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STURM'S COMPARISON THEOREM FOR CLASSICAL DISCRETE ORTHOGONAL POLYNOMIALS

A. SUZUKI

ABSTRACT: In an earlier work [Castillo et al., J. Math. Phys., 61 (2020) 103505], it was established, from a hypergeometric-type difference equation, tractable sufficient conditions for the monotonicity with respect to a real parameter of zeros of classical discrete orthogonal polynomials on linear, quadratic, q-linear, and q-quadratic grids. In this work, we continue with the study of zeros of these polynomials by given a comparison theorem of Sturm type. As an application, we analyze in a simple way some relations between the zeros of certain classical discrete orthogonal polynomials.

KEYWORDS: Classical discrete orthogonal polynomials, discrete Sturm's theorem, difference equations, zeros, nodes.
MATH. SUBJECT CLASSIFICATION (2010): 30C15, 05A30, 33C45, 33D15.

1. Introduction

In a companion paper (see [2]), we give new tractable sufficient conditions for the monotonicity with respect to a real parameter of zeros of classical orthogonal polynomials (COP) on linear, quadratic, q-linear, and q-quadratic grids. In particular, we analyze in a simple and unified way the monotonicity of the zeros of Hahn, Charlier, Krawtchouk, Meixner, Racah, dual Hahn, q-Meixner, quantum q-Krawtchouk, q-Krawtchouk, affine q-Krawtchouk, q-Charlier, Al-Salam-Carlitz, q-Hahn, little q-Jacobi, little q-Laguerre/Wall, q-Bessel, q-Racah and dual q-Hahn polynomials. However, these results do not allow us to compare the zeros of the elements of two different sequences of COP. For this purpose we need a "comparison theorem" of Sturm type. For the linear grid, there is a wide variety of results in this direction, e.g., [1, 4, 5, 14]. However, for the general case, as far as we know, this problem has not yet been considered. The fundamental purpose of this note is to establish the first results in this direction. To achieve this objective, as in [2], our starting point is the hypergeometric-type difference equation introduced by Nikiforov and Uvarov introduced in [10, (5)]

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(see also [11, p. 127] and [9, p. 71]):

$$\widetilde{a}(x(s))\frac{\Delta}{\Delta x(s-1/2)}\left(\frac{\nabla y(x(s))}{\nabla x(s)}\right)+\widetilde{b}(x(s))\left(\frac{\Delta y(x(s))}{\Delta x(s)}+\frac{\nabla y(x(s))}{\nabla x(s)}\right)$$
$$+c\,y(x(s))=0,$$

or, equivalently,

$$a(s)\frac{\Delta}{\Delta x(s-1/2)}\left(\frac{\nabla y(x(s))}{\nabla x(s)}\right)+b(s)\frac{\Delta y(x(s))}{\Delta x(s)}+c\,y(x(s))=0, \tag{1.1}$$

where

$$a(s)=\widetilde{a}(x(s))-\frac{1}{2}\widetilde{b}(x(s))\Delta x(s-1/2), \qquad b(s)=\widetilde{b}(x(s)),$$

$x(s)$ defines class of grids with, generally nonuniform, step $\Delta x(s)=x(s+1)-x(s)$, $\nabla x(s)=x(s)-x(s-1)$, $\widetilde{a}(x(s))$ and $\widetilde{b}(x(s))$ are polynomials of degree at most 2 and 1 in x, respectively, and c is a constant. In what follows, we assume that x is a real-valued function defined on an interval of the real line. For similar purposes, in [2, (2.1)], we rewrite (1.1) in the following useful way:

$$A(s)y(x(s-1))+B(s)y(x(s+1))+C(s)y(x(s))=0, \tag{1.2}$$

where

$$A(s)=\frac{a(s)}{\nabla x(s)\Delta x(s-1/2)}, \quad B(s)=\frac{a(s)+b(s)\Delta x(s-1/2)}{\Delta x(s)\Delta x(s-1/2)},$$
$$C(s)=c-B(s)-A(s). \tag{1.3}$$

Fix $a\in\mathbb{R}$ and $N\in\{3,4,\dots\}\cup\{+\infty\}$. Set $s_i=a+i$ $(i=0,1,\dots,N-1)$, $S=\{s_0,s_1,\dots,s_{N-1}\}$ and $S'=S\setminus\{s_0,s_{N-1}\}$. Assume $A(s)\neq0$ and $B(s)\neq0$ for each $s\in S'$. Set $y=u\,v$ on S, v being the new unknown function and u so that v satisfy

$$\Delta\nabla v(x(s))+\lambda(s)\,v(x(s))=0 \tag{1.4}$$

on S. Direct calculation gives that

$$A(s)u(x(s-1))=B(s)u(x(s+1)) \tag{1.5}$$

for

$$u(x(s_k)) = \begin{cases} u(x(a)) \displaystyle\prod_{j=1}^{k/2} \frac{A(s_{2j-1})}{B(s_{2j-1})}, & k \text{ even}, \\ u(x(a+1)) \displaystyle\prod_{j=1}^{(k-1)/2} \frac{A(s_{2j})}{B(s_{2j})}, & k \text{ odd}, \end{cases} \tag{1.6}$$

with arbitrary initial condition $u(x(a)) \neq 0$ and $u(x(a+1)) \neq 0$. Hence, we can rewrite (1.2) as

$$v(x(s+1)) + v(x(s-1)) + G(s)v(x(s)) = 0, \tag{1.7}$$

with the initial conditions that $v(x(a)) \neq 0$ is arbitrarily chosen and

$$v(x(a+1)) = -\frac{C(a)}{B(a)} \frac{u(x(a))}{u(x(a+1))} v(x(a)),$$

where

$$G(s_k) = \begin{cases} \dfrac{u(x(a))}{u(x(a+1))} \dfrac{C(s_k)}{B(s_k)} \displaystyle\prod_{j=1}^{k/2} \frac{A(s_{2j-1})}{B(s_{2j-1})} \prod_{j=1}^{k/2} \frac{B(s_{2j})}{A(s_{2j})}, & k \text{ even}, \\ \dfrac{u(x(a+1))}{u(x(a))} \dfrac{C(s_k)}{B(s_k)} \displaystyle\prod_{j=1}^{(k-1)/2} \frac{A(s_{2j})}{B(s_{2j})} \prod_{j=1}^{(k+1)/2} \frac{B(s_{2j-1})}{A(s_{2j-1})}, & k \text{ odd}. \end{cases} \tag{1.8}$$

