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MEASURABLE FUNCTIONS ON σ-FRAMES

RAQUEL BERNARDES

Abstract: We study (semi)measurable functions on σ-frames, extending the theory of real-valued functions from frames to σ-frames. The new objects of study are the σ-frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ from the usual frame of reals into the congruence lattice of a σ-frame L, and its subclass of measurable functions $\mathfrak{L}(\mathbb{R}) \to L$. The desired extension faces two obstacles: (1) in general, σ-frames have no uncountable joins and are not pseudocomplemented; (2) σ-sublocales, that is, the subobjects in the dual category of the category of σ-frames and σ-frame homomorphisms, cannot be described as concrete subsets of the σ-frame L, unlike their counterpart in the category of locales, forcing us to work in the congruence lattice of L.

Nevertheless, it is shown that, despite (1), the familiar method for generating real functions on frames via scales can be extended to arbitrary σ-frames. This is achieved by the new notions of σ-scale and finite σ-scale. It is also shown that, despite (2), the familiar insertion, extension and separation results for real-valued functions in several classes of frames (normal, extremally disconnected, $\mathcal{G}$-perfect, $\mathcal{F}$-perfect, perfectly normal) can be proved without involving uncountable joins and pseudocomplements, thus allowing their extension to measurable and semimeasurable functions.

Keywords: σ-frame, σ-locale, σ-frame congruence, σ-scale, measurable function, lower and upper measurable function, normal σ-frame, extremally disconnected σ-frame, $\mathcal{F}$-perfect σ-frame, $\mathcal{G}$-perfect σ-frame, perfectly normal σ-frame.

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Introduction

A *σ-frame* is a (bounded) lattice with countable joins in which binary meets distribute over countable joins. These lattices arise naturally in various contexts (see Banaschewski [3]), and typical examples of σ-frames that are not frames are given by the cozero set lattices of topological spaces and by the Boolean σ-algebras that one encounters in topology, measure theory and logic.

Regarding measure theory, Baboolal-Gosh showed in [1] that the dual category of the category of (Boolean) σ-frames and σ-frame homomorphisms

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extends (a substantial part of) the category of measurable spaces and measurable maps, and later, Simpson used that category as the framework for his point-free approach to measure theory [28].

Simpson's proposal is a surprising remarkable alternative to the classical theory. Indeed, as is well-known, assuming the Axiom of Choice it is impossible to define a measure on all subsets of the Euclidean space $\mathbb{R}^n$ that is invariant under the Euclidean group of isometries; in $\mathbb{R}$ and $\mathbb{R}^3$ this follows from the famous paradoxes found by Vitali [29] and Banach-Tarski [2]. Nonetheless, the point-free approach via σ-frames of Simpson [28] overcomes these restrictions and produces an isometry-invariant measure on all subsets of $\mathbb{R}^n$, which, in particular, agrees with the Lebesgue measure on the measurable sets.

Our aim in this paper is to study measurable and semimeasurable functions in the point-free setting. The classical background of this topic is mainly in the book [9] by Evans-Gariepy and in the articles [12] and [21].

In the point-free setting, measurable real-valued functions are defined as expected in a σ-frame L, as just the σ-frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$ (where $\mathfrak{L}(\mathbb{R})$ is the usual frame of reals, taken as a σ-frame). These functions were originally introduced by Banaschewski-Gilmour [5] as "continuous real-valued functions" for σ-frames.

Our purpose in this paper is twofold. First, to extend the study of measurable functions to more general types of (lower and upper) measurability and to present scaling methods for generating them akin to the ones used for real-valued functions on frames ([13, 17, 18, 26]). The extension of scales from frames to σ-frames is not automatic as we have to overcome the fact that σ-frames are no longer pseudocomplemented lattices. This leads us to the notions of σ-scale and finite σ-scale (which, in the presence of pseudocomplements, reduce to the notions of extended scale and scale in frames). In addition, we also describe some of the basic algebraic operations for measurable functions.

Our second purpose is to extend the study of insertion, separation and extension results for real functions ([13, 14, 15, 17, 6]) from frames to σ-frames. Our results extend, in particular, the results for classical measurable functions of Kotzé-Kubiak [21] and Gutiérrez García-Kubiak [12]. It should be emphasised here that our approach reveals that insertion, extension and separation results on the existence of certain measurable real-valued functions on a frame can be proved after all without involving uncountable joins and pseudocomplements and this is why they hold more generally in σ-frames.

The paper is organised as follows. After reviewing in Section 1 the required general concepts and results, we present in Section 2 the counterpart of (semi)measurable functions on σ-frames. In Section 3, we generalise extended scales and scales in frames to σ-scales and finite σ-scales in σ-frames, and we show how to use them to construct (semi)measurable functions on a σ-frame. Then, in Section 4, we briefly describe the algebraic operations for measurable functions that will be needed throughout the paper. In Section 5, we approach the problem of inserting measurable real functions in-between more general real functions on σ-frames; we obtain as corollaries of our Basic Insertion Theorem, extension and separation results also valid for arbitrary σ-frames. Then, in Section 6, we derive the consequences of the Basic Insertion Theorem for normal σ-frames and extremally disconnected σ-frames. In Section 7, we focus on $\mathcal{F}$-perfect and $\mathcal{G}$-perfect σ-frames. In this case, the insertion results turn out to be easily generalisable from the corresponding results for frames but the separation and extension theorems seem to be new even for frames. Finally, in Section 8, we combine the insertion results for normal and perfect σ-frames in an insertion theorem that characterises perfect normality.

Very recently, we came across the article [19] where some weak variants of measurability in σ-frames are treated (thus intersecting in a few points our presentation in Sections 2 and 4). But while [19] describes measurable functions quite briefly, in a slightly different way and focusing in different aspects of them, our setting appears to be the adequate one to obtain the general formulations for the results we were seeking to.

For future convenience, all insertion results in the paper are formulated for extended measurable functions on σ-frames, in clear contrast with the corresponding literature in frames [13, 14, 15, 17]; yet separation and extension theorems must necessarily involve real-valued functions.

1. Background

Our general reference for point-free topology and lattice theory is Picado-Pultr [26]. For congruences on frames and σ-frames, we use Frith [10], Madden [24] and Manuell [25]. For classical measurable functions, we follow Evans-Gariepy [9] and, in particular, Kotzé-Kubiak [21] and Gutiérrez García-Kubiak [12] on insertion theorems for measurable functions.

1.1. Frames and locales. A *frame* (also, *locale*) is a complete lattice L (with bottom 0 and top 1) satisfying the join-infinite distributive law

$$\Big(\bigvee_{a\in A} a\Big) \wedge b = \bigvee_{a\in A} (a \wedge b)$$

for every $A \subseteq L$ and $b \in L$. This is the same as a complete Heyting algebra, with the Heyting implication given by $a \to b = \bigvee\{x \in L \mid a \wedge x \leq b\}$ and pseudocomplements given by $a^* = a \to 0 = \bigvee\{x \in L \mid a \wedge x = 0\}$. Pseudocomplements in a complete Heyting algebra satisfy the De Morgan law

$$\Big(\bigvee_{a\in A} a\Big)^* = \bigwedge_{a\in A} a^*.$$

A *frame homomorphism* is a map between frames that preserves finite meets and arbitrary joins. We will denote the category of frames and frame homomorphisms by Frm.

There are two well-known relations on frames (in fact, they can be formulated more generally in lattices) that are of particular importance here: a is *rather below* b, denoted $a \prec b$, if $a^* \vee b = 1$ (or, equivalently, if there is some $u \in L$ such that $a \wedge u = 0$ and $u \vee b = 1$); a is *completely below* b, and one writes $a \prec\!\prec b$, when there are $a_q \in L$, $q \in [0,1] \cap \mathbb{Q}$, such that $a_0 = a$, $a_1 = b$, and $a_p \prec a_q$ whenever $p < q$. The latter is the largest interpolative relation contained in $\prec$.

1.2. σ-Frames and σ-locales. A lattice L is *join-σ-complete* if it has countable joins. A join-σ-complete lattice is a *σ-frame* [3, 10] if it satisfies the distributive law

$$\Big(\bigvee_{a\in A} a\Big) \wedge b = \bigvee_{a\in A} (a \wedge b)$$

for every countable $A \subseteq L$ and $b \in L$. A *σ-frame homomorphism* is a map between σ-frames that preserves finite meets and countable joins. σ-frames and σ-frame homomorphisms form a category that will be denoted by $\sigma\mathsf{Frm}$. Of course, Frm is a subcategory of $\sigma\mathsf{Frm}$. Their opposite categories, Loc and $\sigma\mathsf{Loc}$, are the categories of locales and localic maps and of σ-locales and σ-localic maps, respectively. As $\sigma\mathsf{Loc}(L, M) = \sigma\mathsf{Frm}(M, L)$, given a σ-localic map $f\colon L \to M$, we will denote by $f^*\colon M \to L$ the corresponding σ-frame homomorphism that represents f.

An important difference between categories Loc and $\sigma\mathsf{Loc}$ is that, in the latter, one does not have available the concrete description of subobjects of an object L as certain subsets of L ([26]). In fact, σ-sublocales S of a σ-locale L have to be described by σ-frame quotients L/θ_S given by σ-*frame congruences* θ_S on L, that is, equivalence relations on L satisfying the congruence properties

$$(x,y),(x',y')\in\theta_S \;\Rightarrow\; (x\wedge x', y\wedge y')\in\theta_S,$$
$$(x_a,y_a)\in\theta_S \;(a\in A, A=\text{countable set}) \;\Rightarrow\; \Big(\bigvee_{a\in A}x_a, \bigvee_{a\in A}y_a\Big)\in\theta_S.$$

The set $\mathcal{C}(L)$ of all congruences on a σ-frame L ordered by inclusion is a frame ([10, 24]). Hence its dual lattice $\mathsf{S}(L)$ of all σ-sublocales of L equipped with the partial order

$$A\subseteq B \text{ if and only if } \theta_B\subseteq\theta_A$$

is a coframe.

The *open* and *closed* σ-sublocales associated with an element $a\in L$ are the σ-sublocales $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ represented, respectively, by the open and closed congruences

$$\Delta_a \coloneqq \{(x,y)\in L\times L \mid x\wedge a = y\wedge a\}$$
$$\text{and}\quad \nabla_a \coloneqq \{(x,y)\in L\times L \mid x\vee a = y\vee a\}.$$

They are complemented to each other in $\mathcal{C}(L)$.

For each σ-frame L, $\Delta[L]\coloneqq\{\Delta_a\mid a\in L\}$ and $\nabla[L]\coloneqq\{\nabla_a\mid a\in L\}$ have the following properties:

(1) The map $\nabla\colon L\to\nabla[L]$ is an isomorphism of σ-frames and thus one may regard L as embedded in $\mathcal{C}(L)$. This isomorphism preserves all joins that exist in L.
(2) The map $\Delta\colon L^{op}\to\Delta[L]$ is an isomorphism of σ-coframes that preserves all meets that exist in L.
(3) If L is a frame, ∇ is a frame isomorphism and Δ is a coframe isomorphism.

Finally, for any σ-sublocale S of L, each σ-sublocale of S is a σ-sublocale of L. In fact (see [28, 7]):

$$\mathcal{C}(S) = \{\theta_S\vee\theta \mid \theta\in\mathcal{C}(L)\}\subseteq\mathcal{C}(L),$$
$$\nabla[S] = \{\theta_S\vee\nabla_a \mid a\in L\}\subseteq\mathcal{C}(L),$$
$$\text{and}\quad \Delta[S] = \{\theta_S\vee\Delta_a \mid a\in L\}\subseteq\mathcal{C}(L).$$

For any $S\in\mathsf{S}(L)$ and $\theta\in\mathcal{C}(S)$, we will denote by θ^{*_S} the pseudocomplement of θ in $\mathcal{C}(S)$; when $S=L$, we will write just θ^* if there is no ambiguity.

1.3. The frame of reals and the frame of extended reals. The *frame of reals* [4] is the frame $\mathfrak{L}(\mathbb{R})$ generated by all pairs $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ subject to the relations

(R_1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$;
(R_2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$;
(R_3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$;
(R_4) $1 = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

$\mathfrak{L}(\mathbb{R})$ may be presented equivalently ([13]) by generators $(p, —)$ and $(—, q)$, with $p, q \in \mathbb{Q}$, and relations

(R'_1) $(p, —) \wedge (—, q) = 0$ whenever $p \geq q$;
(R'_2) $(p, —) \vee (—, q) = 1$ whenever $p < q$;
(R'_3) $(p, —) = \bigvee\{(r, —) \mid p < r\}$;
(R'_4) $(—, q) = \bigvee\{(—, s) \mid s < q\}$;
(R'_5) $1 = \bigvee\{(p, —) \mid p \in \mathbb{Q}\}$;
(R'_6) $1 = \bigvee\{(—, q) \mid q \in \mathbb{Q}\}$.

The correspondence between the generators of the two presentations is given by the identities

$$(p, —) = \bigvee_{q \in \mathbb{Q}} (p, q), \qquad (—, q) = \bigvee_{p \in \mathbb{Q}} (p, q) \quad \text{and} \quad (p, q) = (p, —) \wedge (—, q).$$

The frame $\mathfrak{L}(\mathbb{R})$ is the same as the σ-frame defined by the same generators and relations since the relations involved only deal with countable joins and any countably generated σ-frame L is automatically a frame (see [5, 28] for more details). A map from the generating set of $\mathfrak{L}(\mathbb{R})$ into a σ-frame L defines a σ-frame homomorphism $f \colon \mathfrak{L}(\mathbb{R}) \to L$ if and only if it sends the relations of $\mathfrak{L}(\mathbb{R})$ into identities in L.

