p(\Omega)} + \|wD\eta\|_{L^p(\Omega)}\big). \tag{11}$$

Using the Sobolev inequality, we get

$$\begin{aligned} \|\eta w\|_{L^{p^*}(\Omega)} \leq & C\|D(\eta w)\|_{L^p(\Omega)} \\ \leq & C\big(\|wD\eta\|_{L^p(\Omega)} + \|\eta Dw\|_{L^p(\Omega)}\big) \\ \leq & C\Big[\|wD\eta\|_{L^p(\Omega)} + Cq\big(\|\eta w\|_{L^p(\Omega)} + \|wD\eta\|_{L^p(\Omega)}\big)\Big] \end{aligned}$$

and so

$$\|\eta w\|_{L^{p^*}(\Omega)} \leq Cq\big(\|\eta w\|_{L^p(\Omega)} + \|wD\eta\|_{L^p(\Omega)}\big). \tag{12}$$

Recall that $\eta = 1$ in B_r and $|D\eta| \leq (R-r)^{-1}$. Hence, (11) becomes

$$\begin{aligned} \|Dw\|_{L^p(B_r)} \leq & Cq\left[\left(\int_{B_R} w^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \frac{1}{R-r}\left(\int_{B_R} w^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right] \\ = & Cq\|w\|_{L^p(B_R)}\left(1 + \frac{1}{R-r}\right) \\ = & Cq\frac{R-r+1}{R-r}\|w\|_{L^p(B_R)} \\ \leq & Cq\frac{\mathrm{diam}(B_1)+1}{R-r}\|w\|_{L^p(B_R)} \\ \leq & Cq\frac{1}{R-r}\|w\|_{L^p(B_R)}. \end{aligned}$$

Similarly, (12) becomes

$$\|w\|_{L^{p^*}(B_r)} \leq Cq\frac{1}{R-r}\|w\|_{L^p(B_R)}. \tag{13}$$

We claim that $F_\ell \le F_{\ell+1}$, for every $\ell \in \mathbb{N}$, $\ell > k$. The only non-trivial case is when $\ell < \bar{t} \le \ell + 1$. In this case, we have

$$F_\ell(\bar{t}) = q\ell^{q-1}\bar{t} - (q-1)\ell^q$$

and

$$F_{\ell+1}(\bar{t}) = \bar{t}^q.$$

Let $f : (\ell, \ell+1] \to \mathbb{R}$ be defined by

$$f(\bar{t}) = \bar{t}^q - q\ell^{q-1}\bar{t} + (q-1)\ell^q.$$

We have $f'(\bar{t}) = q\bar{t}^{q-1} - q\ell^{q-1} > 0$, for every $\bar{t} \in (\ell, \ell+1]$, and hence f is an increasing function. Since $\lim_{\bar{t}\to\ell} f(\bar{t}) = 0$, we have $f \ge 0$ in $(\ell, \ell+1]$, and so $F_\ell \le F_{\ell+1}$. Letting $\ell \to \infty$ in (13), since $0 \le F_\ell \le F_{\ell+1}$ for every $\ell \in \mathbb{N}$, $\ell > k$, by the Monotone Convergence Theorem, we obtain

$$\left(\int_{B_r} \overline{u}^{qp^*} \,\mathrm{d}x \right)^{\frac{1}{p^*}} \le Cq \frac{1}{R-r} \left(\int_{B_R} \overline{u}^{qp} \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Set

$$s := qp \qquad \text{and} \qquad \gamma := p^*/p = d/(d-p);$$

then

$$\left(\int_{B_r} \overline{u}^{s\gamma} \,\mathrm{d}x \right)^{\frac{1}{p\gamma}} \le Cq \frac{1}{R-r} \left(\int_{B_R} \overline{u}^{s} \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Raising both sides of the previous inequality to p/s, one gets

$$\left(\int_{B_r} \overline{u}^{s\gamma} \,\mathrm{d}x \right)^{\frac{1}{s\gamma}} \le C^{\frac{p}{s}} \left(\frac{s}{p} \right)^{\frac{p}{s}} \left(\frac{1}{R-r} \right)^{\frac{p}{s}} \left(\int_{B_R} \overline{u}^{s} \,\mathrm{d}x \right)^{\frac{1}{s}}. \tag{14}$$