(Note that (1.7) can be transform in (1.4) taking $G(s) = \lambda(s) - 2$.)

Although our main motivation are the polynomial solutions of (1.1), the results presented in Section 3 are slightly more general. In Section 2 we give new sufficient conditions for the monotonicity with respect to a real parameter of nodes of COP. Finally, in Section 4 we apply our results to some families of COP.

2. Monotonicity theorem

From now on, we assume that x is a continuous strictly increasing function on an interval of the real line containing the discrete set of points S. To deal with a discrete analogue of Sturm's separation theorem, Hartman (see [6]) introduced the notion of generalized zeros: either an actual zero or where the solution changes its sign. Here we work with some specific generalized zeros, the familiar notion of "node" used by Porter [12] (see also [4, p. 131]). Of course, the interval $(x(s'-1), x(s')]$ ($s' \in S \setminus \{s_0\}$) contains a zero of a COP solution of (1.1) on S if and only if it contains a node (see Lemma 4.1).

Definition 2.1. *Let v be a solution of (1.7) on S. Assume that v changes its sign on the interval $(x(s'-1), x(s')]$ ($s' \in S \setminus \{s_0\}$). The point of intersection of the x-axis with the line segment with endpoints $(x(s'-1), v(x(s'-1)))$ and $(x(s'), v(x(s')))$ is called a node of v (see Figure 1).*

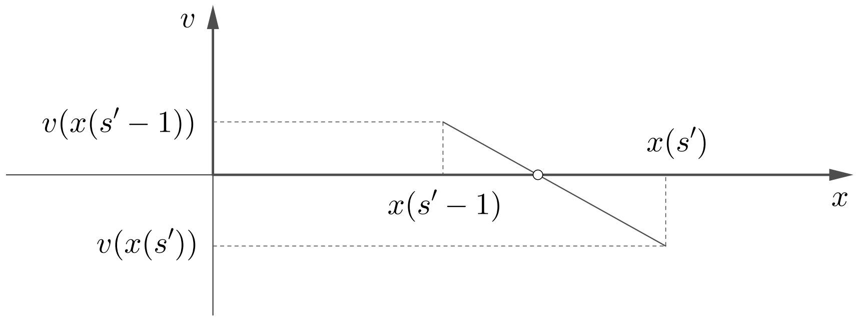

FIGURE 1. The node (white point) between $x(s'-1)$ and $x(s')$.

We assume that the functions A, B, C appearing in (1.2) depend on a parameter t varying in a non-degenerate open interval of the real line. The next theorem was proved by Porter (see [12]) for the linear grid $x(s) = s$.

Theorem 2.1. *For each $s \in S$, let $G(s,t)$ be a decreasing function of a real parameter t varying in a non-degenerate open interval of the real line. Assume that $v(\cdot, t)$ is a nonzero continuous function of t for each $s \in S$ and satisfies*

$$v(x(s+1), t) + v(x(s-1), t) + G(s,t)v(x(s), t) = 0. \tag{2.1}$$

Suppose also that $v(x(a+1), t)/v(x(a), t)$ is an increasing function of t and $v(x(a), t) \neq 0$ for all t. Then the nodes of $v(\cdot, t)$ are increasing functions of t.

Proof: Define $v_\epsilon(x(s), t) = v(x(s), t+\epsilon)$ for $\epsilon > 0$ sufficiently small. Hence,

$$v_\epsilon(x(s+1), t) + v_\epsilon(x(s-1), t) + G(s, t+\epsilon)v_\epsilon(x(s), t) = 0. \tag{2.2}$$

Multiplying (2.1) by $v_\epsilon(x(s), t)$, (2.2) by $v(x(s), t)$ and subtracting the results, we get

$$\begin{aligned} &v(x(s), t)v_\epsilon(x(s+1), t) - v_\epsilon(x(s), t)v(x(s+1), t) \\ &\quad = (G(s,t) - G(s, t+\epsilon))v(x(s), t)v_\epsilon(x(s), t) \\ &\qquad + v(x(s-1), t)v_\epsilon(x(s), t) - v_\epsilon(x(s-1), t)v(x(s), t). \end{aligned} \tag{2.3}$$