Moreover, any σ-frame homomorphism from a countably generated σ-frame L into a frame is a frame homomorphism, as can be easily shown.

Proposition 1.3.1. *Let L be a σ-frame with a countable set of generators. Then L is a frame, and $\sigma\mathsf{Frm}(L, M) = \mathsf{Frm}(L, M)$ for any frame M.*

Hence, in particular, $\sigma\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L)) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L))$.

The frame $\mathfrak{L}(\overline{\mathbb{R}})$ of *extended reals* [6] (the point-free counterpart of the space of extended reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$) is the frame generated by all $(p, —)$ and $(—, q)$, with $p, q \in \mathbb{Q}$, subject to the relations

(R'_1) $(p, —) \wedge (—, q) = 0$ whenever $p \geq q$;

(R'_2) $(p,\text{—}) \vee (\text{—},q) = 1$ whenever $p < q$;
(R'_3) $(p,\text{—}) = \bigvee\{(r,\text{—}) \mid p < r\}$;
(R'_4) $(\text{—},q) = \bigvee\{(\text{—},s) \mid s < q\}$.

It is not possible to present $\mathfrak{L}(\overline{\mathbb{R}})$ alternatively with generators $(p,q) \in \mathbb{Q} \times \mathbb{Q}$. Nonetheless, we will still use the notation (p,q) as an abbreviation for the element $(p,\text{—}) \wedge (\text{—},q)$. Since $\mathfrak{L}(\overline{\mathbb{R}})$ is also a countably generated σ-frame, we have, by 1.3.1, $\sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L)) = \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L))$.

Remark 1.3.2. Consider the element

$$\omega = \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\} = \bigvee_{p\in\mathbb{Q}} \bigvee_{q\in\mathbb{Q}} ((p,\text{—}) \wedge (\text{—},q)) \in \mathfrak{L}(\overline{\mathbb{R}}).$$

The frame $\downarrow \omega$ is isomorphic to $\mathfrak{L}(\mathbb{R})$. Therefore, there exists a onto basic homomorphism $\delta\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(\mathbb{R})$ given by $\delta(p,\text{—}) = (p,\text{—}) \wedge \omega$ and $\delta(\text{—},q) = (\text{—},q) \wedge \omega$ that has as right inverse the inclusion $\mathfrak{L}(\mathbb{R}) \hookrightarrow \mathfrak{L}(\overline{\mathbb{R}})$ (see [6] for more information).

1.4. Localic real and extended real functions. Recall from [13] that a *localic real function* on a frame L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$. Similarly, a *localic extended real function* on a frame L ([6]) is a frame homomorphism $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$.

The set of all real (resp. extended real) functions on L will be denoted by $\mathsf{F}(L)$ (resp. $\overline{\mathsf{F}}(L)$). They are partially ordered by

$$\begin{aligned} f \le g \;&\equiv\; \forall p \in \mathbb{Q},\; f(p,\text{—}) \subseteq g(p,\text{—}) \\ &\Leftrightarrow\; \forall q \in \mathbb{Q},\; g(\text{—},q) \subseteq f(\text{—},q). \end{aligned}$$

An extended real function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ satisfying $f(\omega) = 1$ will be called *finite-valued* (or just *finite*). As $\omega = \big(\bigvee_{p\in\mathbb{Q}}(p,\text{—})\big) \wedge \big(\bigvee_{q\in\mathbb{Q}}(\text{—},q)\big)$, this means that

$$\bigvee_{p\in\mathbb{Q}} f(p,\text{—}) = 1 \quad\text{and}\quad \bigvee_{q\in\mathbb{Q}} f(\text{—},q) = 1,$$

which implies that f is actually a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$.

On the other hand, by Remark 1.3.2, there exists an injective map $\psi\colon \mathsf{F}(L) \to \overline{\mathsf{F}}(L)$ given by $\psi(f) = f \circ \delta$, where $\delta\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(\mathbb{R})$ is the basic homomorphism from 1.3.2. As a consequence, the frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ are in a one-one correspondence with the frame homomorphisms $g\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ such that $g(\omega) = 1$, and this permits us to regard $\mathsf{F}(L)$ as a subset of $\overline{\mathsf{F}}(L)$. Thus, the localic real-valued functions are precisely the finite extended real-valued functions.

In the following, we recall some basic properties of localic extended real functions. Their proofs can be easily obtained from reformulating the ones in [4] or [26] for the finite-valued case.

Proposition 1.4.1. *Let $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ be an extended real function on a frame L. Then:*

1. *If $p \geq q$, then $f(\text{—}, q) \wedge f(p, \text{—}) = 0$.*
2. *If $p < q$, then $f(p, \text{—}) \vee f(\text{—}, q) = 1$.*
3. *$f(p, \text{—}) = \bigvee\{f(r, \text{—}) \mid p < r\}$ and $f(\text{—}, q) = \bigvee\{f(\text{—}, s) \mid s < q\}$.*
4. *If f is finite-valued, then $\bigvee\{f(p, \text{—}) \mid p \in \mathbb{Q}\} = 1 = \bigvee\{f(\text{—}, q) \mid q \in \mathbb{Q}\}$.*
5. *If $p < q$, then $f(q, \text{—}) \subseteq f(\text{—}, q)^* \subseteq f(p, \text{—})$ and $f(\text{—}, p) \subseteq f(p, \text{—})^* \subseteq f(\text{—}, q)$.*

A function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ on a frame L is *upper semicontinuous* (resp. *lower semicontinuous*) if $f(\text{—}, p) \in \nabla[L]$ (resp. $f(p, \text{—}) \in \nabla[L]$) for every $p \in \mathbb{Q}$; f is a *continuous* function whenever it is both upper and lower semicontinuous. Since L is isomorphic to $\nabla[L]$, the continuous extended real functions on a frame L are precisely the frame homomorphisms $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$.

2. Measurable functions

From now on, unless stated otherwise, we will work mainly on a σ-frame L and with the sets

$$\mathsf{F}(L) = \sigma\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L)) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L))$$

and

$$\overline{\mathsf{F}}(L) = \sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L)) = \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{C}(L))$$

of all localic real functions and all extended real functions on L.

We say that an $f \in \overline{\mathsf{F}}(L)$ is *lower measurable* (resp. *upper measurable*) if $f(p, \text{—}) \in \nabla[L]$ (resp. $f(\text{—}, p) \in \nabla[L]$) for every $p \in \mathbb{Q}$, and we denote by $\overline{\mathsf{LM}}(L)$ and $\overline{\mathsf{UM}}(L)$ the corresponding collections of lower measurable and upper measurable extended real functions, respectively. Whenever $f \in \overline{\mathsf{LM}}(L) \cap \overline{\mathsf{UM}}(L)$, we say that f is *measurable*. We shall denote $\overline{\mathsf{LM}}(L) \cap \overline{\mathsf{UM}}(L)$ by $\overline{\mathsf{M}}(L)$. Note that $f \in \overline{\mathsf{M}}(L)$ if $f[\mathfrak{L}(\overline{\mathbb{R}})] \subseteq \nabla[L]$, that is,

$$f(p, q) = f(p, \text{—}) \wedge f(\text{—}, q) \in \nabla[L],$$

and as $\nabla\colon L \to \nabla[L]$ is an isomorphism of σ-frames, identifying each $f \in \overline{\mathsf{LM}}(L) \cap \overline{\mathsf{UM}}(L)$ with $\nabla^{-1} \circ f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$, we conclude that

$$\overline{\mathsf{M}}(L) = \overline{\mathsf{LM}}(L) \cap \overline{\mathsf{UM}}(L) = \sigma\mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), L).$$

In particular, $\overline{\mathsf{M}}(\mathcal{C}(L)) = \overline{\mathsf{F}}(L)$.

In terms of notation, we will work indistinctively with $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ and $\nabla \circ f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \nabla[L]$. Both maps will be denoted by f. The context will distinguish whether we are working in $(L, \leq)$ or rather in $\nabla[L] \subseteq (\mathcal{C}(L), \subseteq)$.

Let $f \in \overline{\mathsf{M}}(L)$. Despite the fact that L has not necessarily arbitrary joins, f preserves all joins:

Proposition 2.1. *Let L be a σ-frame and $f \in \overline{\mathsf{M}}(L)$. For any $A \subseteq \mathfrak{L}(\overline{\mathbb{R}})$,*

$$\bigvee_{a\in A} f(a) \text{ exists in } L \text{ and } \bigvee_{a\in A} f(a) = f\Big(\bigvee_{a\in A} a\Big).$$

Proof: Since $\mathfrak{L}(\overline{\mathbb{R}})$ is a frame, $\bigvee_{a\in A} a$ exists in $\mathfrak{L}(\overline{\mathbb{R}})$. Moreover, since $\mathfrak{L}(\overline{\mathbb{R}})$ has a countable set of generators, there exists a countable set $B \subseteq A$ such that $\bigvee_{a\in A} a = \bigvee_{b\in B} b$. Hence,

$$u := f\Big(\bigvee_{a\in A} a\Big) = f\Big(\bigvee_{b\in B} b\Big) = \bigvee_{b\in B} f(b) \in L,$$

and we only need to show that u is the join of $\{f(a) \mid a \in A\}$ in L. By the monotonicity of f, $u \geq f(a)$ for every $a \in A$. If $v \in L$ is such that $v \geq f(a)$ for all $a \in A$, then, in particular, $v \geq f(b)$ for all $b \in B$, and $v \geq \bigvee_{b\in B} f(b) = u$. ■

Restricting to the finite-valued case, we introduce the classes

$$\begin{aligned} \mathsf{LM}(L) &:= \overline{\mathsf{LM}}(L) \cap \mathsf{F}(L), \\ \mathsf{UM}(L) &:= \overline{\mathsf{UM}}(L) \cap \mathsf{F}(L), \\ \text{and} \quad \mathsf{M}(L) &:= \overline{\mathsf{M}}(L) \cap \mathsf{F}(L) \end{aligned}$$

of *lower measurable*, *upper measurable* and *measurable* real-valued functions, respectively. We have

$$\begin{array}{ccc} \overline{\mathsf{M}}(L) & \subseteq & \overline{\mathsf{F}}(L) \\ \cup\!\!| & & \cup\!\!| \\ \mathsf{M}(L) & \subseteq & \mathsf{F}(L). \end{array}$$

Remark 2.2. In particular, when L is a frame, denoting by $\mathsf{C}(L)$ and $\overline{\mathsf{C}}(L)$ the sets of all localic continuous real functions and all localic continuous extended real functions, respectively, we have that $\mathsf{C}(L) = \mathsf{M}(L)$ and $\overline{\mathsf{C}}(L) = \overline{\mathsf{M}}(L)$, by Proposition 1.3.1.

The novelty of working with these classes of functions in σ-frames, especially with the notion of measurability, is that while frames have pseudocomplements, which are often used in the proofs, σ-frames are not necessarily pseudocomplemented. So if we want to generalise the results to σ-frames, we need to find alternate proofs as we shall see in the next section.

3. σ-Scales

Continuous extended real-valued functions on a frame L can be canonically generated by *extended scales* ([6]), that is, maps $\sigma\colon \mathbb{Q} \to L$ such that $\sigma(r) \prec \sigma(s)$ whenever $r < s$. In order to generate continuous real-valued functions, the extended scale must be a *scale* ([13]), that is, an extended scale $\sigma\colon \mathbb{Q} \to L$ such that $\bigvee\{\sigma(r) \mid r \in \mathbb{Q}\} = 1 = \bigvee\{\sigma(r)^* \mid r \in \mathbb{Q}\}$. Some work has to be done when replacing frames by σ-frames to adapt these notions to measurable functions.

Definition 3.1. Let L be a σ-frame. A map $\varphi\colon \mathbb{Q} \to L$ is an *ascending σ-scale* in L (or just a *σ-scale*) if there exists a family $(c_r)_{r\in\mathbb{Q}}$ of elements of L such that $\varphi(s) \wedge c_r = 0$ whenever $s \le r$, and $c_r \vee \varphi(s) = 1$ whenever $r < s$. Furthermore, we say that φ is *finite* if $\bigvee_{r\in\mathbb{Q}} \varphi(r) = 1 = \bigvee_{r\in\mathbb{Q}} c_r$.

Proposition 3.2. *Let L be a σ-frame. A map $\varphi\colon \mathbb{Q} \to L$ is a σ-scale if and only if $\varphi(r) \prec \varphi(s)$ whenever $r < s$.*

Moreover, φ is a finite σ-scale if and only if $\bigvee\{\varphi(r) \mid r \in \mathbb{Q}\} = 1$ and there are $c_{rs} \in L$ such that $\bigvee\{c_{rs} \mid r, s \in \mathbb{Q}, r < s\} = 1$, with $\varphi(r) \wedge c_{rs} = 0$ and $c_{rs} \vee \varphi(s) = 1$ whenever $r < s$.

Proof: Suppose that φ is a σ-scale. For each pair $r < s$, we have that $\varphi(r) \wedge c_r = 0$ and $c_r \vee \varphi(s) = 1$, hence $\varphi(r) \prec \varphi(s)$. In addition, if φ is finite, taking $c_{rs} := \bigvee\{c_u \mid u \in \mathbb{Q}, r < u < s\}$, we obtain the required family.