Set $s_j = s\gamma^j$ and $r_j = r + 2^{-j}(R-r)$, for every $j \in \mathbb{N}_0$. Iterating (14), which holds for every $s \geq p$, we have

$$
\begin{aligned}
\left(\int_{B_{r_{j+1}}} \overline{u}^{s_j\gamma}\,\mathrm{d}x\right)^{\frac{1}{s_j\gamma}} \leq & C^{\frac{p}{s_j}}\left(\frac{s_j}{p}\right)^{\frac{p}{s_j}} 2^{\frac{p}{s_j}(j+1)}\Big(\frac{1}{R-r}\Big)^{\frac{p}{s_j}}\left(\int_{B_{r_j}} \overline{u}^{s_j}\,\mathrm{d}x\right)^{\frac{1}{s_j}} \\
= & C^{\frac{p}{s_{j-1}\gamma}}\left(\frac{s_{j-1}\gamma}{p}\right)^{\frac{p}{s_{j-1}\gamma}} 2^{\frac{p}{s_{j-1}\gamma}(j+1)}\Big(\frac{1}{R-r}\Big)^{\frac{p}{s_{j-1}\gamma}} \\
& \times \left(\int_{B_{r_j}} \overline{u}^{s_{j-1}\gamma}\,\mathrm{d}x\right)^{\frac{1}{s_{j-1}\gamma}} \\
\leq & C(j,p,s,d)\Big(\frac{1}{R-r}\Big)^{\frac{p}{s}\sum_{k=0}^{j}\gamma^{-k}}\left(\int_{B_R} \overline{u}^{s}\,\mathrm{d}x\right)^{\frac{1}{s}},
\end{aligned}
$$

where

$$
C(j,p,s,d) := C^{\frac{p}{s}\sum_{k=0}^{j}\gamma^{-k}}\left(\frac{s}{p}\right)^{\frac{p}{s}\sum_{k=0}^{j}\gamma^{-k}}\gamma^{\frac{p}{s}\sum_{k=0}^{j}k\gamma^{-k}}2^{\frac{p}{s}\sum_{k=0}^{j}(k+1)\gamma^{-k}}.
$$

Notice that $r < r_j$, for every $j \in \mathbb{N}_0$, the series are convergent and in particular $\sum_{k=0}^{\infty}\gamma^{-k} = d/p$. By letting $j \to \infty$, we get

$$
\sup_{B_r} \overline{u} \leq C\left(\frac{1}{(R-r)^d}\int_{B_R} \overline{u}^s\,\mathrm{d}x\right)^{\frac{1}{s}}. \tag{15}
$$

Step 4 - Now, we can choose some parameters in the former inequalities to complete the proof. By choosing $q = 1$, setting $r := R/2$, and recalling that $\overline{u} = |u| + k$, we get

$$
\|u\|_{L^\infty(B_{R/2})} \leq \|\overline{u}\|_{L^\infty(B_{R/2})} \leq CR^{-\frac{d}{p}}\big(\|u\|_{L^p(B_R)} + R^{\frac{d}{p}}k\big).
$$

The second inequality in the theorem follows by setting $q = 1$ and $r := R/2$ in (13), obtaining

$$
\begin{aligned}
\|Du\|_{L^p(B_{R/2})} = & \|D\overline{u}\|_{L^p(B_{R/2})} \\
\leq & CR^{-1}\|\overline{u}\|_{L^p(B_R)} \\
\leq & CR^{-1}\big(\|u\|_{L^p(B_R)} + \|k\|_{L^p(B_R)}\big) \\
\leq & CR^{-1}\big(\|u\|_{L^p(B_R)} + R^{\frac{d}{p}}k\big).
\end{aligned} \tag{16}
$$

■

4. Gradient regularity estimates in BMO–spaces

In this section, we prove the main result of this paper. We start with a lemma, where we denote by $W^{1,p}_f(\Omega)$ the affine space of functions $w \in W^{1,p}(\Omega)$ such that

$$w - f \in W^{1,p}_0(\Omega).$$

Lemma 3. *Let $w \in W^{1,p}(B_R)$. Let also $h \in W^{1,p}_w(B_R)$ be such that $\Delta_p h = 0$ in B_R, in the weak sense. Then there exists $C = C(d,p) > 0$ such that*

$$\int_{B_R} |Dw|^p - |Dh|^p \,\mathrm{d}x \geq C \int_{B_R} |D(w-h)|^p \,\mathrm{d}x.$$

