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DIAMETER OF A COMMUTATION CLASS ON REDUCED WORDS

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ABSTRACT: Any permutation w of the symmetric group can be generated by a product of adjacent transpositions, and a reduced word for w is a sequence of generators of minimal length whose product is w. The main result in this paper gives a formula to compute the diameter of a commutation class of the graph $G(w)$, whose vertices are reduced words for w and whose edges are braid relations. To do so, we define a metric on the set of all reduced words of a given permutation which turn out to be equal to the usual distance in any commutation class. If a permutation is fully commutative, i.e. it has only one commutation class, then the formula gives the diameter of $G(w)$. The diameter for a Grassmanian permutation is also given in terms of its Lehman code.

KEYWORDS: Reduced words, commutation class, diameter.
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1. Introduction

Given a positive integer $n \geq 1$, let $\mathcal{S}_n$ denote the symmetric group on the alphabet $\{1, 2, \ldots, n\}$. A permutation $w \in \mathcal{S}_n$ will be represented in one-line notation as $w = [w_1 w_2 \cdots w_n]$, where $w_i = w(i)$. It is well known that the symmetric group $\mathcal{S}_n$ is generated by the adjacent transpositions s_i, that swaps the integers i and $i+1$ and fixes all other integers, $1 \leq i < n$, which satisfy the Coxeter relations

$$s_i s_j = s_j s_i, \text{ for } |i-j| > 1, \tag{1.1}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ for } 1 \leq i < n-1, \tag{1.2}$$

and $s_i^2 = e$, the identity permutation. Relations (1.1) are known as *commutations* or short braid relations, and relations (1.2) are called *long braid relations*.

A *reduced word* for $w \in \mathcal{S}_n$ is a sequence $i_1 i_2 \cdots i_\ell$ of minimal length ℓ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. The collection of all reduced words for w will be denoted by $\mathrm{Red}(w)$. The graph $G(w)$, whose vertex set is $\mathrm{Red}(w)$ and where two reduced words are connected by an edge whenever they differ by a single Coxeter

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relation (1.1) or (1.2), has been studied by several authors (see [13, 3, 14, 8] and the references therein). A classical result from Tits [15] shows that $G(w)$ is connected, and some effort was made towards the computation of the diameter of this graph. The diameter of $G(w)$ was first studied asymptotically by Dehornoy and Autord [2] and exact formulas were obtained by Reiner and Roichman [12] for the longest permutation $[n\; n{-}1\; \cdots 1]$, and by Dahlberg and Kim [6] for 12-inflations and several 21-inflations. Lower and upper bounds for the diameter were also considered [6, 9, 12].

By contracting the commutation edges of $G(w)$ we obtain the commutation graph $C(w)$, whose vertices are the commutation classes and the edges are long braid relations between the classes. This graph has also received some attention [7, 5, 10], and in particular a formula for the diameter of $C(w)$ was obtained in [11]. In this paper, we generalize a statistic used to compute the diameter of the commutation graph to define a metric on the set $\mathrm{Red}(w)$ of all reduced words of w. This metric is used to compute the diameter of any commutation class of $G(w)$. As a corollary, we obtain a formula for the diameter of $G(w)$ for any fully commutative permutation w. For the special class of Grassmannian permutations, we get formulas for the diameter based on its Lehman code.

2. A metric for $G(w)$

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The *distance* $d(a, b)$ between two vertices $a, b \in V$ is the number of edges in the shortest path between a and b. The diameter $diam(G)$ of the graph G is the maximum value of $d(a, b)$ over all $a, b \in V$. A subgraph $G' = (V', E')$ of G is called an induced subgraph if it contains all edges of G that join two vertices in V'.

Given a permutation $w \in \mathcal{S}_n$, consider the graph $G(w)$ with vertex set $\mathrm{Red}(w)$, and with an edge between two reduced words if they differ by exactly one Coxeter relation (1.1) or (1.2). If a is a reduced word for w, let $G_{[a]}$ denote the commutation class of a, that is the set of all reduced words obtained from a by a sequence of commutations. The commutation class $G_{[a]}$ is an induced subgraph of $G(w)$, and its reduced words can be ordered by lexicographic order $\leq_{lex}$. We denote by a_- and a_+ the minimum and maximum word in $G_{[a]}$ for the lexicographic order.

Definition 2.1. Given a permutation w and an integer $1 \leq i < \ell(w)$, let $\mathfrak{c}_i$ be the map that acts on $a = a^1 a^2 \cdots a^\ell \in \mathrm{Red}(w)$ by *commuting* the letters a^i and a^{i+1} whenever $|a^i - a^{i+1}| > 1$, and acts as the identity otherwise. Analogously, define the map $\mathfrak{b}_i$ that acts on $a \in \mathrm{Red}(w)$ by *braiding* the factor $a^{i-1}a^i a^{i+1}$ to $a^i a^{i+1} a^i$ whenever $a^{i-1} = a^{i+1} = a^i \pm 1$, and acts as the identity otherwise.

The maps $\mathfrak{c}_i$ and $\mathfrak{b}_i$ are well-defined involutions on $\mathrm{Red}(w)$ [1]. We refer to $\mathfrak{c}_i$ as a commutation, to $\mathfrak{b}_i$ as a braid move, and to either as a Coxeter move.

Figure 1 displays the graph $G(4321)$ and the respective Coxeter moves, with edges corresponding to commutations in red and to long braid relations in blue. The reduced words in the commutation class of $a = 213213$, in lexicographic order, are

$$a_- = 213213 \leq_{lex} 213231 \leq_{lex} 231213 \leq_{lex} 231231 = a_+ \,.$$

An *inversion* of a permutation $w = [w_1 w_2 \cdots w_n] \in \mathcal{S}_n$ is an ordered pair (w_j, w_i) such that $i < j$ and $w_j < w_i$. For simplicity, we represent the inversion (w_j, w_i) just by the word $w_j w_i$. The set of all inversions of w is denoted by $\mathrm{Inv}(w)$.

For $a = a^1 a^2 \cdots a^{\ell(w)}$ a reduced word for w, consider the permutation formed as the left factor $v_p = s_{a^1} s_{a^2} \cdots s_{a^p}$ of $w = s_{a^1} s_{a^2} \cdots s_{a^{\ell(w)}}$. The inversion sets $\mathrm{Inv}(v_p)$ and $\mathrm{Inv}(v_{p-1}) \subset \mathrm{Inv}(v_p)$ differ by exactly one inversion for each $p \geq 1$, with v_0 the identity, and thus we can define a bijection $\iota_a : \{1, 2, \ldots, \ell(w)\} \to \mathrm{Inv}(w)$ where $\iota_a(p) = ij$ if $ij \in \mathrm{Inv}(v_p) \setminus \mathrm{Inv}(v_{p-1})$. This induces a linear ordering on the set $\mathrm{Inv}(w)$, denoted $(\mathrm{Inv}(w), \leq_a)$, with cover relations defined by

$$ij \lessdot_a k\ell \;\text{ if there is } p \geq 1 \text{ such that }\; ij = \iota_a(p) \;\text{ and }\; k\ell = \iota_a(p+1)\,.$$

The *descent set* of a permutation w is the set

$$\mathrm{Des}(w) = \{i : w_i > w_{i+1}\},$$

and its elements are called *descents*. In Algorithms 1 and 2 we use the descent set of w to describe the construction of two reduced words for w, w_{min} and w_{max}, which are, in some sense, opposite to each other.

Algorithm 1 The reduced word w_{min} for the permutation $w = [w_1 w_2 \cdots w_n]$.

Set $v = w$ and $w_{min} = \epsilon$ (emptyword)
for $j = n, n-1, \ldots, 2$ **do**
    while $v^{-1}(j)$ is a descent of v **do**
        Set $w_{min} = v^{-1}(j) \cdot w_{min}$
        Update $v = v \cdot (i\, j)$ where $i = v\left(v^{-1}(j) + 1\right)$

Algorithm 2 The reduced word w_{max} for the permutation $w = [w_1 w_2 \cdots w_n]$.

