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# ON EFFECTIVE DESCENT $\mathcal{V}$-FUNCTORS AND FAMILIAL DESCENT MORPHISMS

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ABSTRACT: We study effective descent $\mathcal{V}$-functors for cartesian monoidal categories $\mathcal{V}$ with finite limits. This study is carried out via the properties enjoyed by the 2-functor $\mathcal{V} \mapsto \mathsf{Fam}(\mathcal{V})$, results about effective descent of bilimits of categories, and the fact that the enrichment 2-functor preserves certain bilimits. Since these results rely on an understanding of (effective) descent morphisms in $\mathsf{Fam}(\mathcal{V})$, we briefly study those epimorphisms when $\mathcal{V}$ is a regular category.

KEYWORDS: effective descent morphisms, enriched category, pseudopullback, Grothendieck descent theory.
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## CONTENTS



# Introduction

Let $\mathcal{C}$ be a category with pullbacks. For each morphism $p\colon x \to y$, we have a change-of-base functor along $p$:

$$p^*\colon \mathcal{C}/y \to \mathcal{C}/x$$

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Via these functors, we are able to provide a description of the *basic bifibration* of $\mathcal{C}$. Thanks to the Bénabou-Roubaud theorem [3] (see also [17, page 258] or [25, Theorems 7.4 and 8.5] for generalisations), the *descent category* for $p$ with respect to the basic bifibration, denoted $\mathsf{Desc}(p)$, is equivalent to the Eilenberg-Moore category for the monad induced by the adjunction $p_! \dashv p^*$. This allows us to say that the morphism $p$ is *effective for descent* if the comparison functor $\mathcal{K}^p$ in the Eilenberg-Moore factorisation (1)

$$\begin{array}{ccc} \mathcal{C}/y & \xrightarrow{\mathcal{K}^p} & \mathsf{Desc}(p) \\ & {\scriptstyle p^*}\searrow \quad \swarrow{\scriptstyle \mathcal{U}^p} & \\ & \mathcal{C}/x & \end{array} \tag{1}$$

is an equivalence of categories; here, $\mathcal{U}^p$ is the functor which forgets descent data.

Janelidze-Galois theory [4] and Grothendieck descent theory [18, 25] feature the use of effective descent morphisms, requiring the knowledge of some (or all) such morphisms in the category of interest, and are the main motivation to undertake the study of finding sufficient conditions or even characterising effective descent; see [16, 17] for introductions to the subject.

As an example, if $\mathcal{C}$ is a locally cartesian closed category, or an exact category (in the sense of Barr [1]), the effective descent morphisms are precisely the regular epimorphisms. However, the characterisation of effective descent morphisms in a given category $\mathcal{C}$ is a notoriously difficult problem in general; for instance, see the characterisation in [29] and a subsequent reformulation [9] for the case $\mathcal{C} = \mathsf{Top}$.

Motivated by this reformulation, [7, 6, 8, 10] study this characterisation problem for more general notions of *spaces*: these works provide various results about effective descent in $(T, \mathcal{V})$*-categories* (originally defined in [12]). Due to their concerns with topological results, the study was restricted to the case in which the enriching category $\mathcal{V}$ is a quantale.

From the perspective of internal structures, we have the work of Le Creurer [21], in which he studies the problem of effective descent morphisms for essentially algebraic structures internal to a category $\mathcal{B}$ with finite limits. In particular, the author provides sufficient conditions for effective descent morphisms in $\mathcal{C} = \mathsf{Cat}(\mathcal{B})$, and confirms these conditions provide a complete characterisation with the added requirement that $\mathcal{B}$ is extensive and has a (regular epi, mono)-factorization system. Generalisations of these results to internal *multicategories* were studied in [28].

Making use of Le Creurer's results, Lucatelli Nunes, via his study on effective descent morphisms for bilimits of categories, provides sufficient conditions for effective descent morphisms in $\mathcal{C} = \mathcal{V}\text{-}\mathsf{Cat}$ via the following pseudopullback (see [25, Lemma 9.10, Theorem 9.11]):

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{-\cdot 1} & \mathsf{Cat}(\mathcal{V}) \\ {\scriptstyle \mathsf{ob}}\downarrow & & \downarrow{\scriptstyle (-)_0} \\ \mathsf{Set} & \xrightarrow[-\cdot 1]{} & \mathcal{V} \end{array} \tag{2}$$

for suitable extensive categories $\mathcal{V}$ with finite limits.

The central contribution of this paper is to extend [25, Theorem 9.11] to all categories $\mathcal{V}$ with finite limits, in Theorem 3.3. We highlight the use of the following three tools, used in the proof of Lemma 3.1, which are the skeleton of the argument: the properties of familial 2-functors, in particular, of the endo-2-functor $\mathsf{Fam}\colon \mathsf{CAT} \to \mathsf{CAT}$ studied in [32]; results about effective descent morphisms in bilimits of categories (see [25, Theorem 9.2 and Corollary 9.5]); and preservation of pseudopullbacks via enrichment (Theorem 2.1).

Since Theorem 3.3 relies on understanding (effective) descent morphisms in $\mathsf{Fam}(\mathcal{V})$, it naturally raises the problem of studying these classes of epimorphisms in the free coproduct completion of $\mathcal{V}$. Lemma 4.1 and Theorem 4.3 provide a couple of improvements, which we first illustrate in Theorem 4.7 for $\mathcal{V}$ a (co)complete Heyting lattice (a new proof direction of [7, Theorem 5.4]), and then we apply to obtain the more general Theorem 4.9, providing an improvement on Theorem 3.3 for regular categories $\mathcal{V}$.

In Section 1, we recall the notion of pseudopullback in the restricted context of the 2-categories $\mathsf{Cat}$ and $\mathsf{MndCat}$, we fix some terminology and notation for (strong) monoidal functors, used in the proofs of the results in Section 2, and we recall a couple of results from [25] and [32], restated in a convenient form, which are part of our toolkit in Section 3.

Section 2 is devoted to establishing some technical results on preservation of pseudopullbacks (Theorem 2.1), full faithfulness (Lemma 2.2) by the 2-functor $(-)\text{-}\mathsf{Cat}\colon \mathsf{MndCat} \to \mathsf{CAT}$, and preservation of descent morphisms by suitable functors (Lemma 2.3), which complete our toolkit.

As alluded to above, we establish our main result in Section 3; this is Theorem 3.3. We restate it here; if $\mathcal{V}$ is a cartesian monoidal category with finite limits, a $\mathcal{V}$-functor $F$ such that

– $F$ is an effective descent morphisms on hom-objects,
– $F$ is a descent morphism on composable pairs of hom-objects,
– $F$ is an almost descent morphism on composable triples of hom-objects,

is an effective descent morphism in $\mathcal{V}$-$\mathsf{Cat}$.

Indeed, these conditions on $F$ are statements about (effective) (almost) descent morphisms in $\mathsf{Fam}(\mathcal{V})$, leading us to studying such morphisms in the coproduct completion of $\mathcal{V}$. We devote Section 4 to provide tractable descriptions of these classes of epimorphisms, with an illustrative application to (co)complete Heyting lattices. We obtain Theorem 4.9, which refines Theorem 3.3 for regular categories, with further simplifications for infinitary coherent categories, exact categories or locally cartesian closed categories.

Finally, we have a couple of concluding remarks in Section 6, where we sketch some possible lines of future research, with regard to extending the result to all symmetric monoidal categories, or to generalized multicategories.

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## 1. Preliminaries

Let $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The *pseudopullback* of $F, G$, denoted by $\mathsf{PsPb}(F, G)$ may be succintly defined as *the full subcategory* of the comma category $(F \downarrow G)$ whose objects are *isomorphisms.* To be explicit, $\mathsf{PsPb}(F, G)$ has

– objects given by isomorphisms $\xi\colon Fc \cong Gd$, where $c \in \mathcal{C}$ and $d \in \mathcal{D}$,
– morphisms $(\zeta\colon Fa \to Gb) \to (\xi\colon Fc \to Gd)$ given by a pair of morphisms $f\colon a \to c$ and $g\colon b \to d$ such that $\xi \circ Ff = Gg \circ \zeta$.
– identities and composition given componentwise from $\mathcal{C}$ and $\mathcal{D}$.