Proof: Fix $\tau \in [0,1]$ and define $v_\tau := \tau w + (1-\tau)h$. We have

$$\begin{aligned}
\int_{B_R} |Dw|^p - |Dh|^p \,\mathrm{d}x =& \int_0^1 \frac{d}{d\tau}\left(\int_{B_R} |Dv_\tau|^p \,\mathrm{d}x\right)\mathrm{d}\tau \\
=& p\int_0^1 \int_{B_R} |Dv_\tau|^{p-2} Dv_\tau \cdot D(w-h)\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (17)
\end{aligned}$$

Since $h \in W^{1,p}_w(B_R)$ and $\Delta_p h = 0$ in B_R, we also get

$$\int_{B_R} |Dh|^{p-2} Dh \cdot D(w-h)\,\mathrm{d}x = 0.$$

Hence,

$$\begin{aligned}
p&\int_0^1 \int_{B_R} |Dv_\tau|^{p-2} Dv_\tau \cdot D(w-h)\,\mathrm{d}x\,\mathrm{d}\tau \\
&=p\int_0^1 \int_{B_R} (|Dv_\tau|^{p-2} Dv_\tau - |Dh|^{p-2}Dh) \cdot D(w-h)\,\mathrm{d}x\,\mathrm{d}\tau \\
&=p\int_0^1 \frac{1}{\tau}\int_{B_R} (|Dv_\tau|^{p-2} Dv_\tau - |Dh|^{p-2}Dh) \cdot D(v_\tau-h)\,\mathrm{d}x\,\mathrm{d}\tau, \qquad (18)
\end{aligned}$$

where the last equality relies on the fact that $v_\tau - h = \tau(w-h)$. Combining (17) and (18), and using a standard monotonicity property (recalling $p > 2$), we obtain

$$\begin{aligned}
\int_{B_R} |Dw|^p - |Dh|^p \,\mathrm{d}x \geq & C \int_0^1 \frac{1}{\tau} \int_{B_R} |D(v_\tau - h)|^p \,\mathrm{d}x \,\mathrm{d}\tau \\
= & C \int_0^1 \tau^{p-1} \,\mathrm{d}\tau \int_{B_R} |D(w-h)|^p \,\mathrm{d}x \\
= & C \int_{B_R} |D(w-h)|^p \,\mathrm{d}x,
\end{aligned}$$

and the proof is complete. ■

The following result concerns the decay of the excess of the gradient of p-harmonic functions with respect to its average. Its proof can be found in [15, Lemma 5.1].

Proposition 1. *Let $h \in W^{1,p}(B_R)$ be a weak solution of the p-Laplace equation in B_R. Then there exist constants $C = C(d,p) > 0$ and $\alpha \in (0,1)$ such that, for every $r \in (0,R]$, we have*

$$\int_{B_r} |Dh - (Dh)_r|^p \,\mathrm{d}x \leq C\left(\frac{r}{R}\right)^{d+p\alpha} \int_{B_R} |Dh - (Dh)_R|^p \,\mathrm{d}x.$$

The next proposition provides a control on the decay of the excess for arbitrary Sobolev functions.

Proposition 2. *Let $w \in W^{1,p}(B_R)$. Let also $h \in W^{1,p}_w(B_R)$ satisfy $\Delta_p h = 0$ in B_R, in the weak sense. Then there exists $C = C(d,p) > 0$ such that, for every $0 < r \leq R$, we have*

$$\begin{aligned}
\int_{B_r} |Dw - (Dw)_r|^p \,\mathrm{d}x \leq & C\left(\frac{r}{R}\right)^{d+p\alpha} \int_{B_R} |Dw - (Dw)_R|^p \,\mathrm{d}x \\
& + C \int_{B_R} |Dw - Dh|^p \,\mathrm{d}x,
\end{aligned}$$

where $\alpha \in (0,1)$ is the same as in Proposition 1.