Set $v = w$ and $w_{max} = \epsilon$ (emptyword)
for $j = 1, 2, \ldots, n-1$ **do**
  while $v^{-1}(j) - 1$ is a descent of v **do**
    Set $w_{max} = \big(v^{-1}(j) - 1\big) \cdot w_{max}$
    Update $v = v \cdot (j\, i)$ where $i = v\,\big(v^{-1}(j) - 1\big)$

To construct w_{max}, we start with the permutation w and apply transpositions in order to send the number 1 to the first position in the one-line notation. Next, we send the number 2 do the second position. We repeat the procedure until we obtain the identity. Reading the descents corresponding to each of the transpositions used, in reverse order of their appearance, we get the word w_{max}. For w_{min}, the procedure is similar, but now we order the numbers in w starting from n.

In Table 2.1 we apply Algorithms 1 and 2 to obtain the reduced words $w_{min} = 13243254$ and $w_{max} = 43451234$ for the permutation $w = [254613]$, reading the left column of each table from bottom to top. The labels on the left column (typed in italics) are descents of the permutation in that line, and in the right column are the corresponding inversions. The corresponding induced orderings of the inversion set of w are:

$$12 \lessdot_{w_{min}} 34 \lessdot_{w_{min}} 14 \lessdot_{w_{min}} 35 \lessdot_{w_{min}} 15 \lessdot_{w_{min}} 45 \lessdot_{w_{min}} 36 \lessdot_{w_{min}} 16$$

and

$$45 \lessdot_{w_{max}} 35 \lessdot_{w_{max}} 34 \lessdot_{w_{max}} 36 \lessdot_{w_{max}} 12 \lessdot_{w_{max}} 15 \lessdot_{w_{max}} 14 \lessdot_{w_{max}} 16.$$

As defined in [12], the set formed by all disjoint pairs of inversions $(ij, k\ell)$, with $i < k$ and $ij, k\ell \in \mathrm{Inv}(w)$ is denoted by $I_2(w)$. Let

$$I_3(w) = \{(ij, k\ell) : ij, k\ell \in \mathrm{Inv}(w) \text{ with } j = k\},$$

and $L_2(w) = I_2(w) \cup I_3(w)$. For simplicity, we often write a pair $(ij, j\ell) \in I_3(w)$ as a triple $ij\ell$.

The next definition generalizes the function defined in [11] on the set $\mathrm{Red}(w) \times I_3(w)$.

descent	permutation						Inv(w)
4	2	5	4	**6**	**1**	3	16
5	2	5	4	1	**6**	**3**	36
2	2	**5**	**4**	1	3	6	45
3	2	4	**5**	**1**	3	6	15
4	2	4	1	**5**	**3**	6	35
2	2	**4**	**1**	3	5	6	14
3	2	1	**4**	**3**	5	6	34
1	**2**	**1**	3	4	5	6	12
	1	2	3	4	5	6	

descent	permutation						Inv(w)
4	2	5	4	**6**	**1**	3	16
3	2	5	**4**	**1**	6	3	14
2	2	**5**	**1**	4	6	3	15
1	**2**	**1**	5	4	6	3	12
5	1	2	5	4	**6**	**3**	36
4	1	2	5	**4**	**3**	6	34
3	1	2	**5**	**3**	4	6	35
4	2	1	3	**5**	**4**	6	45
	1	2	3	4	5	6	

TABLE 2.1. The reduced words w_{min} and w_{max} for $w = [254613]$.

Definition 2.2. Given a permutation $w \in \mathcal{S}_n$, define the family of maps $(\Gamma_a)_{a\in\mathrm{Red}(w)}$ on the set $L_2(w)$ by setting

$$\Gamma_a(ij, k\ell) = \begin{cases} 1, & \text{if } ij >_a k\ell \\ 0, & \text{if } ij <_a k\ell \end{cases},$$

where $(\mathrm{Inv}(w), \leq_a)$ is the order induced by a.

If two reduced words a and b are related by a single commutation, then there is exactly one pair $(ij, k\ell) \in I_2(w)$ for which the image of the Γ-function differs for both words, that is

$$\Gamma_a(ij, k\ell) \neq \Gamma_b(ij, k\ell),$$

with $ij \lessdot_a k\ell$ if and only if $k\ell \lessdot_b ij$. Analogously, if a and b differ by a braid move, then there is exactly one pair $(ij, j\ell) \in I_3(w)$ for which the image of the Γ-function differs for both words, with

$$ij \lessdot_a i\ell \lessdot_a j\ell \quad \text{if and only if} \quad j\ell \lessdot_b i\ell \lessdot_b i\ell.$$

Lemma 2.3. *Let $w \in \mathfrak{S}_n$. Then, the words w_{min} and w_{max}, obtained by Algorithms 1 and 2, are reduced words for w. Moreover, $\Gamma_{w_{min}}(ijk) = 0$ and $\Gamma_{w_{max}}(ijk) = 1$ for any triple $ijk \in I_3(w)$.*

Proof: It is easy to check that for $j = n, \ldots, 2$, each iteration of the **while** cycle in Algorithm 1 produces a auxiliar permutation v with one less inversion in each step. Thus, the permutation obtained after the conclusion of the algorithm is the identity. Since the length of w_{min} is equal to the cardinality of $\mathrm{Inv}(w)$, it follows that w_{min} is a reduced word for w. Next, consider a triple $ijk \in I_3(w)$.

This means that kji is a subword of w with ij and jk inversions of w. Following Algorithm 1, we must have

$$ij \lessdot_{w_{min}} jk.$$

This proves that $\Gamma_{w_{min}}(ijk) = 0$.

The same argument proves that w_{max} is a reduced word for w with $\Gamma_{w_{max}}(ijk) = 1$. ■

The next result, proved in [11, Proposition 4.3], shows that when restricted to $I_3(w)$, the function Γ_a only depends on the commutation class of a.

Proposition 2.4. *Given a commutative class $G_{[a]}$, $\Gamma_a(ijk) = \Gamma_b(ijk)$ for all $ijk \in I_3(w)$ if and only if $b \in G_{[a]}$.*

The commutation classes of w_{min} and w_{max} are the furthest classes in the graph $C(w)$ and thus $diam\,(C(w)) = d\left(G_{[w_{min}]}, G_{[w_{max}]}\right) = |I_3(w)|$ (see [11] for details).

In general, for any two reduced words, one can define the set of all pairs of inversions for which the image of the Γ-functions differs. This set allows us to define a metric on $\mathrm{Red}(w)$, which will be used to prove our main result Theorem 3.9.

Definition 2.5. Given $a, b \in \mathrm{Red}(w)$, let

$$T_{a,b} = \{(ij, k\ell) \in L_2(w) : \Gamma_a(ij, k\ell) \neq \Gamma_b(ij, k\ell)\}$$

and

$$T_{[a]} = \bigcup_{b,c \in [a]} T_{b,c}.$$

Definition 2.6. Given $a, b \in \mathrm{Red}(w)$, let

$$t(a,b) = |T_{a,b}| = \sum_{(ij,k\ell)\in L_2(w)} \Gamma_a(ij, k\ell) \oplus_2 \Gamma_b(ij, k\ell),$$

where $\oplus_2$ represents the sum modulo 2.

It is not difficult to check that t is a metric on $\mathrm{Red}(w)$. The symmetry and reflexivity properties are straightforward. For the triangular inequality, note that if $\Gamma_a(ij, k\ell) \neq \Gamma_b(ij, k\ell)$, then for every $c \in \mathrm{Red}(w)$ either $\Gamma_a(ij, k\ell) \neq \Gamma_c(ij, k\ell)$ or $\Gamma_c(ij, k\ell) \neq \Gamma_b(ij, k\ell)$ and thus $T_{a,b} \subseteq T_{a,c} \cup T_{c,b}$. It follows that $t(a,b) \leq t(a,c) + t(c,b)$.