Let $\mathcal{V}, \mathcal{W}$ be monoidal categories. We recall that a *monoidal functor* $F\colon \mathcal{V} \to \mathcal{W}$ consists of a functor $F$ between the underlying categories, together with

– an isomorphism $\mathsf{e}^F\colon I_{\mathcal{W}} \to FI_{\mathcal{V}}$,
– an isomorphism $\mathsf{m}^F\colon Fx \otimes Fy \to F(x \otimes y)$,

satisfying naturality and coherence conditions (see [2, page 1889]).

Moreover, we will denote the *unit and composition morphisms* of a $\mathcal{V}$-category $\mathcal{C}$ by $\mathsf{u}_\mathcal{C} \colon I \to \mathcal{C}(x,x)$ and $\mathsf{c}_\mathcal{C} \colon \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \to \mathcal{C}(x,z)$.

We denote $\mathsf{MndCat}$ to be the 2-category of monoidal categories, monoidal functors, and their natural transformations. We further highlight that since $\mathsf{MndCat} \to \mathsf{Cat}$ is pseudomonadic [22, Section 3.1], [24, Remark 4.3], it creates bilimits. In particular, the underlying category of the pseudopullback of a cospan of monoidal functors is the pseudopullback of the underlying ordinary functors, and fully faithful morphisms in $\mathsf{MndCat}$ (monoidal functors) are precisely those whose underlying functor is fully faithful in $\mathsf{Cat}$.

Finally, we recall the following results of [25] and [32], restated in a format suitable for our purposes:

**Proposition 1.1** ([25, Corollary 9.6]). *Suppose Diagram* (3) *below is a pseudopullback of categories with pullbacks and pullback-preserving functors*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle K} \\ \mathcal{C} & \xrightarrow[G]{} & \mathcal{D} \end{array} \tag{3}$$

*Let $f$ be a morphism in $\mathcal{A}$. If*

- *$Ff$ is effective for descent,*
- *$Hf$ is effective for descent,*
- *$KFf \cong GHf$ is a descent morphism,*

*then $f$ is effective for descent.*

**Proposition 1.2** ([32, Proposition 5.15]). *The canonical embeddings $\eta_\mathcal{V} \colon \mathcal{V} \to \mathsf{Fam}(\mathcal{V})$, where $\mathcal{V}$ is a category, form a 2-natural cartesian transformation, in the sense that for all functors $F \colon \mathcal{V} \to \mathcal{W}$, Diagram* (4) *below is a 2-pullback.*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\eta_\mathcal{V}} & \mathsf{Fam}(\mathcal{V}) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Fam}(F)} \\ \mathcal{W} & \xrightarrow[\eta_\mathcal{W}]{} & \mathsf{Fam}(\mathcal{W}) \end{array} \tag{4}$$

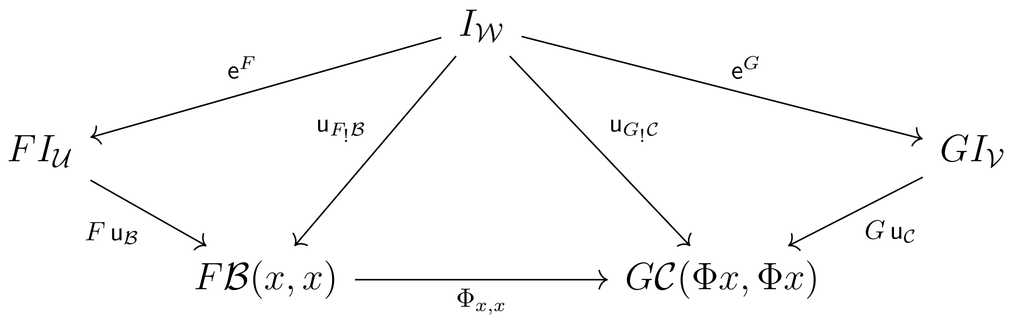
## 2. Preservation of bilimits and descent

**Theorem 2.1.** *The enrichment 2-functor* $(-)\text{-}\mathsf{Cat} \colon \mathsf{MndCat} \to \mathsf{CAT}$ *preserves pseudopullbacks.*

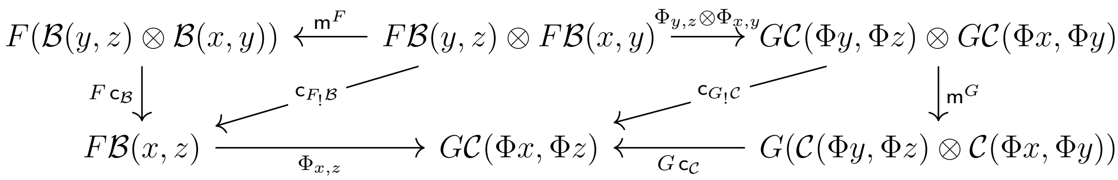
*Proof*: We desire to confirm that $\mathsf{PsPb}(F,G)\text{-}\mathsf{Cat} \simeq \mathsf{PsPb}(F_!, G_!)$; let $\Phi\colon F_!\mathcal{B} \cong G_!\mathcal{C}$ be an isomorphism of $\mathcal{W}$-categories, where $\mathcal{B}$ is a $\mathcal{U}$-category and $\mathcal{C}$ is a $\mathcal{V}$-category. We define a $\mathsf{PsPb}(F,G)$-category $\mathcal{D}_\Phi$ with

- set of objects given by $\mathsf{ob}\,\mathcal{D}_\Phi = \mathsf{ob}\,\mathcal{B}$,
- hom-object given by $\mathcal{D}_\Phi(x,y) = \Phi_{x,y}\colon F\mathcal{B}(x,y) \cong G\mathcal{C}(\Phi x, \Phi y)$ at $x, y \in \mathsf{ob}\,\mathcal{D}_\Phi$,
- unit object and composition given by the pairs $(\mathsf{u}_\mathcal{B}, \mathsf{u}_\mathcal{C})$, $(\mathsf{c}_\mathcal{B}, \mathsf{c}_\mathcal{C})$ of the respective unit objects and compositions from $\mathcal{B}$ and $\mathcal{C}$; these pairs are well-defined morphisms of $\mathsf{PsPb}(F,G)$, since $F, G$ are monoidal functors and $\Phi$ is a $\mathcal{W}$-functor.

To be more precise with this last point, note that the following diagrams commute:




(5)




(6)

Since identity and associativity laws of $\mathcal{D}_\Phi$ are precisely those of $\mathcal{B}$ and $\mathcal{C}$, it follows that $\mathcal{D}_\Phi$ is indeed well-defined.

The underlying $\mathcal{U}$-category of $\mathcal{D}_\Phi$ is $\mathcal{B}$ itself, while the its underlying $\mathcal{V}$-category is isomorphic to $\mathcal{C}$: it is given by $\mathsf{ob}\,\Phi$ on the sets of objects, and identity on the hom-objects.

Moreover, let $\mathcal{X}$, $\mathcal{Y}$ be $\mathsf{PsPb}(F,G)$-categories, and let $H\colon \mathcal{X}_\mathcal{U} \to \mathcal{Y}_\mathcal{U}$ be a $\mathcal{U}$-functor and $K\colon \mathcal{X}_\mathcal{V} \to \mathcal{Y}_\mathcal{V}$ be a $\mathcal{V}$-functor between the underlying $\mathcal{U}$-categories and $\mathcal{V}$-categories of $\mathcal{X}$ and $\mathcal{Y}$ respectively, such that $\mathsf{ob}\,H = \mathsf{ob}\,K$ and

$$GK_{x,y} \circ \mathcal{X}(x,y) = \mathcal{Y}(Hx, Hy) \circ FH_{x,y}. \tag{7}$$

Note that there exists a unique $\mathsf{PsPb}(F,G)$-functor $\Phi\colon \mathcal{X} \to \mathcal{Y}$ with underlying $\mathcal{U}$-functor and $\mathcal{V}$-functor given by $H$ and $K$, respectively. Indeed, let $\Phi\colon \mathcal{X} \to \mathcal{Y}$ be defined as follows:

– $\mathsf{ob}\,\Phi = \mathsf{ob}\,H$,
– $\Phi_{x,y}$ is given by the pair $H_{x,y}$, $K_{x,y}$, which is a morphism

$$(\mathcal{X}(x,y)\colon F\mathcal{X}_\mathcal{U}(x,y) \cong G\mathcal{X}_\mathcal{V}(x,y)) \to (\mathcal{Y}(Hx,Hy)\colon F\mathcal{Y}_\mathcal{U}(Hx,Hy) \cong G\mathcal{Y}_\mathcal{V}(Hx,Hy))$$

in $\mathsf{PsPb}(F,G)$, due to (7).