Proof: Let $r \in (0,R]$. We have

$$\begin{aligned}
\int_{B_r} |Dw - (Dw)_r|^p \,\mathrm{d}x \leq & 2^{p-1} \int_{B_r} |Dw - (Dh)_r|^p \,\mathrm{d}x \\
& + 2^{p-1} \int_{B_r} |(Dw)_r - (Dh)_r|^p \,\mathrm{d}x. \qquad (19)
\end{aligned}$$

Similarly,

$$\int_{B_r} |Dw - (Dh)_r|^p \,\mathrm{d}x \leq 2^{p-1} \int_{B_r} |Dw - Dh|^p \,\mathrm{d}x + 2^{p-1} \int_{B_r} |Dh - (Dh)_r|^p \,\mathrm{d}x. \tag{20}$$

Applying Hölder's inequality, we get

$$\begin{aligned}\int_{B_r} |(Dw)_r - (Dh)_r|^p \,\mathrm{d}x =& |B_r| \left| \frac{1}{|B_r|} \int_{B_r} Dw - Dh \,\mathrm{d}x \right|^p \\ \leq& |B_r|^{1-p} \left[|B_r|^{\frac{p-1}{p}} \left(\int_{B_r} |Dw - Dh|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \right]^p \\ =& \int_{B_r} |Dw - Dh|^p \,\mathrm{d}x. \end{aligned} \tag{21}$$

Combining (19), (20) and (21), we obtain

$$\int_{B_r} |Dw - (Dw)_r|^p \,\mathrm{d}x \leq C \int_{B_r} |Dh - (Dh)_r|^p \,\mathrm{d}x + C \int_{B_r} |Dw - Dh|^p \,\mathrm{d}x.$$

Changing the roles of w and h, and integrating in the ball B_R, we get

$$\int_{B_R} |Dh - (Dh)_R|^p \,\mathrm{d}x \leq C \int_{B_R} |Dw - (Dw)_R|^p \,\mathrm{d}x + C \int_{B_R} |Dw - Dh|^p \,\mathrm{d}x. \tag{22}$$

Now, Proposition 1 implies

$$\int_{B_r} |Dw - (Dw)_r|^p \,\mathrm{d}x \leq C \left(\frac{r}{R} \right)^{d+p\alpha} \int_{B_R} |Dh - (Dh)_R|^p \,\mathrm{d}x + C \int_{B_R} |Dw - Dh|^p \,\mathrm{d}x. \tag{23}$$

Combining (22) with (23), we finally get

$$\begin{aligned}\int_{B_r} |Dw - (Dw)_r|^p \,\mathrm{d}x \leq & C\Big(\frac{r}{R}\Big)^{d+p\alpha} \int_{B_R} |Dw - (Dw)_R|^p \,\mathrm{d}x \\ & + C\Big(\frac{r}{R}\Big)^{d+p\alpha} \int_{B_R} |Dw - Dh|^p \,\mathrm{d}x \\ & + C \int_{B_R} |Dw - Dh|^p \,\mathrm{d}x \\ \leq & C\Big(\frac{r}{R}\Big)^{d+p\alpha} \int_{B_R} |Dw - (Dw)_R|^p \,\mathrm{d}x \\ & + C \int_{B_R} |Dw - Dh|^p \,\mathrm{d}x \end{aligned}$$

and the proof is complete. ■

We now state and prove our main theorem.

Theorem 2 (Gradient regularity in BMO–spaces)**.** *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to* (1)-(2)*. Then $Du \in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$. Moreover, for every $\Omega' \Subset \Omega$,*

$$\|Du\|_{\mathrm{BMO}(\Omega')} \leq C,$$

where $C = C(p, d, \|g\|_{L^\infty(\Gamma)}, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega', \partial\Omega)) > 0$.