Proposition 2.7. *Given $a, b \in Red(w)$, we have $t(a,b) \leq d(a,b)$.*

Proof: If a and b differ by a single commutation or braid relation, then by definition $|T_{a,b}| = 1$. This means that, if $d(a,b) = 1$ then $t(a,b) = 1$.

If $a = a_0, a_1, \ldots, a_n = b$ is a shortest path between a and b, there exists a sequence $\mathfrak{f} = \mathfrak{f}_{i_n} \cdots \mathfrak{f}_{i_2}\mathfrak{f}_{i_1}$ of Coxeter moves such that $\mathfrak{f}_{i_p}(a_{p-1}) = a_p$ with $\mathfrak{f}_{i_p} = \mathfrak{c}_{i_p}$ or $\mathfrak{f}_{i_p} = \mathfrak{b}_{i_p}$ for all $p = 1, \ldots, n$, and thus $|T_{a_r,a_{r+1}}| = 1$, for all $0 \leq r < s \leq n-1$. Since

$$T_{a,b} \subseteq \bigcup_{r=0}^{n-1} T_{a_r,a_{r+1}},$$

it follows that $t(a,b) = |T_{a,b}| \leq n = d(a,b)$. ∎

Note that from this last proposition it follows that the diameter of the metric space $(\mathrm{Red}(w), t)$ is a lower bound for the diameter of the graph $G(w)$, which in many cases is easier to compute.

Next example shows that, in general, the inequality in Proposition 2.7 is strict.

Example 2.1. The inversion set for the longest permutation $w = [4321]$ of $\mathcal{S}_4$ is $\mathrm{Inv}(w) = \{12, 13, 14, 23, 24, 34\}$ and we have

$$I_2(w) = \{(12,34), (13,24), (14,23)\} \text{ and } I_3(w) = \{123, 124, 134, 234\}.$$

The graph $G(w)$ is displayed in Figure 1. Consider the reduced words $a = 132132$ and $b = 231231$ of w, and note that $d(a,b) = 7$. As can be seen in Table 2.2, the orderings of $\mathrm{Inv}(w)$ induced by a and b are:

$$12 \lessdot_a 34 \lessdot_a 14 \lessdot_a 24 \lessdot_a 13 \lessdot_a 23$$

and

$$23 \lessdot_b 24 \lessdot_b 13 \lessdot_b 14 \lessdot_b 12 \lessdot_b 34.$$

We have, $\Gamma_a^{-1}(0) = \{(12,34), (14,23), (12,23), (12,24)\}$ and $\Gamma_b^{-1}(0) = \{(12,34), (13,34), (23,34)\}$, and therefore

$$\begin{aligned} t(a,b) &= |T_{a,b}| \\ &= |\Gamma_a^{-1}(0) \setminus \Gamma_b^{-1}(0)| + |\Gamma_b^{-1}(0) \setminus \Gamma_a^{-1}(0)| \\ &= |\{(14,23), (12,23), (12,24)\}| + |\{(13,34), (23,34)\}| \\ &= 5 < 7 = d(a,b). \end{aligned}$$

descent	permutation	Inv(w)
2	4 3 2 1	23
3	4 2 3 1	13
1	4 2 1 3	24
2	2 4 1 3	14
3	2 1 4 3	34
1	2 1 3 4	12
	1 2 3 4	

descent	permutation	Inv(w)
1	4 3 2 1	34
3	3 4 2 1	12
2	3 4 1 2	14
1	3 1 4 2	13
3	1 3 4 2	24
2	1 3 2 4	23
	1 2 3 4	

TABLE 2.2. The reduced words 132132 and 231231 for $w = [4321]$.

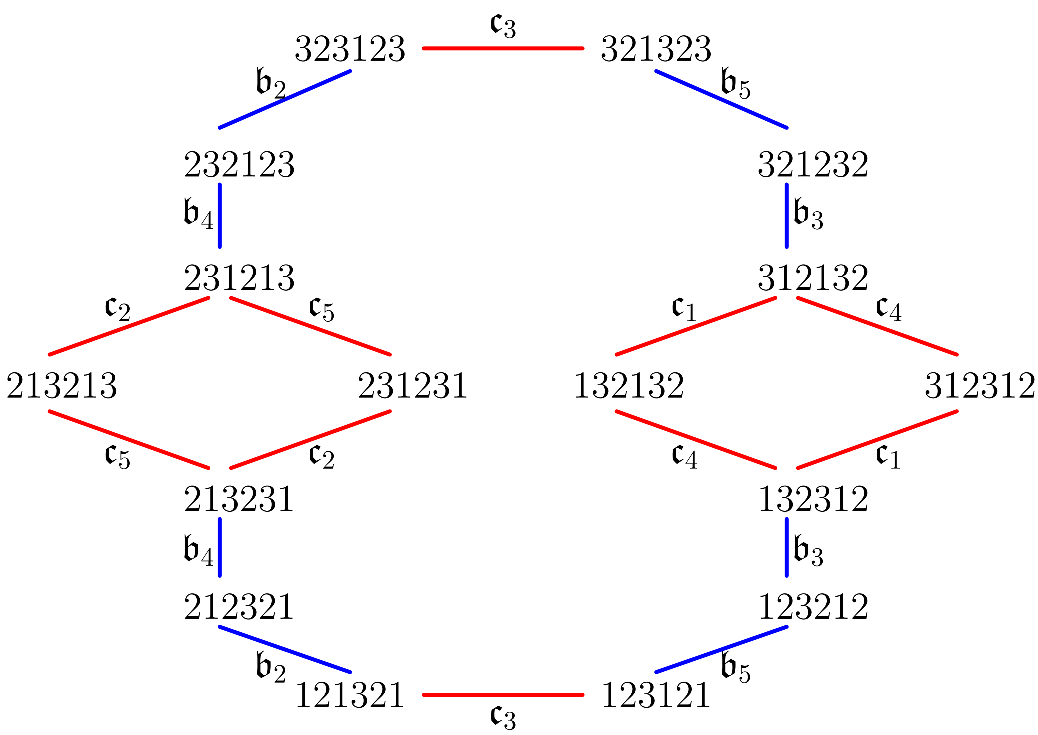

FIGURE 1. The graph $G(4321)$.

3. The main results

Although the metrics t and d are different, in the next result we show that they coincide inside each commutation class of $G(w)$. This coincidence of the metrics will allow us to obtain a formula for the diameter of any commutation class.

Proposition 3.1. *Let $a, b \in Red(w)$. If a and b are in the same commutation class, then $d(a,b) = t(a,b)$.*

Proof: Let $a = a_0, a_1, \ldots, a_n = b$ be a shortest path between a and b. Then, there exists a sequence $\mathfrak{c} = \mathfrak{c}_{i_n} \cdots \mathfrak{c}_{i_2}\mathfrak{c}_{i_1}$ of commutations such that $\mathfrak{c}_{i_p}(a_{p-1}) = a_p$ for all $p = 1, \ldots, n$, and thus $\mathfrak{c}(a) = b$. We claim that $T_{a_r,a_{r+1}} \neq T_{a_s,a_{s+1}}$, for all $0 \leq r < s \leq n-1$.