The laws that make $\Phi$ into a $\mathsf{PsPb}(F,G)$-functor are precisely given by the laws that make $H$ into a $\mathcal{U}$-functor and $K$ into a $\mathcal{V}$-functor.

If $\Psi\colon \mathcal{X} \to \mathcal{Y}$ is a $\mathsf{PsPb}(F,G)$-functor with $H$ as underlying $\mathcal{U}$-functor and $K$ as underlying $\mathcal{V}$-functor, we necessarily get $\Phi = \Psi$ by comparing their hom-morphisms. ■

**Lemma 2.2.** *The enrichment 2-functor* $(-)\text{-}\mathsf{Cat}\colon \mathsf{MndCat} \to \mathsf{CAT}$ *preserves fully faithful functors.*

*Proof*: Let $F\colon \mathcal{V} \to \mathcal{W}$ be a fully faithful monoidal functor. To prove

$$F_!\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{W}\text{-}\mathsf{Cat}$$

is fully faithful, let $\mathcal{C},\mathcal{D}$ be $\mathcal{V}$-categories, and let $\psi\colon F_!\mathcal{C} \to F_!\mathcal{D}$ be a $\mathcal{W}$-functor. It consists of the following data:

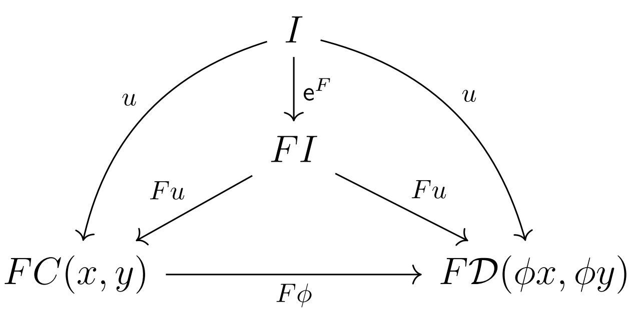
– A function $\psi\colon\ \mathsf{ob}\,\mathcal{C} \to \mathsf{ob}\,\mathcal{D}$,
– A morphism $\psi_{x,y}\colon F\mathcal{C}(x,y) \to F\mathcal{D}(\psi x, \psi y)$ for each pair $x, y \in \mathsf{ob}\,\mathcal{C}$.

Since $F$ is fully faithful, there exists a unique $\phi_{x.y}\colon \mathcal{C}(x,y) \to \mathcal{D}(\psi x, \psi y)$ such that $F\phi_{x,y} = \psi_{x,y}$.

With this, we define a $\mathcal{V}$-functor $\phi\colon \mathcal{C} \to \mathcal{D}$ given

– on objects by the function $\phi = \psi\colon\ \mathsf{ob}\,\mathcal{C} \to \mathsf{ob}\,\mathcal{D}$,
– on morphisms by $\phi_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(\phi x, \phi y)$ for each pair $x, y \in \mathsf{ob}\,\mathcal{C}$.

This is a $\mathcal{V}$-functor: note that the following diagrams commute




(8)

$$
\begin{array}{ccc}
F\mathcal{C}(y,z)\otimes F\mathcal{C}(x,y) & \xrightarrow{F\phi\otimes F\phi} & F\mathcal{D}(\phi y,\phi z)\otimes F\mathcal{D}(\phi x,\phi y) \\
{\scriptstyle \mathsf{m}^F}\downarrow & & \downarrow{\scriptstyle \mathsf{m}^F} \\
F(\mathcal{C}(y,z)\otimes \mathcal{C}(x,y)) & \xrightarrow{F(\phi\otimes\phi)} & F(\mathcal{D}(\phi y,\phi z)\otimes \mathcal{D}(\phi x,\phi y)) \\
{\scriptstyle Fc}\downarrow & & \downarrow{\scriptstyle Fc} \\
F\mathcal{C}(x,z) & \xrightarrow[F\phi]{} & F\mathcal{D}(\phi x,\phi z)
\end{array}
\tag{9}
$$

so, by fully faithfulness of $F$, plus invertibility of $\mathsf{e}^F$ and $\mathsf{m}^F$, we confirm that $\phi$ is a $\mathcal{V}$-functor. Moreover, by definition, it is the unique $\mathcal{V}$-functor such that $F_!\phi = \psi$, which concludes our proof. ■

**Lemma 2.3.** *Given a string of adjoint functors $L \dashv F \dashv R$ between categories with finite limits, if $L$ (and therefore $R$) is fully faithful, then $F$ preserves descent morphisms.*

Before providing the proof, we recall that descent morphisms in categories with finite limits are precisely the pullback-stable regular epimorphisms (see, for instance, [16, 21]).

*Proof*: Let $p\colon x \to y$ be a descent morphism. Since $F$ is a left adjoint, we may conclude that $Fp$ is a regular epimorphism; we just need to prove it is stable under pullback.

To do so, let $f\colon z \to Fy$ be a morphism, and we consider the following pullback diagram:

$$
\begin{array}{ccc}
f^*(Fx) & \xrightarrow{f^*(Fp)} & z \\
\downarrow & & \downarrow{\scriptstyle f} \\
Fx & \xrightarrow[Fp]{} & Fy
\end{array}
\tag{10}
$$

We wish to prove that $f^*(Fp)$ is a regular epimorphism. Indeed, note that $FLf^*(Fp) \cong f^*(Fp)$, and since $F$ reflects pullbacks (via $R$), we have a pullback

$$
\begin{array}{ccc}
Lf^*(Fx) & \xrightarrow{Lf^*(Fp)} & Lz \\
\downarrow & & \downarrow{\scriptstyle f^\sharp} \\
x & \xrightarrow[p]{} & y
\end{array}
\tag{11}
$$

so that $Lf^*(Fp) \cong (f^\sharp)^*(p)$ is a regular epimorphism; which is preserved by $F$, hence $f^*(Fp)$ must be a regular epimorphism, as desired. ■

**Remark 2.4.** We highlight one application of Lemma 2.3: for a category $\mathcal{B}$ with finite limits, the underlying object-of-objects functor

$$(-)_0 \colon \mathsf{Cat}(\mathcal{B}) \to \mathcal{B}$$

has fully faithful left and right adjoints: these assign to each object $b$ of $\mathcal{B}$ its respective *discrete* and *indiscrete* internal categories with $b$ as the underlying object of objects; see [15, 7.2.6]. Thus, we conclude that $(-)_0$ preserves descent morphisms.

This observation can be used to verify that $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathcal{V})$ reflects effective descent morphisms for extensive categories $\mathcal{V}$ with finite limits with $- \cdot 1 \colon \mathsf{Set} \to \mathcal{V}$ fully faithful, *without* requiring $\mathcal{V}$ to have a (regular epi, mono)-factorization system, using the same argument in the proof of [25, Theorem 9.11].

## 3. Descent for enriched categories

Throughout this section, fix a category $\mathcal{V}$ with finite limits, and consider the canonical embedding $\eta \colon \mathcal{V} \to \mathsf{Fam}(\mathcal{V})$, as defined in Proposition 1.2.