Applying recursively (2.3), we have

$$v(x(s_j),t)v_\epsilon(x(s_j+1),t) - v_\epsilon(x(s_j),t)v(x(s_j+1),t)$$

$$= \sum_{i=1}^{j}(G(s_i,t) - G(s_i,t+\epsilon))v(x(s_i),t)v_\epsilon(x(s_i),t)$$

$$+ v(x(a),t)v_\epsilon(x(a),t)\left(\frac{v_\epsilon(x(a+1),t)}{v_\epsilon(x(a),t)} - \frac{v(x(a+1),t)}{v(x(a),t)}\right), \qquad (2.4)$$

for each $s_j \in S$. Thus, under our assumption, it follows easily that

$$v(x(s),t)v_\epsilon(x(s+1),t) - v_\epsilon(x(s),t)v(x(s+1),t) > 0$$

on S. Assume that $v(\cdot,t)$ has a node on $(x(s'),(x(s'+1))$ $(s' \in S)$. Hence $\operatorname{sgn} v(x(s'),t) = -\operatorname{sgn} v(x(s'+1),t)$. We leave it to the reader to verify that from (2.4), and making use of our assumptions, we can conclude that $v(x(s'+1),t)/v(x(s'),t)$ is a strictly increasing function of t. Now we consider the line segment with endpoints $(x(s'),v(x(s'),t))$ and $(x(s'+1),v(x(s'+1),t))$, i.e.,

$$V(X) - v(x(s'),t) = \frac{v(x(s'+1),t) - v(x(s'),t)}{x(s'+1) - x(s')}(X - x(s')).$$

If $V(X') = 0$, then

$$X' = \frac{x(s'+1) - x(s')}{1 - \dfrac{v(x(s'+1),t)}{v(x(s'),t)}} + x(s').$$

Since $v(x(s'+1),t)/v(x(s'),t) < 0$ is a strictly increasing function of t, X' moves to the right when t increases. We reach the same conclusion easily if $v(x(s'),t) = 0$ for some t, which concludes the proof. ■

3. Comparison theorems

Definition 3.1. [3, Definition 7.8] *We say that a solution y of* (1.1) *(or* (1.7)*) has a generalized zero at $x(s')$ $(s' \in S \setminus \{s_0\})$ if if either $y(x(s')) = 0$ or $y(x(s'-1))y(x(s')) < 0$*

The next result is known as a discrete version of Sturm's comparison theorem for solutions of (1.7) (see [1]). It will be used to prove a similar result for solutions of (1.1).

Lemma 3.1. *Let v_1 and v_2 be non-trivial solutions on S of*

$$v_1(x(s+1)) + v_1(x(s-1)) + F_1(s)v_1(x(s)) = 0, \tag{3.1}$$

$$v_2(x(s+1)) + v_2(x(s-1)) + F_2(s)v_2(x(s)) = 0, \tag{3.2}$$

respectively. Let $x(s') < x(s'')$ be two consecutive generalized zeros of v_1. If $F_2(s) \geq F_1(s)$ for each $s \in S$ such that $s' \leq s \leq s'' - 1$, then v_2 has at least one node on $(x(s'-1), x(s''))$.

Proof: Multiplying (3.1) by $v_2(x(s))$, (3.2) by $v_1(x(s))$ and subtracting the results, we obtain

$$v_2(x(s))(v_1(x(s+1)) + v_1(x(s-1))) - v_1(x(s))(v_2(x(s+1)) + v_2(x(s-1)))$$
$$= (F_1(s) - F_2(s))v_1(x(s))v_2(x(s)).$$

We can check that

$$\begin{aligned}
&v_1(x(s))(v_2(x(s+1)) + v_2(x(s-1))) - v_2(x(s))(v_1(x(s+1)) + v_1(x(s-1)))\\
&= v_1(x(s))\Delta\nabla v_2(x(s)) + v_2(x(s))\Delta\nabla v_1(x(s))\\
&= \Delta(v_1(x(s))\nabla v_2(x(s))) - \Delta v_2(x(s))\Delta v_1(x(s))\\
&\quad - \Delta(v_2(x(s))\nabla v_1(x(s))) + \Delta v_1(x(s))\Delta v_2(x(s))\\
&= \Delta(v_1(x(s))\nabla v_2(x(s)) - v_2(x(s))\nabla v_1(x(s)))\\
&= \Delta(v_2(x(s))v_1(x(s-1)) - v_1(x(s))v_2(x(s-1))).
\end{aligned}$$

Hence

$$\begin{aligned}
&\Delta(v_2(x(s))v_1(x(s-1)) - v_1(x(s))v_2(x(s-1)))\\
&\qquad = (F_1(s) - F_2(s))y_1(x(s))y_2(x(s)).
\end{aligned} \tag{3.3}$$

Summing both sides of (3.3) from s' to $s'' - 1$, we get

$$\begin{aligned}
&(v_2(x(s))v_1(x(s-1)) - v_1(x(s))v_2(x(s-1)))\Big|_{s'}^{s''}\\
&= \sum_{s' \leq s \leq s''-1} (F_1(s) - F_2(s))v_1(x(s))v_2(x(s)) = R(s', s'').
\end{aligned} \tag{3.4}$$

There is no loss of generality in assuming $v_1(x(s)) > 0$ for all $s \in [s'+1, s''-1]$. Suppose that v_2 has no nodes on $(x(s'-1), x(s''))$. Hence, without loss of generality we can assume $v_2(x(s)) > 0$ for all $s \in (s'-1, s'')$. Thus, $R(s', s'') \leq 0$ and

$$v_2(x(s''))v_1(x(s''-1)) - v_1(x(s''))v_2(x(s''-1)) \geq 0,$$

$$v_2(x(s'))v_1(x(s'-1)) - v_1(x(s'))v_2(x(s'-1)) < 0,$$

which is a contradiction. ■

Lemma 3.2. *Let y be a solution of* (1.1)*. Set $y = uv$ for $s \in S$, where u is given by* (1.6)*. Assume that $u(x(a))u(x(a+1)) > 0$ and $A(s)B(s) > 0$ on S'. Then, y has a node on $(x(s'-1), x(s'))$ ($s' \in S \setminus \{s_0\}$) if and only v has a node on that interval. Moreover, $y(x(s')) = 0$ if and only if $v(x(s')) = 0$.*

Proof: Follows immediately from (1.6). ■

The following theorem is our main result and will be used for COP. Note that it is written in a more general way.