Recall that $T_{a_r,a_{r+1}}$ is formed by a single element of $I_2(w)$, for each r. By way of contradiction, assume $T_{a_r,a_{r+1}} = T_{a_s,a_{s+1}}$, with $r < s$, where s is the smallest integer for which this equality occurs. Without loss of generality, we may assume $r = 0$ and $s = n-1$, and thus $T_{a_0,a_1} = T_{a_{n-1},a_n}$. Then, writting $a_j = a_j^1 a_j^2 \cdots a_j^{\ell(w)}$ for $j = 0, 1, \ldots, n$, we have

$$\left\{\iota_{a_0}\left(i_1\right), \iota_{a_0}\left(i_1+1\right)\right\} = \left\{\iota_{a_{n-1}}\left(i_n\right), \iota_{a_{n-1}}\left(i_n+1\right)\right\} .$$

In particular this means that the letters $a_0^{i_1}$ and $a_0^{i_1+1}$ are consecutive letters in the words a_0, a_1, a_{n-1} and a_n, and satisfy

$$|a_0^{i_1} - a_0^{i_1+1}| \geq 2\,.$$

Since we can apply $\mathfrak{c}_{i_2}$ to the word a_1, the letters $a_1^{i_2}$ and $a_1^{i_2+1}$ also satisfy

$$|a_1^{i_2} - a_1^{i_2+1}| \geq 2\,.$$

If $\{a_0^{i_1}, a_0^{i_1+1}\} \cap \{a_1^{i_2}, a_1^{i_2+1}\} = \emptyset$, then we have $a_1^{i_2} = a_0^{j}$ and $a_1^{i_2+1} = a_0^{j+1}$ for some $j < i_1 - 1$ or $j > i_1 + 1$ and we can apply $\mathfrak{c}_{i_2}$ to the word a_0. On the other hand, suppose $a_1^{i_2+1} = a_0^{i_1+1}$. Then

$$a_0 = a_0^1 \cdots a_0^{i_1-1}\, a_0^{i_1}\, a_0^{i_1+1} \cdots\, a_0^{\ell}$$

and

$$a_1 = a_0^1 \cdots a_0^{i_1-1}\, a_0^{i_1+1}\, a_0^{i_1} \cdots\, a_0^{\ell}\,,$$

where $a_1^{i_2} = a_0^{i_1-1}$ and $a_1^{i_2+1} = a_0^{i_1+1}$. Since the word $\mathfrak{c}_{i_2}(a_1)$ is well-defined, we have $|a_0^{i_1-1} - a_0^{i_1+1}| \geq 2$ and

$$a_2 = a_0^1 \cdots a_0^{i_1+1}\, a_0^{i_1-1}\, a_0^{i_1} \cdots\, a_0^{\ell}\,.$$

But notice that in the word a_{n-1} the letters $a_0^{i_1+1}$ and $a_0^{i_1}$ are consecutive letters, which means that $|a_0^{i_1-1} - a_0^{i_1}| \geq 2$ since some map $\mathfrak{c}_{i_k}$ must exchange the positions of these letters. These proves that $\mathfrak{c}_{i_2}(a_0)$ is well defined and

$$\mathfrak{c}_{i_2}(a_0) = a_0^1 \cdots a_0^{i_1}\, a_0^{i_1-1}\, a_0^{i_1+1} \cdots\, a_0^{\ell}\,.$$

That is $\mathfrak{c}_{i_2}(a_0)$ differ from a_2 only on the positions of $a_0^{i_1}$ and $a_0^{i_1+1}$. The same argument can be repeated, showing that $\mathfrak{c}_{i_{n-1}} \cdots \mathfrak{c}_{i_2}(a_0)$ is well-defined and differ from a_{n-1} only on the positions of the letters $a_0^{i_1}$ and $a_0^{i_1+1}$ proving that $\mathfrak{c}_{i_{n-1}} \cdots \mathfrak{c}_{i_2}(a_0) = b$. We thus get a contradiction, since this path is shorter than the original one. So, we must have $T_{a_r,a_{r+1}} \neq T_{a_s,a_{s+1}}$, for all $0 \leq r < s \leq n-1$, proving that $d(a,b) = t(a,b)$. ■

Using the coincidence of the metrics t and d inside each commutation class, given a reduced word $a \in \mathrm{Red}(w)$, we can derive a formula for the diameter of a commutation class of $G(w)$.

Theorem 3.2. *Let $a \in Red(w)$. Then $d(a_-, a_+) = |T_{[a]}|$, where a_- (resp. a_+) is the minimum (resp. maximal) word in the commutation class $G_{[a]}$ for the lexicographic order.*

Proof: By Proposition 3.1, we have $d(a_-, a_+) = t(a_-, a_+) = |T_{a_-,a_+}|$. So, it is enough to prove that $T_{a_-,a_+} = T_{[a]}$. By definition of $T_{[a]}$, $T_{a_-,a_+} \subseteq T_{[a]}$.

To prove the other inclusion, it is enough to show that if a pair $ij, k\ell \in \mathrm{Inv}(w)$ satisfies

$$ij <_{a_-} k\ell, \text{ and } ij <_{a_+} k\ell,$$

then $ij <_c k\ell$ for any $c \in G_{[a]}$.

Recall that for any inversion xy, the corresponding letter in a reduced does not change inside a commutation class.

Let $a_- = a_-^1 a_-^2 \cdots a_-^{\ell(w)}$, $a_+ = a_+^1 a_+^2 \cdots a_+^{\ell(w)}$, and let $p = \iota_{a_-}^{-1}(ij)$, $q = \iota_{a_-}^{-1}(k\ell)$, $p' = \iota_{a_+}^{-1}(ij)$, $q' = \iota_{a_+}^{-1}(k\ell)$. By our assumptions $p < q$, $p' < q'$, *i.e.* a_-^p is on the left of a_-^q in the word a_- and $a_+^{p'}$ is on the left of $a_+^{q'}$ in the word a_+. Note also that $a_-^p = a_+^{p'}$ and $a_-^q = a_+^{q'}$.

If $|a_-^p - a_-^q| \leq 1$, then the letters a_-^p and a_-^q maintain their relative positions in all words in the commutation class of a. Suppose now that $|a_-^p - a_-^q| \geq 2$ and assume that $a_-^p < a_-^q$ (otherwise one could argue with a_+ instead of a_-). Since $a_-^p = a_+^{p'}$ is on the left of $a_-^q = a_+^{q'}$ in the word a_+, the definition of lexicographic order implies the existence of the subword $a_+^{p'}\,(a_+^{p'}+1) \cdots (a_+^{q'}-1)\,a_+^{q'}$ in a_+. This subword of consecutive letters must be a subword in any reduced word of the same class. This proves that $ij <_c k\ell$ for any $c \in G_{[a]}$. ■

Corollary 3.3. *Let $a \in Red(w)$. The diameter of the commutation class $G_{[a]}$ is given by*

$$d(a_-, a_+) = |T_{[a]}|.$$

Proof: Given any $b, c \in [a]$, by Proposition 3.1 and Theorem 3.2, $d(b,c) = |T_{b,c}| \leq |T_{a_-,a_+}| = d(a_-, a_+)$. ■

The diameter of a commutative class $G_{[a]}$ is the cardinal of the set $T_{[a]}$ which is contained in $I_2(w)$. In order to give a closed formula for the diameter we need to divide the set $I_2(w)$ into the sets of separated, crossed and nested pairs of inversions.

Definition 3.4. Given $w \in \mathfrak{S}_n$, define the following disjoint subsets of $I_2(w)$:

$$\begin{aligned} I_2^S(w) &:= \{(ij, k\ell) \in I_2(w) : i < j < k < \ell\}, \\ I_2^C(w) &:= \{(ik, j\ell) \in I_2(w) : i < j < k < \ell\}, \\ I_2^N(w) &:= \{(i\ell, jk) \in I_2(w) : i < j < k < \ell\}. \end{aligned}$$

See Example 3.1 for the construction of these sets for the permutation $w = 514632$.

Definition 3.5. Given $a \in \mathrm{Red}(w)$, define the following disjoint subsets of $T_{[a]}$:

$$\begin{aligned} T_{[a]}^S &:= I_2^S(w) \cap T_{[a]}, \\ T_{[a]}^C &:= I_2^C(w) \cap T_{[a]}, \\ T_{[a]}^N &:= I_2^N(w) \cap T_{[a]}. \end{aligned}$$

Next lemmas show that knowing the function Γ_a suffices to compute the diameter of the commutation class $G_{[a]}$.