Conversely, suppose that $\varphi(r) \prec \varphi(s)$ whenever $r < s$. Then there exist elements $c_{rs} \in L$, $r < s$, such that $\varphi(r) \wedge c_{rs} = 0$ and $c_{rs} \vee \varphi(s) = 1$. Setting $c_r := \bigvee\{c_{rs} \mid s \in \mathbb{Q}, s > r\}$, we conclude that φ is a σ-scale. Finally, if $\bigvee\{\varphi(r) \mid r \in \mathbb{Q}\} = 1 = \bigvee\{c_{rs} \mid r, s \in \mathbb{Q}, r < s\}$, then φ is clearly finite. ∎

Given a σ-scale $\varphi\colon \mathbb{Q} \to L$, let $\mathcal{C}_\varphi$ be the set of all families $(x_r)_{r\in\mathbb{Q}} \subseteq L$ satisfying $\varphi(s) \wedge x_r = 0$ whenever $s \leq r$ and $x_r \vee \varphi(s) = 1$ otherwise.

Proposition 3.3. *Any pair of families $(x_r)_{r\in\mathbb{Q}}$, $(y_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$ satisfies*

$$\bigvee_{r>s} x_r = \bigvee_{r>s} y_r \text{ for every } s \in \mathbb{Q}.$$

Proof: As $\nabla\colon L \to \nabla[L]$ is a σ-frame isomorphism, this is equivalent to showing that $\bigvee\{\nabla_{x_r} \mid r > s\} = \bigvee\{\nabla_{y_r} \mid r > s\}$. Therefore, it suffices to check that for any $(x_r)_{r\in\mathbb{Q}}$ in C_φ,

$$\bigvee_{r>s} \nabla_{x_r} = \bigvee_{r>s} \Delta_{\varphi(r)}.$$

Since $\nabla_{\varphi(s)} \wedge \nabla_{x_r} = 0$ whenever $s < r$, that is, $\nabla_{x_r} \subseteq \Delta_{\varphi(s)}$ whenever $s < r$, then $\bigvee_{r>s} \nabla_{x_r} \subseteq \bigvee_{r>s} \Delta_{\varphi(r)}$. On the other hand, as $\nabla_{x_r} \vee \nabla_{\varphi(s)} = 1$ for all $r < s$, then $\Delta_{\varphi(s)} \subseteq \nabla_{x_r}$, hence $\bigvee_{r>s} \Delta_{\varphi(r)} \subseteq \bigvee_{r>s} \nabla_{x_r}$. ■

As a consequence, if there exists $(c_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$ such that $\bigvee\{c_r \mid r \in \mathbb{Q}\} = 1$, then for any other $(b_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$

$$\bigvee_{r\in\mathbb{Q}} b_r = \bigvee_{s\in\mathbb{Q}} \bigvee_{r>s} b_r = \bigvee_{s\in\mathbb{Q}} \bigvee_{r>s} c_r = \bigvee_{r\in\mathbb{Q}} c_r = 1.$$

Thus, in order to show that a σ-scale φ with $\bigvee\{\varphi(r) \mid r \in \mathbb{Q}\} = 1$ is finite, it suffices to find a family $(c_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$ such that $\bigvee\{c_r \mid r \in \mathbb{Q}\} = 1$. In particular, for each $r < s$, it suffices to take c_{rs} as an element satisfying $\varphi(r) \wedge x = 0$ and $x \vee \varphi(s) = 1$ and to check whether $\bigvee\{c_{rs} \mid r, s \in \mathbb{Q}, r < s\} = 1$.

Remark 3.4. In case L is a frame, we can assume that $c_r = \varphi(r)^*$, and it is then clear that the σ-scales in L are precisely the extended scales in L and that the finite σ-scales in L are precisely the scales in L.

Proposition 3.5. *Let L be a σ-frame. Given a σ-scale $\varphi\colon \mathbb{Q} \to L$ and a family $(c_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$, the map $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ determined by*

$$f(p, \text{—}) = \bigvee_{r>p} c_r \text{ and } f(\text{—}, q) = \bigvee_{r<q} \varphi(r) \quad (p, q \in \mathbb{Q})$$

is a measurable function on L. Moreover, if φ is finite, then f is a finite-valued function.

Proof: First of all, note that the definition of f does not depend on the chosen $(c_r)_{r\in\mathbb{Q}}$ in $\mathcal{C}_\varphi$ by the preceding proposition. To show that f is indeed a measurable function, we just need to verify that it sends the relations $(R_1') - (R_4')$ in $\mathfrak{L}(\overline{\mathbb{R}})$ into identities in L:

- (R_1') : For any $p \geq q$, $f(p,\text{—}) \wedge f(\text{—},q) = \bigvee_{s<q\leq p<r} c_r \wedge \varphi(s) = 0$.
- (R_2') : If $p < q$, then $f(p,\text{—}) \vee f(\text{—},q) = \bigvee_{r>p} \bigvee_{s<q} c_r \vee \varphi(s) \geq c_u \vee \varphi(v) = 1$ for some $u, v \in \mathbb{Q}$ such that $p < u < v < q$, which exist because $\mathbb{Q}$ is dense in itself.
- (R_3') : As $\mathbb{Q}$ is dense in itself, $f(p,\text{—}) = \bigvee_{s>p} \bigvee_{r>s} c_r = \bigvee_{s>p} f(s,\text{—})$.
- (R_4') : Again, as $\mathbb{Q}$ is dense in itself, $f(\text{—},q) = \bigvee_{s<q} \bigvee_{r<s} \varphi(r) = \bigvee_{s<q} f(\text{—},s)$.

Finally, if φ is finite, then

- (R_5') : $\bigvee_{p\in\mathbb{Q}} f(p,\text{—}) = \bigvee_{p\in\mathbb{Q}} \bigvee_{r>p} c_r = \bigvee_{r\in\mathbb{Q}} c_r = 1$,
- (R_6') : $\bigvee_{q\in\mathbb{Q}} f(\text{—},q) = \bigvee_{q\in\mathbb{Q}} \bigvee_{r<q} \varphi(r) = \bigvee_{r\in\mathbb{Q}} \varphi(r) = 1$,

hence f is finite-valued, that is, it belongs to $\mathsf{M}(L)$. ■

More precisely, given a σ-scale φ and denoting by f the function determined by φ (provided by 3.5), we have that φ is finite if and only if f is finite-valued. In fact, if f is finite-valued, then

$$\bigvee_{r\in\mathbb{Q}} \varphi(r) = \bigvee_{s\in\mathbb{Q}} \bigvee_{r<s} \varphi(r) = \bigvee_{s\in\mathbb{Q}} f(\text{—},s) = 1,$$
$$\text{and} \quad \bigvee_{r\in\mathbb{Q}} c_r = \bigvee_{s\in\mathbb{Q}} \bigvee_{r>s} c_r = \bigvee_{s\in\mathbb{Q}} f(s,\text{—}) = 1.$$

Proposition 3.6. *Let $f, g \in \overline{\mathsf{M}}(L)$ be determined by σ-scales φ_f and φ_g, respectively. Then $f \leq g$ if and only if $\varphi_g(r) \leq \varphi_f(s)$ for every $r < s$.*

Proof: The proof is similar to the proof of its counterpart result in frames (see e.g. [26], Lemma XIV.5.2.4). ■

So far, we have been using ascending σ-scales to generate measurable functions. But we can generate measurable functions $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ similarly from *descending σ-scales* in L, that is, maps $\varphi\colon \mathbb{Q} \to L$ for which there is a family $(b_r)_{r\in\mathbb{Q}}$ in L satisfying $\varphi(s) \wedge b_r = 0$ whenever $r \leq s$, and $b_r \vee \varphi(s) = 1$ whenever $s < r$. In this case, the measurable function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ determined by φ is given by formulas

$$f(p,\text{—}) = \bigvee_{s>p} \varphi(s) \quad \text{and} \quad f(\text{—},q) = \bigvee_{s<q} b_s$$

for every $p, q \in \mathbb{Q}$. If φ is finite, that is, $\bigvee_{r\in\mathbb{Q}} \varphi(r) = 1 = \bigvee_{r\in\mathbb{Q}} b_r$, then f is finite-valued.

Remark 3.7. A map $\varphi\colon \mathbb{Q} \to L$ is an ascending σ-scale if and only if every $\varphi'\colon \mathbb{Q} \to L$ such that $(\varphi'(r))_{r\in\mathbb{Q}} \in \mathcal{C}_\varphi$ is a descending σ-scale.

Proposition 3.8. *Given a measurable function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ on a σ-frame L, the maps $\varphi_1\colon \mathbb{Q} \to L$ and $\varphi_2\colon \mathbb{Q} \to L$ given by*

$$\varphi_1(r) = f(\text{—}, r) \quad \textit{and} \quad \varphi_2(r) = f(r, \text{—})$$

are, respectively, an ascending σ-scale and a descending σ-scale in L that generate f. Additionally, if $f(r,\text{—})$ is pseudocomplemented for every $r \in \mathbb{Q}$, the map $\varphi_3\colon \mathbb{Q} \to L$ given by

$$\varphi_3(r) = f(r, \text{—})^*$$

is also an ascending σ-scale in L that generates f.

Proof: Define $c_r := f(r,\text{—})$ for any $r \in \mathbb{Q}$. Since $f \in \overline{\mathsf{F}}(L)$, we have that $\varphi_1(s) \wedge c_r = 0$ for $s \leq r$ and $c_r \vee \varphi_1(s) = 1$ otherwise. So φ_1 is indeed a σ-scale, and denoting by f_1 the function generated by φ_1, we have $f_1 = f$:

$$f_1(\text{—}, q) = \bigvee_{r<q} \varphi_1(r) = \bigvee_{r<q} f(\text{—}, r) = f(\text{—}, q) \ \text{ for every } q \in \mathbb{Q}.$$

A similar reasoning applies to φ_2.

If $f(r,\text{—})$ is pseudocomplemented for all $r \in \mathbb{Q}$, let $c_r := f(r,\text{—})$. Once again, φ_3 is a σ-scale because, as $f \in \overline{\mathsf{F}}(L)$, we have $\varphi_3(s) \wedge c_r \leq f(s,\text{—})^* \wedge f(s,\text{—}) = 0$ whenever $s \leq r$ and $c_r \vee \varphi_3(s) \geq f(r,\text{—}) \vee f(\text{—}, s) = 1$ whenever $r < s$. Now, note that $f(\text{—}, r) \leq f(r,\text{—})^* \leq f(\text{—}, s)$ for $r < s$, where $f(r,\text{—})^* \leq f(\text{—}, s)$ comes from the fact that $f(\text{—}, s) \vee f(r,\text{—}) = 1$ and $f(r,\text{—})^* \wedge f(r,\text{—}) = 0$. So denoting by f_3 the function generated by φ_3, we conclude that $f_3 = f$:

$$f_3(\text{—}, q) = \bigvee_{r<q} f(r,\text{—})^* = \bigvee_{r<q} f(\text{—}, r) = f(\text{—}, q) \ \text{ for every } q \in \mathbb{Q}. \qquad \blacksquare$$

Example 3.9. *Extended constant functions.* For each $r \in \mathbb{Q} \cup \{\pm\infty\}$, let $\varphi_r(s) = 0$ if $s \leq r$ and $\varphi_r(s) = 1$ if $s > r$. Setting $c_s := \varphi_r(s)^*$ for any $s \in \mathbb{Q}$, the map $\varphi_r\colon \mathbb{Q} \to L$ is a σ-scale in L. The measurable function $\mathbf{r}_L\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ determined by φ_r is called the *extended constant function associated with* r and is given by the formulas

$$\mathbf{r}_L(p,\text{—}) = \begin{cases} 1 & \text{if } p < r \\ 0 & \text{if } p \geq r \end{cases} \quad \text{and} \quad \mathbf{r}_L(\text{—}, q) = \begin{cases} 0 & \text{if } q \leq r \\ 1 & \text{if } q > r. \end{cases}$$

When there is no ambiguity, we will denote it simply by $\mathbf{r}$.

If $r \in \mathbb{Q}$, the extended constant function $\mathbf{r}$ is trivially finite-valued, and we call it a *constant function*. For $r = +\infty$ or $r = -\infty$, we have

$$\begin{aligned} \boldsymbol{+\infty}\,(p,\text{—}) = 1, &\quad \boldsymbol{+\infty}(\text{—},q) = 0, \\ \text{and } \boldsymbol{-\infty}\,(p,\text{—}) = 0, &\quad \boldsymbol{-\infty}(\text{—},q) = 1, \end{aligned}$$

for every $p, q \in \mathbb{Q}$. The functions $\boldsymbol{+\infty}$ and $\boldsymbol{-\infty}$ are examples of measurable extended real functions which are not finite-valued. So the inclusions $\mathsf{M}(L) \subseteq \overline{\mathsf{M}}(L)$ and $\mathsf{F}(L) \subseteq \overline{\mathsf{F}}(L)$ are actually strict.

Example 3.10. *Characteristic functions.* For each complemented $a \in L$, define $\varphi_a(r) = 0$ if $r \leq 0$, $\varphi_a(r) = a^*$ if $0 < r \leq 1$ and $\varphi_a(r) = 1$ if $r > 1$. Setting $c_r := \varphi_a(r)^*$, the map φ_a is a finite σ-scale with corresponding finite measurable function $\chi_a \colon \mathfrak{L}(\mathbb{R}) \to L$ given by

$$\chi_a(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ a & \text{if } 0 \leq p < 1 \\ 0 & \text{if } p \geq 1 \end{cases} \quad \text{and} \quad \chi_a(\text{—},q) = \begin{cases} 0 & \text{if } q \leq 0 \\ a^* & \text{if } 0 < q \leq 1 \\ 1 & \text{if } q > 1. \end{cases}$$

We call it the *characteristic function associated with $a \in L$*. For any $a \leq b$ in L, $\chi_a \leq \chi_b$.