Theorem 3.1. *Let y_1 and y_2 be non-trivial solutions of the difference equations*

$$A_1(s)y_1(x(s-1)) + B_1(s)y_1(x(s+1)) + C_1(s)y_1(x(s)) = 0 \tag{3.5}$$

and

$$A_2(s)y_2(x(s-1)) + B_2(s)y_2(x(s+1)) + C_2(s)y_2(x(s)) = 0, \tag{3.6}$$

respectively, with $A_1(s)B_1(s) > 0$ and $A_2(s)B_2(s) > 0$ for each $s \in S'$. For $i \in \{1, 2\}$, set

$$F_i(s_k) = \begin{cases} d\,\dfrac{C_i(s_k)}{B_i(s_k)} \displaystyle\prod_{j=1}^{k/2} \dfrac{A_i(s_{2j-1})B_i(s_{2j})}{A_i(s_{2j})B_i(s_{2j-1})}, & k \text{ even}, \\ \dfrac{1}{d}\,\dfrac{C_i(s_k)}{A_i(s_k)} \displaystyle\prod_{j=1}^{(k-1)/2} \dfrac{A_i(s_{2j})B_i(s_{2j-1})}{A_i(s_{2j-1})B_i(s_{2j})}, & k \text{ odd}, \end{cases}$$

where $d > 0$ is an arbitrarily chosen constant. Let $x(s') < x(s'')$ be two consecutive generalized zeros of y_1. If $F_2(s) \geq F_1(s)$ for each $s \in S$ such that $s' \leq s \leq s''-1$, then y_2 has at least one node on $(x(s'-1), x(s''))$.

Proof: Let $i \in \{1, 2\}$. Since $A_i(s)B_i(s) > 0$ for each $s \in S'$, $u_i(x(a)) > 0$ and $u_i(x(a+1)) > 0$, if we consider $y_i = u_i v_i$ as in Section 1 and use Lemma 3.1 for v_1 and v_2, then the result follows from Lemma 3.2. ■

Remark 3.1. *In the conclusion of Theorem 3.1, one may write*

(1) *zero of* y_2 *instead of node, under the additional condition that for any two consecutive points* $z' < z''$ *of* S *there is at most one zero of* y_1 *and at most one zero of* y_2 *on* $(x(z'), x(z'')]$ *(see next section);*
(2) *generalized zero of* y_2 *instead of node, and the result would be that* y_2 *has at least one generalized zero on* $[x(s'), x(s'')]$.

We can also obtain results for extreme nodes of a solution of (1.1) and (1.7).

Lemma 3.3. *Let* v_1 *and* v_2 *be non-trivial solutions of* (3.1) *and* (3.2)*, respectively, where* $x(s)$ *is a continuous and strictly increasing function. Let* $x(s')$ *and* $x(s'')$ *be the smallest and greatest generalized zeros of* v_1*, respectively, with* $s' \geq a+2$ *and* $s'' \leq s_{N-2}$.

(1) *If* $F_2(s) > F_1(s)$ *for each* $a+1 \leq s \leq s'-1$*, and*
$$v_2(x(a+1))v_1(x(a)) - v_1(x(a+1))v_2(x(a)) = 0, \tag{3.7}$$
then v_2 *has at least one node on* $(x(a), x(s'))$*;*
(2) *If* $F_2(s) \geq F_1(s)$ *for each* $s'' \leq s \leq s_{N-2}$*, and*
$$v_2(x(s_{N-1}))v_1(x(s_{N-2})) - v_1(x(s_{N-1}))v_2(x(s_{N-2})) = 0, \tag{3.8}$$
then v_2 *has at least one node on* $(x(s''-1), x(s_N-1))$.

Proof: (1) Summing both sides of (3.3) from $a+1$ to $s'-1$, we obtain
$$\left(v_2(x(s))v_1(x(s-1)) - v_1(x(s))v_2(x(s-1))\right)\Big|_{a+1}^{s'}$$
$$= \sum_{a+1 \leq s \leq s'-1} (F_1(s) - F_2(s))v_1(x(s))v_2(x(s)) = R(s'). \tag{3.9}$$
Without loss of generality, we may assume $v_1(x(s)) > 0$ for all $s \in [a, s'-1]$. Suppose that v_2 has no nodes on $(x(a), x(s'))$. If $v_2(x(s)) > 0$, $\forall s \in (a, s')$, then
$$v_2(x(s'))v_1(x(s'-1)) - v_1(x(s'))v_2(x(s'-1)) \geq 0$$
and $R(s') < 0$, which is a contradiction. Otherwise, if $v_2(x(s)) < 0$, $\forall s \in (a, s')$, then
$$v_2(x(s'))v_1(x(s'-1)) - v_1(x(s'))v_2(x(s'-1)) \leq 0$$
and $R(s') > 0$, which is a contradiction. Therefore, v_2 has at least one node on $(x(a), x(s'))$.

(2) Similar to the proof for i).