Lemma 3.6. *Let a be a reduced word of a permutation w and $(ij, k\ell) \in I_2^S(w)$, with $i < j < k < \ell$. The pair $(ij, k\ell) \in T_{[a]}^S$ if and only if either of the following conditions hold:*

- $ij\ell \notin I_3(w)$ *and* $ik\ell \notin I_3(w)$;
- $ij\ell \notin I_3(w)$ *and* $\Gamma_a(ik\ell) = 1$;
- $\Gamma_a(ij\ell) = 0$ *and* $ik\ell \notin I_3(w)$;
- $\Gamma_a(ij\ell) = 0$ *and* $\Gamma_a(ik\ell) = 1$;
- $\Gamma_a(ij\ell) = 1$ *and* $\Gamma_a(ik\ell) = 0$.

Proof: Recall that by Proposition 2.4, the image of the Γ_a function when restricted to $I_3(w)$ is invariant in each commutation class of $G(w)$, and thus the result of this lemma does not depend on a particular word of $G_{[a]}$.

	$ik\ell \notin I_3$	$\Gamma_a(ik\ell)=0$	$\Gamma_a(ik\ell)=1$
$ij\ell \notin I_3$	✓	✗	✓
$\Gamma_a(ij\ell)=0$	✓	✗	✓
$\Gamma_a(ij\ell)=1$	✗	✓	✗

TABLE 3.1. Separated pairs of inversions $(ij,k\ell) \in I_2^S(w)$

Suppose $(ij,k\ell) \in I_2^S(w)$ and non of these five conditions is fulfilled. Then there are four cases to study, shown in Table 3.1 with ✗, and for each case we have to prove that $(ij,k\ell) \notin T_{[a]}^S$.

1st $ij\ell \notin I_3(w)$ and $\Gamma_a(ik\ell)=0$.
Since $ij\ell \notin I_3(w)$ and $ij \in Inv(w)$, we find that $j\ell$ is not an inversion of the permutation w. Also $ik\ell \in I_3(w)$, and thus $j\ell ki$ must be a subword of the permutation w, which implies that jk is not an inversion. Since $\Gamma_a(ik\ell)=0$, we have $ik <_a i\ell <_a k\ell$. The inversion ik can only occur after either ij or jk. But $jk \notin \mathrm{Inv}(w)$, and so we must have $ij <_a ik$. We conclude that $ij <_a k\ell$, which implies that $(ij,k\ell) \notin T_{[a]}^S$.

2nd $ik\ell \notin I_3(w)$ and $\Gamma_a(ij\ell)=1$.
This case is similar to previous one by symmetry, and therefore we must have $ij >_a k\ell$ for all reduced words a satisfying these conditions. It follows that $(ij,k\ell) \notin T_{[a]}^S$.

3rd $\Gamma_a(ij\ell)=0$ and $\Gamma_a(ik\ell)=0$.
Since $\Gamma_a(ij\ell)=\Gamma_a(ik\ell)=0$, we have $ij <_a i\ell$ and $i\ell <_a k\ell$ and therefore $ij <_a k\ell$. We conclude that $(ij,k\ell) \notin T_{[a]}^S$.

4th $\Gamma_a(ij\ell)=1$ and $\Gamma_a(ik\ell)=1$.
Since $\Gamma_a(ij\ell)=\Gamma_a(ik\ell)=1$, we have $ij >_a i\ell$ and $i\ell >_a k\ell$ and therefore $ij >_a k\ell$. We conclude that $(ij,k\ell) \notin T_{[a]}^S$.

Reciprocally, suppose that $(ij,k\ell) \in I_2^S(w)$ satisfies one of the five conditions, identified by ✓ in Table 3.1. We need to prove that $(ij,k\ell) \in T_{[a]}^S$.

1st $ij\ell \notin I_3(w)$ and $ik\ell \notin I_3(w)$.
Since $ij\ell \notin I_3(w)$ and $ik\ell \notin I_3(w)$, we have $j\ell, ik \notin \mathrm{Inv}(w)$. Therefore, there are only two possibilities for the subword w' of the permutation w formed by the letters $i<j<k<\ell$. Either $w' = ji\ell k$ or $w' = j\ell ik$. In the first case, there are only two inversions on the letters $ijk\ell$, and they can be performed in any order. In the second case, there

is an adicional inversion $i\ell$, which must be done after the previous two inversions. In any case, the reduced words associated with these two choices of ordering the inversions ij and $k\ell$ have distinct Γ values on the pair $(ij, k\ell)$. It follows that $(ij, k\ell) \in T^S_{[a]}$.

2nd $ij\ell \notin I_3(w)$ and $\Gamma_a(ik\ell) = 1$.

Since $ij\ell \notin I_3(w)$, we have $j\ell \notin \text{Inv}(w)$. On the other hand $ik\ell \in I_3(w)$, and therefore the subword of the permutation w formed by the letters $i < j < k < \ell$ is $j\ell ki$. There are four inversions in this letters, namely $ij, k\ell, ik$ and $i\ell$, and since $\Gamma_a(ik\ell) = 1$, we must have $k\ell <_a i\ell <_a ik$. Since $j\ell \notin \text{Inv}(w)$, we must have $ij <_a i\ell$, and thus ij and $k\ell$ must be the first two inversions on these letters to occur, and can be made in any order, followed by $i\ell$. As in the previous case we conclude that $(ij, k\ell) \in T^S_{[a]}$.

3rd $ik\ell \notin I_3(w)$ and $\Gamma_a(ij\ell) = 0$.

This case is similar to previous one by symmetry, and therefore we also have $(ij, k\ell) \in T^S_{[a]}$.

4th $\Gamma_a(ij\ell) = 0$ and $\Gamma_a(ik\ell) = 1$.

Under these conditions, $ij\ell, ik\ell \in I_3(w)$ and then the subword of the permutation w formed by the letters $i < j < k < \ell$ is either ℓkji or ℓjki. Since $\Gamma_a(ij\ell) = 0$ and $\Gamma_a(ik\ell) = 1$, we must have $ij <_a i\ell <_a j\ell$ and $k\ell <_a i\ell <_a ik$. If $jk \in \text{Inv}(w)$, then we must have $j\ell <_a jk$. In any case, the first two inversions in these letters must be ij and $k\ell$, and they can be made in any order, followed by $i\ell$. Therefore $(ij, k\ell) \in T^S_{[a]}$.

5th $\Gamma_a(ij\ell) = 1$ and $\Gamma_a(ik\ell) = 0$.

This case is similar to previous one by symmetry, and therefore we also have $(ij, k\ell) \in T^S_{[a]}$. ■

The proofs of the next two lemmas follows the same reasoning of Lemma 3.6, and some details will be omitted.

Lemma 3.7. *Let a be a reduced word of a permutation w and $(ik, j\ell) \in I^S_2(w)$, with $i < j < k < \ell$. The pair $(ik, j\ell) \in T^C_{[a]}$ if and only if either of the following conditions hold:*

- $ij\ell \notin I_3(w)$ *and* $ik\ell \notin I_3(w)$*;*
- $ij\ell \notin I_3(w)$ *and* $\Gamma_a(ik\ell) = 0$*;*
- $\Gamma_a(ij\ell) = 0$ *and* $\Gamma_a(ik\ell) = 1$*;*
- $\Gamma_a(ij\ell) = 1$ *and* $ik\ell \notin I_3(w)$*;*

- $\Gamma_a(ij\ell) = 1$ *and* $\Gamma_a(ik\ell) = 0$.

	$ik\ell \notin I_3$	$\Gamma_a(ik\ell) = 0$	$\Gamma_a(ik\ell) = 1$
$ij\ell \notin I_3$	✓	✓	✗
$\Gamma_a(ij\ell) = 0$	✗	✗	✓
$\Gamma_a(ij\ell) = 1$	✓	✓	✗

TABLE 3.2. Crossing pairs of inversions $(ik, j\ell) \in I_2^C(w)$

Proof: Suppose $ik, j\ell \in \text{Inv}(w)$ and none of these five conditions is fulfilled. Then there are four cases to study, and for each case we have to prove that $(ik, j\ell) \notin T_{[a]}^C$.

1st $ij\ell \notin I_3(w)$ and $\Gamma_a(ik\ell) = 1$.

Since $ij\ell \notin I_3(w)$ and $j\ell \in Inv(w)$, we find that ij is not an inversion.