To close this section, we remark that as $\overline{\mathsf{F}}(L) = \overline{\mathsf{M}}(\mathcal{C}(L))$ and $\mathsf{F}(L) = \mathsf{M}(\mathcal{C}(L))$, σ-scales and finite σ-scales in $\mathcal{C}(L)$ generate, respectively, extended real-valued and real-valued functions on L.

Example 3.11. Given a σ-locale L and a complemented σ-sublocale $S \in \mathsf{S}(L)$, consider a measurable function $f \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ on S satisfying $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. Let θ_S be the complemented congruence associated with S, and define

$$\varphi_h(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ \theta_S^* \wedge f(\text{—},r) & \text{if } 0 < r \leq 1 \\ 1 & \text{if } r > 1 \end{cases}$$

and

$$\varphi_g(r) = \begin{cases} 1 & \text{if } r < 0 \\ \theta_S^* \wedge f(r,\text{—}) & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r \geq 1. \end{cases}$$

As f is measurable in S, $f(r,\text{—}), f(\text{—},r) \in \nabla[S] = \{\nabla_a \vee \theta_S \mid a \in L\}$. Therefore, as each element $\varphi_h(r)$ is complemented in $\mathcal{C}(L)$ and φ_h is increasing, we have $\varphi_h(r) \prec \varphi_h(s)$ whenever $r < s$, besides $\bigvee_{r \in \mathbb{Q}} \varphi_h(r) =$

$1 = \bigvee_{r\in\mathbb{Q}} \varphi_h(r)^*$. Hence φ_h is a finite σ-scale in $\mathcal{C}(L)$, and the function $h_f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ generated by φ_h is given by the formulas

$$h_f(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ f(p,\text{—}) & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1 \end{cases}$$

and

$$h_f(\text{—},q) = \begin{cases} 0 & \text{if } q \le 0 \\ \theta_S^* \wedge f(\text{—},q) & \text{if } 0 < q \le 1 \\ 1 & \text{if } q > 1. \end{cases}$$

In fact, for $0 < q \le 1$,

$$h_f(\text{—},q) = \bigvee_{r<q} \varphi_h(r) = \bigvee_{0<r<q} [\theta_S^* \wedge f(\text{—},r)] = \theta_S^* \wedge \bigvee_{r<q} f(\text{—},r) = \theta_S^* \wedge f(\text{—},q).$$

On the other hand, for $0 \le p < 1$, and recalling that $f(\text{—},r)^{*_S}$ denotes the pseudocomplement of $f(\text{—},r)$ in $\mathcal{C}(S)$, we have

$$\begin{aligned} h_f(p,\text{—}) &= \bigvee_{r>p} \varphi_h(r)^* = \bigvee_{1\ge r>p} [\theta_S^* \wedge f(\text{—},r)]^* = \bigvee_{1\ge r>p} [\theta_S \vee f(\text{—},r)^*] \\ &= \bigvee_{1\ge r>p} f(\text{—},r)^{*_S} = \bigvee_{r>p} f(\text{—},r)^{*_S} = \bigvee_{r>p} f(r,\text{—}) = f(p,\text{—}), \end{aligned}$$

where $\theta_S \vee f(\text{—},r)^* = f(\text{—},r)^{*_S}$ follows from the fact that as $f(\text{—},r) = \nabla_a \vee \theta_S$ for some $a \in L$,

$$f(\text{—},r)^{*_S} = \Delta_a \vee \theta_S, \quad f(\text{—},r)^* = \Delta_a \wedge \theta_S^*$$

and

$$\theta_S \vee f(\text{—},r)^* = \theta_S \vee (\Delta_a \wedge \theta_S^*) = \theta_S \vee \Delta_a = f(\text{—},r)^{*_S}.$$

Similarly, one may check that the map φ_g is a finite descending σ-scale in $\mathcal{C}(L)$, and the function $g_f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ generated by φ_g is given by the formulas

$$g_f(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ \theta_S^* \wedge f(p,\text{—}) & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1 \end{cases}$$

and

$$g_f(\text{—},q) = \begin{cases} 0 & \text{if } q \le 0 \\ f(\text{—},q) & \text{if } 0 < q \le 1 \\ 1 & \text{if } q > 1. \end{cases}$$

The functions $g_f \leq h_f$ are kind of a “minorant extension” and “majorant extension” of f over L. Later on, they will be useful while studying insertion properties.

Example 3.12. Given a σ-frame L, let $a, b \in L$ be such that $a \vee b = 1$. Set $\theta_S := \nabla_{a\wedge b}$ and define

$$\varphi(r) = \begin{cases} \nabla_{a\wedge b} & \text{if } r \leq 0 \\ \nabla_b \vee \nabla_{a\wedge b} & \text{if } 0 < r \leq 1 \\ 1 & \text{if } r > 1. \end{cases}$$

Since φ is increasing and $\varphi(r)$ is complemented in $\mathcal{C}(S)$ for all $r \in \mathbb{Q}$, $\varphi(r) \prec \varphi(s)$ in $\mathcal{C}(S)$ whenever $r < s$. In addition, $\bigvee_{r\in\mathbb{Q}} \varphi(r) = 1 = \bigvee_{r\in\mathbb{Q}} \varphi(r)^{*_S}$. Thus, φ is a finite σ-scale in $\mathcal{C}(S)$ and the function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ generated by φ is given by

$$f(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ \nabla_a \vee \nabla_{a\wedge b} & \text{if } 0 \leq p < 1 \\ \nabla_{a\wedge b} & \text{if } p \geq 1 \end{cases}$$

and

$$f(\text{—},q) = \begin{cases} \nabla_{a\wedge b} & \text{if } q \leq 0 \\ \nabla_b \vee \nabla_{a\wedge b} & \text{if } 0 < q \leq 1 \\ 1 & \text{if } q > 1, \end{cases}$$

where, for $0 \leq p < 1$, $f(p,\text{—}) = \nabla_a \vee \nabla_{a\wedge b}$ follows from the fact that $a \vee b = 1$ implies $\Delta_b \vee (\nabla_a \wedge \nabla_b) = \nabla_a$.

Clearly, all the results that hold for measurable functions also hold for general functions. For that reason, from now on we will focus mainly on the study of the lattices $\overline{\mathsf{M}}(L)$ and $\mathsf{M}(L)$.

4. Algebraic operations with measurable functions

By similar arguments to the ones used in [4] for the study of the algebra $\mathsf{C}(L)$, one may conclude that the operations in the algebra $\mathsf{M}(L)$ are also determined by the lattice-ordered ring operations of $\mathbb{Q}$.

In this section, we present the algebraic operations in $\overline{\mathsf{M}}(L)$ that we will need throughout the paper, by extending to σ-frames their corresponding description for frames in [6].

Proposition 4.1. *Let $\lambda > 0$ and $f, g \in \overline{\mathsf{M}}(L)$. Then:*

(i) $\varphi_{\lambda\cdot f} \colon r \mapsto f(\text{—}, \frac{r}{\lambda})$ *is a σ-scale that generates the function $\lambda \cdot f \in \overline{\mathsf{M}}(L)$ given by $(\lambda \cdot f)(p, \text{—}) = f(\frac{p}{\lambda}, \text{—})$ and $(\lambda \cdot f)(\text{—}, q) = f(\text{—}, \frac{q}{\lambda})$.*

(ii) $\varphi_{-f} \colon r \mapsto f(-r, \text{—})$ *is a σ-scale that generates the function $-f \in \overline{\mathsf{M}}(L)$ given by $-f(p, \text{—}) = f(\text{—}, -p)$ and $-f(\text{—}, q) = f(-q, \text{—})$.*

(iii) $\varphi_{f\vee g} \colon r \mapsto f(\text{—}, r) \wedge g(\text{—}, r)$ *is a σ-scale that generates the function $f \vee g \in \overline{\mathsf{M}}(L)$ given by $(f \vee g)(p, \text{—}) = f(p, \text{—}) \vee g(p, \text{—})$ and $(f \vee g)(\text{—}, q) = f(\text{—}, q) \wedge g(\text{—}, q)$.*

(iv) $\varphi_{f\wedge g} \colon r \mapsto f(\text{—}, r) \vee g(\text{—}, r)$ *is a σ-scale that generates the function $f \wedge g \in \overline{\mathsf{M}}(L)$ given by $(f \wedge g)(p, \text{—}) = f(p, \text{—}) \wedge g(p, \text{—})$ and $(f \wedge g)(\text{—}, q) = f(\text{—}, q) \vee g(\text{—}, q)$.*

Moreover, if f and g are finite, then $\lambda \cdot f$, $-f$, $f \vee g$ and $f \wedge g$ are also finite.

Things become a bit more complicated with the sum: we need to be cautious with the classical indeterminacy $(-\infty) + (+\infty)$ when working with extended real functions. Following [6], and setting for each $f \in \overline{\mathsf{M}}(L)$

$$a_f^+ := \bigvee_{r\in\mathbb{Q}} f(\text{—}, r), \quad a_f^- := \bigvee_{r\in\mathbb{Q}} f(r, \text{—}) \quad \text{and} \quad a_f := a_f^+ \wedge a_f^- = \bigvee_{r<s} f(r, s) = f(\omega),$$

we say that $f, g \in \overline{\mathsf{M}}(L)$ are *sum compatible* if $a_{f\vee g}^+ \vee a_{f\wedge g}^- = 1$ or, equivalently, if

$$(a_f^+ \vee a_g^-) \wedge (a_g^+ \vee a_f^-) = 1.$$

Proposition 4.2. *Let $f, g \in \overline{\mathsf{M}}(L)$ be sum compatible. Then*

$$\varphi_{f+g} \colon r \mapsto \bigvee_{t\in\mathbb{Q}} f(\text{—}, t) \wedge g(\text{—}, r - t)$$

is a σ-scale in L. It generates the measurable function $f + g \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ given by the formulas

$$(f + g)(\text{—}, q) = \bigvee_{t\in\mathbb{Q}} (f(\text{—}, t) \wedge g(\text{—}, q - t)),$$
$$(f + g)(p, \text{—}) = \bigvee_{t\in\mathbb{Q}} (f(t, \text{—}) \wedge g(p - t, \text{—})).$$

Moreover, if f and g are finite-valued, then f and g are sum compatible and $f + g$ is finite.

Proof: The proof in [6] for the corresponding result in frames can be easily reformulated to a proof of the fact that the family $(c_r)_{r\in\mathbb{Q}}$ defined by

$$c_r := \bigvee_{t\in\mathbb{Q}} (f(t,\text{—}) \wedge g(r-t,\text{—}))$$

is an element of $\mathcal{C}_\varphi$. ∎

We take note that $a_f^+ \vee a_f^- = 1$ for any $f \in \overline{\mathsf{M}}(L)$, and $a_f^+ = a_f^- = a_f = 1$ if and only if $f \in \mathsf{M}(L)$. Besides, for any $f \in \overline{\mathsf{M}}(L)$ and $g \in \mathsf{M}(L)$, f and g are sum compatible.

Finally, concerning the difference, if f and $-g$ are sum compatible, we may define

$$f - g := f + (-g).$$

As $a_{-g}^+ = a_g^-$ and $a_{-g}^- = a_g^+$, then $f, -g \in \overline{\mathsf{M}}(L)$ are sum compatible if and only if

$$(a_f^+ \vee a_g^+) \wedge (a_f^- \vee a_g^-) = 1,$$

and in this case $f - g \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ is given by the formulas

$$(f-g)(\text{—},q) = \bigvee_{t\in\mathbb{Q}} (f(\text{—},t) \wedge g(t-q,\text{—}))$$

and

$$(f-g)(p,\text{—}) = \bigvee_{t\in\mathbb{Q}} (f(t,\text{—}) \wedge g(\text{—},t-p)).$$

5. The Basic Insertion Theorem

In this section, we address the problem of inserting measurable real functions in-between more general real functions on σ-frames. We will deduce, in particular, extension and separation results also valid for general σ-frames.

We need first to recall that a binary relation $\Subset$ on a lattice L is a *Katětov relation* ([20, 22]) whenever it satisfies the following conditions for all $a, b, a', b' \in L$:

(K_1) $a \Subset b \Rightarrow a \leq b$;
(K_2) $a' \leq a, a \Subset b, b \leq b' \Rightarrow a' \Subset b'$;
(K_3) $a \Subset b, a' \Subset b \Rightarrow (a \vee a') \Subset b$;
(K_4) $a \Subset b, a \Subset b' \Rightarrow a \Subset (b \wedge b')$;
(K_5) $a \Subset b \Rightarrow \exists c \in L : a \Subset c \Subset b$.

We will need a result for Katětov relations, known as the Katětov Lemma ([22, 23]), that extends the original basic lemma of Katětov [20, Lemma 1] from power sets to general lattices.

Lemma 5.1. *Let $\Subset$ be a Katětov relation on L and $\lhd$ a transitive and irreflexive (i.e, a relation that is not reflexive) relation on a countable set D. Consider two families $(a_d)_{d\in D}$ and $(b_d)_{d\in D}$ of elements of L such that*

$$d_1 \lhd d_2 \text{ implies } a_{d_2} \le a_{d_1}, b_{d_2} \le b_{d_1} \text{ and } a_{d_2} \Subset b_{d_1}.$$

Then there exists a family $(c_d)_{d\in D}$ in L such that

$$d_1 \lhd d_2 \text{ implies } c_{d_2} \Subset c_{d_1}, a_{d_2} \Subset c_{d_1} \text{ and } c_{d_2} \Subset b_{d_1}.$$

For any $\theta_A, \theta_B \in \mathcal{C}(L)$, define

$$\theta_A \Subset_{\overline{M}} \theta_B \;\equiv\; \exists\, f \in \overline{\mathsf{M}}(L)\colon \theta_A \subseteq f(p,\text{—})^* \text{ and } f(\text{—},q) \subseteq \theta_B \text{ for some } p<q,$$

and write $\theta_A \Subset_M \theta_B$ whenever $f \in \mathsf{M}(L)$.