∎

Theorem 3.2. *Let y_1 and y_2 be non-trivial solutions of the difference equations*

$$A_1(s)y_1(x(s-1)) + B_1(s)y_1(x(s+1)) + C_1(s)y_1(x(s)) = 0 \tag{3.10}$$

and

$$A_2(s)y_2(x(s-1)) + B_2(s)y_2(x(s+1)) + C_2(s)y_2(x(s)) = 0, \tag{3.11}$$

respectively, with $A_1(s)B_1(s) > 0$ and $A_2(s)B_2(s) > 0$ for each $s \in S'$. For $i \in \{1, 2\}$, set

$$F_i(s_k) = \begin{cases} d\,\dfrac{C_i(s_k)}{B_i(s_k)} \displaystyle\prod_{j=1}^{k/2} \dfrac{A_i(s_{2j-1})B_i(s_{2j})}{A_i(s_{2j})B_i(s_{2j-1})}, & k \text{ even}, \\ \dfrac{1}{d}\,\dfrac{C_i(s_k)}{A_i(s_k)} \displaystyle\prod_{j=1}^{(k-1)/2} \dfrac{A_i(s_{2j})B_i(s_{2j-1})}{A_i(s_{2j-1})B_i(s_{2j})}, & k \text{ odd}, \end{cases}$$

where $d > 0$ is an arbitrarily chosen constant. Let $x(s')$ and $x(s'')$ be the smallest and greatest generalized zero of y_1, respectively, with $s' \geq a+2$ and $s'' \leq s_{N-2}$.

(1) *If $F_2(s) > F_1(s)$ for each $a+1 \leq s \leq s'-1$, and*

$$y_2(x(a+1))y_1(x(a)) - y_1(x(a+1))y_2(x(a)) = 0, \tag{3.12}$$

then y_2 has at least one node on $(x(a), x(s'))$;

(2) *If $F_2(s) \geq F_1(s)$ for each $s'' \leq s \leq s_{N-2}$, and*

$$y_2(x(s_{N-1}))y_1(x(s_{N-2})) - y_1(x(s_{N-1}))y_2(x(s_{N-2})) = 0, \tag{3.13}$$

then y_2 has at least one node on $(x(s''-1), x(s_N-1))$.

Proof: Similar to the proof of Theorem 3.1, using Lemma 3.3 instead of Lemma 3.1. ∎

4. Applications

Here we present some examples comparing zeros of two COP. It is known that (1.1) has polynomial solutions in x, whose difference derivatives satisfy equations of the same kind if and only if, for $q \neq 1$ fixed, x is a linear, quadratic, q-linear, or q-quadratic grid of the form

$$x(s) = \begin{cases} C_1 s^2 + C_2 s, \\ C_3 q^{-s} + C_4 q^s, \end{cases}$$

where $(C_1, C_2) \neq (0,0)$ and $(C_3, C_4) \neq (0,0)$. The grids that depend on "q" are called q-linear if C_3 or C_4 is zero; otherwise, it is q-quadratic. By using transformations, we can reduce the expressions for the grids to simpler forms. In what follows, we assume that the grid x takes on the following canonical forms:

$$x(s) = \begin{cases} s & & \text{(I)} \\ s(s+1) & & \text{(II)} \\ q^s & (q>1) & \text{(III)} \\ \frac{1}{2}(q^s - q^{-s}) & (q>1) & \text{(IV)} \\ \frac{1}{2}(q^s + q^{-s}) & (q>1) & \text{(V)} \\ \frac{1}{2}(q^s + q^{-s}) & (q = e^{2i\theta},\ 0<\theta<\pi/2). & \text{(VI)} \end{cases} \tag{4.1}$$

Definition 4.1. *Fix $a \in \mathbb{R} \cup \{-\infty\}$ and $N \in \mathbb{N} \cup \{\infty\}$ and set $b = a + N$. Fix q and let $x(s)$ be a real-valued function given by (4.1), where the variable s ranges over the finite interval $[a,b]$ or the infinity interval $[a,\infty)$. A sequence of polynomials, $(P_n(x(s)))_{n=0}^{N-1}$, is said to be a sequence of classical discrete orthogonal polynomials on the set $\{x(a), x(a+1), \ldots, x(b-1)\}$ or, simply, COP if:*

(1) *P_n satisfy (1.1), with x being a strictly monotone function on $[a,b]$ or $[a,\infty)$ given, up to a linear transformation, by (4.1);*
(2) *there exists a positive weight function ω satisfying the boundary conditions*

$$\omega(s)a(s)x^k\left(s-\frac{1}{2}\right)\Big|_{a,b} = 0 \qquad (k = 0,1,\ldots); \tag{4.2}$$

(3) *the difference equation*

$$\frac{\Delta}{\Delta x\left(x-\frac{1}{2}\right)}(\omega(s)a(s)) = \omega(s)b(s) \tag{4.3}$$

holds.

Remark 4.1. *The Pearson type equation* (4.3) *can be rewritten as*

$$\omega(s+1) = \frac{B(s)\Delta x(s-1/2)}{A(s+1)\Delta x(s+1/2)}\omega(s). \tag{4.4}$$

Applying (4.4) *recursively, we obtain*

$$\prod_{i=1}^{s}\frac{B(i)}{A(i)} = \frac{\omega(s)B(s)\Delta x(s-1/2)}{\omega(a)B(a)\nabla x(1/2)} \tag{4.5}$$

and taking $u(x(a)) = u(x(a+1)) = 1$, *we may rewrite* (1.8) *as*

$$F(s_k) = \frac{\omega(s_k)C(s_k)\Delta x(s_k-1/2)}{\omega(a)B(a)\nabla x(1/2)} \times \begin{cases} \left(\prod_{j=1}^{k/2}\frac{A(s_{2j-1})}{B(s_{2j-1})}\right)^2, & k \text{ even,} \\ \left(\prod_{j=1}^{(k-1)/2}\frac{A(s_{2j})}{B(s_{2j})}\right)^2, & k \text{ odd.} \end{cases} \tag{4.6}$$

From now on, the function F defined by (4.6) will be called *comparison function* of P_n.

Lemma 4.1. *Assume the hypothesis of Lemma 3.2. Assume that there is at most one zero of* y *on* $(x(s-1)), x(s))$ *for each* $s \in S\setminus\{s_0\}$. *Then,* y *has a zero on* $(x(s'-1), x(s'))$ $(s' \in S\setminus\{s_0\})$ *if and only* v *has a node on that interval. Moreover,* $y(x(s')) = 0$ *if and only if* $v(x(s')) = 0$.