Since $\Gamma_a(ik\ell) = 1$, we have $i\ell <_a ik$. The inversion $i\ell$ can only occur after either ij or $j\ell$. But $ij \notin \text{Inv}(w)$, and then we must have $j\ell <_a i\ell$. We conclude that $j\ell <_a ik$ which implies that $(ik, j\ell) \notin T_{[a]}^C$.

2nd $ik\ell \notin I_3(w)$ and $\Gamma_a(ij\ell) = 0$.

This case is similar to previous one by symmetry, and therefore we must have $j\ell >_a ik$ for all reduced words a satisfying these conditions. It follows that $(ik, j\ell) \notin T_{[a]}^C$.

3rd $\Gamma_a(ij\ell) = 0$ and $\Gamma_a(ik\ell) = 0$.

Since $\Gamma_a(ik\ell) = \Gamma_a(ij\ell) = 0$, we have $ik <_a i\ell$ and $i\ell <_a j\ell$ and therefore $ik <_a j\ell$. We conclude that $(ik, j\ell) \notin T_{[a]}^C$.

4th $\Gamma_a(ij\ell) = 1$ and $\Gamma_a(ik\ell) = 1$.

Since $\Gamma_a(ik\ell) = \Gamma_a(ij\ell) = 0$, we have $ik >_a i\ell$ and $i\ell >_a j\ell$ and therefore $ik >_a j\ell$. We conclude that $(ik, j\ell) \notin T_{[a]}^C$.

Reciprocally, suppose that $(ik, j\ell) \in I_2^C(w)$ satisfies one of the five conditions, identified by ✓ in Table 3.2. We need to prove that $(ik, j\ell) \in T_{[a]}^C$.

1st $ij\ell \notin I_3(w)$ and $ik\ell \notin I_3(w)$.

Since $ij\ell \notin I_3(w)$ and $ik\ell \notin I_3(w)$, we have $ij, k\ell \notin \text{Inv}(w)$, which means that jk must be an inversion of w. Therefore, there are only two possibilities for the subword w' of the permutation w formed by the letters $i < j < k < \ell$. Either $w' = ki\ell j$ or $w' = k\ell ij$. In the first case, there is only one option for the first inversion on the letters i, j, k, ℓ, namely jk, and afterwards there are only two possible inversions, ik and $j\ell$, which can be performed in any order, giving rise to w'. In the second case, there is an adicional inversion $i\ell$, which must be done after the previous three inversions. It follows that $(ik, j\ell) \in T^C_{[a]}$.

2nd $ij\ell \notin I_3(w)$ and $\Gamma_a(ik\ell) = 0$.

Since $ij\ell \notin I_3(w)$, we have $ij \notin \mathrm{Inv}(w)$. As in the previous case, this means that jk is an inversion of w. Therefore, the subword of the permutation w formed by the letters i, j, k, ℓ is ℓkij. There are five inversions in this letters, namely $ik, i\ell, jk, j\ell$ and $k\ell$, and since $\Gamma_a(ik\ell) = 0$, we must have $ik <_a i\ell <_a k\ell$. Since $ij \notin \mathrm{Inv}(w)$, we must have $j\ell <_a i\ell$. The first inversion to be performed must be jk. Then, there are only two possible inversions afterwards, ik and $j\ell$, which can be performed in any order, followed by $i\ell$ and $k\ell$. It follows that $(ik, j\ell) \in T^C_{[a]}$.

3rd $ik\ell \notin I_3(w)$ and $\Gamma_a(ij\ell) = 1$.

This case is similar to previous one by symmetry, and therefore we also have $(ik, j\ell) \in T^C_{[a]}$.

4th $\Gamma_a(ij\ell) = 0$ and $\Gamma_a(ik\ell) = 1$.

Since $ij\ell, ik\ell \in I_3(w)$, the subword of the permutation w formed by the letters i, j, k, ℓ is either ℓkji or ℓjki. Since $\Gamma_a(ij\ell) = 0$ and $\Gamma_a(ik\ell) = 1$, we must have $ij <_a i\ell <_a j\ell$ and $k\ell <_a i\ell <_a ik$. If $jk \notin \mathrm{Inv}(w)$, then the first two inversions on the letters $ijk\ell$ must be ij and $k\ell$, and they can be made in any order, followed by $i\ell$, given the word $j\ell ik$. There are now only two possible inversions, ik and $j\ell$, which can be performed in any order. On the other hand, if $jk \in \mathrm{Inv}(w)$, we cannot start by applying jk since this inversion must be followed by either ik or $j\ell$, contradicting the assumptions $\Gamma_a(ij\ell) = 0$ and $\Gamma_a(ik\ell) = 1$. Thus we must apply the inversions as before, followed by jk. It follows that $(ik, j\ell) \in T^C_{[a]}$.

5th $\Gamma_a(ij\ell) = 1$ and $\Gamma_a(ik\ell) = 0$.

This case is similar to previous one by symmetry, and therefore we also have $(ik, j\ell) \in T^C_{[a]}$. ■

Lemma 3.8. *Let a be a reduced word of a permutation w and $(i\ell, jk) \in I_2^S(w)$, with $i < j < k < \ell$. The pair $(i\ell, jk) \in T^N_{[a]}$ if and only if either of the following conditions hold:*

- *$ij\ell \notin I_3(w)$ and $\Gamma_a(ik\ell) = 1$;*

- $\Gamma_a(ijk) = 0$ *and* $\Gamma_a(jk\ell) = 0$*;*
- $\Gamma_a(ijk) = 0$ *and* $jk\ell \notin I_3(w)$*;*
- $\Gamma_a(ijk) = 1$ *and* $\Gamma_a(jk\ell) = 1$*.*

	$jk\ell \notin I_3$	$\Gamma_a(jk\ell) = 0$	$\Gamma_a(jk\ell) = 1$
$ijk \notin I_3$	✗	✗	✓
$\Gamma_a(ijk) = 0$	✓	✓	✗
$\Gamma_a(ijk) = 1$	✗	✗	✓

TABLE 3.3. Nested pairs of inversions $(i\ell, jk) \in I_2^N(w)$

Proof: Suppose $(i\ell, jk) \in I_2^N(w)$ and non of these four conditions is fulfilled. Then there are five cases to study, shown in Table 3.3 with ✗, and for each case we have to prove that $(i\ell, jk) \notin T_{[a]}^N$.

1st $ijk \notin I_3(w)$ and $jk\ell \notin I_3(w)$.
Since $ijk \notin I_3(w)$ and $jk \in Inv(w)$, we find that ij is not an inversion. In the same way, since $(jk\ell) \notin I_3(w)$ and $k\ell \in Inv(w)$, we find that $k\ell$ is not an inversion. Under these conditions the only possible choice for the first inversion between $i < j < k < \ell$ is jk and then $jk <_a i\ell$.
It follows that $(i\ell, jk) \notin T_{[a]}^N$.

2nd $ijk \notin I_3(w)$ and $\Gamma_a(jk\ell) = 0$.
Since $ijk \notin I_3(w)$ and $jk \in Inv(w)$, we find that ij is not an inversion.
From $\Gamma_a(jk\ell) = 0$, we get $jk <_a k\ell$. This means that the only possible choice for the first inversion between $i < j < k < \ell$ is jk and then $jk <_a i\ell$.
It follows that $(i\ell, jk) \notin T_{[a]}^N$.

3rd $jk\ell \notin I_3(w)$ and $\Gamma_a(ijk) = 1$.
Since $jk\ell \notin I_3(w)$ and $jk \in Inv(w)$, we find that $k\ell$ is not an inversion.
From $\Gamma_a(ijk) = 1$, we get $jk <_a ij$. This means that the only possible choice for the first inversion between $i < j < k < \ell$ is jk and then $jk <_a i\ell$.
It follows that $(i\ell, jk) \notin T_{[a]}^N$.

4th $\Gamma_a(ijk) = 0$ and $\Gamma_a(jk\ell) = 1$.