Lemma 5.2. *For any $\theta_A, \theta_B \in \mathcal{C}(L)$ we have:*

(1) *$\theta_A \Subset_{\overline{M}} \theta_B$ if and only if there is some $f \in \overline{\mathsf{M}}(L)$ such that $\theta_A \subseteq f(0,\text{—})^*$ and $f(\text{—},1) \subseteq \theta_B$. Moreover, $\theta_A \Subset_M \theta_B$ if and only if such f is finite-valued.*

(2) *If $\theta_A \Subset_{\overline{M}} \theta_B$ then $\theta_B^* \Subset_{\overline{M}} \theta_A^*$. In particular, if $\theta_A \Subset_M \theta_B$ then $\theta_B^* \Subset_M \theta_A^*$.*

Proof: (1) The implication '$\Leftarrow$' is immediate. To prove the converse, let $p<q$ be a pair such that $\theta_A \subseteq f(p,\text{—})^*$ and $f(\text{—},q) \subseteq \theta_B$, and define $g = \frac{1}{q-p}\cdot(f-\mathbf{p})$. Note that if f is finite-valued, g is finite-valued. As

$$g(0,\text{—}) = \bigvee_{t\in\mathbb{Q}} f(t,\text{—}) \wedge \mathbf{p}(\text{—},t) = \bigvee_{t>p} f(t,\text{—}) \le f(p,\text{—})$$

$$\text{and}\quad g(\text{—},1) = \bigvee_{t\in\mathbb{Q}} f(\text{—},t) \wedge \mathbf{p}(t-(q-p),\text{—}) = \bigvee_{t<q} f(\text{—},t) \le f(\text{—},q),$$

we get $\theta_A \subseteq f(p,\text{—})^* \subseteq g(0,\text{—})^*$ and $g(\text{—},1) \subseteq f(\text{—},q) \subseteq \theta_B$.

(2) If $\theta_A \Subset_{\overline{M}} \theta_B$, there exists $f \in \overline{\mathsf{M}}(L)$ such that $\theta_A \subseteq f(0,\text{—})^*$ and $f(\text{—},1) \subseteq \theta_B$. Recall that $r \mapsto f(r,\text{—})$ is a descending σ-scale in L generating f. So there exists a family $(b_r)_{r\in\mathbb{Q}} \subseteq L$ such that $f(s,\text{—}) \wedge b_r = 0$ if $r \le s$, and $b_r \vee f(s,\text{—}) = 1$ otherwise. Define $\varphi(r) := f(1-r,\text{—})$ for all $r \in \mathbb{Q}$.

Setting $c_r := b_{1-r}$, one easily sees that φ is a σ-scale in L. The measurable function $g\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ generated by φ is given by

$$g(p,\text{—}) = \bigvee_{p<r} c_r = \bigvee_{p<r} b_{1-r} = \bigvee_{1-r<1-p} b_{1-r} = f(\text{—}, 1-p)$$

$$\text{and}\quad g(\text{—},q) = \bigvee_{r<q} \varphi(r) = \bigvee_{r<q} f(1-r,\text{—}) = \bigvee_{1-q<1-r} f(1-r,\text{—}) = f(1-q,\text{—}).$$

Hence, $\theta_B^* \subseteq f(\text{—},1)^* = g(0,\text{—})^*$ and $g(\text{—},1) = f(0,\text{—}) \subseteq \theta_A^*$.

In particular, if f is finite,

$$\bigvee_{p\in\mathbb{Q}} g(p,\text{—}) = \bigvee_{p\in\mathbb{Q}} f(\text{—},1-p) = 1 \quad\text{and}\quad \bigvee_{q\in\mathbb{Q}} g(\text{—},q) = \bigvee_{q\in\mathbb{Q}} f(1-q,\text{—}) = 1.$$

■

Proposition 5.3. *Both $\Subset_{\overline{M}}$ and $\Subset_M$ are Katětov relations in $\mathcal{C}(L)$.*

Proof: The conditions (K_1) and (K_2) are trivially verified.
(K_3): If $\theta_A \Subset_{\overline{M}} \theta_B$ and $\theta_{A'} \Subset_{\overline{M}} \theta_B$, then $\theta_A \subseteq f(0,\text{—})^*$, $f(\text{—},1) \subseteq \theta_B$, and $\theta_{A'} \subseteq g(0,\text{—})^*$, $g(\text{—},1) \subseteq \theta_B$ for some $f, g \in \overline{\mathsf{M}}(L)$; so $\theta_A \vee \theta_{A'} \subseteq (f\wedge g)(0,\text{—})^*$, $(f\wedge g)(\text{—},1) \subseteq \theta_B$.
(K_4): Similar to (K_3), but taking the supremum $f \vee g$ instead.
(K_5): Suppose that $\theta_A \Subset_{\overline{M}} \theta_B$. Then $\theta_A \subseteq f(0,\text{—})^* \subseteq f(\text{—},\frac{1}{2})$ and $f(\frac{1}{2},\text{—})^* \subseteq f(\text{—},1) \subseteq \theta_B$ for some $f \in \overline{\mathsf{M}}(L)$. Hence $\theta_A \Subset_M f(\text{—},\frac{1}{2}) \Subset_M \theta_B$.

To prove that $\Subset_M$ is also a Katětov relation, we just need to proceed similarly as the operations $\vee$ and $\wedge$ on $\overline{\mathsf{M}}(L)$ are closed for finite-valued functions. ■

Given a σ-frame L, we say that a relation $R \subseteq \mathcal{C}(L)$ is *separating* if $\theta_A R\, \theta_B$ implies the existence of $a, b \in L$ such that $\theta_A \subseteq \Delta_a \subseteq \theta_B$ and $\theta_A \subseteq \nabla_b \subseteq \theta_B$.

Proposition 5.4. *Let L be a σ-frame, φ a σ-scale in $\mathcal{C}(L)$ and R a separating relation on $\mathcal{C}(L)$ such that $\varphi(r) R\, \varphi(s)$ whenever $r < s$. Then the function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ generated by φ is measurable. In particular, if φ is finite, then f is finite-valued.*

Proof: Since R is separating and $\varphi(r) R\, \varphi(s)$ whenever $r < s$, there are $\nabla_{rs} \in \nabla[L]$ and $\Delta_{rs} \in \Delta[L]$ (for $r < s$) such that $\varphi(r) \subseteq \nabla_{rs} \subseteq \varphi(s)$ and $\varphi(r) \subseteq \Delta_{rs} \subseteq \varphi(s)$. Consequently, for any $p, q \in \mathbb{Q}$,

$$f(\text{—},q) = \bigvee_{r<q} \varphi(r) = \bigvee\{\nabla_{rs} \mid r < s < q\} \in \nabla[L]$$

$$\text{and}\quad f(p,\text{—}) = \bigvee_{s>p} \varphi(s)^* = \bigvee\{\Delta_{rs}^* \mid s > r > p\} \in \nabla[L].$$

■

Finally, we have the following insertion theorem for a general σ-frame.

Theorem 5.5. [Basic Insertion Theorem] *Given functions $g, h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ on a σ-frame L such that $g \leq h$, the following statements are equivalent:*

(i) *There exists a measurable function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $g \leq f \leq h$.*
(ii) *For each $p < q$, $h(p,\text{—})^* \Subset_{\overline{M}} g(\text{—},q)$.*
(iii) *There exist σ-scales φ_1 and φ_2 generating g and h, respectively, such that $\varphi_2(r) \Subset_{\overline{M}} \varphi_1(s)$ whenever $r < s$.*
(iv) *There exist σ-scales φ_1 and φ_2 generating g and h, respectively, and a separating Katětov relation R on $\mathcal{C}(L)$ such that $\varphi_2(r) R\, \varphi_1(s)$ whenever $r < s$.*

Proof: (i)⇒(ii): If there is a measurable function f such that $g \leq f \leq h$, then for each $p < q$, $h(p,\text{—})^* \subseteq f(p,\text{—})^*$ and $f(\text{—},q) \subseteq g(\text{—},q)$. Hence, $h(p,\text{—})^* \Subset_{\overline{M}} g(\text{—},q)$.
(ii)⇒(iii): Define $\varphi_1(r) \coloneqq g(\text{—},r)$ and $\varphi_2(r) \coloneqq h(r,\text{—})^*$ for all $r \in \mathbb{Q}$. Then φ_1 and φ_2 are σ-scales in $\mathcal{C}(L)$ generating g and h, respectively. Moreover, $\varphi_2(r) = h(r,\text{—})^* \Subset_{\overline{M}} g(\text{—},s) = \varphi_1(s)$ if $r < s$.
(iii)⇒(iv): Immediate, because $\Subset_{\overline{M}}$ is a separating Katětov relation.
(iv)⇒(i): By Katětov Lemma, there is a family $(\varphi(r))_{r\in\mathbb{Q}}$ in $\mathcal{C}(L)$ such that $\varphi_2(r) \subseteq \varphi(s)$, $\varphi(r) R\, \varphi(s)$ and $\varphi(r) \subseteq \varphi_1(s)$ whenever $r < s$. Because R is separating, if $r < s$, there exists a closed congruence ∇_{rs} such that $\varphi(r) \subseteq \nabla_{rs} \subseteq \varphi(s)$. Hence, $\varphi(r) \wedge \nabla_{rs}^* \subseteq \nabla_{rs} \wedge \nabla_{rs}^* = 0$ and $\nabla_{rs}^* \vee \varphi(s) \supseteq \nabla_{rs}^* \vee \nabla_{rs} = 1$. Therefore φ is also a σ-scale in $\mathcal{C}(L)$. The function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ generated by φ not only satisfies $g \leq f \leq h$ but is also measurable by the previous proposition. ■

Note that in case the functions are finite-valued, the obtained σ-scale φ in the proof above of (iv)⇒(i) is finite since

$$\bigvee_{r\in\mathbb{Q}} \bigvee_{s>r} \nabla_{rs}^* \supseteq \bigvee_{r\in\mathbb{Q}} \bigvee_{s>r} \varphi(s)^* = \bigvee_{s\in\mathbb{Q}} \varphi(s)^* \supseteq \bigvee_{s\in\mathbb{Q}} \varphi_1(s)^* = 1$$

and

$$\bigvee_{r\in\mathbb{Q}} \varphi(r) \supseteq \bigvee_{r\in\mathbb{Q}} \varphi_2(r) = 1.$$

As a consequence, we have a similar result for finite-valued functions:

Theorem 5.6. *Given functions $g, h\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ on a σ-frame L such that $g \leq h$, the following statements are equivalent:*

(i) *There exists a measurable function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $g \leq f \leq h$.*

(ii) *For each* $p < q$, $h(p, —)^* \Subset_M g(—, q)$.
(iii) *There exist finite* σ*-scales* φ_1 *and* φ_2 *generating* g *and* h*, respectively, such that* $\varphi_2(r) \Subset_M \varphi_1(s)$ *whenever* $r < s$.
(iv) *There exist finite* σ*-scales* φ_1 *and* φ_2 *generating* g *and* h*, respectively, and a separating Katětov relation* R *on* $\mathcal{C}(L)$ *such that* $\varphi_2(r) R\, \varphi_1(s)$ *whenever* $r < s$.

In particular, inserting a measurable function between two characteristic functions yields the following result:

Corollary 5.7. *Let* θ_A, θ_B *be complemented congruences on a* σ*-frame* L *such that* $\theta_A \subseteq \theta_B$. *There exists a measurable function* $f\colon \mathfrak{L}(\mathbb{R}) \to L$ *satisfying* $\chi_{\theta_A} \le f \le \chi_{\theta_B}$ *if and only if* $\theta_B^* \Subset_M \theta_A^*$.

Proof: It follows from Theorem 5.6, taking $g = \chi_{\theta_A}$ and $h = \chi_{\theta_B}$, and from the fact that $\chi_{\theta_B}(p, —)^* \Subset_M \chi_{\theta_A}(—, q)$ for each $p < q$ if and only if $\chi_{\theta_B}(p, —)^* \Subset_M \chi_{\theta_A}(—, q)$ for $0 \le p < q \le 1$, that is, $\theta_B^* \Subset_M \theta_A^*$. ■

Now, we want to extend a measurable function on a σ-sublocale $S \subseteq L$ to a measurable function on L. For that, consider the σ-frame homomorphism $q_S\colon \mathcal{C}(L) \to \mathcal{C}(S)$ given by $q_S(\theta) = \theta \vee \theta_S$.

Definition 5.8. Let $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(S)$ be a function on S. We say that a function $\widetilde{f}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathcal{C}(L)$ is an *extension of* f *over* L if $f = q_S \circ \widetilde{f}$.

Recalling that L is isomorphic to $\nabla[L]$ and $\nabla[S] = \{\nabla_a \vee \theta_S \colon a \in L\}$ is isomorphic to L/θ_S, the restriction of q_S to $\nabla[L]$ is precisely the quotient map $q_{S|L}\colon L \to L/\theta_S$. As a result, $\widetilde{f}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ is a measurable extension of $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to S$ if it is measurable and $f = q_{S|L} \circ \widetilde{f}$, that is,

$$f(p, —) = \widetilde{f}(p, —) \vee \theta_S \text{ and } f(—, q) = \widetilde{f}(—, q) \vee \theta_S \text{ for every } p, q \in \mathbb{Q}.$$

The Basic Insertion Theorem entails the following counterpart of Mrówka Extension Theorem for σ-frames:

Theorem 5.9. *Let* S *be a complemented* σ*-sublocale of a* σ*-locale* L *and let* $f\colon \mathfrak{L}(\mathbb{R}) \to S$ *be a measurable function such that* $\boldsymbol{0}_S \le f \le \boldsymbol{1}_S$. *The following statements are equivalent:*

(i) f *has a finite-valued measurable extension over* L.
(ii) *For each* $p < q$, $f(p, —)^{*_L} \Subset_M f(—, q)$ *in* $\mathcal{C}(L)$.