Proof: Clearly, if v has a node on $(x(s'-1), x(s'))$ for some $s' \in S\setminus\{s_0\}$, then y has a zero on that interval. Now, assume that y has exactly one zero on $(x(s'-1), x(s'))$ for some $s' \in S\setminus\{s_0\}$. Then, $y(x(s'-1))y(x(s')) < 0$, i.e., y has a node on $(x(s'-1), x(s'))$, and the result follows from Lemma 3.2. ∎

Next, we present examples of applications of Theorem 3.1 on the linear and q-linear grids. On the linear grid, we compare the functions for the Meixner and Charlier polynomials of same degree. On the q-linear grid, we compare the functions for the q-Krawtchouk and Al-Salam-Carlitz II, also of same degree, but different intervals of orthogonality. It is worth mentioning that Theorem 3.1 can also be used for other two polynomials on the grids (4.1) (both polynomials must be on the same grid), even for the cases where their degrees are not the same nor their corresponding orthogonality intervals, as long as they fulfil all the hypothesis. For example, one could use it for the Hahn and Krawtchouk polynomials of different degrees and different orthogonality intervals on the linear grid, under certain conditions.

4.1. The Meixner and Charlier polynomials. The *Meixner polynomials* (see [7, Section 9.10]),

$$y(s) = M_n^{(\gamma,\mu)}(s) = {}_2F_1\left(\begin{matrix}-n, & -s\\ & \gamma\end{matrix}\,\middle|\, 1-\frac{1}{\mu}\right)$$

($n = 1, 2, \ldots, 0 < \mu < 1, \gamma > 0$), satisfy the difference equation (1.2) with $A(s, \gamma, \mu) = s$, $B(s, \gamma, \mu) = \mu(s+\gamma)$ and $C(s, \gamma, \mu) = n(1-\mu) - s - (s+\gamma)\mu$. Note that $A(s, \gamma, \mu)B(s, \gamma, \mu) > 0$ for each $s \in \{1, 2, \ldots\}$.

The *Charlier polynomials* (see [7, Section 9.14]),

$$y(s) = C_n^{(\alpha)}(s) = {}_2F_0\left(\begin{matrix}-n, & -s\\ & -\end{matrix}\,\middle|\, -\frac{1}{\alpha}\right)$$

($n = 1, 2, \ldots, \alpha > 0$), satisfy the difference equation (1.2) with $A(s, \alpha) = s$, $B(s, \alpha) = \alpha$ and $C(s, \alpha) = n - s - \alpha$. Note that $A(s, \alpha)B(s, \alpha) > 0$ for $s \in \{1, 2, \ldots\}$.

Remark 4.2. *(see* [8, Theorem 7]*) For $n \in \{2, 3, \ldots\}$, all the zeros of $M_n^{(\gamma,\mu)}$ are real and lie on the interval $(0, M(n, \gamma, \mu))$, where*

$$M(n,\gamma,\mu) = \mu_2 - \begin{cases} \dfrac{3\mu^{1/6}\mu_2^{1/3}(\mu_2+\gamma)^{1/3}}{2^{2/3}(1-\mu)^{1/3}n^{1/6}(n+\gamma)^{1/6}}, \\ \mu_2 \le \mu_1 + \sqrt{\mu_1(\mu_1+\gamma)}, \\ \dfrac{3\mu^{1/3}(\sqrt{\mu_1}+\sqrt{\mu_1+\gamma})^{2/3}}{(1-\mu)^{2/3}}, \\ \mu_2 > \mu_1 + \sqrt{\mu_1(\mu_1+\gamma)}, \end{cases} \tag{4.7}$$

with

$$\mu_1 = \frac{(\sqrt{n} - \sqrt{\mu(n+\gamma)})^2}{1-\mu}, \qquad \mu_2 = \frac{(\sqrt{n} + \sqrt{\mu(n+\gamma)})^2}{1-\mu}.$$

Lemma 4.2. *Fix $\alpha \in (2, +\infty)$, $\gamma \in (0, +\infty)$, $\mu \in (0, 1)$. Denote the zeros of $M_n^{(\gamma,\mu)}$ and $C_n^{(\alpha)}$ by $x_1 < \cdots < x_n$ and $\widetilde{x}_1 < \cdots < \widetilde{x}_n$, respectively. If $n \in \{2, 3, \dots\}$ is such that $(\mu(\gamma + 1) + 1)/(1 - \mu) \le n < \alpha$, and there are at least two generalized zeros of $C_n^{(\alpha)}$ on $(0, (n(1 - \mu) - \gamma\mu)/(1 + \mu) + 1]$, then there is at least one zero of $M_n^{(\gamma,\mu)}$ on $(s' - 1, s'')$, where $s' - 1 < s''$ are two consecutive generalized zeros of $C_n^{(\alpha)}$ on $(0, (n(1 - \mu) - \gamma\mu)/(1 + \mu) + 1]$.*

Proof: By [2, Lemma 2.1], $M_n^{(\gamma,\mu)}$ and $C_n^{(\alpha)}$ satisfy the hypothesis of Lemma 4.1. Denote the comparison functions of $M_n^{(\gamma,\mu)}$ and $C_n^{(\alpha)}$ by F_M and F_C, respectively. From (4.6), if $n - s - \alpha \le 0$ and $n(1-\mu) - s - \mu(\gamma + s) \ge 0$, then $F_C(s) \le F_M(s)$ for $1 \le s \le (n(1 - \mu) - \gamma\mu)/(1 + \mu)$ and the result follows from Theorem 3.1. ∎