Since $\Gamma_a(ijk) = 0$ and $\Gamma_a(jk\ell) = 1$, we have $ij <_a jk$ and $j\ell <_a jk$. But exactly one of the inversions ij or $j\ell$ must occur after $i\ell$. Therefore $i\ell <_a jk$.

We conclude that $(i\ell, jk) \notin T^N_{[a]}$.

5th $\Gamma_a(ijk) = 1$ and $\Gamma_a(jk\ell) = 0$.

Since $\Gamma_a(ijk) = 1$ and $\Gamma_a(jk\ell) = 0$, we have $jk <_a ij$ and $jk <_a k\ell$. This means that the only possible choice for the first inversion between $i < j < k < \ell$ is jk and then $jk <_a i\ell$.

We conclude that $(i\ell, jk) \notin T^N_{[a]}$.

Reciprocally, suppose that $(i\ell, jk) \in I^N_2(w)$ satisfies one of the four conditions, identified by ✓ in Table 3.3. We need to prove that $(i\ell, jk) \in T^N_{[a]}$.

1st $ijk \notin I_3(w)$ and $\Gamma_a(jk\ell) = 1$.

Since $ijk \notin I_3(w)$, ij is not an inversion and thus the subword w' of the permutation w formed by the letters i, j, k, ℓ is either ℓikj or ℓkij. From the definition, $\Gamma_a(jk\ell) = 1$ means that $k\ell <_a j\ell <_a jk$. Since ij is not an inversion, we must have $j\ell <_a i\ell$. If $w' = \ell ikj$, then the inversions $i\ell$ and jk must be performed after all the other inversions on these letters, and can be done in any order. If $w' = \ell kij$, then there is an extra inversion ik which must be done after both of these two inversions, $i\ell$ and jk, which again can be made in any order. It follows that $(i\ell, jk) \in T^N_{[a]}$.

2nd $jk\ell \notin I_3(w)$ and $\Gamma_a(ijk) = 0$.

This case is similar to previous one by symmetry, and therefore we also have $(i\ell, jk) \in T^N_{[a]}$.

3rd $\Gamma_a(ijk) = 0$ and $\Gamma_a(jk\ell) = 0$.

It follows that $ij <_a ik <_a jk$ and $jk <_a j\ell <_a k\ell$. The sequence of these five inversions impose that the inversion $i\ell$ must occur between inversions ik and $j\ell$, but the relative order to jk is not imposed. Therefore $(i\ell, jk) \in T^N_{[a]}$.

4th $\Gamma_a(ijk) = 1$ and $\Gamma_a(jk\ell) = 1$.

Again, by symmetry to the previous case one conclude $(i\ell, jk) \in T^N_{[a]}$.

■

We can now state our main theorem, which provides a closed formula for the diameter of a commutation class.

Theorem 3.9. *Let $a \in Red(w)$. The diameter of the commutation class $G_{[a]}$ is given by*

$$|T_{[a]}| = |T^D_{[a]}| + |T^C_{[a]}| + |T^N_{[a]}|.$$

Proof: From Corollary 3.3, we have $diam(G_{[a]}) = |T_{[a]}|$. From the definition, $T_{[a]} = T^D_{[a]} \dot{\cup} T^C_{[a]} \dot{\cup} T^N_{[a]}$ is a disjoint union, and therefore, $|T_{[a]}| = |T^D_{[a]}| + |T^C_{[a]}| + |T^N_{[a]}|$. ■

In view of this theorem, Lemmas 3.6, 3.7 and 3.8 give an algorithm to compute the diameter of $G_{[a]}$ for any commutation class. The number of operations used in this algorithm is smaller than those needed to compute the diameter of a commutation class using Corollary 3.3, and gives a characterization of the pairs of inversions in $I_2(w)$ which contributes to the diameter. This can be better seen in the following examples and results.

Example 3.1. Let $w = [514632] \in \mathcal{S}_6$. The graph $G(w)$ has 216 reduced words, 918 Coxeter relations and 20 commutation classes. We will use the formula given by Theorem 3.9 to compute the diameter of the commutation class $G_{[a]}$, for the reduced word $a = 324342154$ of w. The set of inversions is $\mathrm{Inv}(w) = \{15, 23, 24, 25, 26, 34, 35, 36, 45\}$, and $I_3(w) = \{234, 235, 236, 245, 345\}$. The reduced word a induces the following order of $\mathrm{Inv}(w)$:

$$34 \lessdot_a 24 \lessdot_a 35 \lessdot_a 25 \lessdot_a 23 \lessdot_a 45 \lessdot_a 15 \lessdot_a 26 \lessdot_a 36,$$

and thus we have

$$\Gamma_a(234) = 1, \quad \Gamma_a(235) = 1, \quad \Gamma_a(236) = 0, \quad \Gamma_a(245) = 0, \quad \Gamma_a(345) = 0.$$

The sets of separated, crossed and nested pairs of inversions are

$$\begin{aligned} I_2^S(w) =& \{(23, 45)\}, \\ I_2^C(w) =& \{(15, 26), (15, 36), (24, 35), (24, 36), (25, 36)\}, \\ I_2^N(w) =& \{(15, 23), (15, 24), (15, 34), (25, 34), (26, 34), (26, 35), (26, 45), (36, 45)\}. \end{aligned}$$

We have $T^S_{[a]} = \{(23, 45)\}$, since $\Gamma_a(235) = 1$ and $\Gamma_a(245) = 0$, as we can check in Table 3.1. For the sets $T^C_{[a]}$ and $T^C_{[a]}$ we can use Tables 3.2 and 3.3 to obtain:

$(ik, j\ell)$	$\Gamma_a(ij\ell)$	$\Gamma_a(ik\ell)$	$\in T^C_{[a]}$
(15,26)	-	-	yes
(15,36)	-	-	yes
(24,35)	1	0	yes
(24,36)	0	-	no
(25,36)	0	-	no

$(i\ell, jk)$	$\Gamma_a(ijk)$	$\Gamma_a(jk\ell)$	$\in T^N_{[a]}$
(15,23)	-	1	yes
(15,24)	-	0	no
(15,34)	-	0	no
(25,34)	1	0	no
(26,34)	1	-	no
(26,35)	1	-	no
(26,45)	0	-	yes
(36,45)	0	-	yes

Therefore $T^C_{[a]} = \{(15, 26), (15, 36), (24, 35)\}$, $T^N_{[a]} = \{(15, 23), (26, 45), (36, 45)\}$ and thus $|T^C_{[a]}| = |T^N_{[a]}| = 3$. It follows that the diameter of the commutation class $G_{[a]}$ is

$$diam(G_{[a]}) = 1 + 3 + 3 = 7.$$

A permutation having only one commutative class is said to be a fully commutative permutation. A result of Billey, Jockusch and Stanley [4] shows that a permutation is fully commutative if and only if it avoids the pattern 321. That is, w is fully commutative if and only if the set $I_3(w)$ is empty.

Corollary 3.10. *If w is a fully commutative permutation, then the diameter of the graph $G(w)$ is $|I^S_2(w)| + |I^C_2(w)|$.*

Proof: If w is fully commutative, then it has only one commutation class, and the diameter of $G(w)$ is equal to the diameter of its only commutation class. Moreover, w is fully commutative if and only if the set $I_3(w)$ is empty. By Theorem 3.9, the diameter of $G(w)$ is given by $|T^S_{[a]}| + |T^C_{[a]}| + |T^N_{[a]}|$, where a is any reduced word of w. By Lemma 3.8, the set $T^N_{[a]}$ is empty, and by Lemmas 3.6 and 3.7, $T^S_{[a]} = I^S_2(w)$ and $T^C_{[a]} = I^C_2(w)$ (see leftmost top cell of tables 3.1, 3.2 and 3.3). ■

Example 3.2. The permutation $w = 24517386$ is fully commutative, since $I_3(w) = \emptyset$ or equivalently it is 321-avoiding. The graph $G(w)$ has 344 reduced words and 1818 commutations. We now compute the diameter of $G(w)$ using Corollary 3.10.