**Lemma 3.1.** *The functor* $\eta_! \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat}$ *reflects effective descent.*

*Proof*: By Proposition 1.2, Diagram (12) below

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\eta} & \mathsf{Fam}(\mathcal{V}) \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[\eta]{} & \mathsf{Set} \end{array} \tag{12}$$

is a 2-pullback. Since $\mathsf{Fam}(\mathcal{V}) \to \mathsf{Set}$ is an isofibration, it follows that Diagram (12) is a pseudopullback, by [30, Theorem 1]. It is preserved by $(-)\text{-}\mathsf{Cat}$, as shown in Theorem 2.1, so obtain the pseudopullback given in (13) below:

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{\eta_!} & \mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat} \\ {\scriptstyle \mathsf{ob}}\downarrow & & \downarrow \\ \mathsf{Set} & \xrightarrow[\eta_!]{} & \mathsf{Set}\text{-}\mathsf{Cat} \end{array} \tag{13}$$

To conclude the proof, note that since $(-)\text{-}\mathsf{Cat}$ is a 2-functor, it preserves adjoints, which, together with Lemma 2.2, guarantees that the functor

$$\mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Set}\text{-}\mathsf{Cat}$$

has fully faithful left and right adjoints, thus it preserves descent morphisms by Lemma 2.3. It is well-established that $\eta_! \colon \mathsf{Set} \to \mathsf{Set}\text{-}\mathsf{Cat}$ reflects descent morphisms, and descent morphisms in $\mathsf{Set}$ are effective for descent.

This places us under the conditions of Proposition 1.1, so the result follows. ■

**Lemma 3.2.** *The category* $\mathsf{Fam}(\mathcal{V})$ *is extensive with finite limits, and* $- \cdot 1 \colon \mathsf{Set} \to \mathsf{Fam}(\mathcal{V})$ *is fully faithful.*

*Proof*: We have already confirmed that $- \cdot 1 \colon \mathsf{Set} \to \mathsf{Fam}(\mathcal{V})$ is fully faithful in Remark 2.4, because $\mathcal{V}$ has a terminal object. Moreover, extensivity of $\mathsf{Fam}(\mathcal{V})$ is well-established; see, for instance, [5, Proposition 2.4].

Existence of finite limits is a direct corollary of [13, Theorem 4.2]; we consider the fibration $\mathsf{Fam}(\mathcal{V}) \to \mathsf{Set}$. The base category $\mathsf{Set}$ has all (finite) limits, the fiber at a set $X$ is $\mathcal{V}^X$, which has finite limits as well, and these are preserved by the inclusions $\mathcal{V}^X \to \mathsf{Fam}(\mathcal{V})$. See also [4, Sections 6.2, 6.3]. ■

Lemmas 3.1 and 3.2, together with [25, Theorem 9.11] and Remark 2.4, confirm that we have a string of functors which reflect effective descent morphisms:

$$\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat}(\mathsf{Fam}(\mathcal{V})) \tag{14}$$

Thus, a $\mathcal{V}$-functor is effective for descent in $\mathcal{V}\text{-}\mathsf{Cat}$, provided it satisfies suitable "surjectivity" conditions on tuples of composable morphisms. In particular, these concern stable (regular) epimorphisms and effective descent morphisms.

We note that the aforementioned classes $\mathcal{E}$ of epimorphisms are closed under coproducts and stable under pullback. Thus, to verify membership of a given morphism $(f, \phi) \colon (X_i)_{i \in I} \to (Y_j)_{j \in J}$ in $\mathcal{E}$, it is necessary and sufficient to verify that $\phi = (!, \phi) \colon (X_i)_{i \in f^* j} \to Y_j$, is in $\mathcal{E}$ for all $j$. Hence, we restrict our attention to morphisms with $J \cong 1$.

**Theorem 3.3.** *Let* $F \colon \mathcal{C} \to \mathcal{D}$ *be a* $\mathcal{V}$*-functor for* $\mathcal{V}$ *finitely complete. If*

(I) $F$ *induces an effective descent morphism on hom-objects; that is,*
$$F \colon (\mathcal{C}(x_0, x_1))_{x_i \in F^* y_i} \to \mathcal{D}(y_0, y_1)$$
*is effective for descent for all pairs* $y_0, y_1$ *of objects in* $\mathcal{D}$*,*

(II) $F$ *induces a descent morphism on pairs of composable hom-objects; that is, the family*
$$F \times F \colon (\mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1))_{x_i \in F^* y_i} \to \mathcal{D}(y_1, y_2) \times \mathcal{D}(y_0, y_1)$$
*is a descent morphism for all triples* $y_0, y_1, y_2$ *of objects in* $\mathcal{D}$*,*

(III) *$F$ induces an almost descent morphism on triples of hom-objects; that is,*

$$F \times F \times F \colon (\mathcal{C}(x_2, x_3) \times \mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1))_{x_i \in F^* y_i} \\ \to \mathcal{D}(y_2, y_3) \times \mathcal{D}(y_1, y_2) \times \mathcal{D}(y_0, y_1)$$

*is an almost descent morphism for all quadruples $y_0, y_1, y_2, y_3$ of objects in $\mathcal{D}$,*

*then $F$ is effective for descent.*

*Proof*: Let $F$ be a $\mathcal{V}$-functor, and write $\tilde{F}$ for the value of $F$ via the composite (14). One finds that, via explicit calculation, that $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ are precisely the coproducts of the morphisms in $\mathsf{Fam}(\mathcal{V})$ given by (I), (II) and (III).

By [21, Corollary 3.3.1], plus the initial remarks of [28, Section 5], the above hypotheses on $F$ guarantee that $\tilde{F}$ is an effective descent morphism in $\mathsf{Cat}(\mathsf{Fam}(\mathcal{V}))$, and hence so is $F$, first via [25, Theorem 9.11] (together with Remark 2.4 and Lemma 3.2) and then Lemma 3.1. ■

# 4. Familial descent morphisms

Theorem 3.3 raises the question of understanding (stable) regular epimorphisms and effective descent morphisms in $\mathsf{Fam}(\mathcal{V})$ for a category $\mathcal{V}$ with finite limits, with the goal of providing more tractable methods to verify conditions (I), (II), (III).

The key ideas for many of the applications are given in the next couple of lemmas. We begin by noting that the kernel pair of a morphism $\phi \colon (X_i)_{i \in I} \to Y$ in $\mathsf{Fam}(\mathcal{V})$ is calculated by considering the pullback

$$\begin{array}{ccc} X_i \times_Y X_j & \xrightarrow{\pi^0_{i,j}} & X_j \\ {\scriptstyle \pi^1_{i,j}}\downarrow & & \downarrow{\scriptstyle \phi_j} \\ X_i & \xrightarrow[\phi_i]{} & Y \end{array} \tag{15}$$

for each $i, j \in I$. Then, the kernel pair of $\phi$, denoted by $\mathsf{ker}\,\phi$, is given by

$$(X_i \times_Y X_j)_{i,j \in I \times I} \underset{(p_0, \pi^0)}{\overset{(p_1, \pi^1)}{\rightrightarrows}} (X_i)_{i \in I}$$

where $p_n \colon I \times I \to I$ for $n = 0, 1$ is the projection which forgets the $n$th component.

**Lemma 4.1.** *Let* $\phi\colon (X_i)_{i\in I} \to Y$ *be a morphism in* $\mathsf{Fam}(\mathcal{V})$. *We consider the diagram* $D_\phi\colon \mathcal{J}_I \to \mathcal{V}$ *where*

- $\mathsf{ob}\,\mathcal{J}_I = I \times I + I$,
- *for each pair* $i, j \in I$, *we have two arrows* $(i,j) \to i$ *and* $(i,j) \to j$,
- *the values of* $D_\phi$ *at* $(i,j) \to i$ *and* $(i,j) \to j$ *are defined to be* $\pi^1_{i,j}$ *and* $\pi^0_{i,j}$, *respectively.*

$\phi$ *is a (stable) regular epimorphism if and only if* $D_\phi$ *has a (stable) colimit and* $\mathsf{colim}\, D_\phi \cong Y$.

*Proof*: We begin by recalling that a morphism in a category with finite limits is a regular epimorphism if and only if it is the coequalizer of its kernel pair.

The fibration $\mathsf{Fam}(\mathcal{V}) \to \mathsf{Set}$ is a left adjoint functor, hence preserves colimits. In particular, if $\phi\colon (X_i)_{i\in I} \to Y$ is a regular epimorphism, then

$$I \times I \underset{p_0}{\overset{p_1}{\rightrightarrows}} I \longrightarrow 1 \tag{16}$$

must be a coequalizer, and this is the case only when $I$ is non-empty.