Proof: Let $x_0 \in \Gamma$, and let $R > 0$ be such that $B_R := B_R(x_0) \Subset \Omega$. Let $h \in W_u^{1,p}(B_R)$ be the weak solution of $\Delta_p h = 0$ in B_R. Since $h = u$ on ∂B_R in the trace sense, we can extend h in $\Omega \setminus B_R$ such that $h = u$ in $\Omega \setminus B_R$. This implies that $h \in W_0^{1,p}(\Omega)$ and hence, since u is the global minimizer of (4), we have

$$\frac{1}{p}\int_\Omega |Du|^p \,\mathrm{d}x - \int_\Gamma gu \,\mathrm{d}H^{d-1} \leq \frac{1}{p}\int_\Omega |Dh|^p \,\mathrm{d}x - \int_\Gamma gh \,\mathrm{d}\mathcal{H}^{d-1}. \tag{24}$$

Set $\Gamma_R = B_R \cap \Gamma$. Since $h = u$ in $\Omega \setminus B_R$, (24) becomes

$$\frac{1}{p}\int_{B_R} |Du|^p \,\mathrm{d}x - \int_{\Gamma_R} gu \,\mathrm{d}\mathcal{H}^{d-1} \leq \frac{1}{p}\int_{B_R} |Dh|^p \,\mathrm{d}x - \int_{\Gamma_R} gh \,\mathrm{d}\mathcal{H}^{d-1}$$

from which, applying the Trace Theorem, Hölder's inequality and Poincaré's inequality, follows

$$\begin{aligned}\frac{1}{p}\int_{B_R}|Du|^p\,\mathrm{d}x-\frac{1}{p}\int_{B_R}|Dh|^p\,\mathrm{d}x \leq& \int_{\Gamma_R} gu\,\mathrm{d}\mathcal{H}^{d-1}-\int_{\Gamma_R} gh\,\mathrm{d}\mathcal{H}^{d-1}\\ \leq&\|g\|_{L^\infty(\Gamma)}\int_{\Gamma_R}|u-h|\,\mathrm{d}\mathcal{H}^{d-1}\\ \leq& C\int_{B_R}|u-h|\,\mathrm{d}x+C\int_{B_R}|D(u-h)|\,\mathrm{d}x\\ \leq& CR^{d\frac{p-1}{p}}\|u-h\|_{L^p(B_R)}\\ &+CR^{d\frac{p-1}{p}}\|D(u-h)\|_{L^p(B_R)}\\ \leq& CR^{d\frac{p-1}{p}}\|D(u-h)\|_{L^p(B_R)}. \end{aligned} \tag{25}$$

Let us consider the left-hand side of (25). Using Lemma 3, we get

$$\begin{aligned}\frac{1}{p}\int_{B_R}|Du|^p\,\mathrm{d}x-\frac{1}{p}\int_{B_R}|Dh|^p\,\mathrm{d}x \geq& C(d,p)\int_{B_R}|Du-Dh|^p\,\mathrm{d}x\\ =&C\|D(u-h)\|^p_{L^p(B_R)}.\end{aligned} \tag{26}$$

Combining now (25) with (26), we get

$$\|D(u-h)\|^{p-1}_{L^p(B_R)}\leq CR^{d\frac{p-1}{p}}$$

and, raising both sides to $p/(p-1)$, we obtain

$$\int_{B_R}|D(u-h)|^p\,\mathrm{d}x\leq CR^d.$$

From Proposition 2, we get

$$\int_{B_r}|Du-(Du)_r|^p\,\mathrm{d}x\leq C\Big(\frac{r}{R}\Big)^{d+p\alpha}\int_{B_R}|Du-(Du)_R|^p\,\mathrm{d}x+CR^d,\ \forall r\in(0,R]$$

and, applying Lemma 2, we reach

$$\int_{B_r}|Du-(Du)_r|^p\,\mathrm{d}x\leq Cr^d,\quad \forall r\in(0,R].$$

Hence, $Du\in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$ and the proof is complete. ■

As a corollary to Theorem 2, we obtain a modulus of continuity for the solution u in $C^{0,\mathrm{Log-Lip}}$-spaces.

Corollary 1 (Log-Lipschitz continuity estimates)**.** *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to problem* (1)-(2)*. Then $u \in C^{0,\mathrm{Log-Lip}}_{\mathrm{loc}}(\Omega)$. Moreover, for every $\Omega' \Subset \Omega$,*

$$\|u\|_{C^{0,\mathrm{Log-Lip}}(\Omega')} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)}\right),$$

where $C = C(p, d, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega', \partial\Omega)) > 0$.