Proposition 4.1. *Fix $\alpha \in (2, +\infty)$, $\gamma \in (0, +\infty)$, $\mu \in (0, 1)$. Denote the zeros of $M_n^{(\gamma,\mu)}$ and $C_n^{(\alpha)}$ by $x_1 < \cdots < x_n$ and $\widetilde{x}_1 < \cdots < \widetilde{x}_n$, respectively. If $n \in \{2, 3, \dots\}$ is such that $(\mu(\gamma + 1) + 1)/(1 - \mu) \le n < \alpha$ and $M(n, \gamma, \mu) \le (n(1 - \mu) - \gamma\mu)/(1 + \mu)$, with $M(n, \gamma, \mu)$ given by (4.7), then $x_j < \lceil \widetilde{x}_j \rceil$ for each $j \in \{1, \dots, n\}$.*

Proof: Since $M(n, \gamma, \mu) \le (n(1 - \mu) - \gamma\mu)/(1 + \mu)$, all the zeros of $M_n^{(\gamma,\mu)}$ are on $(0, (n(1 - \mu) - \gamma\mu)/(1 + \mu))$. Note that $x_j < \lceil \widetilde{x}_j \rceil$ for the zeros of $C_n^{(\alpha)}$ outside this interval, since they will be greater than any zero of $M_n^{(\gamma,\mu)}$. For the zeros of $C_n^{(\alpha)}$ on $(0, (n(1 - \mu) - \gamma\mu)/(1 + \mu))$, $x_j < \lceil \widetilde{x}_j \rceil$ follows from Lemma 4.2 and the fact that, under our hypothesis, $M_n^{(\gamma,\mu)}(0)M_n^{(\gamma,\mu)}(1) \le 0$, i.e., there is one zero of $M_n^{(\gamma,\mu)}$ on $(0, 1]$ and therefore $x_1 < \lceil \widetilde{x}_1 \rceil$. ∎

From Proposition 4.1, we can obtain some examples.

Example 4.1. *Let $n \in \{2, 3, \dots, 7\}$. Denoting by $x_1 < \cdots < x_n$ and $\widetilde{x}_1 < \cdots < \widetilde{x}_n$ the zeros of $M_n^{(1,1/100)}$ and $C_n^{(10)}$, respectively, we have $x_j < \lceil \widetilde{x}_j \rceil$ for each $j \in \{1, \dots, n\}$.*

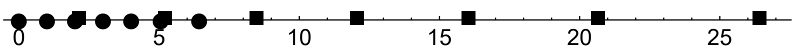

FIGURE 2. Zeros of $C_7^{(10)}$ and $M_7^{(1,1/100)}$(■ and •, respectively).

Example 4.2. *Let $n \in \{2, 3, \dots, 90\}$. Denoting by $x_1 < \cdots < x_n$ and $\widetilde{x}_1 < \cdots < \widetilde{x}_n$ the zeros of $M_n^{(1,1/10000)}$ and $C_n^{(100)}$, respectively, we have $x_j < \lceil \widetilde{x}_j \rceil$ for each $j \in \{1, \dots, n\}$.*

Remark 4.3. *Proposition 4.1 is just one of the applications of Theorem 3.1. For instance, one could follow the same steps to obtain some conditions for* $x_j < \lceil \widetilde{x}_j \rceil$ *for each* $j \in \{1, \dots, n\}$*, where* $x_1 < \cdots < x_n$ *and* $\widetilde{x}_1 < \cdots < \widetilde{x}_n$ *are the zeros of* $C_n^{(\alpha)}$ *and* $M_n^{(\gamma,\mu)}$*, respectively. Note that Lemma 4.2 strongly relies on the signal of the comparison functions and therefore is excluding other cases where the same result could possibly be achieved, i.e., the cases where* $F_C(s)F_M(s) > 0$*. If one prefers, instead of using Lemma 4.2 or Proposition 4.1, they can compute the comparison functions of two polynomials, given their degrees and parameters, check the inequality between both functions and the existence of zeros on the intervals where such inequality holds, and then use Theorem 3.1.*

4.2. The q-Krawtchouk and Al-Salam-Carlitz II polynomials. Now we consider two COP on the q-linear grid $x(s) = q^{-s}$ $(0 < q < 1)$, with different intervals of orthogonality. Note that x is an increasing function of $s \in [0, +\infty)$.

The *q-Krawtchouk polynomials* (see [7, Section 14.15]),

$$y(s) = K_n^{(p)}(q^{-s}; q) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, \ -pq^n, \ q^{-s} \\ 0, \ q^{1-N} \end{matrix} \middle| \, q, \ q \right)$$

$(n = 1, \dots, N-1; p > 0)$, satisfy the difference equation (1.2) with $A(s,p) = p(1-q^s)$, $B(s,p) = q^{s-N+1} - 1$ and $C(s,p) = q^{-n} - q^{s-N+1} - p(q^n - q^s)$. Note that $A(s,p)B(s,p) > 0$ for each $s \in \{1, \dots, N-2\}$.

The *second family of Al-Salam-Carlitz polynomials* (see [7, Section 14.25]),

$$y(s) = V_n^{(\alpha)}(q^{-s}; q) = (-\alpha)^n q^{-\binom{n}{2}} {}_2\phi_0 \left(\begin{matrix} q^{-n}, \ q^{-s} \\ - \end{matrix} \middle| \, q, \ \frac{q^n}{\alpha} \right)$$

$(n = 1, 2, \dots; 0 < \alpha < q^{-1})$, satisfy the difference equation (1.2) with $A(s,\alpha) = (1 - q^{-s})(\alpha - q^{-s})$, $B(s,\alpha) = \alpha q$ and $C(s,\alpha) = \alpha\,(q^{-s} - q - 1) + q^{-2s}\,(q^s - q^n)$. Note that $A(s,\alpha)B(s,\alpha) > 0$ for each $s \in \{1, 2, \dots\}$.