Note we have a natural isomorphism

$$\mathsf{Nat}(\ker\phi, \Delta_{(Z_k)_{k\in K}}) \cong \sum_{k\in K} \mathsf{Nat}(D_\phi, \Delta_{Z_k}),$$

which is fibered over $K$: an element from either is completely determined by an element $k \in K$ and a morphism $\omega\colon (X_i)_{i\in I} \to Z_k$ in $\mathsf{Fam}(\mathcal{V})$ satisfying $\omega_i \circ \pi^1_{i,j} = \omega_j \circ \pi^0_{i,j}$ for all $i, j \in I$. Given such an element, any morphism $(q, \psi)\colon (Z_k)_{k\in K} \to (W_l)_{l\in L}$ provides an element $qk \in L$ and a morphism $\psi_k \circ \omega\colon (X_i)_{i\in I} \to W_{qk}$ satisfying $\psi_k \circ \omega_i \circ \pi^1_{i,j} = \psi_k \circ \omega_j \circ \pi^0_{i,j}$ for all $i, j$.

Thus, if $\ker\phi$ has a colimit, its underlying set is necessarily a singleton by (16), so we denote it as an object $Q$ of $\mathcal{V}$. We have

$$\sum_{k\in K} \mathsf{Nat}(D_\phi, \Delta_{Z_k}) \cong \mathsf{Fam}(\mathcal{V})(Q, (Z_k)_{k\in K}) \cong \sum_{k\in K} \mathcal{V}(Q, Z_k),$$

and since this isomorphism is fibered over $K$, we conclude $Q$ is a colimit of $D_\phi$.

Conversely, if $Q$ is a colimit of $D_\phi$, then we have

$$\mathsf{Nat}(D_\phi, \Delta_{(Z_k)_{k\in K}}) \cong \sum_{k\in K} \mathcal{V}(Q, Z_k) \cong \mathsf{Fam}(\mathcal{V})(Q, (Z_k)_k)$$

which confirms $Q$ is a colimit of $\ker\phi$.

Regarding stability, we assume $\phi$ is a regular epimorphism. Given a morphism $\omega\colon Z \to Y$, the colimits of $\ker\omega^*(\phi)$ and $D_{\omega^*(\phi)}$ are isomorphic whenever either exist, so the stability of one colimit is the equivalent to the other. Taking coproducts in $\mathsf{Fam}(\mathcal{V})$, we confirm the same holds for any morphism $(Z_k)_{k\in K} \to Y$. ■

Understanding effective descent morphisms in $\mathsf{Fam}(\mathcal{V})$ is a more difficult problem, as is to be expected. However, we can reduce the study of the category of descent data of a morphism $\phi\colon (X_i)_{i\in I} \to Y$ to the full subcategory of *connected* descent data, an idea made precise by the following result.

**Lemma 4.2.** *Let $\phi\colon (X_i)_{i\in I} \to Y$ be a morphism in $\mathsf{Fam}(\mathcal{V})$, with $I$ non-empty. We have an equivalence $\mathsf{Desc}(\phi) \simeq \mathsf{Fam}(\mathsf{Desc}_{\mathsf{conn}}(\phi))$, where $\mathsf{Desc}_{\mathsf{conn}}(\phi)$ is the full subcategory of connected objects of $\mathsf{Desc}(\phi)$.*

*Proof*: Given descent data $(f,\gamma)$, $(h,\xi)$ as in the following diagram

$$\begin{array}{ccccc}
(W_k \times_Y X_j)_{k,j\in K\times I} & \underset{(p_0,\pi_0)}{\overset{(h,\xi)}{\rightrightarrows}} & (W_k)_{k\in K} & & \\
\downarrow & & \downarrow{\scriptstyle (f,\gamma)} & & \\
(X_i \times_Y X_j)_{i,j\in I\times I} & \underset{(p_0,\pi_0)}{\overset{(p_1,\pi_1)}{\rightrightarrows}} & (X_i)_{i\in I} & \xrightarrow[\phi]{} & Y
\end{array} \tag{17}$$

be descent data for $\phi$, from which we obtain descent data $(f,h)$ for the unique morphism $I \to 1$. Since $I$ is non-empty, this morphism is effective for descent, so that $K \cong J \times I$ for a set $J$, and we may take $f = p_0\colon J\times I \to I$ and $h = p_2\colon J\times I\times I \to J\times I$ to be projections (recall $p_n$ forgets the $n$th component).

Thus, taking the pullback of this descent data along $((j,-),\mathsf{id})\colon (W_{j,i})_{i\in I} \to (W_{j,i})_{j,i\in J\times I}$, we obtain the following descent data for $\phi$:

$$\begin{array}{ccccc}
(W_{j,i} \times_Y X_k)_{i,k\in I\times I} & \underset{(p_0,\pi_0)}{\overset{(p_1,\xi_{j,-,-})}{\rightrightarrows}} & (W_{j,i})_{i\in I} & & \\
\downarrow & & \downarrow{\scriptstyle (\mathsf{id},\gamma_{j,-})} & & \\
(X_i \times_Y X_k)_{i,k\in I\times I} & \underset{(p_0,\pi_0)}{\overset{(p_1,\pi_1)}{\rightrightarrows}} & (X_i)_{i\in I} & \xrightarrow[\phi]{} & Y
\end{array} \tag{18}$$

for each $j \in J$.

Now, we claim descent data of the form

$$\begin{array}{ccccc} (V_i \times_Y X_j)_{i,j\in I\times I} & \overset{(p_1,\zeta)}{\underset{(p_0,\pi_0)}{\rightrightarrows}} & (V_i)_{i\in I} & & \\ \downarrow & & \downarrow{\scriptstyle (f,\gamma)} & & \\ (X_i \times_Y X_j)_{i,j\in I\times I} & \overset{(p_1,\pi_1)}{\underset{(p_0,\pi_0)}{\rightrightarrows}} & (X_i)_{i\in I} & \xrightarrow[\phi]{} & Y \end{array} \tag{19}$$

is connected in $\mathsf{Desc}(\phi)$. More concretely, we wish to prove that any morphism of descent data $(q,\chi)\colon (V_i)_{i\in I} \to (W_{j,i})_{j,i\in J\times I}$ factors through $((j,-),\mathsf{id})$ for some $j \in J$. This gives a morphism of descent data $q\colon (\mathsf{id}, p_1) \to (p_0, p_2)$ for the unique morphism $I \to 1$. Note that $q$ is uniquely determined by a function $j\colon 1 \to J$, whose value provides the desired factorization.

Having verified all descent data is a coproduct of connected descent data, the result follows. ■

**Theorem 4.3.** *Let $\phi\colon (X_i)_{i\in I} \to Y$ be a morphism in $\mathsf{Fam}(\mathcal{V})$. Then the following are equivalent:*

*(a) $\phi$ is effective for descent.*

*(b) We have an equivalence $\mathcal{V}/Y \simeq \mathsf{Desc}_{\mathsf{conn}}(\phi)$.*

*Proof*: Note that the full subcategory of connected objects of $\mathsf{Fam}(\mathcal{V})/Y$ is precisely $\mathcal{V}/Y$, and any object on $\mathsf{Fam}(\mathcal{V})/Y$ is a coproduct of such connected objects. Thus, $\mathsf{Fam}(\mathcal{V})/Y \simeq \mathsf{Fam}(\mathcal{V}/Y)$, and we may conclude (b) $\implies$ (a), since

$$\mathsf{Fam}(\mathcal{V})/Y \simeq \mathsf{Fam}(\mathcal{V}/Y) \simeq \mathsf{Fam}(\mathsf{Desc}_{\mathsf{conn}}(\phi)) \simeq \mathsf{Desc}(\phi)$$

The converse lies on the fact that the comparsion $\mathcal{K}^\phi\colon \mathsf{Fam}(\mathcal{V})/Y \to \mathsf{Desc}(\phi)$ is of the form $\mathsf{Fam}(\mathcal{K}^\phi_{\mathsf{conn}})$ for a functor $\mathcal{K}^\phi_{\mathsf{conn}}\colon \mathcal{V}/Y \to \mathsf{Desc}_{\mathsf{conn}}(\phi)$. Since $\mathsf{Fam}$ reflects equivalences (because the 2-natural embedding $\mathcal{C} \to \mathsf{Fam}(\mathcal{C})$ is 2-cartesian), we conclude (a) $\implies$ (b). ■

**Frames:** Effective descent morphisms in $\mathcal{V}$-$\mathsf{Cat}$ were studied in [7, Section 5], for Heyting lattices $\mathcal{V}$. As an illustration of our tools, we provide a second proof that *-quotient morphisms in $\mathcal{V}$-$\mathsf{Cat}$ (that is, surjective on objects $\mathcal{V}$-functors that satisfy condition (21) below for all $y_0, y_1, y_2$) are effective for descent when $\mathcal{V}$ is a (co)complete Heyting lattice.