Proof: (i)$\Rightarrow$(ii): If f has a finite-valued measurable extension $\widetilde{f}$ over L, then $f(p,\text{—})^{*_L} \subseteq \widetilde{f}(p,\text{—})^{*_L}$ and $\widetilde{f}(\text{—},q) \subseteq f(\text{—},q)$ for every $p<q$.

(ii)$\Rightarrow$(i): Consider a measurable function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ satisfying $\mathbf{0}_S \le f \le \mathbf{1}_S$ and the corresponding functions $g \coloneqq g_f$ and $h \coloneqq h_f$ given by Example 3.11, that is,

$$h(p,\text{—}) = \begin{cases} 1 & \text{if } p<0 \\ f(p,\text{—}) & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1 \end{cases}, \quad h(\text{—},q) = \begin{cases} 0 & \text{if } q \le 0 \\ \theta_S^* \wedge f(\text{—},q) & \text{if } 0<q\le 1 \\ 1 & \text{if } q>1, \end{cases}$$

$$g(p,\text{—}) = \begin{cases} 1 & \text{if } p<0 \\ \theta_S^* \wedge f(p,\text{—}) & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1 \end{cases}, \quad g(\text{—},q) = \begin{cases} 0 & \text{if } q \le 0 \\ f(\text{—},q) & \text{if } 0<q\le 1 \\ 1 & \text{if } q>1. \end{cases}$$

We have that $\mathbf{0} \le g \le h \le \mathbf{1}$. Moreover, $h(p,\text{—})^* \Subset_M g(\text{—},q)$ in $\mathcal{C}(L)$ for every pair $p<q$. Indeed, if $p<0$ or $p \ge 1$, the relation is trivially verified through the function $\mathbf{0} \in \mathsf{M}(L)$. If $0 \le p < 1$, then there is some $r \in \mathbb{Q}$ such that $p<r<(1 \wedge q)$, and

$$h(p,\text{—})^* \subseteq h(\text{—},r) = \theta_S^* \wedge f(\text{—},r) \subseteq f(r,\text{—})^{*_L} \Subset_M f(\text{—},q) \subseteq g(\text{—},q).$$

Consequently, by Theorem 5.6, there exists a measurable function $\widetilde{f}\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $g \le \widetilde{f} \le h$. All that is left to show is that $f = q_S \circ \widetilde{f}$, that is, $f(\text{—},q) = \widetilde{f}(\text{—},q) \vee \theta_S$ for every $q \in \mathbb{Q}$.

If $q \le 0$, then $\widetilde{f}(\text{—},q) \subseteq g(\text{—},q) = 0$ and $f(\text{—},q) \subseteq \mathbf{0}_S(\text{—},q) = \theta_S$, thus, $f(\text{—},q) = \widetilde{f}(\text{—},q) \vee \theta_S$. If $q>1$, then $1 = h(\text{—},q) \subseteq \widetilde{f}(\text{—},q)$ and $1 = \mathbf{1}_S(\text{—},q) \subseteq f(\text{—},q)$, so $f(\text{—},q) = \widetilde{f}(\text{—},q) \vee \theta_S$. Finally, if $0<q\le 1$, then $\theta_S^* \wedge f(\text{—},q) = h(\text{—},q) \subseteq \widetilde{f}(\text{—},q) \subseteq g(\text{—},q) = f(\text{—},q)$, hence

$$\begin{aligned} f(\text{—},q) \vee \theta_S = (\theta_S^* \wedge f(\text{—},q)) \vee \theta_S &\subseteq \widetilde{f}(\text{—},q) \vee \theta_S \subseteq f(\text{—},q) \vee \theta_S \\ &\Rightarrow \widetilde{f}(\text{—},q) \vee \theta_S = f(\text{—},q) \vee \theta_S = f(\text{—},q). \end{aligned}$$ ∎

Corollary 5.10. *The following statements are equivalent for a complemented σ-sublocale S of a σ-locale L:*

(i) *Each measurable function $f\colon \mathfrak{L}(\mathbb{R}) \to S$ such that $\mathbf{0}_S \le f \le \mathbf{1}_S$ has a finite-valued measurable extension over L.*

(ii) *For each $\theta_A, \theta_B \in \mathcal{C}(S)$, $\theta_A \Subset_M \theta_B$ in $\mathcal{C}(S)$ implies $(\theta_A^{*_S})^{*_L} \Subset_M \theta_B$ in $\mathcal{C}(L)$.*

Proof: (i)⇒(ii): If $\theta_A \Subset_M \theta_B$ in $\mathcal{C}(S)$, then there exists $g \in \mathsf{M}(S)$ such that $\theta_A \subseteq g(0,—)^{*_S}$ and $g(—,1) \subseteq \theta_B$. Set $f := (\mathbf{0}_S \vee g) \wedge \mathbf{1}_S$. We have that $f \in \mathsf{M}(S)$ satisfies $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. Thus f has a finite-valued measurable extension $\widetilde{f}$ over L, and

$$\widetilde{f}(0,—) \subseteq f(0,—) = g(0,—) \subseteq \theta_A^{*_S}, \quad \widetilde{f}(—,1) \subseteq f(—,1) = g(—,1) \subseteq \theta_B.$$

Consequently, $(\theta_A^{*_S})^{*_L} \Subset_M \theta_B$ in $\mathcal{C}(L)$ through $\widetilde{f}$.

(ii)⇒(i): Let $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ be a measurable function on S such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. For each $p < q$, $f(p,—)^{*_S} \Subset_M f(—,q)$ in $\mathcal{C}(S)$. Hence, recalling that $f(p,—)$ is complemented in $\mathcal{C}(S)$, by (ii) we get that $f(p,—)^{*_L} = ((f(p,—)^{*_S})^{*_S})^{*_L} \Subset_M f(—,q)$ in $\mathcal{C}(L)$. Applying Theorem 5.9, we obtain the claimed. ■

Finally, our Basic Insertion Theorem also entails separation results for general σ-frames.

Proposition 5.11. *Given a σ-frame L and closed congruences $\nabla_a \subseteq \nabla_b$, there exists a measurable $f\colon \mathfrak{L}(\mathbb{R}) \to L$ satisfying $\chi_{\nabla_a} \leq f \leq \chi_{\nabla_b}$ if and only if $a \prec\!\prec b$.*

Proof: First of all, notice that analysing the inequalities $\chi_{\nabla_a}(p,—) \subseteq f(p,—) \subseteq \chi_{\nabla_b}(p,—)$ for each $p \in \mathbb{Q}$ yields

$$\chi_{\nabla_a} \leq f \Leftrightarrow \begin{cases} a \leq e_p & \text{if } 0 \leq p < 1 \\ e_p = 1 & \text{if } p < 0 \end{cases} \quad \text{and} \quad f \leq \chi_{\nabla_b} \Leftrightarrow \begin{cases} e_p \leq b & \text{if } 0 \leq p < 1 \\ e_p = 0 & \text{if } p \geq 1, \end{cases}$$

where $e_p := f(p,—)$. Thus, as $\varphi(r) = e_r$ is a finite descending σ-scale in L generating f, it is straightforward to check that there exists an $f \in \mathsf{M}(L)$ such that $\chi_{\nabla_a} \leq f \leq \chi_{\nabla_b}$ if and only if there exists a finite descending σ-scale $\varphi\colon \mathbb{Q} \to L$ such that

$$\begin{cases} \varphi(r) = 1 & \text{if } r < 0 \\ a \leq \varphi(r) \leq b & \text{if } 0 \leq r < 1 \\ \varphi(r) = 0 & \text{if } r \geq 1. \end{cases} \qquad (*)$$

Now, suppose that $a \prec\!\prec b$, that is, that there are $a_p \in L$, $p \in [0,1] \cap \mathbb{Q}$, such that $a_0 = a$, $a_1 = b$ and $a_p \prec a_q$ whenever $p < q$.

Take $\varphi\colon \mathbb{Q} \to L$ where $\varphi(r) = 1$ if $r < 0$, $\varphi(r) = a_{1-r}$ if $0 \leq r < 1$ and $\varphi(r) = 0$ otherwise. It is clear that $\bigvee_{r \in \mathbb{Q}} \varphi(r) = 1$. Moreover, as $\varphi(s) \prec \varphi(r)$ whenever $r < s$, there are $c_{sr} \in L$ such that $\varphi(s) \wedge c_{sr} = 0$ and $c_{sr} \vee \varphi(r) = 1$.

Since $c_{sr} = 1$ for $1 < r < s$ (because $\varphi(r) = 0$ and $c_{sr} = c_{sr} \vee \varphi(r) = 1$), we have $\bigvee_{s\in\mathbb{Q}} \bigvee_{r<s} c_{sr} = 1$. Hence, φ is a finite descending σ-scale in L satisfying $(*)$, and therefore the function f generated by φ is such that $\chi_{\nabla_a} \leq f \leq \chi_{\nabla_b}$.

Conversely, if there exists a measurable function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $\chi_{\nabla_a} \leq f \leq \chi_{\nabla_b}$, $\varphi\colon r \mapsto f(r, -)$ is a finite descending σ-scale in L satisfying $(*)$. Consider the elements $a_p \in L$, $p \in [0,1] \cap \mathbb{Q}$, where $a_0 = a$, $a_1 = b$ and $a_p = f(1-p, \text{—})$ otherwise. For $p < q$, $p, q \in (0,1) \cap \mathbb{Q}$, $a_p = f(1-p, \text{—}) \prec f(1-q, \text{—}) = a_q$ and $a_0 = a \leq a_p \prec a_q \leq b = a_1$. Hence $a_p \prec a_q$ whenever $p < q$, and thus $a \prec\!\prec b$. ■

Combining the previous proposition with Corollary 5.7 and Lemma 5.2 yields a similar result for open congruences.

Proposition 5.12. *Given a σ-frame L and open congruences $\Delta_a \subseteq \Delta_b$, there exists a measurable $f\colon \mathfrak{L}(\mathbb{R}) \to L$ satisfying $\chi_{\Delta_a} \leq f \leq \chi_{\Delta_b}$ if and only if $b \prec\!\prec a$.*

6. Insertion results for normal and extremally disconnected σ-frames

Recall that a σ-frame L is *normal* [8] if for all $a, b \in L$ such that $a \vee b = 1$, there are $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$; dually, we say that L is *extremally disconnected* if for all $a, b \in L$ such that $a \wedge b = 0$, there are $u, v \in L$ such that $u \vee v = 1$ and $a \wedge u = 0 = b \wedge v$.

We will now apply our Basic Insertion Theorem to characterise normality and extremally disconnectedness.

Let L be a σ-frame. Consider the relations $\Subset_N$ and $\Subset_D$ on $\mathcal{C}(L)$ given by

$$\begin{aligned} \theta_A \Subset_N \theta_B \;&\equiv\; \exists\, u, v \in L : \theta_A \subseteq \Delta_u \subseteq \nabla_v \subseteq \theta_B \\ \text{and } \theta_A \Subset_D \theta_B \;&\equiv\; \exists\, u, v \in L : \theta_A \subseteq \nabla_u \subseteq \Delta_v \subseteq \theta_B. \end{aligned}$$

When L is normal, the relation $\Subset_N$ is a separating Katětov relation on $\mathcal{C}(L)$ ([27], Corollary VIII.4.2.2). By complementation, if L is extremally disconnected, then the relation $\Subset_D$ is also a separating Katětov relation.

Lemma 6.1. *For any σ-frame L, $\Subset_M \subseteq \Subset_N \cap \Subset_D$.*

Proof: Suppose that $\theta_A \Subset_M \theta_B$, witnessed by a function $f \in \mathsf{M}(L)$. As $f(0, \text{—})^*, f(\tfrac{1}{2}, \text{—})^*$
$\in \Delta[L]$, $f(\text{—}, \tfrac{1}{2}), f(\text{—}, 1) \in \nabla[L]$ and

$$\theta_A \subseteq f(0, \text{—})^* \subseteq f(\text{—}, \tfrac{1}{2}) \subseteq f(\tfrac{1}{2}, \text{—})^* \subseteq f(\text{—}, 1) \subseteq \theta_B,$$

it is clear that $\theta_A \Subset_N \theta_B$ and $\theta_A \Subset_D \theta_B$. ∎

Theorem 6.2. *A σ-frame L is normal if and only if for any $g \in \overline{\mathsf{UM}}(L)$ and $h \in \overline{\mathsf{LM}}(L)$ such that $g \leq h$, there exists an $f \in \overline{\mathsf{M}}(L)$ such that $g \leq f \leq h$.*

Proof: Suppose that L is normal. Let $g \in \overline{\mathsf{UM}}(L)$ and $h \in \overline{\mathsf{LM}}(L)$ such that $g \leq h$, and define $\varphi_g(r) = g(\text{—}, r)$ and $\varphi_h(r) = h(r, \text{—})^*$ for all $r \in \mathbb{Q}$. Then φ_g and φ_f are σ-scales in $\mathcal{C}(L)$ generating g and h, respectively. Moreover, $g \leq h$ implies

$$\varphi_h(r) = h(r, \text{—})^* \subseteq g(r, \text{—})^* \subseteq g(\text{—}, s) = \varphi_g(s) \text{ for } r < s.$$

Hence, since $h(r, \text{—})^* \in \Delta[L]$ and $g(\text{—}, s) \in \nabla[L]$, we have that $\varphi_h(r) \Subset_N \varphi_g(s)$ whenever $r < s$. Thus, as $\Subset_N$ is a separating Katětov relation (because L is normal), there exists $f \in \overline{\mathsf{M}}(L)$ such that $g \leq f \leq h$, by Theorem 5.5.