We have $\mathrm{Inv}(w) = \{12, 14, 15, 34, 35, 37, 67, 68\}$ and thus,

$$\begin{aligned} I_2^S(w) =& \{(12, 34), (12, 35), (12, 37), (12, 67), (12, 68), (14, 67), (14, 68), \\ & (15, 67), (15, 69), (34, 67), (34, 68), (35, 67), (35, 68)\} \\ I_2^C(w) =& \{(14, 35), (14, 37), (15, 37), (37, 68)\}. \end{aligned}$$

It follows that the diameter of $G(24517386) = |I_2^S(w)|+|I_2^C(w)| = 13+4 = 17$.

Note that in this case, $I_2^N(w) = \{(15, 34)\}$ is not empty, although it is not necessary to compute the diameter.

The minimal lexicographic reduced word for the only commutation class is 13243657 and the maximal lexicographic reduced word is 67345123. By Corollary 3.3 we need 17 commutations to obtain one word from the other.

To see another application of our main theorem, recall that a permutation $w \in \mathcal{S}_n$ is called Grassmannian if it has at most one descent. In other words, there is a unique integer $1 \leq r \leq n-1$ such that $w_1 < w_2 < \cdots < w_r > w_{r+1} < \cdots < w_n$, or w is the identity. Grassmannian permutations can be defined using their Lehmer code $L(w) = (c_1, c_2, \ldots, c_n)$, where

$$c_i = \#\{j : j > i \text{ e } w_j < w_i\},$$

i.e. c_i counts the number of inversions $w_j w_i$. It is easy to see that w is a Grassmannian permutation with descent r if and only if $c_1 \leq c_2 \leq \cdots c_r \leq n-r$ and $c_{r+1} = \cdots = c_n = 0$.

Having at most one descent, Grassmannian permutations are a special type of the more general class of fully commutative permutations. In this particular case, we can compute its diameter using the Lehmer code.

Corollary 3.11. *Let $w \in \mathcal{S}_n$ be a Grassmannian permutation with Lehmar code $L(w) = (c_1, \ldots, c_n)$. Then, the diameter of the graph $G(w)$ is given by*

$$\sum_{1 \leq i < k \leq r} c_i(c_k - c_i) + \sum_{i=1}^{r-1} \binom{c_i}{2}(r-i).$$

Proof: If w is the identity, then its diameter is clearly zero. Otherwise, let r be the only descent of the Grassmannian permutation w. By Corollary 3.10, the diameter of $G(w)$ is given by $|I_2^S(w)| + |I_2^C(w)|$. Note that if $w_j w_i$ is an inversion of w, then $j > r$ and $i \leq r$.

Recall that a pair of inversions $(w_j w_i, w_\ell w_k)$ is in the set $I_2^S(w)$ if $w_i < w_\ell$, and thus by the definition of Lehman code, we have $c_i < c_k$. For each i there are c_i inversions $w_j w_i$. We now need the number of inversions $w_\ell w_k$ such that $w_i < w_\ell$ and therefore $w_\ell w_i$ is not an inversion. From the Lehman code, we find that there are $c_k - c_i$ such inversions. Thus

$$|I_2^S(w)| = \sum_{1 \le i < k \le r} c_i(c_k - c_i)\,.$$

Note that a pair of inversions $(w_j w_i, w_\ell w_k)$ is in the set $I_2^C(w)$ if $w_j < w_\ell < w_i < w_k$. For a Grassmannian permutation this implies that $w_\ell w_i$ is also an inversion, since $\ell > r$ and $i \le r$. To compute the number of pairs of inversions in $I_2^C(w)$ we need first to choose, for each i, two different inversion $w_j w_i$ and $w_\ell w_i$, with $w_j < w_\ell$. This can be done in $\binom{c_i}{2}$ ways. Next, for each k such that $i < k \le r$, $w_\ell w_k$ is an inversion with $w_j < w_\ell < w_i < w_k$. Therefore

$$|I_2^C(w)| = \sum_{i=1}^{r-1} \binom{c_i}{2}(r-i)\,. \qquad \blacksquare$$

Example 3.3. The permutation $w = [45681237] \in \mathcal{S}_8$ is Grassmanian and its only descent is $r = 4$. The graph $G(w)$ has 3432 reduced words and 24948 commutations. To compute the diameter using Corollary 3.11 we start with the Lehman code, $L(w) = (3,3,3,4,0,0,0,0)$. The diameter is

$$\begin{aligned}
&\sum_{1 \le i < k \le 4} c_i(c_k - c_i) + \sum_{i=1}^{3} \binom{c_i}{2}(4-i) = \\
&= 0 + 0 + 3 + 0 + 3 + 3 + \binom{3}{2}(4-1) + \binom{3}{2}(4-2) + \binom{3}{2}(4-3) = 27.
\end{aligned}$$

As a final application of Theorem 3.9, we give formulas for the diameter of the commutation classes of the reduced words w_{min} and w_{max}, for any permutation $w \in \mathcal{S}_n$.

Corollary 3.12.

(1) *The diameter of the commutation class $G_{[w_{min}]}$ is given by*

$$\begin{aligned}
|T_{[w_{min}]}| &= |\{(ij, k\ell) \in I_2^S(w) \,:\, ik\ell \notin I_3(w)\}| \\
&\quad + |\{(ik, j\ell) \in I_2^C(w) \,:\, ij\ell \notin I_3(w)\}| \\
&\quad + |\{(i\ell, jk) \in I_2^N(w) \,:\, ijk \in I_3(w)\}|.
\end{aligned}$$

(2) *The diameter of the commutation class* $G_{[w_{max}]}$ *is given by*

$$\begin{aligned}|T_{[w_{max}]}| = &|\{(ij, k\ell) \in I_2^S(w) \,:\, ij\ell \notin I_3(w)\}| \\ &+ |\{(ik, j\ell) \in I_2^C(w) \,:\, ik\ell \notin I_3(w)\}| \\ &+ |\{(i\ell, jk) \in I_2^N(w) \,:\, jk\ell \in I_3(w)\}|.\end{aligned}$$

Proof: Recall that for all $xyz \in I_3(w)$, $\Gamma_{w_{min}}(xyz) = 0$ and $\Gamma_{w_{max}}(xyz) = 1$. Then the results follow from Theorem 3.9 and Lemmas 3.6, 3.7, 3.8. ∎

Notice that in the case of a fully commutative permutation the classes of w_{min} and w_{max} coincide, and thus Corollary 3.10 is a special case of this result.

Example 3.4. Consider the permutation $w = [426513] \in \mathcal{S}_6$. The graph $G(w)$ has 384 reduced words and 1898 Coxeter relations. The commutation classes of $w_{min} = 132143543$ and $w_{max} = 534523412$ have 70 and 44 reduced words, respectively. We have $\mathrm{Inv}(w) = \{12, 14, 15, 16, 24, 34, 35, 36, 56\}$ and thus,

$$\begin{aligned}I_2^S(w) =&\{(12, 34), (12, 35), (12, 36), (12, 56), (14, 56), (24, 56), (34, 56)\},\\ I_2^C(w) =&\{(14, 35), (14, 36), (15, 36), (24, 35), (24, 36)\},\\ I_2^N(w) =&\{(15, 24), (15, 34), (16, 24), (16, 34), (16, 35)\}.\end{aligned}$$

Since $I_3(w) = \{124, 156, 356\}$, it follows that the diameter of $G_{[w_{min}]}$ is equal to:

$$\begin{aligned}|\{(12, 34), (12, 35), (12, 36), (24, 56)\}| + |I_2^C(w)| + |\{(15, 24), (16, 24)\}| =&\\ = 4 + 5 + 2 = 11,&\end{aligned}$$

and the diameter of $G_{[w_{max}]}$ is equal to:

$$|I_2^S(w) \setminus \{12, 34\}| + |I_2^C(w) \setminus \{(15, 36)\}| + |\{(16, 35)\}| = 6 + 4 + 1 = 11.$$

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