Lemma 4.3. *Fix* $q \in (0,1)$, $N \in \{3, 4, \dots\}$, $p \in (0, +\infty)$ *and* $\alpha \in (0, q^{-1})$*. Denote the zeros of* $K_n^{(p)}(\cdot; q)$ *and* $V_n^{(\alpha)}(\cdot; q)$ *by* $q^{-x_1} < \cdots < q^{-x_n}$ *and* $q^{-\widetilde{x}_1} < \cdots < q^{-\widetilde{x}_n}$*, respectively. If* $n \in \{2, \dots, N-1\}$ *is such that* $\alpha\,(q^{-s} - q - 1) + q^{-2s}\,(q^s - q^n) \leq 0$ *and* $q^{-n} - q^{s-N+1} - p(q^n - q^s) \geq 0$ *for each* $s \in \{1, \dots, N-2\}$ *and there are at least two zeros of* $V_n^{(\alpha)}(\cdot; q)$ *on* $(1, q^{1-N}]$*, then there is at least one zero of* $K_n^{(p)}(\cdot; q)$ *on* $(q^{-s'+1}, q^{-s''})$*, where* $q^{1-s'} < q^{-s''}$ *are two consecutive generalized zeros of* $V_n^{(\alpha)}(\cdot; q)$ *on* $(1, q^{1-N}]$.

Proof: By [2, Lemma 2.1], $V_n^{(\alpha)}(\cdot;q)$ and $K_n^{(p)}(\cdot;q)$ satisfy the hypothesis of Lemma 4.1. Denote the comparison functions of $V_n^{(\alpha)}(\cdot;q)$ and $K_n^{(p)}(\cdot;q)$ by $F_V(s)$ and $F_K(s)$, respectively. By (4.6), if $\alpha\,(q^{-s}-q-1)+q^{-2s}\,(q^s-q^n)\leq 0$ and $q^{-n}-q^{s-N+1}-p(q^n-q^s)\geq 0$, then $F_V(s)\leq F_K(s)$. The result follows from Theorem 3.1 and Lemma 4.1. ■

Proposition 4.2. *Fix $q\in(0,1)$, $N\in\{3,4,\dots\}$, $p\in(0,+\infty)$ and $\alpha\in(0,q^{-1})$. Denote the zeros of $K_n^{(p)}(\cdot;q)$ and $V_n^{(\alpha)}(\cdot;q)$ by $q^{-x_1}<\cdots<q^{-x_n}$ and $q^{-\widetilde{x}_1}<\cdots<q^{-\widetilde{x}_n}$, respectively. If $n\in\{2,\dots,N-1\}$ is such that $\alpha\,(q^{-s}-q-1)+q^{-2s}\,(q^s-q^n)\leq 0$ and $q^{-n}-q^{s-N+1}-p(q^n-q^s)\geq 0$ for each $s\in\{1,\dots,N-2\}$, and $q^{N-n}-q+p(q^N-q^{N+n})\geq 0$, then $q^{-x_j}<q^{-\lceil\widetilde{x}_j\rceil}$ for each $j\in\{1,\dots,n\}$.*

Proof: By [13, Theorem 3.3.1], all the zeros of $K_n^{(p)}(\cdot;q)$ are on $(1,q^{1-N})$. Note that $q^{-x_j}<q^{-\lceil\widetilde{x}_j\rceil}$ for the zeros of $V_n^{(\alpha)}(\cdot;q)$ outside this interval, since they will be greater than any zero of $K_n^{(p)}(\cdot;q)$. For the zeros of $V_n^{(\alpha)}(\cdot;q)$ on $(1,q^{1-N}]$, $q^{-x_j}<q^{-\lceil\widetilde{x}_j\rceil}$ follows from Lemma 4.3 and the fact that, since $q^{N-n}-q+p(q^N-q^{N+n})\geq 0$, we have $K_n^{(p)}(1;q)K_n^{(p)}(q^{-1};q)\leq 0$, i.e., there is one zero of $K_n^{(p)}(\cdot;q)$ on $(1,q^{-1}]$, and therefore $q^{-x_1}<q^{-\lceil\widetilde{x}_1\rceil}$. ■

Some of the examples that can be obtained from Proposition 4.2 are the following:

Example 4.3. *Let $q=99/100$, $M=10$ and $n\in\{5,6,\dots,9\}$. Denoting by $q^{-x_1}<\cdots<q^{-x_n}$ and $q^{-\widetilde{x}_1}<\cdots<q^{-\widetilde{x}_n}$ the zeros of $K_n^{(1)}(\cdot;q)$ and $V_n^{(10)}(\cdot;q)$, respectively, we have $q^{-x_j}<q^{-\lceil\widetilde{x}_j\rceil}$ for each $j\in\{1,\dots,n\}$.*

Example 4.4. *Let $q=99/100$, $M=50$ and $n\in\{32,33,\dots,49\}$. Denoting by $q^{-x_1}<\cdots<q^{-x_n}$ and $q^{-\widetilde{x}_1}<\cdots<q^{-\widetilde{x}_n}$ the zeros of $K_n^{(1)}(\cdot;q)$ and $V_n^{(10)}(\cdot;q)$, respectively, we have $q^{-x_j}<q^{-\lceil\widetilde{x}_j\rceil}$ for each $j\in\{1,\dots,n\}$.*

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A. Suzuki

University of Coimbra, CMUC, Dep. Mathematics, 3001-501 Coimbra, Portugal

E-mail address: asuzuki@uc.pt