Conversely, consider $a, b \in L$ such that $a \vee b = 1$. Then $\chi_{\Delta_b} \leq \chi_{\nabla_a}$, with $\chi_{\Delta_b} \in \overline{\mathsf{UM}}(L)$ and $\chi_{\nabla_a} \in \overline{\mathsf{LM}}(L)$. Hence, there is a measurable function $f \in \overline{\mathsf{M}}(L)$ such that $\chi_{\Delta_b} \leq f \leq \chi_{\nabla_a}$; that is, $\Delta_a \Subset_M \nabla_b$ by Corollary 5.7. But as $\Subset_M \subseteq \Subset_D$, there exist $u, v \in L$ such that $\Delta_a \subseteq \nabla_u \subseteq \Delta_v \subseteq \nabla_b$, which means that $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. ∎

Theorem 6.3. *The following statements are equivalent for a σ-frame L.*

(i) *L is normal.*

(ii) (Normal Insertion) *For any $g \in \mathsf{UM}(L)$ and $h \in \mathsf{LM}(L)$ such that $g \leq h$, there exists an $f \in \mathsf{M}(L)$ such that $g \leq f \leq h$.*

(iii) (Normal Separation) *For every $a, b \in L$, $a \vee b = 1$ implies that $\Delta_a \Subset_M \nabla_b$.*

(iv) (Normal Extension) *For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has a finite-valued measurable extension over L.*

Proof: (i)⇒(ii): By Theorem 6.2, there exists $f \in \overline{\mathsf{M}}(L)$ such that $g \leq f \leq h$. As g and h are finite-valued, f is also finite-valued.

(ii)⇒(iii): If Δ_a, Δ_b are disjoint, then $\chi_{\Delta_b} \leq \chi_{\nabla_a}$, with $\chi_{\Delta_b} \in \mathsf{UM}(L)$ and $\chi_{\nabla_a} \in \mathsf{LM}(L)$. Therefore, there exists $f \in \mathsf{M}(L)$ such that $\chi_{\Delta_b} \leq f \leq \chi_{\nabla_a}$; in other words, $\Delta_a \Subset_M \nabla_b$, by Corollary 5.7.

(iii)⇒(iv): Let S be a closed σ-sublocale of L, and consider an $f \in \mathsf{M}(S)$ such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. The fact that f is measurable and θ_S is closed implies that

$$f(p, \text{—}), f(\text{—}, q) \in \nabla[S] \subseteq \nabla[L]$$

for all $p, q \in \mathbb{Q}$. As a result, for each $p < q$, $f(p,\text{—})^{*_L}$ and $f(\text{—},q)^{*_L}$ are disjoint open congruences because

$$f(p,\text{—})^{*_L} \wedge f(\text{—},q)^{*_L} = (f(p,\text{—}) \vee f(\text{—},q))^{*_L} = 1^{*_L} = 0_L.$$

Thus $f(p,\text{—})^{*_L} \Subset_M f(\text{—},q)$; equivalently, f has a finite measurable extension over L (by Theorem 5.9).
(iv)⇒(i): Let $a, b \in L$ be such that $a \vee b = 1$. Set $\theta_S := \nabla_{a\wedge b}$, and consider the function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ defined in Example 3.12:

$$f(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ \nabla_a & \text{if } 0 \le p < 1 \\ \nabla_{a\wedge b} & \text{if } p \ge 1 \end{cases} \quad \text{and} \quad f(\text{—},q) = \begin{cases} \nabla_{a\wedge b} & \text{if } q \le 0 \\ \nabla_b & \text{if } 0 < q \le 1 \\ 1 & \text{if } q > 1. \end{cases}$$

Since $f \in \mathsf{M}(S)$ and it is such that $\mathbf{0}_S \le f \le \mathbf{1}_S$, there exists a measurable extension $\widetilde{f} \in \mathsf{M}(L)$ of f to L. Therefore, $\Delta_a \Subset_M \nabla_b$ through $\widetilde{f}$, but as $\Subset_M \subseteq \Subset_D$, there are $u, v \in L$ such that $\Delta_a \subseteq \nabla_u \subseteq \Delta_v \subseteq \nabla_b$. In other words, $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. ■

We point out that the Normal Separation result is equivalent to saying that for all $a, b \in L$ such that $a \vee b = 1$, there exists $f\colon \mathfrak{L}(\mathbb{R}) \to L$ satisfying $f((\text{—},0) \vee (1,\text{—})) = 0$, $f(0,\text{—}) \le a$ and $f(\text{—},1) \le b$ since we can choose $f \in \mathsf{M}(L)$ in a way that $\mathbf{0} \le f \le \mathbf{1}$.

Reformulating the previous proofs, we can obtain similar (almost "dual") results for extremally disconnected σ-frames, by replacing $\Subset_N$ with $\Subset_D$ and closed congruences with open congruences. In general, such reformulation is straightforward; but some exceptions arise in some details (arguments), which are described in the following results.

Theorem 6.4. *A σ-frame L is extremally disconnected if and only if for any $g \in \overline{\mathsf{LM}}(L)$ and $h \in \overline{\mathsf{UM}}(L)$ such that $g \le h$, there exists an $f \in \overline{\mathsf{M}}(L)$ such that $g \le f \le h$.*

Theorem 6.5. *The following statements are equivalent for a σ-frame L.*

(i) *L is extremally disconnected.*
(ii) (ED Insertion) *For any $g \in \mathsf{LM}(L)$ and $h \in \mathsf{UM}(L)$ such that $g \le h$, there exists an $f \in \mathsf{M}(L)$ such that $g \le f \le h$.*
(iii) (ED Separation) *For every $a, b \in L$, $a \wedge b = 0$ implies that $\nabla_a \Subset_M \Delta_b$.*

(iv) (ED Extension) *For each open σ-sublocale S of L, every $f \in \mathsf{M}(S)$ such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has a finite-valued measurable extension over L.*

Proof: We can easily show that (i)⇒(ii) and (ii)⇒(iii) using similar arguments to the ones used to prove Theorem 6.3. To prove that (iii)⇒(iv), take an open S and an $f \in \mathsf{M}(S)$ such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. Note that $f(p,\text{—})^{*_S}, f(\text{—},q)^{*_S} \in \Delta[L]$ for all $p,q \in \mathbb{Q}$. Hence, for each $p < q$, picking $r,s \in \mathbb{Q}$ such that $p < r < s < q$, we verify that $(f(s,\text{—})^{*_S})^{*_L}$, $(f(\text{—},r)^{*_S})^{*_L}$ are disjoint closed congruences, and therefore

$$f(p,\text{—})^{*_L} \subseteq (f(\text{—},r)^{*_S})^{*_L} \Subset_M f(s,\text{—})^{*_S} \subseteq f(\text{—},q).$$

The claim then follows from Theorem 5.9.

Finally, to show (iv)⇒(i), taking $a,b \in L$ such that $a \wedge b = 0$, set $\theta_S := \Delta_{a\vee b}$. Then $\nabla_b \Subset_M \Delta_a$ through the measurable extension of f over L, where $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ is the measurable function on S generated by the finite σ-scale in $\mathcal{C}(S)$ given by

$$\varphi(r) = \begin{cases} \Delta_{a\vee b} & \text{if } r \leq 0 \\ \nabla_b \vee \Delta_{a\vee b} & \text{if } 0 < r \leq 1 \\ 1 & \text{if } r > 1. \end{cases}$$

And because $\Subset_M \subseteq \Subset_N$, we obtain the claimed. ■

7. Insertion results for $\mathcal{G}$-perfect and $\mathcal{F}$-perfect σ-frames

Extending the notions of perfectness in [16] from locales to σ-locales, we say that a σ-locale L is *$\mathcal{F}$-perfect* [16] if each open σ-sublocale is a countable join of closed σ-sublocales, that is, for each $a \in L$ there is a sequence $(a_i)_{i\in\mathbb{N}} \subseteq L$ such that

$$\Delta_a = \bigwedge_{i\in\mathbb{N}} \nabla_{a_i}.$$

Similarly, L is *$\mathcal{G}$-perfect* if each closed σ-sublocale is a countable meet of open σ-sublocales, that is, for each $a \in L$ there is a sequence $(a_i)_{i\in\mathbb{N}} \subseteq L$ such that

$$\nabla_a = \bigvee_{i\in\mathbb{N}} \Delta_{a_i}.$$

$\mathcal{G}$-perfectness is, in general, stronger than $\mathcal{F}$-perfectness, but the equivalence happens in some cases, namely, when L is normal (see [16] for more details).

Proposition 7.1. *Let L be a σ-frame.*

(1) *If L is $\mathcal{G}$-perfect, then L is $\mathcal{F}$-perfect.*
(2) *If L is normal, then L is $\mathcal{F}$-perfect if and only if L is $\mathcal{G}$-perfect.*

We start this section by characterising $\mathcal{F}$-perfect σ-frames L in terms of an insertion result for upper measurable and lower measurable functions on L. The proof of the following lemma is a straightforward reformulation of the proof of its counterpart for frames in [17].

Lemma 7.2. *Given a σ-frame L, consider a function $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{C}(L)$ such that $\boldsymbol{0} \leq f$. If there exists an increasing sequence $(\Delta_n)_{n\in\mathbb{N}}$ in $\Delta[L]$ such that*

$$\Delta_n \subseteq f(\tfrac{1}{n}, \text{—}),$$

then there exists $u\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L)$ such that $\boldsymbol{0} \leq u \leq f$ and $u \in \mathsf{UM}(L)$.

Moreover,

(1) *If $\bigwedge_{n\in\mathbb{N}} \nabla_n \subseteq f(0,\text{—})^*$, then $u(0,\text{—})^* = f(0,\text{—})^* = (f-u)(0,\text{—})^*$.*
(2) *If $f(0,\text{—}) \subseteq \bigvee_{n\in\mathbb{N}} \Delta_n$, then $u(0,\text{—}) = f(0,\text{—}) = (f-u)(0,\text{—})$.*

Theorem 7.3. *A σ-frame L is $\mathcal{F}$-perfect if and only if for any $-u, l \in \overline{\mathsf{LM}}(L)$ such that l and $-u$ are sum compatible and $\boldsymbol{0} \leq l-u$, there exist $u_1 \in \mathsf{UM}(L)$ and $l_1 \in \mathsf{LM}(L)$ such that $\boldsymbol{0} \leq u_1 \leq l-u$, $u-l+u_1 \leq l_1 \leq \boldsymbol{0}$ and*

$$(l-u)(0,\text{—})^* = u_1(0,\text{—})^* = (-l_1)(0,\text{—})^*.$$

Proof: To prove the sufficient condition, for any $a \in L$, take $u := \mathbf{0}$ and $l := \chi_{\nabla_a}$. As l and $-u$ are sum compatible and $\mathbf{0} \leq l-u$, with $-u, l \in \mathsf{LM}(L)$, there exists a $u_1 \in \mathsf{UM}(L)$ satisfying $u_1(0,\text{—})^* = \chi_{\nabla_a}(0,\text{—})^*$, and

$$\Delta_a = \nabla_a^* = \chi_{\nabla_a}(0,\text{—})^* = u_1(0,\text{—})^* = \Big(\bigvee_{p>0} u_1(p,\text{—})\Big)^*$$
$$= \Big(\bigvee_{q>0} u_1(\text{—},q)^*\Big)^* = \bigwedge_{q>0} u_1(\text{—},q) = \bigwedge_{n\in\mathbb{N}} u_1(\text{—},\tfrac{1}{n}).$$

Hence, L is $\mathcal{F}$-perfect since $u_1(\text{—},\frac{1}{n})$ is closed for all $n \in \mathbb{N}$.

Conversely, since $l, -u \in \overline{\mathsf{LM}}(L)$, then $(l-u) \in \overline{\mathsf{LM}}(L)$ and $(l-u)(\frac{1}{i},\text{—})^* \in \Delta[L]$ for all $i \in \mathbb{N}$. Consequently, by the $\mathcal{F}$-perfectness of L, there exists a sequence $(\nabla_{ij})_{j\in\mathbb{N}}$ in $\nabla[L]$ such that

$$(l-u)(\tfrac{1}{i},\text{—})^* = \bigwedge_{j\in\mathbb{N}} \nabla_{ij}.$$

Setting $F_n \coloneqq \bigwedge_{i,j\leq n} \nabla_{ij}$, $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence of closed congruences such that

$$F_n \supseteq \bigwedge_{i\leq n}\bigwedge_{j\in\mathbb{N}} \nabla_{ij} = \bigwedge_{i\leq n}(l-u)(\tfrac{1}{i},\text{—})^* = \Big(\bigvee_{i\leq n}(l-u)(\tfrac{1}{i},\text{—})\Big)^* = (l-u)(\tfrac{1}{n},\text{—})^*,$$
$$\bigwedge_{n\in\mathbb{N}} F_n = \bigwedge_{i,j\in\mathbb{N}} \nabla_{ij} = \bigwedge_{i\in\mathbb{N}}(l-u)(\tfrac{1}{i},\text{—})^* = \Big(\bigvee_{i\in\mathbb{N}}(l-u)(\tfrac{1}{i},\text{—})\Big)^* = (l-u)(0,\text{—})^*.$$

Therefore, Lemma 7.2 implies the existence of some $u_1 \in \mathsf{UM}(L)$ such that $\mathbf{0} \leq u_1 \leq (l-u)$ and

$$(l-u)(0,\text{—})^* = u_1(0,\text{—})^* = (l-u-u_1)(0,\text{—})^*.$$

Now, since u_1 is finite-valued, $(l-u)$ and $-u_1$ are sum compatible and $\mathbf{0} \leq l-u-u_1$, with $l-u, -u_1 \in \overline{\mathsf{LM}}(L)$. So applying a similar argument to the inequality $\mathbf{0} \leq l-u-u_1$, we obtain a function $v_1 \in \mathsf{UM}(L)$ such that $\mathbf{0} \leq v_1 \leq l-u-u_1$ and

$$(l-u-u_1)(0,\text{—})^* = v_1(0,\text{—})^* = (l-u-u_1-v_1)(0,\text{—})^*.$$

Consider $l_1 \coloneqq -v_1 \in \mathsf{LM}(L)$. Then $u_1+u-l \leq l_1 \leq \mathbf{0}$ and $(l-u)(0,\text{—})^* = u_1(0,\text{—})^* = (-l_1)(0,\text{—})^*$. ■

We are now ready to characterise $\mathcal{F}$-perfectness via insertion, extension and separation conditions for semimeasurable functions.