Let $\mathcal{V}$ be a thin category (ordered set). A morphism $(X_i)_{i\in I} \to Y$ in $\mathsf{Fam}(\mathcal{V})$ is simply the assertion "for all $i \in I$, $X_i \leqslant Y$". Thus, we simply write $(X_i)_{i\in I} \leqslant Y$ in this context.

**Lemma 4.4.** *Let $(X_i)_{i\in I} \leqslant Y$ be a morphism in $\mathsf{Fam}(\mathcal{V})$.*

- *It is an epimorphism if and only if $I$ is non-empty.*
- *If it is an epimorphism, it is also stable.*
- *It is a regular epimorphism if and only if $\bigvee_{i\in I} X_i \cong Y$.*
- *If it is a regular epimorphism, it is stable if and only if the above join is distributive, that is,*

$$Z \wedge \bigvee_{i\in I} X_i \cong \bigvee_{i\in I} Z \wedge X_i \tag{20}$$

*for all $Z \leqslant Y$.*

*Proof*: Note that $(X_i)_{i\in I} \leqslant Y$ is an epimorphism if and only if the underlying function $I \to 1$ is surjective, and this is the case exactly when $I$ be non-empty.

So, if $I$ is non-empty, we confirm $(X_i)_{i\in I} \leqslant Y$ is a stable epimorphism: given $Z \leqslant Y$ we can produce an epimorphism $(Z \wedge X_i)_{i\in I} \leqslant Z$, since $\mathcal{V}$ has meets. By taking coproducts, the same holds for all $(Z_j)_{j\in J} \leqslant Y$.

We immediately deduce from Lemma 4.1, that $(X_i)_{i\in I} \leqslant Y$ is a regular epimorphism if and only if $\bigvee_{i\in I} X_i \cong Y$, and stability under pullbacks is exactly the condition (20), so there's nothing to verify. ■

We say a thin category $\mathcal{V}$ is a *Heyting semilattice* (also known as *implicative semilattices* [27] and *Brouwerian semilattices* [19]) if it has finite limits (has meets and is bounded) and is cartesian closed (has implication). In particular, this means that $a \wedge -$ is a left adjoint functor for each $a \in \mathcal{V}$, which must preserve colimits (joins). As a corollary, we conclude that:

**Corollary 4.5.** *If $\mathcal{V}$ is a Heyting semi-lattice, regular epimorphisms in $\mathsf{Fam}(\mathcal{V})$ are stable.*

*Proof*: Condition (20) is automatically satisfied, by the previous remark. ■

**Corollary 4.6.** *If $\mathcal{V}$ is a (co)complete Heyting (semi-)lattice, then regular epimorphisms in $\mathsf{Fam}(\mathcal{V})$ are effective for descent.*

*Proof*: Let $(X_i)_{i\in I} \leqslant Y$ be a regular epimorphism. Given connected descent data $\mathsf{id}\colon (W_i)_{i\in I} \to (X_i)_{i\in I}$, $\pi_2\colon (W_i \wedge X_j)_{i,j\in I\times I} \to (W_i)_{i\in I}$, we may define $Z = \bigvee_{i\in I} W_i$.

Note that it is enough to prove that $X_i \wedge Z \leqslant W_i$ for all $i \in I$. Indeed, by distributivity, we have

$$X_i \wedge Z \cong \bigvee_{j\in I} X_i \wedge W_j \leqslant W_i.$$

Now, Theorem 4.3 and [28, Corollary 2.3] complete our proof. ■

With this, we obtain one direction of [7, Theorem 5.4]:

**Theorem 4.7.** *Let $\mathcal{V}$ be a (co)complete Heyting (semi-)lattice, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor. If $F$ is surjective on objects and we have an isomorphism*

$$\bigvee_{x_i \in F^* y_i} \mathcal{C}(x_1, x_2) \wedge \mathcal{C}(x_0, x_1) \cong \mathcal{D}(y_1, y_2) \wedge \mathcal{D}(y_0, y_1) \tag{21}$$

*for all $y_0, y_1, y_2$, then $F$ is effective for descent.*

*Proof*: Due to Lemma 4.4, we may conclude that

- condition (III) is satisfied, since $F$ is surjective on objects, and
- condition (II) is given by (21), plus the stability of regular epimorphisms provided by Corollary 4.5.

Condition (I) remains to be verified. Taking $y_1 = y_2$ above, so that $\mathcal{D}(y_1, y_2) \cong 1$, we have

$$\mathcal{D}(y_0, y_1) \cong \bigvee_{x_i \in F^* y_i} \mathcal{C}(x_1, x_2) \wedge \mathcal{C}(x_0, x_1) \leqslant \bigvee_{x_i \in F^* y_1} \mathcal{C}(x_0, x_1),$$

and since we have $\mathcal{C}(x_0, x_1) \leqslant \mathcal{D}(y_0, y_1)$ for all $x_i \in F^* y_i$, we conclude that $(\mathcal{C}(x_0, x_1))_{x_i \in F^* y_i} \leqslant \mathcal{D}(y_0, y_1)$ is a regular epimorphism in $\mathsf{Fam}(\mathcal{V})$, and therefore is effective for descent by Corollary 4.6. ■

**Regular categories:** The ideas behind the previous results generalize to regular categories $\mathcal{V}$, via their (regular epi, mono)-factorization system, which allow us to reduce statements about epimorphisms $\phi\colon (X_i)_{i \in I} \to Y$ in $\mathsf{Fam}(\mathcal{V})$ to families of monomorphisms. The following result makes this precise:

**Lemma 4.8.** *Suppose $\mathcal{V}$ is a regular category, and let $\phi\colon (X_i)_{i \in I} \to Y$ be a morphism in $\mathsf{Fam}(\mathcal{V})$. For each $i \in I$, consider the following factorization*

$$X_i \xrightarrow{\pi_i} M_i \xrightarrow{\iota_i} Y \tag{22}$$

*where $\pi_i$ is a regular epimorphism and $\iota_i$ is a monomorphism for all $i \in I$.*

- *$\phi$ is a (stable) epimorphism if and only if $\iota$ is a (stable) epimorphism.*
- *$\phi$ is a (stable) regular epimorphism if and only if $\bigvee_{i \in I} M_i \cong Y$ (and the join is stable).*
- *If $\pi_i$ is an effective descent morphism in $\mathcal{V}$ for all $i \in I$, then $\phi$ is an effective descent morphism if and only if $\iota$ is an effective descent morphism.*

*Proof*: The factorizations (22) for each $i \in I$ give a factorization $\phi = \iota \circ (\mathsf{id}, \pi)$ in $\mathsf{Fam}(\mathcal{V})$. Note that $(\mathsf{id}, \pi)$ is a coproduct of stable regular epimorphisms, hence $\phi$ is a (stable) (regular) epimorphism if and only if $\iota$ is a (stable) (regular) epimorphism (see [16, Propositions 1.3, 1.5]).

Moreover, if $\pi_i$ is effective for descent for all $i \in I$, taking coproducts will guarantee $(\mathsf{id}, \pi)$ is effective for descent as well, meaning that $\phi$ is effective for descent if and only if $\iota$ is effective for descent (see [18, Section 4]).