Indeed, a function whose partial derivatives are in BMO belongs to the Zygmund class (cf. [22]). Because functions in the latter have a $C^{0,\mathrm{Log-Lip}}$ modulus of continuity, the corollary follows.

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References

[1] Ellen Shiting Bao, YanYan Li, and Biao Yin. Gradient estimates for the perfect conductivity problem. *Arch. Ration. Mech. Anal.*, 193(1):195–226, 2009.

[2] Ellen Shiting Bao, YanYan Li, and Biao Yin. Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. *Comm. Partial Differential Equations*, 35(11):1982–2006, 2010.

[3] Eric Bonnetier and Michael Vogelius. An elliptic regularity result for a composite medium with "touching" fibres of circular cross-section. *SIAM J. Math. Anal.*, 31(3):651–677, 2000.

[4] Mikhail V. Borsuk. A priori estimates and solvability of second order quasilinear elliptic equations in a composite domain with nonlinear boundary condition and conjugacy condition. *Trudy Mat. Inst. Steklov.*, 103:15–50. (loose errata), 1968.

[5] Mikhail V. Borsuk. *Transmission problems for elliptic second-order equations in non-smooth domains*. Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2010.

[6] Marc Briane, Yves Capdeboscq, and Luc Nguyen. Interior regularity estimates in high conductivity homogenization and application. *Arch. Ration. Mech. Anal.*, 207(1):75–137, 2013.

[7] Luis Caffareli, María Soria-Carro, and Pablo R. Stinga. Regularity for $C^{1,\alpha}$ interface transmission problems. *Arch. Ration. Mech. Anal.*, 240(1):265–294, 2021.

[8] Sergio Campanato. Sul problema di M. Picone relativo all'equilibrio di un corpo elastico incastrato. *Ricerche Mat.*, 6:125–149, 1957.

[9] Sergio Campanato. Sui problemi al contorno per sistemi di equazioni differenziali lineari del tipo dell'elasticità. I. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 13:223–258, 1959.

[10] Sergio Campanato. Sui problemi al contorno per sistemi di equazioni differenziali lineari del tipo dell'elasticità. II. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 13:275–302, 1959.
[11] Hongjie Dong. A simple proof of regularity for $C^{1,\alpha}$ interface transmission problems. *arXiv Preprint*, arxiv.2004.09365, 2020.
[12] Vladimir A. Il'in and Il'ya A. Šišmarev. The method of potentials for the problems of Dirichlet and Neumann in the case of equations with discontinuous coefficients. *Sibirsk. Mat. Ž.*, pages 46–58, 1961.
[13] YanYan Li and Louis Nirenberg. Estimates for elliptic systems from composite material. *Comm. Pure Appl. Math.*, 56(7):892–925, 2003. Dedicated to the memory of Jürgen K. Moser.
[14] YanYan Li and Michael Vogelius. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Arch. Ration. Mech. Anal.*, 153(2):91–151, 2000.
[15] Gary Lieberman. The natural generalisation of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Comm. in Partial Differential Equations*, 16(2-3):311–361, 1991.
[16] Mauro Picone. Sur un problème nouveau pour l'équation linéaire aux dérivées partielles de la théorie mathématique classique de l'élasticité. In *Colloque sur les équations aux dérivées partielles, CBRM, Bruxelles*, pages 9–11, 1954.
[17] Martin Schechter. A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 14:207–236, 1960.
[18] James Serrin. Local behaviour of solutions of quasi-linear equations. *Acta Math.*, 111:247–302, 1964.
[19] Zinovi G. Šeftel'. Estimates in L_p of solutions of elliptic equations with discontinuous coefficients and satisfying general boundary conditions and conjugacy conditions. *Soviet Math. Dokl.*, 4:321–324, 1963.
[20] María Soria-Carro and Pablo R. Stinga, Regularity of viscosity solutions to fully nonlinear elliptic transmission problems. *arXiv Preprint*, arxiv.org/abs/2207.13772, 2022.
[21] Guido Stampacchia. Su un problema relativo alle equazioni di tipo ellittico del secondo ordine. *Ricerche Mat.*, 5:3–24, 1956.
[22] Antoni Zygmund. *Trigonometric series. Vol. I, II.* Cambridge University Press, Cambridge, 2002.

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