Theorem 7.4. *The following statements are equivalent for a σ-frame L.*

(i) *L is $\mathcal{F}$-perfect.*

(ii) *For any $u \in \mathsf{UM}(L)$ and $l \in \mathsf{LM}(L)$ such that $u \leq l$, there exist $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $u \leq u' \leq l' \leq l$ and*

$$(u'-u)(0,\text{—})^* = (l-l')(0,\text{—})^* = (l-u)(0,\text{—})^*.$$

(iii) *For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ with $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ has an upper measurable extension $u'\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L)$ and a lower measurable extension $l'\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L)$ such that $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$ and*

$$\theta_S^* \vee u'(0,\text{—})^* = \theta_S^* \vee l'(\text{—},1)^* = \theta_S^*.$$

(iv) *For any* $a, b \in L$ *such that* $a \vee b = 1$, *there are* $u' \in \mathsf{UM}(L)$ *and* $l' \in \mathsf{LM}(L)$ *such that* $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$,

$$\begin{aligned} & u'(0,\text{—})^* \vee l'(\text{—},1)^* \subseteq \Delta_{a\wedge b}, \\ & \Delta_a \subseteq u'(p,\text{—}) \wedge l'(\text{—},q) \textit{ for all } p<0, q>0, \\ \textit{and } \ & \Delta_b \subseteq u'(p,\text{—}) \wedge l'(\text{—},q) \textit{ for all } p<1, q>1. \end{aligned}$$

Proof: (i)⇔(ii): This is a consequence of Theorem 7.3: if u, l are finite-valued, then all the involved functions are in $\mathsf{F}(L)$, where all the operations are compatible. So taking $u' = u_1 + u$ and $l' = l + l_1$, we obtain the claimed.

(ii)⇒(iii): Let $\theta_S \coloneqq \nabla_a$ for some $a \in L$, and pick a function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(S)$ measurable on S such that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. Recall Example 3.11, and consider the functions $u, l\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ given by $u = g_f$ and $l = h_f$:

$$l(p,\text{—}) = \begin{cases} 1 & \text{if } p<0 \\ f(p,\text{—}) & \text{if } 0 \leq p < 1 \\ 0 & \text{if } p \geq 1 \end{cases}, \quad l(\text{—},q) = \begin{cases} 0 & \text{if } q \leq 0 \\ \Delta_a \wedge f(\text{—},q) & \text{if } 0 < q \leq 1 \\ 1 & \text{if } q > 1, \end{cases}$$

$$u(p,\text{—}) = \begin{cases} 1 & \text{if } p<0 \\ \Delta_a \wedge f(p,\text{—}) & \text{if } 0 \leq p < 1 \\ 0 & \text{if } p \geq 1 \end{cases}, \quad u(\text{—},q) = \begin{cases} 0 & \text{if } q \leq 0 \\ f(\text{—},q) & \text{if } 0 < q \leq 1 \\ 1 & \text{if } q > 1. \end{cases}$$

Observe that $\mathbf{0} \leq u \leq l \leq \mathbf{1}$. Moreover, because f is measurable on S, we have that $l \in \mathsf{LM}(L)$ and $u \in \mathsf{UM}(L)$. Consequently, there exist $u', l' \in \mathsf{F}(L)$ such that

$$\begin{aligned} & \mathbf{0} \leq u \leq u' \leq l' \leq l \leq \mathbf{1}, \quad u' \in \mathsf{UM}(L), l' \in \mathsf{LM}(L), \\ \text{and } \ & (u'-u)(0,\text{—})^* = (l-l')(0,\text{—})^* = (l-u)(0,\text{—})^*. \end{aligned}$$

Now, because $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ and $u \leq u' \leq l$, u' is an upper measurable extension of f over L. Similarly, as $\mathbf{0}_S \leq f \leq \mathbf{1}_S$ and $u \leq l' \leq l$, l' is a lower measurable extension of f over L. Hence, all that is left to show is that $\Delta_a \vee u'(0,\text{—})^* = \Delta_a \vee l'(\text{—},1)^* = \Delta_a$. But

$$\begin{aligned} (l-u)(0,\text{—})^* &= \Big(\bigvee_{t \in \mathbb{Q}} l(t,\text{—}) \wedge u(\text{—},t) \Big)^* \\ &= \Big(\bigvee_{0<t<1} f(t,\text{—}) \wedge f(\text{—},t) \Big)^* = 0_S^* = (\nabla_a)^* = \Delta_a, \end{aligned}$$

and since $u' - u \leq u'$, $l - l' \leq 1 - l'$ and $(1 - l')(0, \text{—}) = l'(\text{—}, 1)$, we have

$$u'(0, \text{—})^* \subseteq (u' - u)(0, \text{—})^* = (l - u)(0, \text{—})^* = \Delta_a$$
$$\text{and} \quad l'(\text{—}, 1)^* = (1 - l')(0, \text{—})^* \subseteq (l - l')(0, \text{—})^* = (l - u)(0, \text{—})^* = \Delta_a.$$

(iii)⇒(iv): Suppose $a \vee b = 1$. Let $\theta_S := \nabla_{a \wedge b}$, and consider the function $f \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(S)$ defined in Example 3.12:

$$f(p, \text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ \nabla_a \vee \nabla_{a \wedge b} & \text{if } 0 \leq p < 1 \\ \nabla_{a \wedge b} & \text{if } p \geq 1 \end{cases}, \quad f(\text{—}, q) = \begin{cases} \nabla_{a \wedge b} & \text{if } q \leq 0 \\ \nabla_b \vee \nabla_{a \wedge b} & \text{if } 0 < q \leq 1 \\ 1 & \text{if } q > 1. \end{cases}$$

It is clear that $\mathbf{0}_S \leq f \leq \mathbf{1}_S$. The upper measurable and lower measurable extensions $u' \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L)$ and $l' \colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}(L)$ obtained by (iii) satisfy all the required conditions.

(iv)⇒(i): For any $a \in L$, $a \vee 1 = 1$. Thus, there exist $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $\mathbf{0} \leq u' \leq l' \leq \mathbf{1}$, $l'(\text{—}, 1)^* \vee u'(0, \text{—})^* \subseteq \Delta_a$ and $\Delta_a \subseteq u'(p, \text{—}) \wedge l'(\text{—}, q)$ for all $p < 1$, $q > 1$.

Since $\Delta_a \subseteq u'(p, \text{—}) \subseteq l'(p, \text{—})$ for $p < 1$, and

$$\Delta_a \supseteq l'(\text{—}, 1)^* = \big(\bigvee_{q<1} l'(\text{—}, q) \big)^* = \big(\bigvee_{p<1} l'(p, \text{—})^* \big)^* = \bigwedge_{0<p<1} l'(p, \text{—}),$$

we get that

$$\Delta_a = \bigwedge_{0<p<1} l'(p, \text{—}) = \bigwedge_{n \in \mathbb{N}} l'(1 - \tfrac{1}{n}, \text{—}). \qquad \blacksquare$$

Reformulating the previous proofs yields similar characterisations for $\mathcal{G}$-perfect σ-frames.

Theorem 7.5. *A σ-frame L is $\mathcal{G}$-perfect if and only if for any $-u, l \in \overline{\mathsf{LM}}(L)$ such that l and $-u$ are sum compatible and $\mathbf{0} \leq l - u$, there exist $u_1 \in \mathsf{UM}(L)$ and $l_1 \in \mathsf{LM}(L)$ such that $\mathbf{0} \leq u_1 \leq l - u$, $u - l + u_1 \leq l_1 \leq \mathbf{0}$ and*

$$(l - u)(0, \text{—}) = u_1(0, \text{—}) = (-l_1)(0, \text{—}).$$

Theorem 7.6. *The following statements are equivalent for a σ-frame L.*

(i) *L is $\mathcal{G}$-perfect.*

(ii) *For any $u \in \mathsf{UM}(L)$ and $l \in \mathsf{LM}(L)$ such that $u \leq l$, there exist $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $u \leq u' \leq l' \leq l$ and*

$$(u' - u)(0, \text{—}) = (l - l')(0, \text{—}) = (l - u)(0, \text{—}).$$

(iii) *For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ with $\boldsymbol{0}_S \leq f \leq \boldsymbol{1}_S$ has an upper measurable extension $u' \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ and a lower measurable extension $l' \colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ such that $\boldsymbol{0} \leq u' \leq l' \leq \boldsymbol{1}$ and*

$$\theta_S \wedge u'(0, \text{—}) = \theta_S \wedge l'(\text{—}, 1) = \theta_S.$$

(iv) *For any $a, b \in L$ such that $a \vee b = 1$, there are $u' \in \mathsf{UM}(L)$ and $l' \in \mathsf{LM}(L)$ such that $\boldsymbol{0} \leq u' \leq l' \leq \boldsymbol{1}$,*

$$\begin{aligned} &\nabla_{a\wedge b} \subseteq u'(0, \text{—}) \wedge l'(\text{—}, 1), \\ &\Delta_a \subseteq u'(p, \text{—}) \wedge l'(\text{—}, q) \textit{ for all } p < 0, q > 0, \\ \textit{and } \; &\Delta_b \subseteq u'(p, \text{—}) \wedge l'(\text{—}, q) \textit{ for all } p < 1, q > 1. \end{aligned}$$

Remark 7.7. We emphasise that although the study of insertion theorems for perfectness has already been done in [17] for frames, with proofs that also hold for σ-frames, the results above on separation and extension conditions are novel in the point-free setting.

8. Insertion theorem for perfectly normal σ-frames

In this final section, we combine Theorems 6.3 and 7.6 to characterise perfect normality of a σ-frame in terms of insertion, extension and separation conditions. We will get, in particular, a σ-frame version of the point-free Michael's insertion theorem [15].

Perfectly normal σ-frames were originally introduced by Charalambous [8] as the normal σ-frames L with the following property: for each $a \in L$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in L such that for any $b, c \in L$, $b \wedge a = c \wedge a$ if and only if $b \vee a_n = c \vee a_n$ for all $n \in \mathbb{N}$. Gilmour [11] showed that these σ-frames are precisely the *regular* σ-frames, that is, the σ-frames L such that each $a \in L$ can be written as $a = \bigvee_{n\in\mathbb{N}} a_n$, with $a_n \prec a$.

It is easy to check that perfectly normal σ-frames are exactly the normal σ-frames with the $\mathcal{F}$-perfectness property (or, equivalently, the $\mathcal{G}$-perfectness property), and this is the formulation that we adopt here as the definition of a perfectly normal σ-frame.

Theorem 8.1. *The following statements are equivalent for a σ-frame L.*

(i) *L is regular.*
(ii) *L is perfectly normal.*

(iii) *For any $u \in \mathsf{UM}(L)$ and $l \in \mathsf{LM}(L)$ such that $u \le l$, there exists an $f \in \mathsf{M}(L)$ such that $u \le f \le l$ and*

$$(f-u)(0,\text{—}) = (l-f)(0,\text{—}) = (l-u)(0,\text{—}).$$

(iv) *For each closed σ-sublocale S of L, every $f \in \mathsf{M}(S)$ with $\boldsymbol{0}_S \le f \le \boldsymbol{1}_S$ has a measurable extension $\widetilde{f}\colon \mathfrak{L}(\mathbb{R}) \to L$ such that*

$$\theta_S \subseteq \widetilde{f}(0,\text{—}) \wedge \widetilde{f}(\text{—},1).$$

(v) *For every $a, b \in L$ such that $a \vee b = 1$, there exists an $f \in \mathsf{M}(L)$ such that $\boldsymbol{0} \le f \le \boldsymbol{1}$,*

$$\Delta_b \subseteq f(p,\text{—}) \wedge f(\text{—},q) \text{ for all } p<1, q>1,$$
$$\Delta_a \subseteq f(p,\text{—}) \wedge f(\text{—},q) \text{ for all } p<0, q>0,$$
$$\text{and } \nabla_{a\wedge b} \subseteq f(0,\text{—}) \wedge f(\text{—},1).$$

Summing up, we have:

$\mathcal{F}$-perfectness	+ Normality	= Perfect normality
$\mathcal{F}$-perfect insertion	normal insertion	perfectly normal insertion
$\mathcal{F}$-perfect separation	normal insertion	perfectly normal separation
$\mathcal{F}$-perfect extension	normal insertion	perfectly normal extension

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Raquel Bernardes
CMUC, Department of Mathematics, University of Coimbra, 3000-143 Coimbra, Portugal
E-mail address: rbernardes@mat.uc.pt