Under this light, the results are immediate consequences of Lemma 4.1. ∎

**Theorem 4.9.** *Let $\mathcal{V}$ be a regular category, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor. We consider the following (regular epi, mono)-factorizations*

$$\mathcal{C}(x_0, x_1) \xrightarrow{P_{x_0,x_1}} M_{x_0,x_1} \xrightarrow{I_{x_0,x_1}} \mathcal{D}(Fx_0, Fx_1)$$

$$\mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1) \longrightarrow M_{x_0,x_1,x_2} \xrightarrow{I_{x_0,x_1,x_2}} \mathcal{D}(Fx_1, Fx_2) \times \mathcal{D}(Fx_0, Fx_1)$$

$$\begin{array}{l}\mathcal{C}(x_2, x_3) \times \mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1) \\ \qquad\qquad\downarrow \\ \qquad M_{x_0,x_1,x_2,x_3} \xrightarrow{I_{x_0,x_1,x_2,x_3}} \mathcal{D}(Fx_2, Fx_3) \times \mathcal{D}(Fx_1, Fx_2) \times \mathcal{D}(Fx_0, Fx_1)\end{array}$$

*of the hom-morphisms $F_{x_i,x_{i+1}}\colon \mathcal{C}(x_i, x_{i+1}) \to \mathcal{D}(Fx_i, Fx_{i+1})$ (and respective products), for each quadruple $x_0, x_1, x_2, x_3$ of objects. If*

- *(i) $P_{x_0,x_1}$ is an effective descent morphism for each pair of objects $x_0, x_1$,*
- *(ii) $\mathcal{V}/\mathcal{D}(y_0, y_1) \simeq \mathsf{Desc}_{\mathsf{conn}}(I)$; that is, $I\colon (M_{x_0,x_1})_{x_i \in F^*y_i} \to \mathcal{D}(y_0, y_1)$ is an effective descent morphism for all $y_0, y_1$,*
- *(iii) The join $\bigvee_{x_i \in F^*y_i} M_{x_0,x_1,x_2}$ exists, is stable and is isomorphic to $\mathcal{D}(y_1, y_2) \times \mathcal{D}(y_0, y_1)$.*
- *(iv) $I\colon (M_{x_0,x_1,x_2,x_3})_{x_i \in F^*y_i} \to \mathcal{D}(y_2, y_3) \times \mathcal{D}(y_1, y_2) \times \mathcal{D}(y_0, y_1)$ is an almost descent morphism for all $y_0, y_1, y_2, y_3$,*

*then $F$ is effective for descent.*

*Proof*: The goal is to verify that properties (I), (II) and (III) are satisfied, so we can apply Theorem 3.3. Indeed, due to Lemma 4.8, we have that

- (I) follows as a consequence of (i) and (ii),
- (II) is a consequence of (iii),
- (III) is a consequence of (iv),

so the result follows. ∎

In case $\mathcal{V}$ satisfies further properties, we can simplify the above list:

– If $\mathcal{V}$ is infinitary coherent (has stable, arbitrary unions of subobjects), then the join in (iii) exists and is stable; one only needs to verify if the isomorphism exists.
– If $\mathcal{V}$ is exact, or locally cartesian closed, then (i) is redundant, since regular epimorphisms are effective for descent.

## 5. Enrichment in cartesian monoidal categories

Theorem 3.3 extends Lucatelli's result [25] about effective descent $\mathcal{V}$-functors, leaving out the extensivity requirement. Thus, we restrict our examples to such non-extensive, cartesian monoidal categories $\mathcal{V}$ with finite limits (excluding examples such as $\mathcal{V} = \mathsf{Set}, \mathsf{Top}, \mathsf{Cat}$), dedicating this section to the study of such categories $\mathcal{V}$-$\mathsf{Cat}$.

**Thin categories:** Thin categories $\mathcal{V}$ with cartesian monoidal structures are (essentially) bounded meet-semilattices, which we have previously discussed in Section 4, as an illustrative example. We only briefly repeat here that the result for (co)complete Heyting lattices $\mathcal{V}$ admit a particularly nice description (Theorem 4.7), which was already provided in [7] using other techniques.

**Colax-pointed categories:** We consider the colax comma category $1//\mathsf{Cat}$. To be explicit, this has

– objects: pairs $(\mathcal{C}, c)$ where $\mathcal{C}$ is a category and $c \in \mathsf{ob}\,\mathcal{C}$.
– morphisms $(\mathcal{C}, c) \to (\mathcal{D}, d)$: pairs $(F, f)$ where $F$ is a functor and $f \colon Fc \to d$ is a morphism in $\mathcal{D}$.
– identity on $(\mathcal{C}, c)$: the pair $(\mathsf{id}, \mathsf{id})$.
– composite of $(F, f) \colon (\mathcal{C}, c) \to (\mathcal{D}, d)$ with $(G, g) \colon (\mathcal{D}, d) \to (\mathcal{E}, e)$: the pair $(G \circ F, g \circ Gf)$.

This is the category of strict algebras and colax morphisms for the 2-monad $1 + -$ on $\mathsf{Cat}$ (the dual and codual notion is present in [14, 31, 11]). Hence, by [20, Corollary 4.9], $1//\mathsf{Cat} \to \mathsf{Cat}$ creates products, hence $1//\mathsf{Cat}$ admits a cartesian monoidal structure.

However, $1//\mathsf{Cat}$ is not an extensive category, since it doesn't have an initial object. It doesn't even have coproducts for any pair of objects: let $(\mathcal{C}_1, c_1)$ and $(\mathcal{C}_2, c_2)$ be pointed categories, and we assume this pair has a coproduct $(\tilde{\mathcal{C}}, \tilde{c})$ in $1//\mathsf{Cat}$, with coprojections $(I_i, \iota_i) \colon (\mathcal{C}_i, c_i) \to (\tilde{\mathcal{C}}, \tilde{c})$ for $i = 1, 2$.

Let $F_i\colon \mathcal{C}_i \to \mathcal{D}$ be functors, and suppose we have morphisms $f_i\colon Fc_i \to d$ for $i = 1, 2$. These define morphisms $(F_i, f_i)\colon (\mathcal{C}_i, c_i) \to (\mathcal{D}, d)$ for $i = 1, 2$, so the universal property guarantees there exists a unique morphism $(G, g)\colon (\tilde{\mathcal{C}}, \tilde{c}) \to (\mathcal{D}, d)$ satisfying $GI_i = F_i$ and $f_i = g \circ G\iota_i$.

In fact, we can prove that $G\tilde{c} \cong Fc_1 + Fc_2$: if $h\colon G\tilde{c} \to d$ is such that $f_i = h \circ G\iota_i$ for $i = 1, 2$, then $(G, h) \circ (I_i, \iota_i) = (F_i, f_i)$, and by the universal property, $(G, g) = (G, h)$, hence $g = h$.

But there is no reason for $\mathcal{D}$ to have such a coproduct: consider the category $\mathcal{D}$ given by the following graph

$$\begin{array}{ccccc} & & x & & \\ & \overset{f_1}{\nearrow} & & \overset{f_2}{\nwarrow} & \\ d_1 & & & & d_2 \\ & \searrow & & \swarrow & \\ & & y & & \end{array}$$

and observe that the pair $d_1, d_2$ does not have a coproduct. Thus, we obtain the desired contradiction by letting $F_i\colon \mathcal{C}_i \to \mathcal{D}$ be the constant functor to $d_i$, for $i = 1, 2$.

A $(1 \Downarrow \mathsf{Cat})$-category is a 2-category $\mathbb{B}$ and

- for each $x, y$, an object $\mathsf{hom}(x, y) \in \mathbb{B}(x, y)$,
- for each $x$, a morphism $e_x\colon 1_x \to \mathsf{hom}(x, x)$,
- for each $x, y, z$, a morphism $m_{x,y,z}\colon \mathsf{hom}(y, z) \cdot \mathsf{hom}(x, y) \to \mathsf{hom}(x, z)$,
- the following diagrams commute for all $w, x, y, z$:

$$\begin{array}{ccc} \mathsf{hom}(x,y)\cdot 1_x & & \\ {\scriptstyle \mathsf{id}\cdot e_x}\downarrow & \diagdown\!\!\diagdown & \\ \mathsf{hom}(x,y)\cdot\mathsf{hom}(x,x) & \xrightarrow[m_{x,x,y}]{} & \mathsf{hom}(x,y) \end{array} \qquad \begin{array}{ccc} & & 1_y\cdot\mathsf{hom}(x,y) \\ & \diagup\!\!\diagup & \downarrow{\scriptstyle e_y\cdot\mathsf{id}} \\ \mathsf{hom}(x,y) & \xleftarrow[m_{x,y,y}]{} & \mathsf{hom}(y,y)\cdot\mathsf{hom}(x,y) \end{array}$$

$$\begin{array}{ccccc} & & p_{y,z}\cdot(p_{x,y}\cdot p_{w,x}) & \xrightarrow{\mathsf{id}\cdot m_{w,x,y}} & p_{y,z}\cdot p_{w,y} \\ & \diagup\!\!\diagup & & & \\ (p_{y,z}\cdot p_{x,y})\cdot p_{w,x} & & & & \downarrow{\scriptstyle m_{w,y,z}} \\ {\scriptstyle m_{x,y,z}\cdot\mathsf{id}}\downarrow & & & & \\ p_{x,z}\cdot p_{w,x} & & \xrightarrow[m_{w,x,z}]{} & & p_{w,z} \end{array}$$

which is a $\mathbb{B}$-enriched category on the same set of objects.

**Categories with zero object:** Let $\mathcal{V}$ be a category with a zero object, which we denote by 1. Such categories are usually not extensive: if the zero object were strict, we would have $\mathcal{V} \simeq 1$.

For a $\mathcal{V}$-category $\mathcal{C}$, we write $p_{x,y} \colon 1 \to \mathcal{C}(x,y)$ for the uniquely determined morphism. In particular, this implies $\mathsf{u}_x = p_{x,x}$ for all $x$, and

$$1 \xrightarrow{(p_{y,x}, p_{x,y})} \mathcal{C}(y,x) \times \mathcal{C}(x,y) \xrightarrow{\mathsf{c}_{x,y,x}} \mathcal{C}(x,x)$$

must also equal $\mathsf{u}_x$.

With this, we can confirm that all hom-objects must be isomorphic: the isomorphism is given by:

$$\mathcal{C}(x,y) \xrightarrow{p_{y,z} \times \mathsf{id} \times p_{w,x}} \mathcal{C}(y,z) \times \mathcal{C}(x,y) \times \mathcal{C}(w,x) \xrightarrow{\mathsf{c}_{w,y,z} \circ (\mathsf{id} \times \mathsf{c}_{w,x,y})} \mathcal{C}(w,z)$$

Thus, we conclude $\mathcal{V}$-$\mathsf{Cat}$ has objects the empty $\mathcal{V}$-category plus pairs (non-empty set, $\mathcal{V}$-monoid).

**Eckmann-Hilton:** Suppose $\mathcal{V}$ is the category of unital magmas. By the Eckmann-Hilton argument, a $\mathcal{V}$-monoid is precisely a commutative monoid. Since $\mathcal{V}$ has a zero object, we conclude $\mathcal{V}$-$\mathsf{Cat}$ essentially has objects the empty $\mathcal{V}$-category plus pairs (non-empty set, commutative monoid).

**Coextensive products:** We say a category $\mathcal{V}$ with finite limits

- has *codisjoint* products if $\mathcal{V}^{\mathsf{op}}$ has disjoint coproducts,
- has *a strict terminal object* if $\mathcal{V}^{\mathsf{op}}$ has a strict initial object,
- is *finitely coextensive* if $\mathcal{V}^{\mathsf{op}}$ is finitely extensive.

As expected, finitely coextensive categories $\mathcal{V}$ have codisjoint products and a strict terminal object. This is the case for the categories of commutative $R$-algebras for a ring $R$, as a class of examples.

We verify that categories $\mathcal{V}$ with codisjoint products and strict terminal object do not provide an interesting enriching base with the cartesian monoidal structure: we shall confirm that $\mathcal{V}$-$\mathsf{Cat} \simeq \mathsf{Set}$.

Let $\mathcal{C}$ be a $\mathcal{V}$-category. For each $x \in \mathsf{ob}\,\mathcal{C}$, the unit morphism $1 \to \mathcal{C}(x,x)$ is an isomorphism, and for each pair $x, y \in \mathsf{ob}\,\mathcal{C}$, the composition morphism $\mathcal{C}(x,y) \times \mathcal{C}(y,x) \to 1$ is uniquely determined. Thus, the associativity condition for $\mathcal{C}(x,y) \times \mathcal{C}(y,x) \times \mathcal{C}(x,y)$ translates to saying that the projections on the first and third component

$$\mathcal{C}(x,y) \times \mathcal{C}(y,x) \times \mathcal{C}(x,y) \underset{!\times \mathsf{id}}{\overset{\mathsf{id}\times !}{\rightrightarrows}} \mathcal{C}(x,y)$$

are equal. But since products are codisjoint, we must have $\mathcal{C}(x,y) \cong 1$, for all $x, y$.

**Categories of spaces:** Since most varieties of algebras $\mathcal{V}$ seem to have an uninteresting $\mathcal{V}$-$\mathsf{Cat}$ for the cartesian monoidal structure, we turn our attention to categories of spaces. Our results can be instanciated with $\mathcal{V} = \mathsf{CHaus}$ of compact Hausdorff spaces (which is a pretopos, and therefore exact, but not (infinitary) extensive) or $\mathcal{V} = \mathsf{Stn}$ of Stone spaces (which is regular [26]).

## 6. Future work

Having established sufficient conditions for effective descent in $\mathcal{V}$-$\mathsf{Cat}$ for cartesian monoidal categories $\mathcal{V}$, an obvious continuation would be to extend this result to suitable monoidal categories $\mathcal{V}$. We describe a strategy which would rely on the present work; we denote $\mathsf{CartCat}$ and $\mathsf{SymMndCat}$ for the 2-categories of cartesian (monoidal) categories and symmetrical monoidal categories.

Provided $\mathsf{CartCat}$ has needed (strict) codescent objects, the left 2-adjoint (biadjoint) (pseudo)functor of the forgetful 2-functor $\mathsf{CartCat} \to \mathsf{SymMndCat}$ exists; the existence and an explicit description of such a left 2-adjoint (biadjoint) would be provided via the biadjoint triangle theorem [23, Theorem 4.4] (see also [24, Theorem 2.3]):

$$\begin{array}{ccc} \mathsf{CartCat} & \longrightarrow & \mathsf{SymMndCat} \\ & \searrow \quad \swarrow & \\ & \mathsf{Cat} & \end{array}$$

where every 2-functor is forgetful. Both functors to $\mathsf{Cat}$ have left 2-adjoints which are easy to describe.

So, if the existence of the left biadjoint $F\colon \mathsf{SymMndCat} \to \mathsf{CartCat}$ is guaranteed, we need to study the following questions:

- What conditions on $\mathcal{V}$ guarantee existence of pullbacks in $F\mathcal{V}$?
- Is the unit $\eta\colon \mathcal{V} \to F\mathcal{V}$ fully faithful?

After obtaining solutions to the above questions, we could then study the functor

$$\eta_!\colon \mathcal{V}\text{-}\mathsf{Cat} \to F\mathcal{V}\text{-}\mathsf{Cat},$$

which raises the ultimate question: does it reflect effective descent morphisms? An affirmative answer would provide a string of functors

$$\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow F\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Fam}(F\mathcal{V})\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat}(\mathsf{Fam}(F\mathcal{V}))$$

that reflect effective descent. Then, since $F\mathcal{V}$ is hypothetically a cartesian monoidal category with finite limits, we obtain a more general result via Theorem 3.3. Combined with an adequate study of effective descent morphisms in $F\mathcal{V}$, these results can be applied in the study of effective descent morphisms in $\mathcal{V}$-$\mathsf{Cat}$ for any symmetrical monoidal category $\mathcal{V}$.

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